

# On lensing by a cosmological constant

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Several recent papers have suggested that the cosmological constant  $\Lambda$  directly influences the gravitational deflection of light. We place this problem in a cosmological context, deriving an expression for the linear potentials which control the cosmological bending of light, finding that it has no explicit dependence on the cosmological constant. To explore the physical origins of the apparent  $\Lambda$ -dependent potential that appears in the static Kottler metric, we highlight the two classical effects which lead to the aberration of light. The first relates to the observer's motion relative to the source, and encapsulates the familiar concept of angular-diameter distance. The second term, which has proved to be the source of debate, arises from cosmic acceleration, but is rarely considered since it vanishes for photons with radial motion. This apparent form of light-bending gives the appearance of curved geodesics even within a flat and homogeneous universe. However this cannot be construed as a real lensing effect, since its value depends on the observer's frame of reference. Our conclusion is thus that standard results for gravitational lensing in a universe containing  $\Lambda$  do not require modification, with any influence of  $\Lambda$  being restricted to negligible high-order terms.

## I. INTRODUCTION

Conventional wisdom (e.g. [1, 2]) states that the cosmological constant plays no direct role in gravitational lensing, other than the inevitable modification to the angular diameter distance. This is reinforced by the intuition that lensing is sourced by inhomogeneities in the density field, whereas the cosmological constant is wholly uniform.

This position was challenged by Rindler & Ishak [3], who presented a term associating the cosmological constant with a diminished bending angle for a photon. This was followed by a further two papers [4, 5] analysing this phenomenon in greater detail. Indeed the former claims to place observational constraints on the value of  $\Lambda$  based on applying this result to strong lensing by clusters, in a 'Swiss-cheese' model, where the matter in a spherical vacuole collapses to the centre to form the lensing object. Although the effects are relatively small, they are certainly large enough to be important in next-generation applications of lensing as a tool for precision cosmology. However, opinion seems divided as to whether the Ishak-Rindler analysis is correct: Park [6] and Khriplovich & Pomeransky [7] have shed doubt on these calculations, although Schücker [8, 9, 10], and Lake [11] are in agreement. Work by Gibbons et al. [12] explore the properties of the Kottler optical metric, while Sereno [13, 14] revealed a different term contributing to the deflection angle.

In this work we aim to clarify the source of these discrepancies and to investigate the bending of light in an expanding Universe. In §II, we translate the metric inside a vacuole from the static Kottler [15] form to a perturbed Friedmann-Robertson-Walker (FRW) metric. We

do not exclude a contribution of  $\Lambda$  to the lensing equations at some level, but show that the linear potential is unaffected by  $\Lambda$ , with the apparent  $\Lambda r^2/3$  contribution appearing as a consequence of the choice of a static metric. We verify this with numerical solutions in §III.

The remainder of this work aims to clarify the physical interpretation of the apparent light bending. We revisit the analysis of Ishak [5] in §IV, before extending this to evaluate the photon's deflection angle from different perspectives within the Kottler metric. The source of the extra term is revealed in §V, and its relation to the angular-diameter distance is outlined in §VI. Final discussions are presented in §VIII.

## II. VACUOLE MODEL IN THE NEWTONIAN GAUGE

We now consider the Ishak-Rindler vacuole from the point of view of the standard approach to cosmological perturbations, as described by e.g. Dodelson [16] or Mukhanov [17]. Our goal is to find an explicit linear expression for the perturbing potentials responsible for the cosmological bending of light within the vacuole model. In order to avoid coordinate-dependent artefacts, one looks for gauge-independent measures of inhomogeneity; in practice, this is achieved by working in the Newtonian gauge. Scalar metric fluctuations are then described by scalar potentials,  $\Phi$  and  $\Psi$ , which act to modify the Robertson-Walker metric:

$$ds^2 = (1 + 2\Phi) dt^2 - a^2(t) (1 - 2\Psi) (d\chi^2 + \chi^2 d\psi^2). \quad (1)$$

We take  $c = G = 1$  throughout.  $\chi$  is comoving radius, and  $d\psi$  is an element of angle on the sky. We also restrict attention to the case of a flat universe, and no anisotropic stresses, so that  $\Psi = \Phi$ . We will always be interested in the case where the fluctuations causing lensing are well within the horizon, in which case the potential  $\Phi$  obeys

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the Poisson equation, sourced by the fractional matter fluctuation  $\delta_m$ . In this apparatus, a homogeneous density from  $\Lambda$  appears only implicitly, through its contribution to the scale factor  $a(t)$ . Conventionally, light deflection would be computed by integrating twice the component of  $-\nabla\Phi$  perpendicular to the line of sight, and the conclusion would be that  $\Lambda$  has no direct lensing effect. Clearly this is true in a homogeneous universe that contains  $\Lambda$ , since the FRW metric defines the path of unperturbed light rays. Indeed, no true lensing can arise from a homogeneous background: the photon would require a preferential direction in which to bend – and doing so would break the symmetry of the cosmology.

How does the perturbed FRW metric compare with the exact Kottler metric inside the vacuole? The comparison can only be made if we understand the relation between the coordinates used in the two forms. The key to doing this is the transverse part of the metric, which would be  $-r^2 d\psi^2$  in the Kottler form:

$$ds^2 = f(r) dT^2 - f(r)^{-1} dr^2 - r^2 d\psi^2, \quad (2)$$

for some time coordinate  $T$ , and where

$$f(r) = 1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}. \quad (3)$$

This means that the proper radius in the perturbed FRW form is, to first order in  $\Phi$ ,

$$r \equiv (1 - \Phi)a(t)\chi, \quad (4)$$

and the metric is

$$ds^2 = (1 + 2\Phi) dt^2 - a^2(t) (1 - 2\Phi) d\chi^2 - r^2 d\psi^2. \quad (5)$$

In order to eliminate  $d\chi$ , we must differentiate the definition of  $r$ . To first order in  $\Phi$ , this gives

$$a^2(1 - 2\Phi)d\chi^2 = [(1 + \chi\Phi') dr - Y dt]^2, \quad (6)$$

where  $Y \equiv \dot{a}\chi(1 - \Phi) - a\chi\dot{\Phi} + \dot{a}\chi^2\Phi'$ , where we have defined  $\Phi' \equiv d\Phi/d\chi$  and  $\dot{\Phi} \equiv d\Phi/dt$ . Note that we always work to first order in the perturbation, so that e.g.  $(1 - \chi\Phi')^{-1}$  can be replaced by  $(1 + \chi\Phi')$ . Note also that differentiating the definition of  $r$  introduces  $\dot{a}$ :  $\Lambda$  enters at this point, since it is related to  $\dot{a}$  via the Friedmann equation.

In order to eliminate the  $dr dt$  cross term, we define the time coordinate  $dT = A dt + B dr$ , and solve for  $A$  and  $B$  by requiring that the metric be written in the desired Kottler form,

$$ds^2 = f(r) dT^2 - f(r)^{-1} dr^2 - r^2 d\psi^2. \quad (7)$$

Solving this problem as written gives

$$f = \frac{1 + 2\Phi - Y^2}{(1 + \chi\Phi')^2(1 + 2\Phi)}. \quad (8)$$

To first order in  $\Phi$ , our expression for  $f$  is

$$\begin{aligned} f &= 1 - 2\chi\Phi' - \dot{a}^2\chi^2(1 - 4\Phi - 2\chi\Phi') \\ &\quad + 2\dot{a}a\chi^2\dot{\Phi} - 2\dot{a}^2\chi^3\Phi' \\ &= 1 - 2\chi\Phi' - \dot{a}^2\chi^2 \left[ 1 - 2\Phi \left( \frac{\partial \ln |\Phi|}{\partial \ln a} + 2 \right) \right]. \end{aligned} \quad (9)$$

We proceed initially by ignoring the terms proportional to  $\Phi$  in the square brackets, in comparison to unity. This is justified because the vacuole should be small compared with the Hubble length:  $\dot{a}\chi \ll 1$ . Thus,  $\dot{a}^2\chi^2\Phi$  will be negligible in comparison with  $\chi\Phi' = \Phi(\partial \ln \Phi / \partial \ln \chi)$ . We show below that this assumption yields a consistent solution for  $\Phi$ , with the neglected term shown to be second order.

In practice, therefore, the perturbed FRW metric reduces to the Kottler form with

$$f = 1 - 2\chi\Phi' - \dot{a}^2\chi^2. \quad (10)$$

The Kottler metric is expressed in terms of  $r = (1 - \Phi)a(t)\chi$ , but we have already treated  $\dot{a}^2\chi^2\Phi$  as negligible in deriving this expression for  $f$ , so the same level of approximation allows us to set  $r = a(t)\chi$  here:

$$f = 1 - 2r\Phi'/a - (\dot{a}^2/a^2)r^2. \quad (11)$$

The Friedmann equation says that

$$\dot{a}^2/a^2 = \Lambda/3 + 2m/R^3, \quad (12)$$

where  $R = a(t)R_v$  is the proper radius of the vacuole of fixed comoving radius  $R_v$ . Using this and the Kottler metric yields

$$f = 1 - 2r\Phi'/a - 2mr^2/R^3 - \Lambda r^2/3. \quad (13)$$

Note that the Friedmann equation has yielded a term  $-\Lambda r^2/3$ , which will cancel the corresponding term in the Kottler expression for  $f(r)$ .

Recalling that  $\Phi'$  denotes the derivative of  $\Phi$  with respect to comoving radius, we solve this using the Kottler form for  $f(r)$ , equation (3), which requires

$$\Phi' = a \left( \frac{m}{r^2} - \frac{mr}{R^3} \right) = \frac{m}{a\chi^2} - \frac{m\chi}{aR_v^3}, \quad (14)$$

where again the error in writing  $r = a(t)\chi$  is of second order in  $\Phi$ . The solution is

$$\Phi = -\frac{m}{a\chi} - \frac{m\chi^2}{2aR_v^3} + \frac{3m}{2aR_v} = -\frac{m}{r} - \frac{mr^2}{2R^3} + \frac{3m}{2R}, \quad (15)$$

where the additive constant is determined by requiring  $\Phi = 0$  at the boundary of the vacuole. This expression for  $\Phi$  agrees with what one would expect from a simple Newtonian calculation with a point mass and a spherical vacuole underdensity. We can dispose of the technical issue that for a point mass,  $\Phi \rightarrow -\infty$  as  $\chi \rightarrow 0$ , by considering a spherical mass of finite radius, and appealing

to Birkhoff's theorem so that our solution for  $\Phi$  applies in the vacuole outside the mass.

To verify that we have a consistent linear solution for  $\Phi$ , we note that  $\partial \ln |\Phi| / \partial \ln a = -1$ , and substitution into (9), retaining the full relation (4) to linear order in  $\Phi$ , gives a complete cancellation of the  $\Lambda$  terms:

$$\Phi' = \frac{m}{a\chi^2} - \frac{m\chi}{aR_v^3} + \Phi \left( \frac{m}{a\chi^2} + \frac{2m\chi}{aR_v^3} \right). \quad (16)$$

The last term on the right, which we neglected, is indeed seen to be second-order in  $\Phi$ .

Having now described the vacuole metric as a perturbed FRW metric, we find no evidence for  $\Lambda$ -dependence in the linear peculiar gravitational potential, and the standard cosmological lensing results follow. The potential-like term  $\Lambda r^2/3$  in the Kottler  $f(r)$  arises simply by virtue of the introduction of  $\dot{a}$  in the coordinate transformation from the FRW form, plus the fact that  $\dot{a}$  is related to  $\Lambda$  through the Friedmann equation. But this term does not arise from the true potential  $\Phi$ , and thus it should not be taken to cause lensing. From this point of view, it seem fair to assert that the appearance of a lensing effect from  $\Lambda r^2$  in the Ishak–Rindler analysis is purely a gauge artefact.

Our expression for the potential  $\Phi$  is correct only to lowest order, and we have neglected corrections of order  $(H^2 r^2)\Phi$ . Nevertheless, it is clear that the disputed  $\Lambda r^2/3$  term is of an altogether larger magnitude. In most of the volume of the vacuole,  $r \sim R$  and  $\Phi \sim m/R$ . The ratio between the disputed term and  $\Phi$  is then  $\sim \Lambda R^3/m$ , which is of order the ratio between the vacuum and matter densities – i.e. an order unity correction at the present epoch. While  $\Lambda$  may appear in higher order corrections to  $\Phi$ , it is clear that such corrections cannot involve a  $\Lambda r^2/3$  term in the potential.

### III. NUMERICAL SOLUTIONS

We now derive expressions for the linear peculiar gravitational potential for a spherical vacuole in a universe

containing arbitrary densities of non-relativistic matter and a cosmological constant, in which the mass  $m$  in the vacuole is concentrated at the centre. Otherwise, the universe is assumed to be homogeneous and isotropic.

The perturbed FRW metric is written in the Newtonian gauge as

$$ds^2 = (1 + 2\Phi)dt^2 - a^2(t)(1 - 2\Phi)[d\chi^2 + S_k^2(\chi)d\psi^2] \quad (17)$$

where  $S_k(\chi) = R_0 \sin(\chi/R_0)$ ,  $\chi$ ,  $R_0 \sinh(\chi/R_0)$  for  $k = 1, 0, -1$ , and  $R_0 = (1/H_0)\sqrt{k/(\Omega - 1)}$ .

We wish to find  $\Phi(\chi, a)$  by coordinate transformation from the known Kottler metric (3) in terms of static coordinates  $r$  and time coordinate  $T$ . Equating the coefficients of  $d\psi^2$  gives (working always only to linear order in  $\Phi$ )

$$r(\chi, a) = a(t)(1 - \Phi)S_k(\chi). \quad (18)$$

We also have  $T = T(\chi, a)$  and hence  $dT = T_t dt + T_\chi d\chi$  where  $T_t = \partial T / \partial t$  etc. Substitution into the Kottler metric (3), equating coefficients of  $dt^2$ ,  $d\chi^2$  and  $dt d\chi$ , eliminating  $T_t$  and  $T_\chi$  gives

$$r_\chi^2(1 + 2\Phi) - a^2(1 - 2\Phi)r_t^2 - a^2 f(r) = 0. \quad (19)$$

Since

$$\begin{aligned} r_\chi &= a(1 - \Phi)C_k - a\Phi_\chi S_k \\ r_t &= \dot{a}(1 - \Phi)S_k - a\Phi_t S_k, \end{aligned} \quad (20)$$

where  $C_k(\chi) \equiv \cos(\chi/R_0)$ ,  $1$ ,  $\cosh(\chi/R_0)$  for  $k = 1, 0, -1$ , we obtain (to  $O(\Phi)$ ),

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$$1 - \frac{2m}{aS_k}(1 + \Phi) - \frac{a^2\Lambda S_k^2}{3}(1 - 2\Phi) = C_k^2 - 2\Phi_\chi S_k C_k - \dot{a}^2 S_k^2(1 - 4\Phi) + 2\dot{a}^2 a S_k^2 \Phi_a. \quad (21)$$

Since  $1 - C_k^2 = kS_\chi^2/R_0^2$ , we have the equation for  $\Phi(\chi, a)$ :

$$\Phi_\chi = \frac{S_k}{C_k} \left[ \frac{-k}{2R_0^2} - \frac{\dot{a}^2}{2}(1 - 4\Phi - 2a\Phi_a) + \frac{m}{aS_k^3}(1 + \Phi) + \frac{\Lambda a^2}{6}(1 - 2\Phi) \right]. \quad (22)$$

Substituting Friedmann's equation  $\dot{a}^2/a^2 = 8\pi\rho_m/3 + \Lambda/3 - k/(a^2 R_0^2)$  gives

$$\Phi_\chi = \frac{S_k}{C_k} \left[ -\frac{4\pi m}{3V_b a}(1 - 4\Phi - 2a\Phi_a) + \frac{m}{aS_k^3}(1 + \Phi) + \left( \frac{\Lambda a^2}{3} - \frac{k}{R_0^2} \right) (\Phi + a\Phi_a) \right], \quad (23)$$

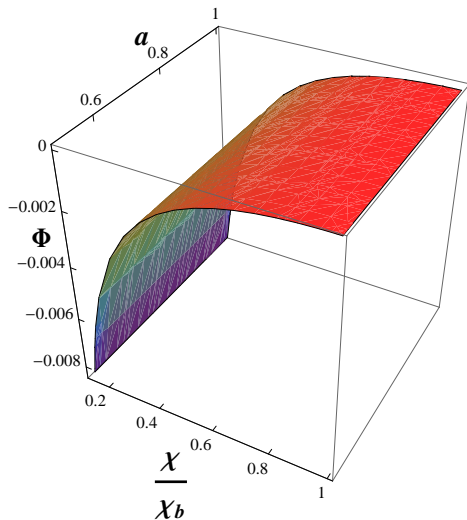


FIG. 1: The potential  $a\Phi(\chi/\chi_b, a)$  of an open universe, with  $\Omega_m = 0.27, \Omega_\Lambda = 0$ . Here we have taken the vacuole size to be  $\chi_b = 0.1$ .

where the comoving volume of the vacuole in terms of its boundary coordinate  $\chi_b$  is given by

$$V_b = \pi R_0^3 \left[ \sinh\left(\frac{2\chi_b}{R_0}\right) - \frac{2\chi_b}{R_0} \right], \quad (24)$$

resulting in a vacuole of mass

$$m = \frac{3V_b\Omega_m h^2}{8\pi}. \quad (25)$$

Note that when modifying the cosmological parameters such that  $R_0$  changes, we iteratively adjust the vacuole radius  $x_b$  in order to maintain a constant enclosed mass  $m$ .

Figure 1 shows Mathematica solutions for  $a\Phi$  against  $\chi/\chi_b$  (plotted from 0.1 to 1) and  $a$  (plotted from 0.5 to 1) for  $\Omega_m = 0.27$  and  $\Omega_\Lambda = 0$ , with  $\chi_b = 0.1cH_0^{-1}$ . The leading-order term is given by

$$\Phi(r) \simeq -\frac{m}{r}. \quad (26)$$

When introducing a contribution from the cosmological constant, Ishak & Rindler predict a change in the potential given by a new term

$$\begin{aligned} \Phi_\Lambda(r) &= -\frac{\Lambda r^2}{12} \\ &= -\Omega_\Lambda h^2 \frac{(r - r_b)^2}{4}, \end{aligned} \quad (27)$$

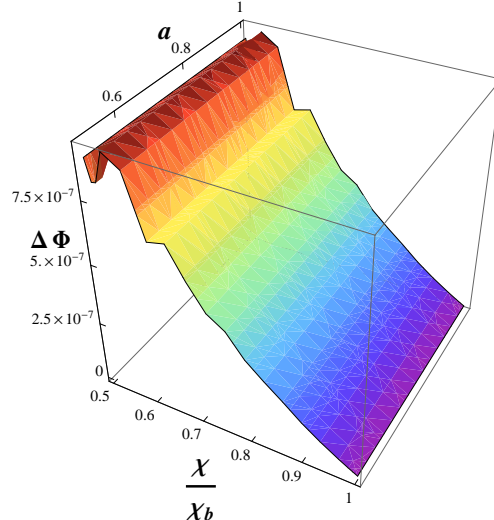


FIG. 2: The difference between the matter-only potential presented in Figure 1, and that of a cosmology with both matter and a cosmological constant:  $\Omega_m = 0.27, \Omega_\Lambda = 0.27$ .

where the  $r_b$  term arises due to the boundary condition imposed at the edge of the vacuole. For a cosmology with  $\Omega_\Lambda = 0.27$  this leads us to predict  $\Phi_\Lambda(r_b/2) = -1.7 \times 10^{-4}$ , where we have adopted  $r = r_b/2$  as the benchmark value. However the signal in Figure 2 is more than two orders of magnitude smaller than this, an amplitude consistent with the  $O(\Phi^2)$  terms we discarded earlier.

In the following sections we will explore why the Ishak-Rindler term does not manifest itself in the potential.

#### IV. DEFLECTION IN THE STATIC METRIC

Now let us see how the above section can be made consistent with the Ishak and Rindler computation of deflection within a static Kottler metric.

We begin by reassessing the treatment outlined by Ishak [5], which relies on superposing two metrics, and applying classical Newtonian dynamics, a valid approximation provided we restrict ourselves to the weak field regime. For a flat universe consisting of non-relativistic matter and a cosmological constant, its evolution may be well described by an appropriate choice of potential. This was applied by Ishak [5] to evaluate the deflection angle of a photon by considering the gradient of the Newtonian potential.

$$\alpha = \int \nabla_\perp(\Phi + \Psi) dx, \quad (28)$$

where we integrate along the path of the photon. The potentials  $\Phi$  and  $\Psi$  are extracted from the space and

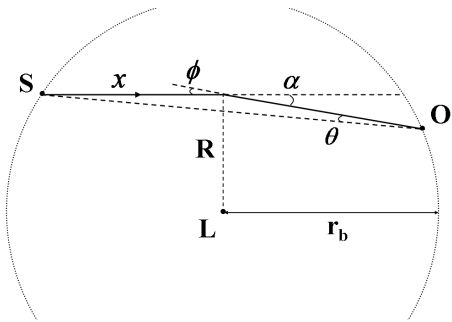


FIG. 3: The weak gravitational lensing of light by a vacuole, where all matter within a sphere of radius of radius  $r_b$  is collected to a central mass. The observer is located at  $O$ .

time components of the metric, and differ from the FRW potentials in equation (1).

Ishak superposes the Schwarzschild and de-Sitter metrics in static coordinates,

$$ds^2 = (1 + 2\Phi) dt^2 - (1 - 2\Psi) (dr^2 + r^2 d\Omega^2) \quad (29)$$

leading to the potentials (see eg [5, 18]).

$$\begin{aligned} \Phi &\simeq -\frac{m}{r} - \frac{\Lambda r^2}{6}, \\ \Psi &\simeq -\frac{m}{r} + \frac{\Lambda r^2}{12}. \end{aligned} \quad (30)$$

The mass terms are only an approximation, but are accurate to first order provided  $m/R \ll 1$ . It is the terms involving the cosmological constant to which we will pay the greater attention.

In the frame of the lens, the gradients of the potentials are

$$\begin{aligned} \nabla_{\perp} \Phi &= \frac{mR}{r^3} - \frac{\Lambda R}{3}, \\ \nabla_{\perp} \Psi &= \frac{mR}{r^3} + \frac{\Lambda R}{6}. \end{aligned} \quad (31)$$

Therefore a photon passing within a distance  $R$  becomes deflected by

$$\begin{aligned} \alpha &= \int_{-x_b}^{x_b} \nabla_{\perp} (\Phi + \Psi) dx \\ &= \left( \frac{4M}{R} - \frac{\Lambda R r_b}{3} \right) \sqrt{1 - R^2/r_b^2} \end{aligned} \quad (32)$$

where  $r = \sqrt{R^2 + x^2}$ , and  $r_b$  is the radius of the vacuole. This result corrects for the discontinuity which arises at the boundary  $R = r_b$  from Ishak [5].

From (32), the first term within the parentheses is readily recognisable as the conventional result for gravitational lensing. However it is the second term which is of greater interest. Why does the cosmological constant now appear to reduce the value of  $\alpha$ ? Part of the reason

is that, unlike the mass  $m$ , the cosmological constant has a potential which appears centred on whichever frame we choose. If the potential associated with the cosmological constant is now centred on the observer at  $O$ , then for the limit of a very weak lens the photon's motion is purely radial, with no component perpendicular to the potential's gradient, and thus the integral in (32) trivially vanishes. Yet we have not fully resolved the anomaly, since for non-negligible deflection angles, the photon's path *does* have a transverse component before reaching the lens, as illustrated in Figure 3. Note however that the observable,  $\theta$ , remains constant.

If the conventional deflection angle induced by the matter in the lens,  $\alpha_m$ , is small, then our modified deflection angle requires the inclusion of an extra term given by

$$\begin{aligned} \alpha_{\Lambda} &= \int_0^{r_S} \nabla_{\perp} (\Phi + \Psi) dx \\ &= \int_0^{r_S} \nabla_{\perp} \left( -\frac{\Lambda r^2}{6} + \frac{\Lambda r^2}{12} \right) dx \\ &= \int_{r_L}^{r_S} -\frac{\Lambda}{6} r \sin \phi dx \\ &= \int_{r_L}^{r_S} -\frac{\Lambda}{6} r_L \sin \alpha_m dx \\ &\simeq -\frac{\Lambda}{6} \sin \alpha_m r_L (r_S - r_L), \end{aligned} \quad (33)$$

where  $r_L$  and  $r_S$  are the distances to the lens and source respectively. The angle between the observer's line of sight and the path of the photon is denoted by  $\phi$ , and we have used  $r \sin \phi = r_L \sin \alpha_m$ .

To clarify,  $\alpha_{\Lambda}$  corresponds to the extra angle through which the photon appears to be deflected when tracking its motion in a physical coordinate system in the frame of  $O$ . The local deflection  $\alpha_m$  occurs at the closest approach to  $L$ , while the extra  $\Lambda$  contribution is a cumulative effect. The  $\Lambda$ -dependent terms given in (32) and (33) appear particularly problematic since they do not vanish when the lensing mass is taken to be zero. Ordinarily the local deflection  $\sin \alpha_m$  would vanish, yet it could equally have been a mirror we used to deflect the photon, maintaining a non-zero  $\alpha_m$  in an otherwise pure de Sitter universe.

How can we reconcile this modified deflection angle with the concept that light travels in straight lines within FRW models? The difference is highlighted in Figure 4 where we see the influence of our coordinate system. The lines plotted in Figure 4 have a physical interpretation as those which would be marked on a sheet of graph paper centred on the observer. Aside from the deflection, the photon follows a Euclidean trajectory in comoving space, since the geometry is conformally flat. However when we map this trajectory onto a coordinate system with proper distances – that measured by a ruler – the transverse motion of a photon appears bent by the acceleration of the cosmology, and the deflection angle induced

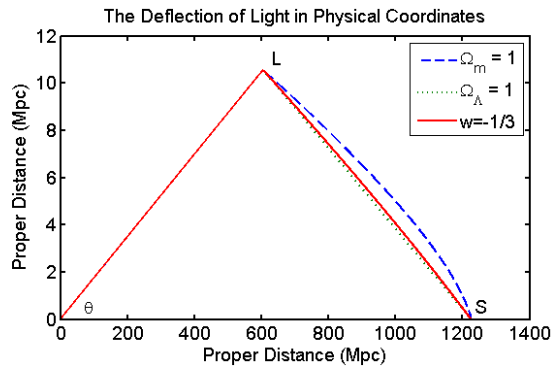


FIG. 4: A comparison of a photon's path in different flat cosmologies, using the physical distances from the perspective of the observer at the origin. The lens  $L$  and source  $S$  appears at the same angular-diameter distance in all cases, and the true deflection angle is taken to be  $\alpha = 1^\circ$ . The lines correspond to de Sitter ( $\Omega_\Lambda = 1$ , dotted), Einstein-de Sitter ( $\Omega_m = 1$ , dot-dash), and the neutral  $w = -1/3$  cosmology ( $\Omega_m = 2/3$ ,  $\Omega_\Lambda = 1/3$ , solid).

at  $L$  appears to change. Yet the observable angle  $\theta$  remains constant, as does the deflection angle for a set of comoving observers, since we have fixed the angular diameter distances. Note that in de Sitter space, the same distance corresponds to a substantially lower redshift, so the magnitude of the apparent bending is reduced.

Now we quantify the relation between the angle in a comoving coordinate system, and that observed in terms of proper distances. Consider a photon at a comoving coordinate  $(x, y)$  travelling at a small angle  $\alpha$  with respect to the radial direction (which we take to be the  $x$ -axis, so  $y = 0$ ). The proper distance of the photon, and its derivative with respect to the scale factor, is given by

$$\begin{aligned} x_p &= ax \\ x'_p &= x + ax' \end{aligned} \quad (34)$$

and similarly for  $y$ . The angle of the trajectory in terms of comoving and proper distances is given by

$$\begin{aligned} \tan \alpha &= \frac{y'}{x'}, \\ \tan \alpha_p &= \frac{y + ay'}{x + ax'}. \end{aligned} \quad (35)$$

To first order in  $x/ax'$  this leads to

$$\begin{aligned} \tan \alpha_p &= \frac{y'}{x'} - \frac{y'x}{ax'^2} \\ &= \left(1 - \frac{x}{ax'}\right) \tan \alpha. \end{aligned} \quad (36)$$

Note that for  $\alpha = 0$  then  $\alpha_p = 0$ , and as we expect, radial trajectories remain radial in terms of proper distances. This may be simplified to leave

$$\tan \alpha_p = \left(1 - r \frac{da}{dt}\right) \tan \alpha, \quad (37)$$

For a homogeneous cosmology with equation of state  $w$  then this may be expressed as

$$\tan \alpha_p = \left(1 - rH_0 a^{-\frac{1+3w}{2}}\right) \tan \alpha, \quad (38)$$

highlighting the significance of  $w = -1/3$ , which corresponds to a cosmology with zero acceleration.

The photon trajectories plotted in Figure 4 appear distorted as a result of the aforementioned coordinate transformation. However we stress that any observable such as  $\theta$  will remain unaltered, since only distant motion with some transverse component may be affected. The deflection angle is modified, as given by (33), yet remains unobservable since we cannot measure the angle of emission. We consider the physical interpretation of these results in the following section.

## V. ABERRATION AND THE ORIGIN OF THE ISHAK-RINDLER TERM

To illustrate the meaning of this bent trajectory, consider a photon reflecting off the interior walls of a box of proper size  $\ell$  within a de Sitter background. In the frame of the box the photon simply bounces back and forth, only subject to a small degree of blue- and red-shifting depending on which side of the box we sit. Yet for an observer located at a distance  $y$  transverse to this motion, as illustrated in Figure V, both the box and photon must appear to be accelerated. This can be thought of in terms of an angular aberration, which to first order in  $v$  and  $\theta$  is given by

$$\theta' = \theta(1 + v_x) + v_y, \quad (39)$$

where  $v_x$  and  $v_y$  denotes the vertical and horizontal velocities of the frame of  $S'$  with respect to  $S$ , and the photon is travelling in the positive  $x$  direction.

The proper distance  $y$  is simply related to the comoving coordinate  $\chi$  by

$$y(t) = \chi a(t). \quad (40)$$

While the value of  $\chi$  between two points remains fixed, our distance  $y$  changes at a rate governed by the Friedmann equations.

$$\begin{aligned} \ddot{y} &= \chi \ddot{a} \\ &= \chi a \left( -\frac{4\pi\rho_m}{3} + \frac{\Lambda}{3} \right). \end{aligned} \quad (41)$$

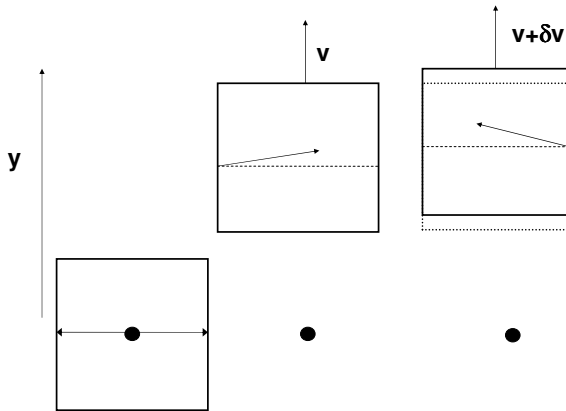


FIG. 5: The bending of a photon trajectory in the absence of any lensing mass, in de Sitter space. To begin on the left we sit in the frame of a reflective box and observe the photon bouncing back and forth. The central illustration shows the appearance of the same box now receding from some displaced viewpoint. As the box accelerates in de Sitter space, the angle of incidence must enlarge to match this, giving the illusion of lensing. This demonstrates the main problem associated with measuring the deflection of light in the rest frame of the lens.

For de Sitter space  $\rho_m = 0$  and so  $v_y = Hy$ . Therefore at the first bounce the photon appears to have a trajectory given by

$$\theta' = \sqrt{\frac{\Lambda}{3}}\chi, \quad (42)$$

but accumulates an additional vertical velocity after travelling a distance  $\ell$  across the width of the box,

$$\begin{aligned} v_y &= v_0 + \int \ddot{y} dt \\ &= \sqrt{\frac{\Lambda}{3}}\chi + \frac{\Lambda\ell\chi}{3}, \end{aligned} \quad (43)$$

provided  $v \ll c$  and  $\ell \ll \chi$ . The photon's angle of incidence exceeds the angle of the previous bounce, and this is interpreted as a bend angle of

$$\alpha_\Lambda = \frac{\Lambda\ell\chi}{3}. \quad (44)$$

This deflection is actually an angular aberration arising from acceleration, due to the change in velocity attained by a freefalling body at a distance  $\chi$ . Note that the term relates to that in (32), but here the bend angle has been defined as positive since in this case the photon is receding from the observer. The analogy here is that one side of the box is the source, the other side the observer, and the reference frame is the lens (which in this case is massless). In the frame of the box - be it source or

observer - no deflection is observed, yet the lens frame shows this anomalous bending.

Switching between different inertial frames therefore leaves us with an angular aberration associated with the relative velocity between the two frames. But of what physical significance are the aberrations? They are “real” angles in the sense that if a physical tube were to be constructed down which light could be shone over cosmological distances, it would need to be bent in this manner. However it would also be in a non-inertial reference frame, as the tube would need to be continuously accelerated in order to counteract the gravitational forces which would otherwise have led to it joining the Hubble flow. Therefore locally it would appear that the photon is travelling straight while the bent physical structure is accelerated in just the right manner so as to allow the photon to pass. Despite the temptation to consider the intuitive physical coordinates, this example illustrates that it is much more natural to think in terms of a comoving framework, such that a fixed point corresponds to an appropriate inertial reference frame.

Recently Sereno [14] highlighted the presence of an additional term given by  $2mb\Lambda/3$ . In contrast to the aberration terms outlined above, which appear to reduce the deflection angle due to cosmic acceleration, this increases the deflection angle due to the cosmic expansion rate. In the Appendix we present a heuristic approach that provides a physical interpretation of the Sereno term. Essentially any transverse motion by the lens will modify the deflection angle, due to the time-dependent impact parameter. This term is found to be consistent with the motion associated with the cosmological expansion. One might worry that peculiar velocities may modify the cosmic shear signal via this mechanism, though this contribution has been shown to be too small to be of concern [19].

## VI. ANGULAR-DIAMETER DISTANCE

We have already established the angle at which the photon appears to impact an observer, from a distant perspective. Now we assess the relative appearance of a more physically meaningful angle, the path crossing of two photons. The setup involves two sources  $A$  and  $B$  separated by a fixed distance  $R$ , and an observer  $O$  at a distance  $r/(1+z) \gg R$ . Therefore the initial angle of interest is  $\theta_i = R(1+z)/r$ , as illustrated in Figure 6. By the time the photon reaches  $O$ , the physical angle separating the bodies is  $\theta_f = R/r$ .

Consider the angle of incidence as determined by the source at  $B$  (bottom-right panel). In the time taken for the photon to travel the distance  $r$ , the photon from  $A$  acquires a vertical velocity, leading to a bending angle such that

$$\theta_B = \theta_f - \frac{\Lambda Rr}{3}, \quad (45)$$

## VII. STRONG LAMBDA

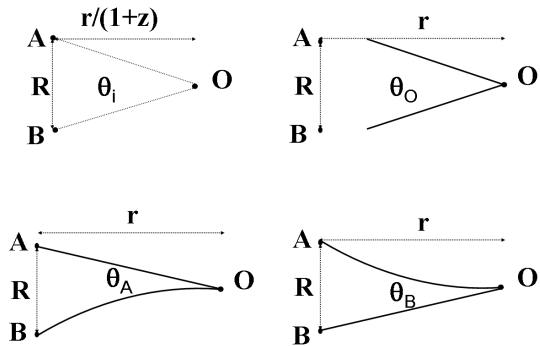


FIG. 6: Photons from sources  $A$  and  $B$ , located at either end of a rigid body of length  $R$ , propagate to the observer  $O$  at a proper distance  $r$  in de Sitter space. As viewed from the reference frame of  $O$  (top-right),  $A$  (bottom-left), and  $B$  (bottom right). The top-left panel illustrates the configuration at the time of emission. Note that from the perspective of the observer, the distance traversed by the photons does not match the source-observer separation.  $r$ .

From the perspective of  $A$ , its own photon now appears straight yet the photon from  $B$  is deflected, and by symmetry we have  $\theta_A = \theta_B$ .

The relative motion of the comoving observer is

$$v_x = \sqrt{\frac{\Lambda}{3}}r, \quad (46)$$

so once again by utilising (39) we arrive at

$$\begin{aligned} \theta_O &= \theta_f(1 + v_x) \\ &= \theta_f \left( 1 + \sqrt{\frac{\Lambda}{3}}r \right) \\ &= R(1 + z)/r, \end{aligned} \quad (47)$$

where the redshift  $z$  represents the horizontal recession velocity. So we recover the expression for the angular-diameter distance.

Alternatively, consider an observer on  $A$  studying the angle at which a telescope on  $O$  is pointing in order to detect the source at  $B$ . What angle must the telescope be pointed in order to let the photon pass? The Lorentz contraction of the telescope actually increases its apparent inclination, though this is an  $O(v^2)$  effect. The problem is resolved simply by the motion of the telescope, which allows the photon to pass at an angle of approximately  $\theta_t = \theta_p(1 + v)$ . This process provides a alternative picture of how the factor of  $(1 + z)$  arises in the angular-diameter distance.

As an aside, since we have been considering the limit of a weak field, let us address the stronger regime. If  $\Lambda$  were to modify the deflection of light, this would be expected to become most apparent in a scenario were the length scales involved exceeded the event horizon,  $r_\Lambda = \sqrt{3/\Lambda}$ . Naturally the Source-Observer distance is restricted to a sub-horizon scale, but suppose a mass was positioned with a large transverse displacement  $R$ , beyond the event horizon. One might be tempted to believe that, for reasons of causality, the event horizon “shields” the photon from the distant mass, thereby nulling the lens. This is evidently not the case, as the Kottler metric is still valid for the regime  $r > \sqrt{3/\Lambda}$ , in much the same way as a black hole gravitates beyond its event horizon. Causality is preserved since no information is transmitted – for instance any gravitational waves emitted by the mass will remain confined within the horizon. Reassuringly, this scenario also suggests that in the context of gravitational lensing, no terms involving  $\Lambda r^2$  arise.

## VIII. CONCLUSIONS

In this work we have placed light bending by a spherically symmetric mass distribution with Lambda into a cosmological context, and attempted to reconcile the apparent bending of light as described in [3, 4, 5, 8, 9, 10, 11, 13, 14, 20], with the conventional view that the cosmological constant does not directly influence gravitational lensing.

To confirm that the cosmological constant does not contribute at linear order to the deflection of light by a density fluctuation, we explicitly transformed a perturbed FRW metric into the Kottler metric. In the former metric, the linear lensing potential has no explicit dependence on  $\Lambda$ , so the  $\Lambda$ -dependent bending claimed to exist in the Kottler metric appears to be a gauge artefact, with no direct implications for observations.

The source of explicit  $\Lambda$ -dependence primarily arises by adopting physical distances to define the angles, and doing so in the frame of reference of the lens rather than the observer merely exacerbates the problem. Whilst physical scales provide an intuitive picture of the photon’s trajectory, it fails to take into account the relative motion between local and distant comoving observers, and the frame-dependence of the metric.

Terms involving  $\Lambda r^2$  essentially arise from measuring the photon’s trajectory within a non-inertial frame of reference – that of a particular physical coordinate, within the context of an accelerating cosmology. This effect should therefore be considered distinct from genuine gravitational lensing effects, where the deflection angle is gauge invariant.

Broadly, we are in agreement with the analyses of Sereno [13, 14] in concluding that at linear order, there is no influence of Lambda on light bending. Indeed, for



light bending on cluster scales, Sereno [14] shows that the influence of Lambda on the bend angle is third-order in the two mass- and Lambda-related small quantities he introduces (of comparable size for clusters). This term is several orders of magnitude smaller than second-order mass terms which are routinely neglected.

Of course, the cosmological constant does still influence the lens geometry, and it is primarily this modification to the distance-redshift relation which allows weak lensing surveys to constrain dark energy models. Our belief is that this application can proceed without requiring modification of the basic lensing theory.

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## APPENDIX A: LENSES IN MOTION

In the case of Newtonian Gravitational Lensing, we can evaluate the deflection angle of a particle of arbitrary velocity by considering the integrated force perpendicular to the direction of motion.

Consider a particle of horizontal motion  $v_x = v$  and initial vertical motion  $v_y = 0$ . After passing a gravitating mass the particle accrues a vertical velocity determined by

$$v_y = \int F_y dt, \quad (\text{A1})$$

where

$$F_y = \frac{GM}{r^2} \sin \theta. \quad (\text{A2})$$

A change of variable leads to

$$\begin{aligned} v_y &= \frac{GM}{v} \int_{-\infty}^{\infty} \frac{R}{r^3} dx. \\ &= \frac{2GM}{Rv} \end{aligned} \quad (\text{A3})$$

since  $r^2 = R^2 + x^2$ . For small angles, the Newtonian deflection is then simply given by

$$\begin{aligned} \alpha_N &= \frac{v_y}{v_x} \\ &= \frac{2GM}{Rv^2} \end{aligned} \quad (\text{A4})$$

For the case  $v = c$ , this result is of course half that predicted by General Relativity, and this discrepancy can be attributed to the matching contribution from the spatial component of the metric, which is otherwise negligible for  $v \ll c$ . The Newtonian approach allows us to gain some intuition on the influence of a lens in transverse motion. A lens travelling with a velocity  $v_L$  introduces a time-dependence to the vertical displacement  $R$  which we parameterise as  $R = R_0 + v_L x$  and we have set  $v = 1$ . In this case we find (A3) becomes

$$\begin{aligned} v_y &= GM \int_{-\infty}^{\infty} \frac{R_0 + v_L x}{[(R_0 + v_L x)^2 + x^2]^{3/2}} dx, \\ &= \frac{2GM}{R\sqrt{1 - v_L^2}} \end{aligned} \quad (\text{A5})$$

This corresponds to a modification to the Newtonian deflection angle given by

$$\delta\alpha_N \simeq \frac{GMv_L^2}{R} \quad (\text{A6})$$

If we define the velocity to correspond to the Hubble flow, then the transverse lens velocity is given by  $v_L = HR$ , and reintroducing the factor of two from General Relativity leaves us with

$$\delta\alpha \simeq 2GMRH^2. \quad (\text{A7})$$

Finally taking  $H^2 = \Lambda/3$  corresponds to the term from Sereno

$$\delta\alpha \simeq 2GMRA/3. \quad (\text{A8})$$

In most practical cases the peculiar motion of the lens will likely far exceed this influence.

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