



Extended Bell and Stirling Numbers From Hypergeometric Exponentiation

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Abstract

Exponentiating the hypergeometric series ${}_0F_L(1, 1, \dots, 1; z)$, $L = 0, 1, 2, \dots$, furnishes a recursion relation for the members of certain integer sequences $b_L(n)$, $n = 0, 1, 2, \dots$. For $L > 0$, the $b_L(n)$'s are generalizations of the conventional Bell numbers, $b_0(n)$. The corresponding associated Stirling numbers of the second kind are also investigated. For $L = 1$ one can give a combinatorial interpretation of the numbers $b_1(n)$ and of some Stirling numbers associated with them. We also consider the $L \geq 1$ analogues of Bell numbers for restricted partitions.

The conventional Bell numbers [1] $b_0(n)$, $n = 0, 1, 2, \dots$, have a well-known exponential generating function

$$B_0(z) \equiv e^{(e^z - 1)} = \sum_{n=0}^{\infty} b_0(n) \frac{z^n}{n!}, \quad (1)$$

which can be derived by interpreting $b_0(n)$ as the number of partitions of a set of n distinct elements. In this note we obtain recursion relations for related sequences of positive integers, called $b_L(n)$, $L = 0, 1, 2, \dots$,

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obtained by exponentiating the hypergeometric series ${}_0F_L(1, 1, \dots, 1; z)$ defined by [2]:

$${}_0F_L(\underbrace{1, 1, \dots, 1}_L; z) = \sum_{n=0}^{\infty} \frac{z^n}{(n!)^{L+1}}, \quad (2)$$

(which we shall denote by ${}_0F_L(z)$) and which includes the special cases ${}_0F_0(z) \equiv e^z$ and ${}_0F_1(z) \equiv I_0(2\sqrt{z})$, where $I_0(x)$ is the modified Bessel function of the first kind. For $L > 1$, the functions ${}_0F_L(z)$ are related to the so-called hyper-Bessel functions [3], [4], [5], which have recently found application in quantum mechanics [6], [7]. Thus we are interested in $b_L(n)$ given by

$$e^{[{}_0F_L(z)-1]} = \sum_{n=0}^{\infty} b_L(n) \frac{z^n}{(n!)^{L+1}}, \quad (3)$$

thereby defining a *hypergeometric* generating function for the numbers $b_L(n)$. From eq. (3) it follows formally that

$$b_L(n) = (n!)^L \cdot \frac{d^n}{dz^n} \left(e^{[{}_0F_L(z)-1]} \right) \Big|_{z=0}. \quad (4)$$

For $L = 0$ the r.h.s of eq. (4) can be evaluated in closed form:

$$b_0(n) = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!} = \left\{ \frac{1}{e^z} \left[\left(z \frac{d}{dz} \right)^n e^z \right] \right\}_{z=1}. \quad (5)$$

The first equality in (5) is the celebrated Dobinski formula [1], [8], [9]. The second equality in eq. (5) follows from observing that for a power series $R(z) = \sum_{k=0}^{\infty} A_k z^k$ we have

$$\left(z \frac{d}{dz} \right)^n R(z) = \sum_{k=0}^{\infty} A_k k^n z^k \quad (6)$$

and applying eq. (6) to the exponential series ($A_k = (k!)^{-1}$).

The reason for including the divisors $(n!)^{L+1}$ rather than $n!$ as in the usual exponential generating function arises from the fact that only by using eq. (3) are the numbers $b_L(n)$ actually integers. This can be seen from general formulas for exponentiation of a power series [8], which employ the (exponential) Bell polynomials, complicated and rather unwieldy objects. It cannot however be considered as a proof that the $b_L(n)$ are integers. At this stage we shall use eq. (3) with $b_L(n)$ real and apply to it an efficient method, described in [9], which will yield the recursion relation for the $b_L(n)$. (For the proof that the $b_L(n)$ are integers, see below eq. (11)). To this end we first obtain a result for the multiplication of two power-series of the type (3). Suppose we wish to multiply $f(x) = \sum_{n=0}^{\infty} a_L(n) \frac{x^n}{(n!)^{L+1}}$ and $g(x) = \sum_{n=0}^{\infty} c_L(n) \frac{x^n}{(n!)^{L+1}}$. We get $f(x) \cdot g(x) = \sum_{n=0}^{\infty} d_L(n) \frac{x^n}{(n!)^{L+1}}$, where

$$d_L(n) = (n!)^{L+1} \sum_{r+s=n}^{\infty} \frac{a_L(r) c_L(s)}{(r!)^{L+1} (s!)^{L+1}} = \sum_{r=0}^n \binom{n}{r}^{L+1} a_L(r) c_L(n-r). \quad (7)$$

Substitute eq. (2) into eq. (3) and take the logarithm of both sides of eq. (3):

$$\sum_{n=1}^{\infty} \frac{z^n}{(n!)^{L+1}} = \ln \left(\sum_{n=0}^{\infty} b_L(n) \frac{z^n}{(n!)^{L+1}} \right). \quad (8)$$

Now differentiate both sides of eq. (8) and multiply by z :

$$\left(\sum_{n=0}^{\infty} b_L(n) \frac{z^n}{(n!)^{L+1}} \right) \left(\sum_{n=0}^{\infty} n \frac{z^n}{(n!)^{L+1}} \right) = \sum_{n=0}^{\infty} n b_L(n) \frac{z^n}{(n!)^{L+1}}, \quad (9)$$

which with eq. (7) yields the desired recurrence relation

$$b_L(n+1) = \frac{1}{n+1} \sum_{k=0}^n \binom{n+1}{k}^{L+1} (n+1-k) b_L(k), \quad n = 0, 1, \dots \quad (10)$$

$$= \sum_{k=0}^n \binom{n}{k} \binom{n+1}{k}^L b_L(k), \quad (11)$$

$$b_L(0) = 1. \quad (12)$$

Since eq. (11) involves only positive integers, it follows that the $b_L(n)$ are indeed positive integers. For $L = 0$ one gets the known recurrence relation for the Bell numbers [9]:

$$b_0(n+1) = \sum_{k=0}^n \binom{n}{k} b_0(k). \quad (13)$$

We have used eq. (11) to calculate some of the $b_L(n)$'s, listed in Table I, for $L = 0, 1, \dots, 6$. Eq.(11), for n fixed, gives closed form expressions for the $b_L(n)$ directly as a function of L (columns in Table I): $b_L(2) = 1 + 2^L$, $b_L(3) = 1 + 3 \cdot 3^L + (3!)^L$, $b_L(4) = 1 + 4 \cdot 4^L + 3 \cdot 6^L + 6 \cdot 12^L + (4!)^L$, etc.

The sets of $b_L(n)$ have been checked against the most complete source of integer sequences available [10]. Apart from the case $L = 0$ (conventional Bell numbers) only the first non-trivial sequence $L = 1$ is listed:¹ it turns out that this sequence $b_1(n)$, listed under the heading A023998 in [10], can be given a combinatorial interpretation as the number of block permutations on a set of n objects which are uniform, i.e. corresponding blocks have the same size [12].

Eq.(1) can be generalized by including an additional variable x , which will result in ‘‘smearing out’’ the conventional Bell numbers $b_0(n)$ with a set of integers $S_0(n, k)$, such that for $k > n$, $S_0(n, k) = 0$, and $S_0(0, 0) = 1$, $S_0(n, 0) = 0$. In particular,

$$B_0(z, x) \equiv e^{x(e^z - 1)} = \sum_{n=0}^{\infty} \left[\sum_{k=1}^n S_0(n, k) x^k \right] \frac{z^n}{n!}, \quad (14)$$

which leads to the (exponential) generating function of $S_0(n, l)$, the conventional Stirling numbers of the second kind, (see [1], [8]), in the form

$$\frac{(e^z - 1)^l}{l!} = \sum_{n=l}^{\infty} \frac{S_0(n, l)}{n!} z^n, \quad (15)$$

and defines the so-called exponential or Touchard polynomials $l_n^{(0)}(x)$ as

$$l_n^{(0)}(x) = \sum_{k=1}^n S_0(n, k) x^k. \quad (16)$$

They satisfy

$$l_n^{(0)}(1) = b_0(n), \quad (17)$$

¹(others have since been added)

justifying the term “smearing out” used above.

The appearance of integers in eq. (3) suggests a natural extension with an additional variable x :

$$B_L(z, x) \equiv e^{x[{}_0F_L(z)-1]} = \sum_{n=0}^{\infty} \left[\sum_{k=1}^n S_L(n, k) x^k \right] \frac{z^n}{(n!)^{L+1}}, \quad (18)$$

where we include the right divisors $(n!)^{L+1}$ in the r.h.s of (18).

This in turn defines “hypergeometric” polynomials of type L and order n through

$$l_n^{(L)}(x) = \sum_{k=1}^n S_L(n, k) x^k, \quad (19)$$

which satisfy

$$l_n^{(L)}(1) = b_L(n), \quad (20)$$

with the $b_L(n)$ of eq. (10). Thus the polynomials of eq. (19) “smear out” the $b_L(n)$ with the generalized Stirling numbers of the second kind, of type L , denoted by $S_L(n, k)$ (with $S_L(n, k) = 0$, if $k > n$, $S_L(n, 0) = 0$ if $n > 0$ and $S_L(0, 0) = 1$), which have, from eq. (18) the “hypergeometric” generating function

$$\frac{({}_0F_L(z) - 1)^l}{l!} = \sum_{n=l}^{\infty} \frac{S_L(n, l)}{(n!)^{L+1}} z^n, \quad L = 0, 1, 2, \dots \quad (21)$$

Eq.(21) can be used to derive a recursion relation for the numbers $S_L(n, k)$, in the same manner as eq. (3) yielded eq. (12). Thus we take the logarithm of both sides of eq. (21), differentiate with respect to z , multiply by z and obtain:

$$\left(\sum_{n=0}^{\infty} \frac{S_L(n, l-1)}{(n!)^{L+1}} z^n \right) \left(\sum_{n=0}^{\infty} \frac{n}{(n!)^{L+1}} z^n \right) = \sum_{n=0}^{\infty} \frac{n S_L(n, l)}{(n!)^{L+1}} z^n, \quad (22)$$

which, with the help of eq. (7), produces the required recursion relation

$$S_L(n+1, l) = \sum_{k=l-1}^n \binom{n}{k} \binom{n+1}{k}^L S_L(k, l-1), \quad (23)$$

$$S_L(0, 0) = 1, \quad S_L(n, 0) = 0, \quad (24)$$

which for $L = 0$ is the recursion relation for the conventional Stirling numbers of the second kind [1], [8], and in eq. (23) the appropriate summation range has been inserted. Since the recursions of eq. (23) and eq. (24) involve only integers we conclude that $S_L(n, l)$ are positive integers.

We have calculated some of the numbers $S_L(n, l)$ using eq. (21) and have listed them in Tables II and III, for $L = 1$ and $L = 2$ respectively. Observe that $S_1(n, 2) = \binom{2n+1}{n+1} - 1$ and $S_L(n, n) = (n!)^L$, $L = 1, 2$. Also, by fixing n and l , the individual values of $S_L(n, l)$ have been calculated as a function of L with the help of eq. (23), see Table IV, from which we observe

$$S_L(n, n) = (n!)^L, \quad L = 1, 2, \dots \quad (25)$$

which is the lowest diagonal in Table IV. We now demonstrate that the repetitive use of eq. (23) permits one to establish closed-form expressions for any supra-diagonal of order p , i.e. the sequence $S_L(n+p, n)$,

for $p = 1, 2, 3, \dots$, if one knows the expression for all $S_L(n+k, n)$ with $k < p$. We shall illustrate it here for $p = 1, 2$. To this end fix $l = n$ on both sides of eq. (23). It becomes, upon using eq. (25), and defining $\alpha_L(n) \equiv S_L(n+1, n)$, a linear recursion relation

$$\alpha_L(n) = \frac{n[(n+1)!]^L}{2^L} + (n+1)^L \alpha_L(n-1), \quad \alpha_L(0) = 0, \quad (26)$$

with the solution

$$\alpha_L(n) = S_L(n+1, n) = \frac{n(n+1)}{2} \left[\frac{(n+1)!}{2} \right]^L \quad (27)$$

$$= \left[\frac{(n+1)!}{2} \right]^L S_0(n+1, n), \quad (28)$$

which gives the second lowest diagonal in Table IV. Observe that for any L , $S_L(n+1, n)$ is proportional to $S_0(n+1, n) = n(n+1)/2$. The sequence $S_1(n+1, n) = 1, 9, 72, 600, 5400, 8564480, \dots$ is of particular interest: it represents the sum of inversion numbers of all permutations on n letters [10]. For more information about this and related sequences see the entry A001809 in [10]. The $S_L(n+1, n)$ for $L > 1$ do not appear to have a simple combinatorial interpretation. A recurrence equation for $\beta_L(n) \equiv S_L(n+2, n)$ is obtained upon substituting eq. (25) and eq. (27) into eq. (23):

$$\beta_L(n) = \frac{n(n+1)}{2!} \left[\frac{(n+2)!}{2!} \right]^L \left(\frac{n-1}{2^L} + \frac{1}{3^L} \right) + (n+2)^L \beta_L(n-1), \quad \beta_L(0) = 0. \quad (29)$$

It has the solution

$$S_L(n+2, n) = \frac{n(n+1)(n+2)}{3 \cdot 2^3} \left[\frac{(n+2)!}{2} \right]^L \left(\frac{3}{2^L} (n-1) + \frac{4}{3^L} \right) \quad (30)$$

which is a closed form expression for the second lowest diagonal in Table IV. Clearly, eq. (30) for $L = 0$ gives the combinatorial form for the series of conventional Stirling numbers

$$S_0(n+2, n) = \frac{n(n+1)(n+2)(3n+1)}{4!}. \quad (31)$$

In a similar way we obtain

$$\begin{aligned} S_L(n+3, n) &= \frac{n(n+1)(n+2)(n+3)}{3 \cdot 2^4} \left[\frac{(n+3)!}{3} \right]^L \\ &\times \left(n^2 \left(\frac{3}{8} \right)^L + n \left(\frac{1}{4^{L-1}} - \frac{3^{L+1}}{8^L} \right) + \frac{2+2 \cdot 3^L}{8^L} - \frac{1}{4^{L-1}} \right) \end{aligned} \quad (32)$$

which for $L = 0$ reduces to

$$S_0(n+3, n) = \frac{1}{48} n^2 (n+1)^2 (n+2)(n+3). \quad (33)$$

Combined with the standard definition [8], [9]

$$S_0(n, l) = \frac{(-1)^l}{l!} \sum_{k=1}^l (-1)^k \binom{l}{k} k^n. \quad (34)$$

eqs.(28), (31) and (33) give compact expressions for the summation form of $S_0(n + p, n)$. Further, from eq. (34), use of eq. (6) gives the following generating formula

$$S_0(n, l) = \frac{(-1)^l}{l!} \left[\left(z \frac{d}{dz} \right)^n \left(\sum_{k=1}^l (-1)^k \binom{l}{k} z^k \right) \right]_{z=1} \quad (35)$$

$$= \frac{(-1)^l}{l!} \left[\left(z \frac{d}{dz} \right)^n [(1-z)^l - 1] \right]_{z=1}, \quad n \geq l. \quad (36)$$

The formula (1) can be generalized by putting restrictions on the type of resulting partitions. The generating function for the number of partitions of a set of n distinct elements without singleton blocks $b_0(1, n)$ is [8], [14], [15],

$$B_0(1, z) = e^{e^z - 1 - z} = \sum_{n=0}^{\infty} b_0(1, n) \frac{z^n}{n!}, \quad (37)$$

or more generally, without singleton, doubleton \dots , p -blocks ($p = 0, 1, \dots$) is [15]

$$B_0(p, z) = e^{e^z - \sum_{k=0}^p \frac{z^k}{k!}} = \sum_{n=0}^{\infty} b_0(p, n) \frac{z^n}{n!}, \quad (38)$$

with the corresponding associated Stirling numbers defined by analogy with eq. (14) and eq. (22). The numbers $b_0(1, n)$, $b_0(2, n)$, $b_0(3, n)$, $b_0(4, n)$ can be read off from the sequences A000296, A006505, A057837 and A057814 in [10], respectively. For more properties of these numbers see [11].

We carry over this type of extension to eq. (3) and define $b_L(p, n)$ through

$$B_L(p, z) \equiv e^{e^{F_L(z)} - \sum_{k=0}^p \frac{z^k}{(k!)^{L+1}}} = \sum_{n=0}^{\infty} b_L(p, n) \frac{z^n}{(n!)^{L+1}}, \quad (39)$$

where $b_L(0, n) = b_L(n)$ from eq. (3). (We know of no combinatorial meaning of $b_L(p, n)$ for $L \geq 1$, $p > 0$). The $b_L(p, n)$ satisfy the following recursion relations:

$$b_L(p, n) = \sum_{k=0}^{n-p} \binom{n}{k} \binom{n+1}{k}^L b_L(p, k), \quad (40)$$

$$b_L(p, 0) = 1, \quad (41)$$

$$b_L(p, 1) = b_L(p, 2) = \dots = b_L(p, p) = 0, \quad (42)$$

$$b_L(p, p+1) = 1. \quad (43)$$

That the $b_L(p, n)$ are integers follows from eq. (40). Through eq. (39) additional families of integer Stirling-like numbers $S_{L,p}(n, k)$ can be readily defined and investigated.

The numbers $b_0(p, n)$ are collected in Table V, and Tables VI and VII contain the lowest values of $b_1(p, n)$ and $b_2(p, n)$, respectively.

Formula (1) can be used to express e in terms of $b_0(n)$ in various ways. Two such lowest order (in differentiation) forms are

$$e = 1 + \ln \left(\sum_{n=0}^{\infty} \frac{b_0(n)}{n!} \right) = \quad (44)$$

$$= \ln \left(\sum_{n=0}^{\infty} \frac{b_0(n+1)}{n!} \right). \quad (45)$$

In the very same way, eq. (3) can be used to express the values of ${}_0F_L(z)$ and its derivatives at $z = 1$ in terms of certain series of $b_L(n)$'s. For $L = 1$, the analogues of eq. (44) and eq. (45) are

$$I_0(2) = 1 + \ln \left(\sum_{n=0}^{\infty} \frac{b_1(n)}{(n!)^2} \right), \quad (46)$$

$$I_0(2) + \ln(I_1(2)) = 1 + \ln \left(\sum_{n=0}^{\infty} \frac{b_1(n+1)}{(n+1)(n!)^2} \right) \quad (47)$$

and for $L = 2$ the corresponding formulas are

$${}_0F_2(1, 1; 1) = 1 + \ln \left(\sum_{n=0}^{\infty} \frac{b_2(n)}{(n!)^3} \right), \quad (48)$$

$${}_0F_2(1, 1; 1) + \ln({}_0F_2(2, 2; 1)) = 1 + \ln \left(\sum_{n=0}^{\infty} \frac{b_2(n+1)}{(n+1)^2(n!)^3} \right). \quad (49)$$

By fixing z_0 at values other than $z_0 = 1$, one can link the numerical values of certain combinations of ${}_0F_L(1, 1, \dots; z_0)$, ${}_0F_L(2, 2, \dots; z_0), \dots$ and their logarithms, with other series containing the $b_L(n)$'s.

The above considerations can be extended to the exponentiation of the more general hypergeometric functions of type ${}_0F_L(k_1, k_2, \dots, k_L; z)$ where k_1, k_2, \dots, k_L are positive integers. We conjecture that for every set of k_n 's a different set of integers will be generated through an appropriate adaptation of eq. (3). We quote one simple example of such a series. For

$${}_0F_2(1, 2; z) = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)(n!)^3} \quad (50)$$

eq. (3) extends to

$$e^{[{}_0F_2(1,2;z)-1]} = \sum_{n=0}^{\infty} f_2(n) \frac{z^n}{(n+1)(n!)^3} \quad (51)$$

where the numbers

$$f_2(n) = (n+1)(n!)^2 \left[\frac{d^n}{dz^n} e^{[{}_0F_2(1,2;z)-1]} \right]_{z=0} \quad (52)$$

turn out to be integers: $f_2(n)$, $n = 0, 1, \dots, 8$ are: 1, 1, 4, 37, 641, 18276, 789377, 48681011, etc. (A061683). The analogue of equations (23) and (44) is:

$${}_0F_2(1, 2; 1) = 1 + \ln \left(\sum_{n=0}^{\infty} \frac{f_2(n)}{(n+1)(n!)^3} \right). \quad (53)$$

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Table I: Table of $b_L(n)$: $L, n = 0, 1, \dots, 6$. (The rows give sequences A000110, A023998, A061684–A061688.)

L	$b_L(0)$	$b_L(1)$	$b_L(2)$	$b_L(3)$	$b_L(4)$	$b_L(5)$	$b_L(6)$
0	1	1	2	5	15	52	203
1	1	1	3	16	131	1 496	22 482
2	1	1	5	64	1 613	69 026	4 566 992
3	1	1	9	298	25 097	4 383 626	1 394 519 922
4	1	1	17	1 540	461 105	350 813 126	573 843 627 152
5	1	1	33	8 506	9 483 041	33 056 715 626	293 327 384 637 282
6	1	1	65	48 844	209 175 233	3 464 129 078 126	173 566 857 025 139 312

Table II: Table of $S_L(n, l)$: for $L = 1$ and $l, n = 1, 2, \dots, 8$. (The triangle, read by columns, gives A061691, the rows and diagonals give A017063, A061690, A000142, A001809, A061689.)

l	$S_1(1, l)$	$S_1(2, l)$	$S_1(3, l)$	$S_1(4, l)$	$S_1(5, l)$	$S_1(6, l)$	$S_1(7, l)$	$S_1(8, l)$
1	1	1	1	1	1	1	1	1
2		2	9	34	125	461	1 715	6 434
3			6	72	650	5 400	43 757	353 192
4				24	600	10 500	161 700	2 361 016
5					120	5 400	161 700	4 116 000
6						720	52 920	2 493 120
7							5 040	564 480
8								40 320

Table III: Table of $S_L(n, l)$: for $L = 2$ and $l, n = 1, 2, \dots, 8$. (The triangle, read by columns, gives A061692, the rows and diagonals give A061693, A061694, A001044, A061695.)

l	$S_2(1, l)$	$S_2(2, l)$	$S_2(3, l)$	$S_2(4, l)$	$S_2(5, l)$	$S_2(6, l)$	$S_2(7, l)$	$S_2(8, l)$
1	1	1	1	1	1	1	1	1
2		4	27	172	1 125	7 591	52 479	369 580
3			36	864	17 500	351 000	7 197 169	151 633 440
4				576	36 000	1 746 000	80 262 000	3 691 514 176
5					14 400	1 944 000	191 394 000	17 188 416 000
6						518 400	133 358 400	23 866 214 400
7							25 401 600	11 379 916 800
8								1 625 702 400

Table IV: Table of $S_L(n, l)$: $l, n = 1, 2, \dots, 6$.

l	$S_L(1, l)$	$S_L(2, l)$	$S_L(3, l)$	$S_L(4, l)$	$S_L(5, l)$	$S_L(6, l)$
1	1	1	1	1	1	1
2		$(2!)^L$	$3 \cdot 3^L$	$4 \cdot 4^L + 3 \cdot 6^L$	$5 \cdot 5^L + 10 \cdot 10^L$	$6 \cdot 6^L + 15 \cdot 15^L + 10 \cdot 20^L$
3			$(3!)^L$	$6 \cdot 12^L$	$10 \cdot 20^L + 15 \cdot 30^L$	$15 \cdot 30^L + 60 \cdot 60^L + 15 \cdot 90^L$
4				$(4!)^L$	$10 \cdot 60^L$	$20 \cdot 120^L + 45 \cdot 180^L$
5					$(5!)^L$	$15 \cdot 360^L$
6						$(6!)^L$

Table V: Table of $b_0(p, n)$: $p = 0, 1, 2, 3$; $n = 0, \dots, 10$. (The columns give A000110, A000296, A006505, A057837.)

n	$b_0(0, n)$	$b_0(1, n)$	$b_0(2, n)$	$b_0(3, n)$
0	1	1	1	1
1	1	0	0	0
2	2	1	0	0
3	5	1	1	0
4	15	4	1	1
5	52	11	1	1
6	203	41	11	1
7	877	162	36	1
8	4 140	715	92	36
9	21 147	3 425	491	127
10	115 975	17 722	2 557	337

Table VI: Table of $b_1(p, n)$: $p = 0, 1, 2$; $n = 0, \dots, 9$. (The columns give A023998, A061696, A061697.)

n	$b_1(0, n)$	$b_1(1, n)$	$b_1(2, n)$
0	1	1	1
1	1	0	0
2	3	1	0
3	16	1	1
4	131	19	1
5	1 496	101	1
6	22 482	1 776	201
7	426 833	23 717	1 226
8	9 934 563	515 971	5 587
9	277 006 192	11 893 597	493 333

Table VII: Table of $b_2(p, n)$: $p = 0, 1, 2$; $n = 0, \dots, 8$. (The columns give A061698–A061700.)

n	$b_2(0, n)$	$b_2(1, n)$	$b_2(2, n)$
0	1	1	1
1	1	0	0
2	5	1	0
3	64	1	1
4	1 613	109	1
5	69 026	1 001	1
6	4 566 992	128 876	4 001
7	437 665 649	4 682 637	42 876
8	57 903 766 800	792 013 069	347 117

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