Backward Stochastic Differential Equations in Finance

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Abstract

The dissertation is built on the paper "Backward Stochastic Dynamics on a filtered probability Space" done by G. Liang, T. Lyons and Z. Qian [27]. They demonstrate that backward stochastic differential equations (BSDE) may be reformulated as ordinary functional equations on certain path spaces. In this dissertation, we use the new approach to study the following general type of backward stochastic differential equations

$$dY_t^j = -f^j(t, Y_t, L(M)_t)dt + dM_t^j$$

with $Y_T = \xi$, on a general filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, wher L is a prescribed non-linear mapping which sends M to an adapted process $L(M)$, and M, a correction term, is a martingale to be determined.

In G. Liang, T. Lyons and Z. Qian's paper, the existence and uniqueness of $\mathbb{L}^2$ solutions were proved under certain technical conditions. We extended the results to $\mathbb{L}^p$ solutions and proved the existence and uniqueness of $\mathbb{L}^p$ solutions under these conditions. Furthermore, we established the Comparison Theorem for this type of BSDEs under $\mathbb{L}^2$ solutions. Last, based on the idea in G. Liang, T. Lyons and Z. Qian's paper, we revisited the Malliavin derivatives of $\mathbb{L}^2$ solutions of BSDEs in the typical form:

$$dY_t = -f(t, Y_t, Z_t) + Z_t^* dB_t, \quad Y_T = \xi.$$

Based on our theorems, some standard and famous theorems in the literature were revisited and proved. For example, the existence and uniqueness $\mathbb{L}^2$ solution [3] and the Comparison Theorem [9] for the typical BSDEs in the above form.

Last but not least, we briefly illustrated how the backward stochastic dynamics problem is to determine the price of a standard European contingent claim $\xi \geq 0$ of maturity T in a complete market, which pays an amount ξ at time T. In addition, the pricing problem of a contingent claim in a constrained case was studied by the Malliavin derivatives of solutions of BSDEs.

Key words: Brownian motion, backward SDE, SDE, semimartingale, comparison theorem, Malliavin derivative

Content

Chapter 1: Introduction **1**

Chapter 2: Literature Review **4**

Chapter 3: Generalised Backward Stochastic Differential Equations **7**

3.1 Preliminaries … 7
3.2 Existence and Uniqueness of Local Solutions … … … … … … … … … … … .8
3.3 Global solutions … .13
3.4 Generalised Comparison Theorem … … … … … … … … … … … … … … . .21
3.5 Application to Standard European Option Pricing… … … … … … … … … .26

Chapter 4: Malliavin Derivatives of BSDE solutions **31**

4.1 Preliminaries … .31
4.2 Differentiation on Wiener Space of BSDE Solutions … … … … … … … … . .32
4.3 Application to European Option Pricing in the constrained Case… … … . .40

Chapter 5: Conclusions **43**

Reference **44**

Chapter 1

Introduction

Backward stochastic differential equations (BSDEs) are a new class of stochastic differential equations, whose value is prescribed at the terminal time T. BSDEs have received considerable attention in the probability literature in last 20 years because BSDEs provide a probabilistic formula for the solution of certain classes of quasilinear parabolic PDEs of second order, and have connection with viscosity solutions of PDEs. The theory of BSDEs has found wide applications in areas such as stochastic control, theoretical economics and mathematical finance problems. Especially in mathematical finance, the theory of the hedging and pricing of a contingent claim is typically expressed in terms of a linear BSDE.

In the introduction chapter, we would like to start with the following type of BSDEs and provide you a main idea about how BSDEs may be reformulated as ordinary functional equations. Certainly, this main idea is the essence of the paper, and greatly applied through the paper.

The important class of backward stochastic differential equations considered in the literature introduced by (Pardoux and Peng, 1990) [3] are Itô′s type equations such as

$$dY_t = -f(t, Y_t, Z_t)dt + Z_t^* dB_t, \quad Y_T = \xi \quad (1.1)$$

where B=$(B_t)_{t\geq 0}$ is Brownian motion in $\mathbb{R}^n$, $\{Y_t : t \in [0,T]\}$ is a continuous $\mathbb{R}^d$-valued adapted process, $\{Z_t : t \in [0,T]\}$ is an $\mathbb{R}^{n\times d}$-valued predictable process and $\xi \in \mathbb{L}_T^2(\mathbb{R}^d)$, and $(\mathcal{F}_t)_{t\geq 0}$ is the Brownian filtration. Equation (1.1) is equivalent to

$$dY^j{}_t = -f^j(t, Y_t, Z_t)dt + \sum_{i=1}^{n} Z_t^{j,i}\, dB_t^i, \quad Y_T^j = \xi^j \quad (1.2)$$

where $j = 1, \dots, d$. For simplicity, we eliminate the superscript j and equation (1.2) becomes

$$dY_t = -f(t, Y_t, Z_t)dt + \sum_{i=1}^{n} Z_t^i\, dB_t^i, \qquad Y_T = \xi. \quad (1.3)$$

The differential equation (1.3) can be interpreted as the integral equation

$$\xi - Y_t = -\int_t^T f(s, Y_s, Z_s)ds + \sum_{i=0}^{n} \int_t^T Z_s^i dB_s^i. \quad (1.4)$$

Our main idea is based on the following simple observation. Suppose that $\{Y_t: t \in [\tau, T]\}$ is a solution of (1.4) back to time $\tau < T$, then Y must be a special semimartingale whose variation part is continuous. Let $Y_t = M_t - V_t$ be the Doob-Meyer's decomposition into its martingale part M and its finite variation part $-V$. The decomposition is unique up to a random variable measurable with respect to $\mathcal{F}_\tau$. Let us assume that local martingale part M is indeed a martingale up to T. We have $Y_T = \xi$, which implies $\xi = M_T - V_T$. Since M is a true martingale, it follows that

$$M_t = \mathbb{E}(\xi + V_T|\mathcal{F}_t) \text{ and } Y_t = \mathbb{E}(\xi + V_T|\mathcal{F}_t) - V_t, \quad \text{for } t \in [\tau, T].$$

Hence the integral equation (1.4) can be rewritten as

$$\xi - M_t + V_t = -\int_t^T f(s, Y_s, Z_s)ds + \sum_{i=0}^n \int_t^T Z_s^i dB_s^i,$$

for any $t \in [\tau, T]$. Taking expectations both sides conditional on $\mathcal{F}_t$, we obtain

$$\mathbb{E}(\xi|\mathcal{F}_t) - M_t + V_t = -\mathbb{E}\left[\int_\tau^T f(s, Y_s, Z_s)ds \middle| \mathcal{F}_t\right] + \int_\tau^t f(s, Y_s, Z_s)ds.$$

By identifying the martingale parts and variational parts on both sides, we have

$$V_t - V_\tau = \int_\tau^T f(s, Y_s, Z_s)ds, \quad (1.5)$$

where Y and Z are considered as functionals of V, namely

$$M_t = \mathbb{E}(\xi + V_T|\mathcal{F}_t) \text{ and } Y_t = \mathbb{E}(\xi + V_T|\mathcal{F}_t) - V_t, \quad (1.6)$$

and Z is determined uniquely by the martingale representation by

$$M_T - M_t = \sum_{i=1}^n \int_t^T Z_s^i dB_s^i.$$

Thus, by writing Y and Z as $Y(V)$ and $Z(V)$, we have the functional differential equation from (1.5)

$$\frac{dV}{dt} = f(t, Y(V)_t, Z(V)_t),$$

which can be solved by Picard iteration applying to V alone, instead of the pair (Y, Z).

Now by setting $\mathrm{M}_{\mathrm{T}}-\mathrm{M}_{\tau}=\sum_{\mathrm{i}=1}^{\mathrm{n}} \int_{\tau}^{\mathrm{t}} \mathrm{Z}_{\mathrm{s}}^{\mathrm{i}} \mathrm{dB}_{\mathrm{s}}^{\mathrm{i}}$, and regarding Z as a function of M, so denoted by L(M), then equation (1.3) can be rewritten as

$$\mathrm{dY}_{\mathrm{t}}=-\mathrm{f}(\mathrm{t}, \mathrm{Y}_{\mathrm{t}}, \mathrm{L}(\mathrm{M})_{\mathrm{t}}) \mathrm{dt}+\mathrm{dM}_{\mathrm{t}}, \quad \mathrm{Y}_{\mathrm{T}}=\xi \quad (1.7)$$

which is equivalent to the functional integral equation

$$\mathrm{V}_{\mathrm{t}}-\mathrm{V}_{\tau}=\int_{\tau}^{\mathrm{T}} \mathrm{f}(\mathrm{s}, \mathrm{Y}(\mathrm{V})_{\mathrm{s}}, \mathrm{L}(\mathrm{M}(\mathrm{V}))_{\mathrm{s}}) \mathrm{ds}, \quad (1.8)$$

where $\mathrm{Y}(\mathrm{V})$ and $\mathrm{M}(\mathrm{V})$ are defined by (1.6).

In this paper, we constraint ourselves the study of the following type of backward stochastic differential equations

$$\mathrm{dY}_{\mathrm{t}}^{\mathrm{j}}=-\mathrm{f}^{\mathrm{j}}(\mathrm{t}, \mathrm{Y}_{\mathrm{t}}, \mathrm{L}(\mathrm{M})_{\mathrm{t}}) \mathrm{dt}+\mathrm{dM}_{\mathrm{t}}^{\mathrm{j}}, \quad \mathrm{Y}_{\mathrm{T}}^{\mathrm{j}}=\xi^{\mathrm{j}}, \quad (1.9)$$

on the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_{\mathrm{t}}, \mathbb{P})$ and $\mathrm{j}=1, \ldots, \mathrm{d}$, where $\xi^{\mathrm{j}} \in \mathbb{L}^{\mathrm{p}}(\Omega, \mathcal{F}_{\mathrm{T}}, \mathbb{P})$, f^{j} is locally bounded and Borel measurable, and L is a prescribed mapping which sends a vector of martingales $\mathrm{M}=(\mathrm{M}^{\mathrm{j}})$ to a progressively measurable process $\mathrm{L}(\mathrm{M})$.

Chapter 2

Literature Review

Backward Stochastic Differential Equations (BSDEs) is an interesting field attracting lots of well-known researchers' investigation especially in last twenty years, because BSDEs have important connections with the pricing of contingent claims and stochastic optimizations problems in mathematical finance. Therefore, it is quite interesting to know how the theory of BSDE and its applications is being developed with time.

Started from 1973, the linear Backward stochastic differential equations were first introduced by (Bismut, 1973) [1], who used these BSDEs to study stochastic optimal control problems in the stochastic version of the Pontryagin's maximum principle. 5 years later, (Bismut, 1978) [2] extended his theory and showed the existence of a unique bounded solution of the Riccati BSDE.

In 1990s, the theory of BSDE was greatly developed by many academic researchers, and there were a large number of published articles devoted to the theory of BSDE and its applications. Among these authors, the most famous ones are (Pardoux and Peng, 1990) [3] who considered general BSDEs in the following form

$$dY_t = -f(t, Y_t, Z_t)dt + Z_t^* dB_t, \qquad Y_T = \xi,$$

and showed the existence and uniqueness of BSDEs under some assumptions such as the Lipschitz condition of the driver f. Meanwhile, based on the theory of BSDEs, (Peng, 1990) [4] suggested a general stochastic maximum principle with first and second order adjoint equations. Then Peng's stochastic optimal control theory was further developed by (Kohlmann and Zhou, 2000) [5] who interpreted BSDE as equivalent to stochastic control problems.

In 1992, an important theorem called comparison theorem is introduced by (Peng, 1992b) [6]. This theorem provides a sufficient condition for the wealth process to be nonnegative. Besides it, several articles were written by Peng or Pardoux and Peng about BSDEs. For example, (Pardoux and Peng, 1992) [7] showed the solution of the BSDE in the Markovian case corresponds to a probabilistic solution of a non-linear PDE, and gave a generalization of the Feynman Kac formula. Furthermore, Peng (1991, 1992a, 1992c together with 1992b) [8], [9], [10], [6] stated the connection between the BSDEs associated with a state process satisfying some forward classical SEDs, and PDEs in the Markovian cases. The theories behind can be used for European option pricing in the constrained Markovian case. In addition to Peng and Pardoux, (Duffine and Epstein,

1992) [11] investigated a class of non-linear BSDEs to give a stochastic differential formulation of recursive utilities and their properties in the case of information generated by Brownian motion.

The theory and applications of BSDEs were further explored after 1992. (Antoelli, 1993) [12] was the first to study BSDE coupled with a forward stochastic differential equation(a Forward-Backward Stochastic Differential Equations), without a density process Z in the driver. Some more general forward- backward stochastic differential equations of the type, in which the parameters of the forward and backward equations depend on the solution (X, Y, Z) of the system, were investigated in a deep level by (Ma, Protter and Yong, 1994) [13]. Simultaneously, a study about BSDE with random jumps was done in (Tang and Li, 1994) [14]. Similarly, (Barles, Buckdahn and Pardoux, 1997) [15] investigated the relation between BSDE with random jumps and some parabolic integro-differential equations, followed by the existence and uniqueness under non-Lipschitz continuous coefficients for this type of BSDE which were proved by (Rong, 1997) [16]. In the same year, there were some other findings or development in BSDEs achieved. For instance, (Lepeltier and San Martin, 1997) [17] stated that a square integrable solution is existed under the only assumption of continuous driver with linear growth in the one-dimensional case though the uniqueness does not hold in general. And also the existence of solution for FBSDEs with arbitrary time was proved by (Yong, 1997) [18] using the method of continuation. Finally, it was claimed in (El Karoui, Peng and Quenez, 1997) [19] that the solution of a linear BSDE is in fact the pricing and hedging strategy of the contingent claim ξ. This paper suggests that BSDEs can be applied to option pricing problems, and demonstrate a general framework for the application of BSDEs in finance.

In the 21st century, the theories behind BSDEs are getting more and more mature. However, there are still plenty of research areas in BSDEs that appeal to many researchers. For example, the utility maximization problems with backward stochastic dynamics are very popular among researchers in finance nowadays. In 2000, a class of BSDEs was introduced by (Rouge and El Karoui, 2000) [20] who tried solving the utility maximization problems in incomplete market by the class of BSDEs they introduced. Then their results were generalised by (Hu, Imkeller, and Müller, 2005) [21]. Meanwhile, by using the monotonicity method adopted from PDE theory, (Kobylanski, 2000) [22] solved a type of BSDEs with drivers which are quadratic growth of Z. This particular class of BSDEs, but with unbound terminal values, were further studied by (Briand and Hu, 2006) [23].

After people had known more and more about the strong solutions of BSDEs under an underlying filtered probability space, the weak solutions of BSDEs were full of interests to researchers. (Buckdahn, Engelbert and Rascanu, 2004) [24] were three of the pioneers in the introduction of weak solutions for BSDEs. Moreover, the uniqueness of weak solutions of BSDEs, whose coefficients are independent of the density process Z, was proved by (Buckdahn and Engelbert, 2007) [25] 3 years later. In the field of FBSDEs, (Antonelli and Ma, 2003) also introduced the week solutions. By using the martingale problem approach, the theory of weak solutions for FBSDEs was studied deeply by (Ma et al, 2008) [26].

Recently, it was discovered by (Liang, Lyons and Qian, 2009) [27] that one can reformulate a backward stochastic differential equation as a functional ordinary differential equation on path spaces. This discovery allows many extensions of the classical BSDE theory and to explore more applications in mathematical finance.

In this article, we apply the idea in Liang, Lyons and Qi's paper to the class of BSDEs such as (1.9) which is more general form of BSDEs than that introduced by (Pardoux and Peng, 1990) [3]. Our main contributions in this article is to generalise or revisited some theory of BSDEs in both Liang, Lyons and Qi's paper and the literature, and also to establish the Comparison Theorem corresponding to the kind of BSDEs like (1.9).

Chapter 3

Generalised Backward Stochastic Differential Equations

3.1 Preliminaries

First we fix some notions. Given a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we consider for $0 \leq \tau < T$

- $\{(\mathcal{F}_t); t \in [0, T]\}$ is a right-continuous filtration, each $\mathcal{F}_t$ contains all events in $\mathcal{F}$ with probability zero, and $\mathcal{F} = \sigma\{\mathcal{F}_t : t \geq 0\}$.
- $\mathbb{L}^p(\Omega, \mathcal{F}_T, \mathbb{P})$, the space of all $\mathcal{F}_T$ –measurable random variables X satisfying $\|X\|^{\,p} = \mathbb{E}(|X|^p) < +\infty$.
- $C^p([\tau, T];\ \mathbb{R}^d)$, the space of all continuous adapted processes $(V_t)_{t\in[\tau,T]}$ satisfying $\max_j \sup_{t\in[\tau,T]} \left|V_t^j\right| \in \mathbb{L}^p(\Omega, \mathcal{F}_T, \mathbb{P})$, and $\|V\|_{C^p[\tau,T]} = \sqrt[p]{\sum_{j=1}^{d} \mathbb{E} \sup_{t\in[\tau,T]} \left|V_t^j\right|^p} < +\infty.$
- $C_0^p([\tau, T];\ \mathbb{R}^d)$, the space of all processes $(V_t)_{t\in[\tau,T]}$ in $C^p([\tau, T];\ \mathbb{R}^d)$ with initial data $V_\tau = 0$.
- $\mathcal{M}^p([\tau, T];\ \mathbb{R}^d)$, the space of $\mathbb{R}^d$-valued p –intergrable martingale.
- $S^p\big([\tau, T];\ \mathbb{R}^d\big)$, the direct sum space of $\mathcal{M}^p([\tau, T];\ \mathbb{R}^d)$ and $C^p([\tau, T];\ \mathbb{R}^d)$ with its norm $\|\cdot\|_{C^p[\tau,T]} < +\infty$. If $Y \in S^p\big([\tau, T];\ \mathbb{R}^d\big)$, then there exists $M \in M^p\big([\tau, T];\ \mathbb{R}^d\big)$ and $V \in C^p\big([\tau, T];\ \mathbb{R}^d\big)$ such that $Y = M - V$.
- $\mathcal{H}^p\big([\tau, T];\ \mathbb{R}^{n\times d}\big)$, the space of all predictable processes Z with its norm $\|Z\|_{\mathcal{H}^p[\tau,T]} = \sqrt[p]{\sum_{j=1}^{n} \sum_{i=1}^{d} \mathbb{E} \int_\tau^T \left|Z_s^{i,j}\right|^p ds}.$

Consider the BSDE

$$dY_t^j = -f^j(t, Y_t, L(M)_t)dt + dM_t^j, \quad Y_T^j = \xi^j, \quad (3.11)$$

on the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ and $j = 1, \dots, d$, where $\xi^j \in \mathbb{L}^p(\Omega, \mathcal{F}_T, \mathbb{P})$, f^j is locally bounded and Borel measurable, and
$L: \mathcal{M}^p([\tau, T];\ \mathbb{R}^d) \longrightarrow \mathcal{H}^p\big([\tau, T];\ \mathbb{R}^{n\times d}\big)$ is a prescribed mapping on $\mathcal{M}^p([\tau, T];\ \mathbb{R}^d)$ valued in $\mathcal{H}^p\big([\tau, T];\ \mathbb{R}^{n\times d}\big)$.

A solution of (3.11) backward to τ is a pair of adapted processes $(Y_t^j, M_t^j)_{t\in[\tau,T]}$ where $M^j = (M_t^j)_{t\in[\tau,T]}$ are martingales and $Y^j = (Y_t^j)_{t\in[\tau,T]}$ are special semimartingales with continuous variation parts $V^j = (V_t^j)_{t\in[\tau,T]}$, which satisfies the integral equations

$$Y_t^j - \xi^j = \int_t^T f^j(s, Y_s, L(M)_s)\, ds + M_t^j - M_T^j, \quad (3.12)$$

for $t \in [\tau, T], j = 1, \dots, d$.

As we have done in the introduction chapter, by setting $Y_t = M_t - V_t$, where $M_t = \mathbb{E}(\xi + V_T|\mathcal{F}_t)$ and $Y_t = \mathbb{E}(\xi + V_T|\mathcal{F}_t) - V_t$. We find a solution (Y, M) of (3.12) is equivalent to a solution V of the functional integral equation

$$V_t - V_\tau = \int_\tau^t f\left(s, Y(V)_s, L\big(M(V)\big)_s\right) ds, \quad (3.13)$$

for $t \in [\tau, T]$. We are going to study the integral equation (3.13) and prove the existence and uniqueness of the BSDE (3.11) by its functional integral equation (3.13).

3.2 Existence and Uniqueness of Local Solutions

In this section, we are going to prove the uniqueness and the existence of a local solution to (3.11) under the assumption that f and L are Lipschitz continuous:

1) $f = (f^j)_{j\le d}$ are Lipschitz conditions on $[0,\infty) \times \mathbb{R}^d \times \mathbb{R}^{n\times d}$, if there exists a constant C_2 such that
$$|f(t,y,z)| \le C_2(1 + t + |y| + |z|)$$
$$|f(t,y,z) - f(t,y',z')| \le C_2(|y - y'| + |z - z'|).$$
2) $L: \mathcal{M}^p([\tau,T];\ \mathbb{R}^d) \to \mathcal{H}^p([\tau,T];\ \mathbb{R}^{n\times d})$ is Lipschitz continuous if there exists a constant C_1 such that
$$\| L(M) \|_{\mathcal{H}^p} \le C_1 \| M \|_{C^p}$$
$$\| L(M) - L(\widetilde{M}) \|_{\mathcal{H}^p} \le C_1 \| M - \widetilde{M} \|_{C^p}.$$

Define $\mathbb{L}: C_0^p([\tau,T];\mathbb{R}^d) \longrightarrow C^p([\tau,T];\ \mathbb{R}^d)$ by $\mathbb{L}(V)_t = \int_\tau^t f(s, Y(V)_s, L(M(V))_s) ds$, where $M(V)_t = \mathbb{E}(\xi + V_T)|\mathcal{F}_t)$ and $Y(V)_t = M(V)_t - V_t$ for $t \in [\tau, T]$, which implied $Y(V)_t = \xi$. By setting V= $\mathbb{L}(V)$, then $\mathbb{L}(V)_t = \int_\tau^t f(s, Y(V)_s, L(M(V))_s) ds$ is equivalent to the following BSDE:

$$dY_t^j = -f^j(t, Y_t, L(M)_t)dt + dM_t^j, \quad Y_T^j = \xi^j,$$

which is the same as equation (3.11).

Thereom 3.2.1 Under the assumptions on f and L, let

$$l_1 = \frac{1}{C_2^q\left[2C_1\left(\frac{p}{p-1}\right) + 2\left(1 + \frac{p}{p-1}\right)\right]^q} \wedge 1,$$

which depends on the Lipschitz constants C_1, C_2, but is independent of the terminal data ξ. Suppose that $\delta = T - \tau \leq l_1$, then $\mathbb{L}$ admits a unique fixed point on $C_0^p([\tau, T]; \mathbb{R}^d)$.

Proof:

Basically, we apply the fixed theorem to $\mathbb{L}$ in order to show that $\mathbb{L}$ is a contraction on $C_0^p([\tau, T]; \mathbb{R}^d)$ as long as $\delta = T - \tau \leq l_1$. It starts with

$$\| \mathbb{L}(V) \|_{C^p} \leq \sqrt[p]{\mathbb{E}\left[\int_\tau^T f\left[s, Y(V)_s, L\big(M(V)\big)_s\right] ds\right]^p}.$$

By the Hölder's inequality,

$$\| \mathbb{L}(V) \|_{C^p} \leq \sqrt[q]{\delta} \sqrt[p]{\mathbb{E}\int_\tau^T \left|f\left[s, Y(V)_s, L\big(M(V)\big)_s\right]\right|^p ds},$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Since f is Lipschitz continuous,

$$\| \mathbb{L}(V) \|_{C^p} \leq \sqrt[q]{\delta} \sqrt[p]{\mathbb{E}\int_\tau^T C_2^p \left(1 + s + |Y(V)_s| + \left|L\big(M(V)\big)_s\right|\right)^p ds}.$$

By the Minkowski inequality,

$$\| \mathbb{L}(V) \|_{C^p} \leq C_2 \sqrt[q]{\delta}\left[\sqrt[p]{\int_\tau^T (1+s)^p ds} + \sqrt[p]{\int_\tau^T \mathbb{E}|Y(V)_s|^p ds} + \sqrt[p]{\int_\tau^T \mathbb{E}\left|L\big(M(V)\big)_s\right|^p ds}\right]$$

$$\leq K + C_2\delta \| Y(V) \|_{C^p} + C_2 \sqrt[q]{\delta} \| L(M(V)) \|_{\mathcal{H}^p}$$

$$\leq K + C_2\delta \parallel Y(V) \parallel_{C^p} + C_1 C_2 \sqrt[q]{\delta} \parallel M(V) \parallel_{C^p},$$

where $K = C_2 \sqrt[q]{\delta} \left[\sqrt[p]{\int_\tau^T (1+s)^p ds} \right],$

Together with the elementary estimates

$$\parallel Y(V) \parallel_{C^p} \leq \frac{p}{p-1} \sqrt[p]{\mathbb{E}|\xi|^p} + \frac{p}{p-1} \parallel V \parallel_{C^p}$$

and

$$\parallel M(V) \parallel_{C^p} \leq \frac{p}{p-1} \sqrt[p]{\mathbb{E}|\xi|^p} + \left(\frac{p}{p-1} + 1\right) \parallel V \parallel_{C^p},$$

we deduce that

$$\parallel \mathbb{L}(V) \parallel_{C^p} \leq K + \frac{p}{p-1}\left(C_2\delta + C_1 C_2 \sqrt[q]{\delta}\right) \sqrt[p]{\mathbb{E}|\xi|^p}$$

$$+ \left[\frac{p}{p-1}\left(C_2\delta + C_1 C_2 \sqrt[q]{\delta}\right) + C_1 C_2 \sqrt[q]{\delta}\right] \parallel V \parallel_{C^p}.$$

Similarly, for $V, \widetilde{V} \in C^P[\tau, T]$ such that $V_\tau = \widetilde{V}_\tau = 0$ one has

$$\parallel \mathbb{L}(V) - \mathbb{L}(\widetilde{V}) \parallel_{C^p}$$

$$\leq \sqrt[p]{\mathbb{E}(\int_\tau^T \left| f(s, Y_s, L(M)_s) - f\left(s, \widetilde{Y}_s, L(\widetilde{M})_s\right) \right| ds)^p}$$

$$\leq C_2 \sqrt[p]{\mathbb{E}[\int_\tau^T \left(|Y_s - \widetilde{Y}_s| + \left| L(M)_s - L(\widetilde{M})_s \right| \right) ds]^p} \text{ (f is Lipschitz continuous)}$$

$$\leq C_2 \sqrt[q]{\delta} \sqrt[p]{\mathbb{E} \int_\tau^T \left(|Y_s - \widetilde{Y}_s| + \left| L(M)_s - L(\widetilde{M})_s \right| \right)^p ds} \text{ (Hölder's inequality inequality)}$$

$$\leq C_2 \sqrt[q]{\delta} \left[\sqrt[p]{\int_\tau^T \mathbb{E} |Y_s - \widetilde{Y}_s|^p ds} + \sqrt[p]{\int_\tau^T \mathbb{E} \left| L(M)_s - L(\widetilde{M})_s \right|^p ds} \right] \text{ (Minkowski inequality)}$$

$$\leq C_2\delta \parallel Y - \widetilde{Y} \parallel_{C^p} + C_2 \sqrt[q]{\delta} \left\| L(M) - L(\widetilde{M}) \right\|_{\mathcal{H}^p}$$

$$\leq C_2\delta \parallel Y - \widetilde{Y} \parallel_{C^p} + C_1 C_2 \sqrt[q]{\delta} \left\| M - \widetilde{M} \right\|_{\mathcal{H}^p}. \quad (3.21)$$

On the other hand, it is easy to see that

$$\| M-\widetilde{M} \|_{C^P}=\sqrt[p]{\mathbb{E}\sup_{t\in[\tau,T]}\mathbb{E}\big(V_T-\widetilde{V}_T\big|\mathcal{F}_t\big)^p}\leq\frac{p}{p-1}\left\|V-\widetilde{V}\right\|_{C^P}$$

and

$$\left\|Y-\widetilde{Y}\right\|_{C^P}\leq\left(1+\frac{p}{p-1}\right)\left\|V-\widetilde{V}\right\|_{C^P}.$$

Substituting these estimates into (3.21), we finally have

$$\| \mathbb{L}(V)-\mathbb{L}(\widetilde{V}) \|_{C^P}\leq[C_2\delta(1+\frac{p}{p-1})+C_1C_2\sqrt[q]{\delta}(\frac{p}{p-1})]\left\|V-\widetilde{V}\right\|_{C^P}$$

$$=C_2\sqrt[q]{\delta}[\sqrt[p]{\delta}(1+\frac{p}{p-1})+C_1(\frac{p}{p-1})]\left\|V-\widetilde{V}\right\|_{C^P}.$$

Since $\delta\leq l_1$, the constant in front of the norm on the right-hand side is less than $\frac{1}{2}$, so that

$$\| \mathbb{L}(V)-\mathbb{L}(\widetilde{V}) \|_{C^P}\leq\frac{1}{2}\left\|V-\widetilde{V}\right\|_{C^P}.$$

Therefore, $\mathbb{L}$ is a contraction on $C_0^p([\tau,T];\mathbb{R}^d)$ as long as $T-\tau\leq l_1$, so there is unique fixed point in $C_0^p([\tau,T];\mathbb{R}^d)$. ■

We are about to show the local existence and uniqueness of solutions to BSDE (3.11).

Theorem 3.2.2 Let L, f be Lipschitz continuous with Lipschitz constants C_1, C_2, and

$$l_1=\frac{1}{C_2^q\left[2C_1\left(\frac{p}{p-1}\right)+2\left(1+\frac{p}{p-1}\right)\right]^q}\wedge 1,$$

which is independent of the terminal data $\xi\in\mathbb{L}_T^p(\mathbb{R}^d)$. Suppose that $T-\tau\leq l_1$, and $L(M)=L(M-M_\tau)$ for $\forall\, M\in\mathcal{M}^p([\tau,T];\ \mathbb{R}^d)$. Then there exists a pair (Y,M), where $Y=(Y_t)_{t\in[\tau,T]}$ is a special semimartingale, $M=(M_t)_{t\in[\tau,T]}$ is a p −integrable martingale, which solves the backward stochastic differential equation (3.11) to time τ. Furthermore, such a pair of solution is unique in the sense if $(Y,M)$and $(\widetilde{Y},\widetilde{M})$ are two pairs of solutions, then $Y=\widetilde{Y}$ and $M-M_\tau=\widetilde{M}-\widetilde{M}_\tau$ on $[\tau,T]$.

Proof for Existence:

By theorem 3.2.1, there is a unique $V \in C_0^p[\tau, T]$ such that

$$V_t = \int_\tau^t f(s, Y(V)_s, L(M(V))_s) ds, \quad \forall t \in [\tau, T]$$

where $M(V)_t = \mathbb{E}[(\xi + V_T)|\mathcal{F}_t]$and $Y(V)_t = M(V)_t - V_t$. It implies that $Y_T = \xi$ and

$$Y_t - \xi = \int_t^T f(s, Y(V)_s, L(M(V))_s) ds + M_t - M_T, \quad (3.22)$$

for $\forall t \in [\tau, T]$. Hence there exists a pair of (Y, M) which solves the backward stochastic differential equation (3.11).

Proof for Uniqueness:

Suppose that (Y, M) and $(\widetilde{Y}, \widetilde{M})$ are two solutions satisfying (3.22), where Y and $\widetilde{Y}$ are two special semimartingales. Thus,

$$Y_t - \xi = \int_t^T f(s, Y(V)_s, L(M(V))_s) ds + M_t - M_T, \quad (3.23)$$

for $t \in [\tau, T]$. Taking conditional expectation on both sides, we have

$$\mathbb{E}(Y_t - \xi|\mathcal{F}_t) = \mathbb{E}\left(\int_t^T f\left(s, Y(V)_s, L\big(M(V)\big)_s\right) ds \,\middle|\, \mathcal{F}_t\right) + \mathbb{E}(M_t - M_T|\mathcal{F}_t).$$

It implies

$$Y_t = \mathbb{E}[(\xi + V_T)|\mathcal{F}_t] - V_t,$$

where

$$V_t = \int_\tau^t f(s, Y(V)_s, L(M(V))_s) ds, \quad \forall\, t \in [\tau, T]$$

Therefore, the integral equation (3.23) becomes

$$Y_t = V_T - M_T + \xi - V_t + M_t \quad (3.24)$$

Since $V_\tau = 0$, we have

$$V_T - M_T + \xi = Y_\tau - M_\tau$$

By substituting this into (3.24), it follows that

$$Y_t = Y_\tau + (M_t - M_\tau) - V_t.$$

We have shown that

$$V_t = \int_\tau^t f[s, Y(V)_s, L(M(V))_s]ds.$$

The same argument applies to $(\widetilde{Y}, \widetilde{M})$, so that we also have

$$\widetilde{V}_t = \int_\tau^t f\left[s, Y(\widetilde{V})_s, L(M(\widetilde{V}))_s\right]ds.$$

By theorem 3.2.1, $V_t = \widetilde{V}_t$, which implies $Y_t = \widetilde{Y}_t$. It follows then

$$M_t - M_\tau = \widetilde{M}_t - \widetilde{M}_\tau.$$

■

3.3 Global solutions

For T sufficient small, the existence and uniqueness of the solution have been shown by using a fixed point theorem. In this section, for arbitrary T, the global solution to (3.11) is obtained by subdividing the interval [0,T] into a finite number of small intervals if L satisfies further regularity conditions.

Firstly, define the restriction for any $[T_2, T_1] \subset [0, T]$,

$$L_{[T_2,T_1]}: \mathcal{M}^p([T_2, T_1]; \mathbb{R}^d) \longrightarrow \mathcal{H}^p([T_2, T_1]; \mathbb{R}^{n\times d})$$

by $L_{[T_2,T_1]}(N)_t = L(\widetilde{N})_t$ for any $N \in \mathcal{M}^p([T_2, T_1]; \mathbb{R}^d)$ and $t \in [T_2, T_1]$, where $\widetilde{N} \in \mathcal{M}^p([0, T]; \mathbb{R}^d)$ defined by $\widetilde{N}_t = \mathbb{E}(N_{T_1}|\mathcal{F}_t)$ for $t \le T_1$and $\widetilde{N}_t = N_{T_1}$ for $t \ge T_1$.

Theorem 3.3.1 Assume that L satisfies the following conditions:

a) (Local-in-time property) For every pair of non-negative rational numbers $T_2 < T_1 < T$, and for any $M \in \mathcal{M}^p([0, T]; \mathbb{R}^d)$, $L(M)_t = L_{[T_2,T_1]}(\widetilde{M})_t$ on (T_2, T_1), where $\widetilde{M} = (M_t)_{t\in[T_2,T_1]}$ is restriction of M on $[T_2, T_1]$. The local-in-time property requires that $L(M)_t$ is locally defined, i.e. $L(M)_t$ depends only on $(M_s)_{s\in[t,t+\epsilon)}$ for whatever how small the $\epsilon > 0$.
b) (Differential property) For every pair of non-negative rational numbers $T_2 < T_1 \le T$, and $M \in \mathcal{M}^p([T_2, T_1]; \mathbb{R}^d)$, one has $L_{[T_2,T_1]}(M - M_{T_2})_t =$

$L_{[T_2,T_1]}(M)_t$ on (T_2, T_1). The differential property requires that $L_{[T_2,T_1]}(M)_t$ depends only on the increments $\{M_s - M_{T_2}: s \geq t\}$ for $t \in [T_2, T_1]$.

c) (Lipschitz continuity) $L: \mathcal{M}^p([\tau, T];\ \mathbb{R}^d) \longrightarrow \mathcal{H}^p\big([\tau, T];\ \mathbb{R}^{n\times d}\big)$ is bounded and Lipschitz continuous: there is a constant C_1 such that

$$\| L(M) \|_{\mathcal{H}^p_{[T_2,T_1]}} \leq C_1 \| M \|_{C^p_{[T_2,T_1]}}$$

and

$$\| L(M) - L(\widetilde{M}) \|_{\mathcal{H}^p_{[T_2,T_1]}} \leq C_1 \| M - \widetilde{M} \|_{C^p_{[T_2,T_1]}}$$

for any $M, \widetilde{M} \in \mathcal{M}^p([0, T];\ \mathbb{R}^d)$ and for any rationales T_1 and T_2 such that $0 \leq T_2 < T_1 \leq T$. That is to say $L_{[T_2,T_1]}$ are Lipschitz continuous with Lipschitz constant independent of $[T_2, T_1] \subset [0, T]$.

Then there exists a pair of processes (Y, M), where $Y = (Y_t)_{t\in[0,T]}$ is a special semimartingale, and $M = (M_t)_{t\in[0,T]}$ is a p −integrable martingale, which solves the backward equation

$$dY_t = -f(t, Y_t, L(M)_t)dt + dM_t, \quad Y_T = \xi, \quad (3.31)$$

The solution Y is unique, its martingale term M is unique up to a random variable measurable with respect to $\mathcal{F}_0$.

Proof:

Recall that

$$l_1 = \frac{1}{C_2^q[2C_1\left(\frac{p}{p-1}\right) + 2\left(1 + \frac{p}{p-1}\right)]^q} \wedge 1,$$

which is positive and independent of ξ.

By theorem 3.2.1, if the terminal time $T \leq l_1$, the mapping $\mathbb{L}$ on $C_0^p([0, T]; \mathbb{R}^d)$ admits a unique fixed point, where

$$\mathbb{L}(V)_t = \int_0^t f(s, Y(V)_s, L(M(V))_s)ds.$$

Next we consider the case $T > l_1$. In this case we divide the interval [0, T] into subintervals with length not exceeding l_1 . More precisely, let

$$T = T_0 > T_1 > \cdots > T_k = 0,$$

so that $0 < T_{i-1} - T_i \leq l_1$ where T_i are rationales except $T_0 = T$.

Begin with the top interval $[T_1, T_0]$, together with the terminal value $Y_{T_0} = \xi$ and the filtration starting from $\mathcal{F}_{T_1}$. Applying theorem 3.2.1 to the interval $[T_1, T_0]$ and $\mathbb{L}_1$, we obtain

$$(\mathbb{L}_1 V)_t = \int_{T_1}^{t} f\big(s, Y_1(V)_s, L_{[T_1,T_0]}(M_1(V))_s\big) ds,$$

where

$$M_1(V)_t = \mathbb{E}\big[\big(\xi + V_{T_0}\big)\big|\mathcal{F}_t\big] \text{ and } Y_1(V)_t = M_1(V)_t - V_t.$$

for any $V \in C^p([T_1, T_0]; \mathbb{R}^d)$ and $t \in [T_1, T_0]$. Then there exists a unique $V(1) \in C_0^p([T_1, T_0]; \mathbb{R}^d)$ such that $\mathbb{L}_1 V(1) = V(1)$.

Repeat the same argument to each interval $\left[T_j, T_{j-1}\right]$ (for $2 \le j \le k$) with the terminal value $Y_{j-1}(V(j-1))_{T_{j-1}}$, the filtration starting from $\mathcal{F}_{T_j}$, and the mapping $\mathbb{L}_j$ defined on $C_0^p(\left[T_j, T_{j-1}\right]; \mathbb{R}^d)$ by

$$(\mathbb{L}_j V)_t = \int_{T_1}^{t} f\left(s, Y_j(V)_s, L_{[T_j,T_{j-1}]}(M_j(V))_s\right) ds,$$

where $V \in C(\left[T_j, T_{j-1}\right]; \mathbb{R}^d)$ and

$$M_j(V)_t = \mathbb{E}\left[\left(Y_{j-1}(V)_{T_{j-1}} + V_{T_{j-1}}\right) |\mathcal{F}_t\right]$$

$$Y_j(V)_t = M_j(V)_t - (V_j)_t$$

for $t \in [T_j, T_{j-1}]$.

Therefore, for $1 \le j \le k$, there exists a unique $V(j) \in C(\left[T_j, T_{j-1}\right]; \mathbb{R}^d)$ such that

$$V(j)_t = \int_{T_j}^{t} f\left(s, Y_j(V)_s, L_{[T_j,T_{j-1}]}(M_j(V))_s\right) ds,$$

for $t \in \left[T_j, T_{j-1}\right]$, where $Y(0)_{T_0} = \xi$, $Y(j-1)_{T_{j-1}} = Y(j)_{T_{j-1}}$, for $2 \le j \le k$,

and

$$M(j)_t = \mathbb{E}\left[\left(Y(j-1)_{T_{j-1}} + V(j)_{T_{j-1}}\right) |\mathcal{F}_t\right]$$

$$Y(j)_t = M(j)_t - V(j)_t,$$

for $t \in [T_j, T_{j-1}]$.

Since $Y(j-1)_{T_{j-1}} = Y(j)_{T_{j-1}}$, for $2 \leq j \leq k$, $Y = (Y_t)_{t\in[0,T]}$ given by

$$Y_t = Y(j)_t, \quad \text{if } t \in [T_j, T_{j-1}],$$

for $1 \leq j \leq k$, is well defined. Define V by shifting it at the partition points:

$$V_t = \begin{cases} V(k)_t & \text{if } t \in [0, T_{k-1}] \\ V(k-1)_t + V(k)_{T_{k-1}} & \text{if } t \in [T_{k-1}, T_{k-2}] \\ \quad \cdots & \\ V(j)_t + \sum_{l=j+1}^{k} V(l)_{T_{l-1}} & \text{if } t \in [T_j, T_{j-1}] \\ \quad \cdots & \\ V(1)_t + \sum_{l=2}^{k} V(l)_{T_{l-1}} & \text{if } t \in [T_1, T]. \end{cases}$$

Then $V \in C^p([0,T]; \mathbb{R}^d)$.

Now we define

$$M_t = Y_t + V_t, \quad t \in [0, T],$$

and it remains to show that M is a martingale, so that $M_t = \mathbb{E}(\xi + V_T | \mathcal{F}_t)$.

Since for $1 \leq j \leq k$,

$$Y_t = Y(j)_t = M(j)_t - V(j)_t, \quad \text{if } t \in [T_j, T_{j-1}],$$

so that if $t \in [T_j, T_{j-1}]$,

$$\begin{aligned} M_t &= M(j)_t - V(j)_t + V_t \\ &= M(j)_t - V(j)_t + V(j)_t + \sum_{l=j+1}^{k} V(l)_{T_{l-1}} \\ &= M(j)_t + \sum_{l=j+1}^{k} V(l)_{T_{l-1}}, \quad (3.32) \end{aligned}$$

It is clear that M is adapted to $(\mathcal{F}_t)$, so we only need to show $\mathbb{E}(M_t|\mathcal{F}_s) = M_s$ for any $0 \leq s \leq t \leq T$. If $s, t \in [T_j, T_{j-1}]$ for some j, then

$$\mathrm{M_t - M_s = M(j)_t - M(j)_s}$$

so that

$$\mathbb{E}[(\mathrm{M_t - M_s})|\mathcal{F}_\mathrm{s}] = \mathbb{E}[(\mathrm{M(j)_t - M(j)_s})|\mathcal{F}_\mathrm{s}] = 0$$

If $\mathrm{s} \in [\mathrm{T_i, T_{i-1}}]$ and $\mathrm{t} \in [\mathrm{T_j, T_{j-1}}]$ for some $\mathrm{i} > j$, then by (3.32)

$$\mathrm{M_s = M(i)_s} + \sum_{\mathrm{l=i+1}}^{\mathrm{k}} \mathrm{V(l)_{T_{l-1}}}$$

and

$$\mathrm{M_t = M(j)_t} + \sum_{\mathrm{l=j+1}}^{\mathrm{k}} \mathrm{V(l)_{T_{l-1}}}.$$

Since $\mathrm{M(j)}$ is a martingale on $[\mathrm{T_j, T_{j-1}}]$ so that

$$\mathbb{E}\left(M_t \middle| \mathcal{F}_{T_j}\right) = \mathrm{M(j)_{T_j}} + \sum_{\mathrm{l=j+1}}^{\mathrm{k}} \mathrm{V(l)_{T_{l-1}}}.$$

By conditional on $\mathcal{F}_{\mathrm{T_{j+1}}} \subset \mathcal{F}_{\mathrm{T_j}}$, we obtain

$$\mathbb{E}\left(M_t \middle| \mathcal{F}_{T_{j+1}}\right) = \mathbb{E}\left[\left(M(j)_{T_j} + V(j+1)_{T_j}\right) \middle| \mathcal{F}_{T_{j+1}}\right] + \sum_{l=j+2}^{k} V(l)_{T_{l-1}}. \quad (3.33)$$

On the other hand, $\mathrm{M(j)_{T_j} = Y(j)_{T_j} + V(j)_{T_j} = Y_{T_j}}$ so that

$$\mathbb{E}\left[\left(\mathrm{M(j)_{T_j} + V(j+1)_{T_j}}\right) \middle| \mathcal{F}_{\mathrm{T_{j+1}}}\right] = \mathbb{E}\left[\left(\mathrm{Y_{T_j} + V(j+1)_{T_j}}\right) \middle| \mathcal{F}_{\mathrm{T_{j+1}}}\right] = \mathrm{M(j+1)_{T_{j+1}}}.$$

Substitute it into (3.33) we have

$$\mathbb{E}\left(\mathrm{M_t} \middle| \mathcal{F}_{\mathrm{T_{j+1}}}\right) = \mathrm{M(j+1)_{T_{j+1}}} + \sum_{\mathrm{l=j+2}}^{\mathrm{k}} \mathrm{V(l)_{T_{l-1}}}.$$

By repeating the same argument we may establish

$$\mathbb{E}\left(\mathrm{M_t} \middle| \mathcal{F}_{\mathrm{T_{i-1}}}\right) = \mathrm{M(i-1)_{T_{i-1}}} + \sum_{\mathrm{l=i}}^{\mathrm{k}} \mathrm{V(l)_{T_{l-1}}}.$$

Since $\mathrm{s} \in [\mathrm{T_i, T_{i-1}}]$, by conditional on $\mathcal{F}_\mathrm{s}$,

$$\mathbb{E}(M_t|\mathcal{F}_s) = \mathbb{E}\left[\left(M(i-1)_{T_{i-1}} + V(i)_{T_{i-1}}\right)\middle|\mathcal{F}_s\right] + \sum_{l=i+1}^{k} V(l)_{T_{l-1}}$$

$$= \mathbb{E}\left[\left(Y_{T_{i-1}} + V(i)_{T_{i-1}}\right)\middle|\mathcal{F}_s\right] + \sum_{l=i+1}^{k} V(l)_{T_{l-1}}$$

$$= M(i)_s + \sum_{l=i+1}^{k} V(l)_{T_{l-1}}$$

$$= M_s,$$

which proves M is an $\mathcal{F}_t$-adapted martingale up to T, so that $M_t = \mathbb{E}(\xi + V_T|\mathcal{F}_t)$.

Since L satisfies the local-in-time property and the differential property, so that

$$L_{[T_j,T_{j-1}]}\left(M\left(V_j\right)\right)_s = L(M)_s, \quad \text{for } s \in \left[T_j, T_{j-1}\right].$$

Hence

$$V(j)_t = \int_{T_j}^{t} f(s, Y_s, L(M)_s)ds,$$

For any $t \in [T_j, T_{j-1}]$ and $j = 2, \dots, k$. Therefore,

$$V_t = \int_0^t f(s, Y_s, L(M)_s)ds, \quad \forall t \in [0, T].$$

And $Y = M - V$, $Y_T = \xi$, which together imply that

$$M_t - Y_t = \int_0^t f(s, Y_s, L(M)_s)ds, \quad \forall t \in [0, T].$$

Thus (Y, M) solves the backward equation (3.11). Uniqueness follows the fact the solution $(Y(j), M(j) - M(j)_{T_j})$ is unique for any j. ■

Corollary 3.3.2 (Pardoux-Peng 1990 [3]) There exists a unique pair $(Y, Z) \in C_T^2(\mathbb{R}^d) \times \mathcal{H}_T^2(\mathbb{R}^{n\times d})$ which solves the BSDE:

$$dY_t = -f(t, Y_t, Z_t)dt + Z_t^* dB_t, \qquad Y_T = \xi,$$

where f satisfies the Lipschitz condition defined as before.

Proof:

As explained in the chapter 1, the above BSDE is equivalent to the functional differential equation:

$$dV_t = f(t, Y_t, L(M)_t)dt, \quad Y_T = \xi,$$

where $L(M)$ is defined by $L(M)_t = Z_t$, where the density process Z_t is given by

$$M_t - M_\tau = \int_\tau^t Z_s^* dB_s$$

in this particular case.

Clearly, $L(M)$ satisfies the the Lipschitz condition, the local-in-time property and the differential property, and the generator f is Lipschitz continuous. Therefore, by theorem 3.3.1, there exists a unique pair of solution (Y, M) in the sense if $(Y, M)$and $(\widetilde{Y}, \widetilde{M})$ are two pairs of solutions, then $Y = \widetilde{Y}$ and $M - M_\tau = \widetilde{M} - \widetilde{M}_\tau$ on $[\tau, T]$. Since Z_t only depends on the increments $\{M_t - M_\tau\}$ for $t \in [\tau, T]$, Z_t is uniquely determined. We conclude that there exists a unique pair $(Y, Z) \in C_T^2(\mathbb{R}^d) \times \mathcal{H}_T^2(\mathbb{R}^{n\times d})$ which solves the BSDE:

$$dY_t = -f(t, Y_t, Z_t)dt + Z_t^* dB_t, \qquad Y_T = \xi.$$

■

Proposition 3.3.3 Let (β, γ) be a bounded $(\mathbb{R}, \mathbb{R}^n)$-valued predictable process, φ an element of $\mathcal{H}_T^2(\mathbb{R})$, and ξ an element of $\mathbb{L}_T^2(\mathbb{R})$. Then the LBSDE

$$dY_t = -[\varphi_t + Y_t\beta_t + Z_t^*\gamma_t]dt + Z_t^* dW_t, \quad Y_T = \xi, \quad (3.46)$$

has a unique solution (Y, Z) in $C_T^2(\mathbb{R}) \times \mathcal{H}_T^2(\mathbb{R}^n)$, and Y_t is given by the closed formula

$$Y_t = \frac{1}{\Gamma_t}\mathbb{E}\left[\xi\Gamma_T + \int_0^T \Gamma_s\varphi_s ds \middle| \mathcal{F}_t\right],$$

where Γ_t is a process defined by the forward LSDE

$$d\Gamma_t = \Gamma_t[\beta_t dt + \gamma_t^* dW_t], \quad \Gamma_0 = 1.$$

In particular, if ξ and φ are nonnegative, the process Y is nonnegative. If, in addition, $Y_0 = 0$, then, for any t, $Y_t = 0$ a.s., and $\varphi_t = 0$ a.s.

Proof:

Since β, γ are bounded processes, the linear generator $f(t, Y_t, Z_t) = [\varphi_t + Y_t\beta_t + Z_t^*\gamma_t]$ is uniformly Lipschitz. By corollary 3.3.3, there exists a unique square-integrable solution (Y, Z) of the linear BSDE.

Now we need to prove $Y_t = \frac{1}{\Gamma_t}\mathbb{E}\left[\xi\Gamma_T + \int_t^T \Gamma_s\varphi_s ds\,|\mathcal{F}_t\right]$ satisfies the linear BSDE.

Firstly, check the terminal condition: $Y_T = \frac{1}{\Gamma_T}\mathbb{E}[\xi\Gamma_T] = \xi$.

Then check if

$$Y_t = \frac{1}{\Gamma_t}\mathbb{E}\left[\xi\Gamma_T + \int_t^T \Gamma_s\varphi_s ds \middle| \mathcal{F}_t\right]$$

satisfies

$$dY_t = -[\varphi_t + Y_t\beta_t + Z_t^*\gamma_t]dt + Z_t^*dW_t.$$

Now consider $X_t = Y_t\Gamma_t + \int_0^t \Gamma_s\varphi_s ds$. By the Itô's lemma,

$$\begin{aligned} d(X_t) &= Y_t d\Gamma_t + \Gamma_t dY_t + \langle Y, \Gamma\rangle_t + \Gamma_t\varphi_t dt \\ &= Y_t[\Gamma_t(\beta_t dt + \gamma_t^* dW_t)] + \Gamma_t[-(\varphi_t + Y_t\beta_t + Z_t^*\gamma_t)dt + Z_t^* dW_t] \\ &\quad + Z_t^*\Gamma_t\gamma_t^* dt + \Gamma_t\varphi_t dt \\ &= Y_t\Gamma_t\gamma_t^* dW_t + \Gamma_t Z_t^* dW_t \\ &= (Y_t\Gamma_t\gamma_t^* + \Gamma_t Z_t^*)dW_t. \end{aligned}$$

Thus, it implies X_t is a local martingale. Moreover, since $\sup_{s\le T}|Y_s|$ and $\sup_{s\le T}|\Gamma_s|$ belong to $\mathbb{L}_T^{2,1}$, it follows that $\sup_{s\le T}|Y_s| \times \sup_{s\le T}|\Gamma_s|$ belongs to $\mathbb{L}_T^{1,1}$. Then we conclude that

$$X_t = Y_t\Gamma_t + \int_0^t \Gamma_s\varphi_s ds$$

is uniformly integrable, so X_t is a martingale.

Hence we have

$$Y_t\Gamma_t + \int_0^t \Gamma_s\varphi_s ds = \mathbb{E}\left[\xi\Gamma_T + \int_0^T \Gamma_s\varphi_s ds \,|\mathcal{F}_t\right]$$

which implies

$$Y_t = \frac{1}{\Gamma_t}\mathbb{E}\left[\xi\Gamma_T + \int_t^T \Gamma_s\varphi_s ds \middle| \mathcal{F}_t\right].$$

Hence Y_t satisfies

$$dY_t = -[\varphi_t + Y_t\beta_t + Z_t^*\gamma_t]dt + Z_t^* dW_t.$$

Next by considering

$$d\Gamma_t = \Gamma_t[\beta_t dt + \gamma_t^* dW_t], \quad \Gamma_0 = 0,$$

we have

$$\Gamma_t = \exp\left[\int_0^t \beta_s ds + \int_0^t \gamma_t^* dW_s - \frac{1}{2}\int_0^t |\gamma_s|^2 ds\right]$$

which is a nonnegative process.

Therefore, if ξ and φ are nonnegative, Y_t is also nonnegative.

If, in addition,

$$Y_0 = 0 = \mathbb{E}\left[\xi\Gamma_T + \int_0^T \Gamma_s\varphi_s ds\right],$$

Then

$$\xi = 0, \varphi_t = 0 \quad a.s.$$

It follows

$$Y_t = 0 \quad a.s.$$

■

3.4 Generalised Comparison Theorem

Starting from the section, we only consider the case when $p = 2$.

The Comparison theorem for BSDEs turns to be one of the classic and important results of properties of BSDEs. It was first introduced by (Peng, 1992) [6] under the Lipschitz hypothesis on the coefficient, then further studied by (Cao and Yan, 1999) [28] with a special diffusion coefficient. The comparison theorem plays the same role that the maximum principle in the theory of partial differential equation. In mathematical finance, it gives a sufficient condition for the wealth process to be nonnegative and yields the classical properties of utilities.

In this section, we use the martingale approach other than the pure probabilistic approach to establish the comparison theorem for the kind of BSDEs like (3.11).

Lemma 3.4.1 (Cao and Yan, 1999) [28] Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a filtered probability space which satisfies the conditions: $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete space; $(\mathcal{F}_t)_{t\geq 0}$ is continuous; all local martingales on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ is continuous. Also, let $X_t = X_0 + M_t + V_t$ be a continuous semimartingale on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, where M_t is a continuous $\mathcal{F}_t$ local martingale with $M_0 = 0$ and V_t is a continuous $\mathcal{F}_t$-adapted process of finite variation with $V_0 = 0$. Then

$$X_t^{+2} = X_0^{+2} + 2\int_0^t X_s^+ dM_s + 2\int_0^t X_s^+ dV_s + \int_0^t I_{(X_s>0)} d\langle M\rangle_s.$$

Proof: Applying Itô's formula to the Tanaka-Meyer formula, we refer to (Cao and Yan, 1999) for details. Alternatively, we can get the result directly by Itô-Tanaka formula (Trotter, Meyer).

Theorem 3.4.2 (Comparison Theorem)

Consider the following BSDEs

$$dY_t = -f(t, Y_t, L(M)_t)dt + dM_t, \quad Y_T = \xi \quad (3.41)$$

and

$$d\overline{Y}_t = -\overline{f}(t, \overline{Y}_t, L(\overline{M})_t)dt + d\overline{M}_t, \quad \overline{Y}_T = \overline{\xi} \quad (3.42)$$

on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ which satisfies the conditions: $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete space; $(\mathcal{F}_t)_{t\geq 0}$ is continuous; all martingales on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ is continuous. $L: \mathcal{M}^2([\tau, T];\ \mathbb{R}^d) \longrightarrow \mathcal{H}^2([\tau, T];\ \mathbb{R}^{n\times d})$ is a prescribed mapping on $\mathcal{M}^p([\tau, T];\ \mathbb{R}^d)$

valued in $\mathcal{H}^p([\tau, T];\ \mathbb{R}^{n\times d})$. Moreover, f satisfies the Lipschitz condition, and L satisfies the Lipschitz condition, the local-in-time property and the differential property to ensure the existence and uniqueness of solutions.

Let (Y, M) and $(\overline{Y}, \overline{M}) \in C^2(\mathbb{R}^d) \times \mathcal{M}^2(\mathbb{R}^d)$ be the unique adapted solutions of (3.41) and (3.42) respectively. Assume that L also satisfies the following condition:

For any $\widetilde{M}_1, \widetilde{M}_2 \in \mathcal{M}^2(\mathbb{R}^d)$ and $t \in [0, T]$,

$$\mathbb{E}\left[\langle \widetilde{M}_1 - \widetilde{M}_2\rangle_T - \langle \widetilde{M}_1 - \widetilde{M}_2\rangle_t\right] \geq \alpha \mathbb{E}\left[\int_t^T \left|L(\widetilde{M}_1)_s - L(\widetilde{M}_2)_s\right|^2 ds\right],$$

where α is a positive constant.

(1) If $\xi \leq \bar{\bar{\xi}}$ a.s., $f(t, \overline{Y}_t, L(\overline{M})_t) \leq \bar{f}(t, \overline{Y}_t, L(\overline{M})_t)$ a.s., then $Y_t \leq \overline{Y}_t$, a.s.,$\forall 0 \leq t \leq T$;
(2) If $\xi \geq \bar{\bar{\xi}}$ a.s., $f(t, \overline{Y}_t, L(\overline{M})_t) \geq \bar{f}(t, \overline{Y}_t, L(\overline{M})_t)$ a.s., then $Y_t \geq \overline{Y}_t$, a.s.,$\forall 0 \leq t \leq T$.

Proof:

We only need to prove (1), since (2) can be easily deduced from (1).

Let $\widehat{Y}_t = Y_t - \overline{Y}_t$, $\hat{\xi} = \xi - \bar{\bar{\xi}}$, $\widehat{M}_t = M_t - \overline{M}_t$, from (3.41) and (3.42) we obtain

$$\widehat{Y}_t = \hat{\xi} + \int_t^T [f(s, Y_s, L(M)_s) - \bar{f}(t, \overline{Y}_s, L(\overline{M})_s)]ds - \int_t^T d\widehat{M}_s$$

and

$$\widehat{Y}_t = \widehat{Y}_0 - \int_0^t [f(s, Y_s, L(M)_s) - \bar{f}(t, \overline{Y}_s, L(\overline{M})_s)]ds + \int_0^t d\widehat{M}_s.$$

Therefore, $\widehat{Y}_t$ is a continuous semimartingale. By Lemma 3.4.1, it follows for $t \in [0, T]$,

$$\widehat{Y}_t^{+2} = \hat{\xi}^{+2} + 2\int_t^T \widehat{Y}_s^+ \left[f(s, Y_s, L(M)_s) - \bar{f}(t, \overline{Y}_s, L(\overline{M})_s)\right]ds - 2\int_t^T \widehat{Y}_s^+ \, d\widehat{M}_s$$
$$- \int_t^T I_{(Y_s>0)} d\langle \widehat{M}\rangle_s.$$

Rearranging the equation above, we have

$$\widehat{Y}_t^{+2} + \int_t^T I_{(Y_s>0)} d\langle \widehat{M}\rangle_s = \hat{\xi}^{+2} + 2\int_t^T \widehat{Y}_s^+ \left[f(s, Y_s, L(M)_s) - \bar{f}(t, \overline{Y}_s, L(\overline{M})_s)\right]ds$$

$$-2\int_t^T \widehat{Y}_s^+ \, d\widehat{M}_s. \quad (3.43)$$

Next we show that $\int_t^T \widehat{Y}_s^+ \, d\widehat{M}_s$ is a martingale. By the Burkholder–Davis–Gundy inequality, we have

$$\mathbb{E}\left[\sup_{0\le t\le T}\left|\int_0^t \widehat{Y}_s^+ \, d\widehat{M}_s\right|\right] \le \hat{C}\mathbb{E}\left[\left(\int_0^T \left|\widehat{Y}_s^+\right|^2 d\langle\widehat{M}\rangle_s\right)^{\frac{1}{2}}\right]$$

$$\le \hat{C}\mathbb{E}\left[\sup_{0\le t\le T}\left|\widehat{Y}_s^+\right|\left(\int_0^T 1 \, d\langle\widehat{M}\rangle_s\right)^{\frac{1}{2}}\right]$$

$$\le \hat{C}\,\mathbb{E}\left[\sup_{0\le t\le T}\left|\widehat{Y}_s^+\right|\langle\widehat{M}\rangle_T^{\frac{1}{2}}\right]$$

$$\le \frac{\hat{C}}{2}\left\{\mathbb{E}\left[\sup_{0\le t\le T}\left|\widehat{Y}_s^+\right|^2\right] + \mathbb{E}\left[\langle\widehat{M}\rangle_T\right]\right\}$$

$$\le \frac{\hat{C}}{2}\left\{2\mathbb{E}[\sup_{0\le t\le T}|Y_s|^2] + 2\mathbb{E}[\sup_{0\le t\le T}|\overline{Y}_s|^2] + \mathbb{E}\left[\langle\widehat{M}\rangle_T\right]\right\}$$

$$< \infty,$$

where $\hat{C}$ is a positive constant.

Hence $\int_t^T \widehat{Y}_s^+ \, d\widehat{M}_s$ is a martingale. Taking expectation on the both sides of the equation (3.43), we have

$$\mathbb{E}\left({\widehat{Y}_t^+}^2\right) + \mathbb{E}\left(\int_t^T I_{(Y_s>0)} d\langle\widehat{M}\rangle_s\right) = 2\mathbb{E}\left\{\int_t^T \widehat{Y}_s^+ \left[f(s, Y_s, L(M)_s) - \bar{f}(t, \overline{Y}_s, L(\overline{M})_s)\right]ds\right\}.$$

It follows

$$\mathbb{E}\left({\widehat{Y}_t^+}^2\right) + \mathbb{E}\left(\int_t^T I_{(Y_s>0)} d\langle\widehat{M}\rangle_s\right)$$

$$\le \mathbb{E}\left\{\int_t^T \frac{2C_2^2}{\alpha}{\widehat{Y}_s^+}^2 + I_{(Y_s>0)}\frac{\alpha}{2C_2^2}\left|f(s, Y_s, L(M)_s) - \bar{f}(t, \overline{Y}_s, L(\overline{M})_s)\right|^2 ds\right\}$$

$$\le \mathbb{E}\left\{\int_t^T \frac{2C_2^2}{\alpha}{\widehat{Y}_s^+}^2 + I_{(Y_s>0)}\frac{\alpha}{2C_2^2}\left[C_2\left(\left|\widehat{Y}_s\right| + |L(M)_s - L(\overline{M})_s|\right)\right]^2 ds\right\}$$

$$\leq \mathbb{E}\left\{\int_t^T \frac{2C_2^2}{\alpha}\hat{Y}_s^{+^2} + I_{(Y_s>0)}\alpha\left[\left|\hat{Y}_s\right|^2 + |L(M)_s - L(\overline{M})_s|^2\right] ds\right\},$$

where C_2 is the Lipschitz constant.

By the assumption, we have

$$\mathbb{E}\left[\int_t^T d\langle\hat{M}\rangle_s\right] \geq \alpha\mathbb{E}\left[\int_t^T |L(M)_s - L(\overline{M})_s|^2\, ds\right].$$

It implies

$$\mathbb{E}\left(\hat{Y}_t^{+^2}\right) \leq \mathbb{E}\left\{\int_t^T \frac{2C_2^2}{\alpha}\hat{Y}_s^{+^2} + \alpha\left|\hat{Y}_s\right|^2 ds\right\}$$

$$\leq \mathbb{E}\left\{\int_t^T \left(\frac{2C_2^2}{\alpha} + \alpha\right)\hat{Y}_s^{+^2}\, ds\right\}$$

$$\leq \left(\frac{2C_2^2}{\alpha} + \alpha\right)\int_t^T \mathbb{E}\left(\hat{Y}_s^{+^2}\right) ds.$$

We put $g(t) = \mathbb{E}\left(\hat{Y}_{T-t}^{+^2}\right)$, $0 \leq t \leq T$. Then from the above inequality, we get

$$g(t) \leq \left(\frac{2C_2^2}{\alpha} + \alpha\right)\int_{T-t}^T \mathbb{E}\left(\hat{Y}_s^{+^2}\right) ds$$

$$= \left(\frac{2C_2^2}{\alpha} + \alpha\right)\int_{T-t}^T g(T-s)\, ds$$

$$= \left(\frac{2C_2^2}{\alpha} + \alpha\right)\int_0^t g(u)\, du$$

Thus, by Gronwall's inequality, we have that $g(t) = 0, 0 \leq t \leq T$, i.e., $\hat{Y}_t^{+} = 0, 0 \leq t \leq T$.
■

Corollary 3.4.3 (Peng, 1992a) Consider the following BSDEs

$$dY_t = -f(t, Y_t, Z_t)dt + Z_t dB_t, \quad Y_T = \xi, \quad (3.44)$$

and

$$d\overline{Y}_t = -\overline{f}(t, \overline{Y}_t, \overline{Z}_t)dt + \overline{Z}_t dB_t, \quad \overline{Y}_T = \overline{\xi}, \quad (3.45)$$

where f satisfies the Lipschitz condition. Let (Y, Z) and $(\overline{\mathrm{Y}}, \overline{\mathrm{Z}}) \in \mathrm{C}^2(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R}^n)$ be the unique adapted solution of (3.44) and (3.45) respectively.

If $\xi \leq \bar{\xi}$ a.s., $\mathrm{f}(\mathrm{t}, \overline{\mathrm{Y}}_t, \overline{\mathrm{Z}}_t) \leq \bar{\mathrm{f}}(\mathrm{t}, \overline{\mathrm{Y}}_t, \overline{\mathrm{Z}}_t)$ a.s., then $\mathrm{Y}_t < \overline{\mathrm{Y}}_t$, a.s., $\forall 0 \leq t \leq T$.

Proof:

In this particular case, $\mathrm{L}(\mathrm{M}): \mathcal{M}^2(\mathbb{R}) \longrightarrow \mathcal{H}^2(\mathbb{R}^n)$ is defined by $\mathrm{L}(\mathrm{M})_t = \mathrm{Z}_t$, where the density process Z_t is given by $\mathrm{M}_t - \mathrm{M}_\tau = \int_\tau^t \mathrm{Z}_s d\mathrm{B}_s$, for any $t \in [\tau, \mathrm{T}]$. Without losing generality, let $\mathrm{M}_\tau = 0$, so we have $\mathrm{M}_t = \int_\tau^t \mathrm{Z}_s d\mathrm{B}_s$.

For any $\widetilde{\mathrm{M}}_1, \widetilde{\mathrm{M}}_2 \in \mathcal{M}^2(\mathbb{R})$,

$$\mathbb{E}\left[\langle \widetilde{\mathrm{M}}_1 - \widetilde{\mathrm{M}}_2 \rangle_\mathrm{T} - \langle \widetilde{\mathrm{M}}_1 - \widetilde{\mathrm{M}}_2 \rangle_t\right] = \mathbb{E}\left\{\int_t^\mathrm{T} \left[\left(\tilde{\mathrm{Z}}_1\right)_s - \left(\tilde{\mathrm{Z}}_2\right)_s\right]^2 ds\right\},$$

On the other hand,

$$\mathbb{E}\left[\int_t^\mathrm{T} \left|\mathrm{L}(\widetilde{\mathrm{M}}_1)_s - \mathrm{L}(\widetilde{\mathrm{M}}_2)_s\right|^2 ds\right] = \mathbb{E}\left\{\int_t^\mathrm{T} \left[\left(\tilde{\mathrm{Z}}_1\right)_s - \left(\tilde{\mathrm{Z}}_2\right)_s\right]^2 ds\right\},$$

Therefore,

$$\mathbb{E}\left[\langle \widetilde{\mathrm{M}}_1 - \widetilde{\mathrm{M}}_2 \rangle_\mathrm{T} - \langle \widetilde{\mathrm{M}}_1 - \widetilde{\mathrm{M}}_2 \rangle_t\right] = \mathbb{E}\left[\int_t^\mathrm{T} \left|\mathrm{L}(\widetilde{\mathrm{M}}_1)_s - \mathrm{L}(\widetilde{\mathrm{M}}_2)_s\right|^2 ds\right].$$

By Theorem 3.42, with $\alpha = 1$, it follows $\mathrm{Y}_t < \overline{\mathrm{Y}}_t$. ■

3.5 Application to Standard European Option Pricing

BSDEs are widely used in numerous problems in finance. Primarily, the theory of contingent claim valuation in a complete market can be expressed in terms of BSDEs. The problem is to determine the price of a contingent claim $\xi \geq 0$ of maturity T, which is a contract that pays an amount ξ at time T. In a complete market, it is possible to construct a portfolio which attains as final wealth the amount ξ. Therefore, the corresponding BSDE gives the dynamics of the value of the replication portfolio which is the fair price of the contingent claim.

In this section, we illustrate how the existence and uniqueness of square-integrable solutions of BSDEs, comparison theorem, etc can help price a standard European contingent claim ξ in a complete market.

We begin with the typical setup for continuous-time asset pricing. Suppose in a general market there $n+1$ assets, where the first one is riskless while the others are risky. The riskless asset price S_t^0 satisfies $dS_t^0 = r_t S_t^0 dt$, where $r(\cdot)$ is a one dimensional $\mathcal{F}_t -$ adapted measurable process. The $i-$th risky asset price $S_t^i, i = 1,2,..,n$, satisfies $dS_t^i = S_t^i[a_t^i dt + \sum_{j=1}^m \sigma_t^{i,j}\, dB_t^j]$, where $B = (B^1, \dots, B^m)$ is a standard Brownian motion on $\mathbb{R}^m$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)$ and where $a^i(\cdot)$ and $\sigma^{i,j}(\cdot)$ are called the appreciation rate and volatility rate respectively, and $\mathcal{F}_t -$ adapted measurable processes.

Consider an agent with an initial endowment $X_0 \in \mathbb{R}$, and an investment horizon $T > 0$. Let his allocation to the $i-$th asset, $i = 0,1, \dots, n$ be π_t^i, the corresponding number of shares be N_t^i, and the total wealth be X_t at time t. It follows $X_t = \sum_{i=0}^n N_t^i\, S_t^i$.

Suppose the agent's strategy is self-financing. Then we have

$$
\begin{aligned}
dX_t &= \sum_{i=0}^{n} N_t^i\, dS_t^i \\
&= r_t N_t^0 S_t^0 dt + \sum_{i=1}^{n} N_t^i\, S_t^i \left[a_t^i dt + \sum_{j=1}^{m} \sigma_t^{i,j}\, dB_t^j\right] \\
&= r_t \left\{\left[X_t - \sum_{i=1}^{n} \pi_t^i\right] + \sum_{i=1}^{n} a_t^i \pi_t^i\right\} + \sum_{i=1}^{n}\sum_{j=1}^{m} \sigma_t^{i,j} \pi_t^i\, dB_t^j \\
&= \left[r_t X_t + \sum_{i=1}^{n} (a_t^i - r_t) \pi_t^i\right] dt + \sum_{i=1}^{n}\sum_{j=1}^{m} \sigma_t^{i,j} \pi_t^i\, dB_t^j \\
&= [r_t X_t + \pi_t^* (a_t - r_t 1)] dt + \pi_t^* \sigma_t dB_t,
\end{aligned}
$$

which is called a wealth equation.

Assumptions:

- The nonnegative risky rate r is a predictable and bounded process.
- The appreciation rate $a = (a^1, \dots, a^n)^*$ is a predictable and bounded process.

- The volatility rate $\sigma = (\sigma^{i,j})$ is a predictable and bounded process. σ_t has full rank a.s. for all $t \in [0, T]$ and the inverse σ^{-1} has a bounded process.
- There exists a predictable and bounded valued process vectors θ, called a risk premium, such that

$$a_t - r_t 1 = \sigma_t \theta_t , \quad d\mathbb{P} \otimes dt \text{ a. s.}$$

Under these assumptions, the market is arbitrage-free and complete on $[0, T]$, and the wealth equation becomes

$$dX_t = [r_t X_t + \pi_t^* \sigma_t \theta_t] dt + \pi_t^* \sigma_t dB_t.$$

Definition 3.5.1 $\pi(\cdot)$ is called an admissible portfolio if it is self-financing and $\pi(\cdot) \in \mathbb{L}_T^2(\mathbb{R}^n)$.

Definition 3.5.2 A European contingent claim ξ settled at time T is a $\mathcal{F}_T -$ measurable random variable. The claim ξ is called replicable if there exists an initial X_0 and an admissible portfolio $\pi(\cdot)$ such that the corresponding $X(\cdot)$ satisfies $X(T) = \xi$.

Remark: The European contingent claim can be thought of as a contract which pays ξ at maturity T. The arbitrage-free pricing of a positive contingent claim is based on the following principle: if we start with the price of the claim as initial endowment and invest it in the $n + 1$ assets, the values of the portfolio at time T must be just enough to guarantee ξ.

Definition 3.5.3 A market is called complete on [0,T] if any claim $\xi \in \mathbb{L}_T^2(\mathbb{R})$ is replicable.

Definition 3.5.4 A self-financing trading strategy is a pair (X, π), where X is the market value and $\pi = (\pi^1, \dots, \pi^n)^*$ is the portfolio process, such that (X, π) satisfies

$$dX_t = [r_t X_t + \pi_t^* \sigma_t \theta_t] dt + \pi_t^* \sigma_t dB_t,$$

$$\int_0^T |\pi_t^* \sigma_t|^2 dt < +\infty \ , \mathbb{P} \text{ a. s.}$$

Theorem 3.5.5 Let ξ be a positive square-integrable contingent claim. Under the assumptions mentioned above, then there exists a unique replication strategy (X, π) of ξ such that

$$dX_t = [r_t X_t + \pi_t^* \sigma_t \theta_t] dt + \pi_t^* \sigma_t dB_t, \quad X_T = \xi.$$

Hence X is the fair price of the claim, and X_t is given by

$$X_t = \mathbb{E}(H_T \xi | \mathcal{F}_t), \text{a.s.},$$

where H_t is the process defined by the forward LSDE

$$dH_t = -H_t[r_t dt + \theta_t^* dW_t], \quad H_0 = 1.$$

Proof:

By assumptions, we get that the market is arbitrage-free and complete on $[0, T]$. Then it implies ξ is replicable, which follows that there exists an initial X_0 and an admissible portfolio $\pi(\cdot) \in \mathbb{L}_T^2(\mathbb{R})$ such that the corresponding $X(\cdot)$ satisfies $X(T) = \xi$.

Since r_t and θ_t are bounded, by proposition 3.3.3, it follows there exists an unique solution pair $(X, \sigma^*\pi) \in C_T^2(\mathbb{R}) \times \mathcal{H}_T^2(\mathbb{R}^n)$, satisfying

$$dX_t = [r_t X_t + \pi_t^* \sigma_t \theta_t]dt + \pi_t^* \sigma_t dB_t, \quad X_T = \xi,$$

such that $\int_0^T |\pi_t^* \sigma_t|^2 dt < +\infty$, $\mathbb{P}$ a.s.

Therefore, (X, π) is the unique replication strategy of ξ. And also it easily followed by proposition 3.3.3 that

$$X_t = \mathbb{E}(H_T \xi | \mathcal{F}_t), \text{a.s.},$$

where

$$dH_t = -H_t[r_t dt + \theta_t^* dW_t], \quad H_0 = 1.$$

Proposition 3.5.6 Consider the general setting of the wealth equation:

$$dX_t = -f(t, X_t, \pi_t^* \sigma_t) + \pi_t^* \sigma_t dB_t,$$

where f is a real process defined on $\Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^n$ satisfying the Lipschitz condition. Suppose X is the wealth process associated with an admissible strategy which finances the contingent claim ξ; i.e. $(X, \sigma^*\pi)$ is the square-integrable solution of the following BSDE:

$$dX_t = -f(t, X_t, \pi_t^* \sigma_t) + \pi_t^* \sigma_t dB_t, \quad X_T = \xi.$$

Then the following properties hold:

1. The price X is increasing with respect to the contingent claim ξ.

2. Suppose $\xi \geq 0$ and $f(t, 0,0) \geq 0, d\mathbb{P} \otimes dt$ a. s., then the price is nonnegative.

Proof 1:

Let $(X, \sigma^*\pi)$ and $(\overline{X}, \overline{\sigma^*\pi}) \in C^2(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R}^n)$ be the unique adapted solutions of

$$dX_t = -f(t, X_t, \pi_t^*\sigma_t) + \pi_t^*\sigma_t dB_t, \quad X_T = \xi$$

and

$$d\overline{X}_t = -f\big(t, \overline{X}_t, \overline{\pi_t^*\sigma_t}\big) + \overline{\pi_t^*\sigma_t} dB_t, \quad \overline{X}_T = \overline{\xi}.$$

Suppose that $\xi \leq \overline{\xi}$, together with $f\big(t, \overline{X}_t, \overline{\pi_t^*\sigma_t}\big) = f\big(t, \overline{X}_t, \overline{\pi_t^*\sigma_t}\big)$. Hence by comparison theorem, it follows that $X_t \leq \overline{X}_t$. Hence the price X is increasing with respect to the contingent claim ξ. ■

Proof 2:

Let $(X, \sigma^*\pi)$ and $(0, 0) \in C^2(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R}^n)$ be the unique adapted solutions of

$$dX_t = -f(t, X_t, \pi_t^*\sigma_t) + \pi_t^*\sigma_t dB_t, \quad X_T = \xi$$

and

$$d\overline{X}_t = 0, \quad \overline{X}_T = 0.$$

Since $\xi \geq 0$ and $f(t, 0,0) \geq 0$, by comparison theorem, it follows that $X_t \geq \overline{X}_t = 0$. Hence the price is nonnegative. ■

Chapter 4

Malliavin Derivatives of BSDE solutions

In mathematics, the Malliavin derivative is the notion of derivative appropriate to paths in Wiener space, which are not differentiable in the usual sense. In this chapter, we still use Liang, Lyons and Qian's idea (reformulating BSDE as ordinary differential equations) to show that the Malliavin derivative of the solution of

$$dY_t = -f(t, Y_t, Z_t) + Z_t dB_t\,, \quad Y_T = \xi \quad (4.1)$$

is still the solution of a linear BSDE(see El Karoui, Peng and Quenez, 1997 [19]). Then this property together with the comparison theorem is applied to the European option pricing in two constrained cases.

4.1 Preliminaries

Initially, we recall briefly the notion of differentiation on Wiener space (see Nualart, 2006 [29], El Karoui, Peng and Quenez, 1997 [19])

- $C_b^k(\mathbb{R}^k, \mathbb{R}^q)$ will denote the set of functions of class C^k from $\mathbb{R}^k$ into $\mathbb{R}^q$ whose partial derivative of order less than or equal to k are bound.
- Let $\mathcal{S}$ denote the set of random variables ξ of the form
$$\xi = \varphi\left(W(h^1), \dots, W\left(h^k\right)\right),$$
where
$$\varphi \in C_b^\infty\left(\mathbb{R}^k, \mathbb{R}\right), h^1, \dots, h^k \in \mathbb{L}^2([0,T]; \mathbb{R}^n), \text{and } W\left(h^i\right) = \int_0^T \langle h^i, dW_s \rangle.$$
- If $\xi \in \mathcal{S}$ is of the above form, we define its derivative as being the $n-$dimensional process
$$D_\theta \xi = \sum_{j=1}^{k} \frac{\partial \varphi}{\partial x_j}\left(W(h^1), \dots, W\left(h^k\right)\right) h_\theta^j, \quad 0 \le \theta \le T.$$
For $\xi \in \mathcal{S}, p > 1$, we define the norm
$$\|\xi\|_{1,p} = \left[\mathbb{E}\left\{|\xi|^p + \left(\int_0^T |D_\theta \xi|^2 d\theta\right)^{\frac{p}{2}}\right\}\right]^{\frac{1}{p}}.$$

It can be shown (Nualart, 2006) [29] that the operator D has a closed extension to the space $D^{1,p}$, the closure of $\mathcal{S}$ with respect to the norm $\|\cdot\|_{1,p}$. Observe that if ξ is $\mathcal{F}_t - measurable$, then $D_\theta \xi$=0 for $\theta \in (t, T]$. We denote by $D_\theta^i \xi, 1 \le i \le n$, the ith component of $D_\theta \xi$.

Let $\mathbb{L}^a_{1,p}(\mathbb{R}^d)$ denote the set of $\mathbb{R}^d -$ valued progressively measurable processes $\{u(t,\omega), 0 \le t \le T; \omega \in \Omega\}$ such that

1. For a.e. $t \in [0,T], u(t,\cdot) \in \left(D_{1,p}\right)^d$.
2. $(t,\omega) \longrightarrow Du(t,\omega) \in \left(\mathbb{L}^2([0,T])\right)^{n\times d}$ admits a progressively measurable version.
3. $\|u\|^a_{1,p} = \mathbb{E}\left[\left(\int_0^T |u(t)|^2 dt\right)^{\frac{p}{2}} + \left(\int_0^T \int_0^T |D_\theta u(t)|^2 d\theta dt\right)^{\frac{p}{2}}\right] < \infty.$

Observe that for each (θ, t, ω), $D_\theta u(t)$ is an $n \times d$ matrix. Thus

$$|D_\theta u(t)\ |^2 = \sum_{i,j} \left|D^i_\theta u_j(t)\right|^2.$$

Clearly, $D_\theta u(t,w)$ is defined uniquely up to sets of $d\theta \otimes dt \otimes dP$ measurable zero.

4.2 Differentiation on Wiener Space of BSDE Solutions

We now show that the solution of (4.1) is differential in Malliavin's sense and that the derivative is the solution of a linear BSDE. Although the result was stated and proved by (El Karoui, Peng and Quenez, 1997 [19]), in this paper we show this by reformulating the BSDE as an ordinary differential equation and using a different estimation rather than the priori estimation.

Lemma 4.2.1 (Pardoux and Peng, 1992 [7]) Let $Z \in \mathbb{H}^2_T(\mathbb{R}^n)$ be such that be such that $\xi = \int_t^T Z^*_s dB_s$ satisfies $\xi \in \mathbb{D}^{1,2}$. Then $Z^i \in \mathbb{L}^2(t,T;\ \mathbb{D}^{1,2}), 1 \le i \le n,$ and $d\theta \otimes d\mathbb{P}$ a.s.,

$$D^i_\theta \xi = \int_t^T D^i_\theta Z_r dB_r\,, \quad \theta \le t,$$

$$D^i_\theta \xi = Z^i_\theta + \int_\theta^T D^i_\theta Z_r dB_r\,, \quad \theta > t.$$

Proposition 4.2.2 (El Karoui, Peng and Quenez, 1997 [19]) Consider the BSDE:

$$dY_t = -f(t, Y_t, Z_t) + Z_t dB_t\,, \quad Y_T = \xi$$

Suppose that $\xi \in \mathbb{D}^{1,2}$ and $f: \Omega \times [0,T] \times \mathbb{R}^d \times \mathbb{R}^{n\times d} \longrightarrow \mathbb{R}^d$ is continuously differentiable in (y,z), with uniformly bounded and continuous derivatives and such that, for each (y,z), $f(\cdot\,,y,z)$ is in $\mathbb{L}^a_{1,2}(\mathbb{R}^d)$ with Malliavin derivative denoted by $D_\theta f(t,y,z)$. Let (Y,Z) be the solution of the associated BSDE.

Also, suppose that

- $\mathbb{E}\left[\int_0^T |f(t,0,0)|^2 ds\right] < \infty$ and $\xi \in \mathbb{L}^2_T(\mathbb{R}^d)$.
- $\int_0^T \mathbb{E}(|D_\theta \xi|^2) d\theta < +\infty$, $\int_0^T \|D_\theta f(t,Y,Z)\|_2^2 d\theta < +\infty$, and for any $t \in [0,T]$ and for and (y^1,z^1,y^2,z^2),
  $$|D_\theta f(t,\omega,y^1,z^1) - D_\theta f(t,\omega,y^2,z^2)| \le K_\theta(t,\omega)(|y^1-y^2| + |z^1-z^2|)$$
  where for a.e. θ, $\{K_\theta(t,\cdot), 0 \le t \le T\}$ is an $\mathbb{R}^+$-valued adapted and bounded process satisfying $|K_\theta| < +\infty$.

Then $(Y,Z) \in \mathbb{L}^2(0,T;(\mathbb{D}^{1,2})^d \times (\mathbb{D}^{1,2})^{n\times d})$, and for each $1 \le i \le n$, a version of $\{(D^i_\theta Y_t, D^i_\theta Z_t); 0 \le \theta, t \le T\}$ is given by

$$D^i_\theta Y_t = 0, \quad D^i_\theta Z_t = 0, \quad 0 \le t < \theta \le T$$

$$D^i_\theta Y_t = D^i_\theta \xi + \int_t^T [\partial_y f(s,Y_s,Z_s) D^i_\theta Y_s + \partial_z f(s,Y_s,Z_s) D^i_\theta Z_s + D^i_\theta f(s,Y_s,Z_s)]\, ds$$
$$- \int_t^T D^i_\theta Z_s dB_s \,. \quad \theta \le t \le T$$

Moreover,

$$D_t Y_t = Z_t. \quad 0 \le t \le T$$

Proof:

Without losing generality, let us set $d = 1$. Let (Y^k, Z^k) be the Picard iterative sequence defined recursively by $Y^0 = 0, Z^0 = 0$ and

$$dY^{k+1}_t = -f(t, Y^k_t, Z^k_t)dt + \left(Z^{k+1}_t\right)^* dB_t, \quad Y^{k+1}_T = \xi. \quad (4.21)$$

As we have shown in the introduction chapter, (4.21) is equivalent to

$$dV^{k+1}_t = f\left(t, Y^k(V^k)_t, L\left(M^k(V^k)\right)_t\right) dt,$$

where

$$M^k_t = \mathbb{E}(\xi + V^k_T | \mathcal{F}_t), Y^k_t = \mathbb{E}(\xi + V^k_T | \mathcal{F}_t) - V^k_t \text{ and } Y^k_t = M^k_t - V^k_t.$$

$V^0 = 0$ and $L\left(M^k(V^k)\right) = Z^k$, where the density process Z^k is given by

$$M_T^k - M_t^k = \int_t^T (Z_s^k)^* dB_t. \quad (4.22)$$

By the contraction mapping proved before, we know that the sequence (V^k) converges in $C_T^2(\mathbb{R})$ to (V) as $k \to +\infty$, the unique solution of BSDE.

By induction, we show that

$$(V^k, Y^k, Z^k) \in \mathbb{L}^2(0, T; (\mathbb{D}^{1,2}) \times (\mathbb{D}^{1,2}) \times (\mathbb{D}^{1,2})^n).$$

Suppose that $(V^k) \in \mathbb{L}^2(0, T; \mathbb{D}^{1,2})$. Since $\xi \in \mathbb{D}^{1,2}$ and $M_t^k = \mathbb{E}(\xi + V_T^k | \mathcal{F}_t), Y_t^k = \mathbb{E}(\xi + V_T^k | \mathcal{F}_t) - V_t^k$, we conclude $M^k, Y^k \in \mathbb{D}^{1,2}$. Then according to (4.22) and by lemma 4.2.1, it implies $Z^k \in \mathbb{D}^{1,2}$. Since $\int_t^T f(s, Y_s^k, Z_s^k)\, ds \in \mathbb{D}^{1,2}$, then $V^{k+1} \in \mathbb{D}^{1,2}$, which follows that $Y^{k+1}, Z^{k+1} \in \mathbb{D}^{1,2}$.

By differentiating (4.21), it follows that for $0 \le \theta \le t, 1 \le i \le n$,

$$dD_\theta^i Y_t^{k+1} = -\left[\partial_y f(t, Y_t^k, Z_t^k) D_\theta^i Y_t^k + \partial_z f(t, Y_t^k, Z_t^k) D_\theta^i Z_t^k + D_\theta^i f(t, Y_t^k, Z_t^k)\right] dt + \left(D_\theta^i Z_t^{k+1}\right)^* dB_t, \quad (4.23)$$

$$D_\theta^i Y_T^{k+1} = D_\theta^i \xi,$$

Reformulating (4.23) into an ordinary differential equation, we obtain by the same argument that (4.23) is equivalent to

$$dD_\theta^i V_t^{k+1} = \left[\partial_y f(t, Y_t^k, Z_t^k) D_\theta^i Y_t^k + \partial_z f(t, Y_t^k, Z_t^k) D_\theta^i Z_t + D_\theta^i f(t, Y_t^k, Z_t^k)\right] dt, \quad (4.24)$$

$$D_\theta^i Y_T^{k+1} = D_\theta^i \xi,$$

where

$$D_\theta M_t^k = \mathbb{E}(D_\theta \xi + D_\theta V_T^k | \mathcal{F}_t), D_\theta Y_t^k = \mathbb{E}(D_\theta \xi + D_\theta V_T^k | \mathcal{F}_t) - D_\theta V_t^k \text{ and } D_\theta Y_t^k = D_\theta M_t^k - D_\theta V_t^k.$$

The density process $D_\theta Z^k$ is given by

$$D_\theta^i M_T^k - D_\theta^i M_t^k = \int_t^T D_\theta^i Z_s^k dB_t, \quad 1 \le i \le n, \quad 0 \le \theta \le t \le T.$$

Without losing generality and to simply notation, we assume that the Brownian is one-dimension.

We need to show that $(D_\theta V^k)$ converges to (V^θ) in $\mathbb{L}^2(0,T;\mathbb{D}^{1,2})$, where V^θ satisfies

$$dV_t^\theta = \left[\partial_y f(t,Y_t,Z_t)Y_t^\theta + \partial_z f(t,Y_t,Z_t)Z_t^\theta + D_\theta f(t,Y_t,Z_t)\right]dt, \quad (4.25)$$

$$Y_T^\theta = D_\theta \xi.$$

where

$$M_t^\theta = \mathbb{E}\left(D_\theta\xi + V_T^\theta \middle| \mathcal{F}_t\right), Y_t^\theta = \mathbb{E}\left(D_\theta\xi + V_T^\theta \middle| \mathcal{F}_t\right) - V_t^\theta \text{ and } Y_t^\theta = M_t^\theta - V_t^\theta.$$

and $L\left(M^\theta(V^\theta)\right) = Z^\theta$, where the density process Z^θ is given by

$$M_T^\theta - M_t^\theta = \int_t^T Z_s^\theta dB_t.$$

Subtracting between (4.24) and (4.25), we have

$$d\left(V_t^\theta - D_\theta V_t^{k+1}\right)$$

$$= \{\left[\partial_y f(t,Y_t,Z_t)Y_t^\theta + \partial_z f(t,Y_t,Z_t)Z_t^\theta + D_\theta f(t,Y_t,Z_t)\right]$$

$$-[\partial_y f(t,Y_t^k,Z_t^k)D_\theta Y_t^k + \partial_z f\left(t,Y_t^k,Z_t^k\right)D_\theta Z_t^k + D_\theta f\left(t,Y_t^k,Z_t^k\right)]\}dt, \qquad Y_T^\theta - D_\theta Y_T^{k+1} = 0.$$

By integrating on both sides over the interval $[\theta, t]$ and with initial data $V_\theta = 0$, it follows

$$V_t^\theta - D_\theta V_t^{k+1}$$

$$= \int_\theta^t \{\left[\partial_y f(s,Y_s,Z_s)Y_s^\theta + \partial_z f(s,Y_s,Z_s)Z_s^\theta + D_\theta f(s,Y_s,Z_s)\right]$$

$$-[\partial_y f(t,Y_s^k,Z_s^k)D_\theta Y_s^k + \partial_z f\left(s,Y_s^k,Z_s^k\right)D_\theta Z_s^k + D_\theta f\left(s,Y_s^k,Z_s^k\right)]\}ds.$$

Then we obtain for almost all $\theta \in [0,T]$ that

$$\left\|V^\theta - D_\theta V^{k+1}\right\|_{C^2}^2$$

$$\leq C\left[\mathbb{E}\left(\int_\theta^t \left[\partial_y f(s,Y_s,Z_s)Y_s^\theta + \partial_l f(s,Y_s,Z_s)Z_s^\theta + D_\theta f(s,Y_s,Z_s)\right]\right.\right.$$

$$\left.\left. - \left[\partial_y f(t,Y_s^k,Z_s^k)D_\theta Y_s^k + \partial_z f\left(s,Y_s^k,Z_s^k\right)D_\theta Z_s^k + D_\theta f\left(s,Y_s^k,Z_s^k\right)\right]ds\right)^2\right]$$

$$\leq C\left[A_k^\theta(T) + B_k^\theta(T) + C_k^\theta(T)\right],$$

where C is a positive constant, and

$$A_k^\theta(T) = \mathbb{E}\left(\int_\theta^T \left|D_\theta\, f(s, Y_s, Z_s) - D_\theta f\left(s, Y_s^k, Z_s^k\right)\right| ds\right)^2$$

$$B_k^\theta(T) = \mathbb{E}\left(\int_\theta^T \left|\partial_y f\left(t, Y_s^k, Z_s^k\right)\left(Y_s^\theta - D_\theta Y_s^k\right)\right| ds\right)^2$$

$$+\mathbb{E}\left(\int_\theta^T \left|\partial_z f\left(t, Y_s^k, Z_s^k\right)(Z_s^\theta - D_\theta Z_s^k)\right| ds\right)^2$$

$$C_k^\theta(T) = \mathbb{E}\left(\int_\theta^T \left|\left[\partial_y f(s, Y_s, Z_s) - \partial_y f\left(t, Y_s^k, Z_s^k\right)\right] Y_s^\theta\right| ds\right)^2$$

$$+\mathbb{E}\left(\int_\theta^T \left|[\partial_z f(s, Y_s, Z_s) - \partial_z f\left(t, Y_s^k, Z_s^k\right)] Z_s^\theta\right| ds\right)^2$$

Firstly, we consider $A_k^\theta(T)$.

$$A_k^\theta(T) \leq \mathbb{E}\left(\int_\theta^T |K_\theta(s)|(\left|\, Y_s - Y_s^k\right| + \left|\, Z_s - Z_s^k\right| ds\right)^2$$

$$\leq C_3^2 \mathbb{E}\left(\int_\theta^T (\left|\, Y_s - Y_s^k\right| + \left|Z_s - Z_s^k\right| ds\right)^2$$

$$\leq C_3^2 (T-\theta) \mathbb{E}\left(\int_\theta^T \left(\left|\, Y_s - Y_s^k\right| + \left|Z_s - Z_s^k\right|\right)^2 ds\right)$$

$$\leq 2C_3^2(T-\theta)\left[\mathbb{E}\left(\int_\theta^T \left|\, Y_s - Y_s^k\right|^2 ds\right) + \mathbb{E}\left(\int_\theta^T \left|\, Z_s - Z_s^k\right|^2 ds\right)\right]$$

$$\leq 2C_3^2(T-\theta)^2 \left\|Y - Y^k\right\|_{C^2}^2 + 2C_3^2(T-\theta)\left\|Z - Z^k\right\|_{C^2}^2$$

$$\leq 2C_3^2(T-\theta)^2 \left\|Y - Y^k\right\|_{C^2}^2 + 2C_3^2 C_1^2 (T-\theta)\left\|M - M^k\right\|_{C^2}^2$$

$$\leq [18C_3^2(T-\theta)^2 + 8C_3^2 C_1^2(T-\theta)]\left\|V - V^k\right\|_{C^2}^2$$

where C_3 is a positive constant and C_1 is the Lipschitz constant.

The last inequality is followed by

$$\| M - M^k \|_{C^2} = \sqrt[2]{\mathbb{E}\sup_{t\in[\theta,T]} \mathbb{E}\left(V_T - V_T^k \middle| \mathcal{F}_t\right)^2} \leq 2\left\|V - V^k\right\|_{C^2}$$

and

$$\left\|Y - Y^k\right\|_{C^2} \leq 3\left\|V - V^k\right\|_{C^2}.$$

Since $\left(V^k\right)$ converges to (V), it follows that $\lim_{k\to\infty}\left\|V - V^k\right\|_{C^2}^2 = 0$. Hence

$$\lim_{k\to\infty}\int_0^T A_k^\theta(T)\,d\theta = 0.$$

Secondly, we consider $C_k^\theta(T)$.

Since (Y^θ, Z^θ) is the solution of (4.25), it follows

$$\int_0^T \|Y\|_{C^2}^2 + \|Z\|_{\mathcal{H}^2}^2 d\theta < +\infty.$$

Furthermore, since $\partial_y f$ and $\partial_z f$ are bounded and continuous with respect to y and z, it implies by the Lebesgue theorem that

$$\lim_{k\to\infty}\int_0^T C_k^\theta(T)\,d\theta = 0.$$

Finally, we consider$B_k^\theta(T)$. Since the derivatives of f are bounded,

$B_k^\theta(T)$

$$\leq C_4^2\mathbb{E}\left(\int_\theta^T\left|\left(Y_s^\theta - D_\theta Y_s^k\right)\right|ds\right)^2 + C_5^2\mathbb{E}\left(\int_\theta^T\left|\left(Z_s^\theta - D_\theta Z_s^k\right)\right|ds\right)^2$$

$$\leq C_4^2(T-\theta)\mathbb{E}\left(\int_\theta^T\left|\left(Y_s^\theta - D_\theta Y_s^k\right)\right|^2 ds\right) + C_5^2(T-\theta)\mathbb{E}\left(\int_\theta^T\left|\left(Z_s^\theta - D_\theta Z_s^k\right)\right|^2 ds\right)$$

$$\leq C_4^2(T-\theta)^2\left\|Y^\theta - D_\theta Y^k\right\|_{C^2}^2 + C_5^2(T-\theta)\left\|Z^\theta - D_\theta Z^k\right\|_{\mathcal{H}^2}^2$$

$$\leq C_4^2(T-\theta)^2\left\|Y^\theta - D_\theta Y^k\right\|_{C^2}^2 + C_1^2C_5^2(T-\theta)\left\|M^\theta - D_\theta M^k\right\|_{\mathcal{H}^2}^2$$

$$\leq [9C_4^2(T-\theta)^2 + 4C_1^2C_5^2(T-\theta)]\left\|V^\theta - D_\theta V^k\right\|_{C^2}^2$$

where C_4, C_5 is positive constants and C_1 is the Lipschitz constant.

The last inequalities is followed by

$$\| M^{\theta} - D_{\theta}M^{k} \|_{C^2} = \sqrt[2]{\mathbb{E}\sup_{t\in[\theta,T]}\mathbb{E}\left(V_T^{\theta} - D_{\theta}V_T^{k}|\mathcal{F}_t\right)^2} \leq 2\left\|V^{\theta} - D_{\theta}V^{k}\right\|_{C^2}$$

and

$$\left\|Y^{\theta} - D_{\theta}Y^{k}\right\|_{C^2} \leq 3\left\|V^{\theta} - D_{\theta}V^{k}\right\|_{C^2}.$$

Choose T so that $\alpha = 9C_4^2(T-\theta)^2 + 4C_1^2C_5^2(T-\theta) < 1$. Fix a positive real $\epsilon > 0$. There exists $N > 0$ such that, for any $k \geq N$,

$$\int_0^T \left\|V^{\theta} - D_{\theta}V^{k+1}\right\|_{C^2}^2 d\theta \leq \epsilon + \alpha \int_0^T \left\|V^{\theta} - D_{\theta}V^{k}\right\|_{C^2}^2 d\theta$$

Therefore, we inductively obtain, for every $k \geq N$,

$$\int_0^T \left\|V^{\theta} - D_{\theta}V^{k}\right\|_{C^2}^2 d\theta \leq \frac{\epsilon}{1-\alpha} + \alpha^k \int_0^T \left\|V^{\theta} - D_{\theta}V^{0}\right\|_{C^2}^2 d\theta$$

$$\leq \frac{\epsilon}{1-\alpha} + \alpha^k K \longrightarrow 0,$$

since $0 \leq \alpha \leq 1$, where K is a positive constant.

Thus, the sequence $(D_{\theta}V^{k})$ converges to (V^{θ}) in $\mathbb{L}^2(0,T;\mathbb{D}^{1,2})$=$\mathbb{L}_{1,2}^{a}$. Consequently, since $\mathbb{L}_{1,2}^{a}$ is closed for the norm $\|\cdot\|_{1,2}^{a}$, it follows that the limit (V) belongs to $\mathbb{L}_{1,2}^{a}$ and that a version of $(D_{\theta}V)$ is given by $\left(V^{\theta}\right)$. Therefore,

$$dD_{\theta}V_t = \left[\partial_y f(t,Y_t,Z_t)D_{\theta}Y_t + \partial_z f(t,Y_t,Z_t)D_{\theta}Z_t + D_{\theta}f(t,Y_t,Z_t)\right]dt, \quad D_{\theta}Y_T = D_{\theta}\xi$$

where

$$D_{\theta}M_t = \mathbb{E}(D_{\theta}\xi + D_{\theta}V_T|\mathcal{F}_t), D_{\theta}Y_t = \mathbb{E}(D_{\theta}\xi + D_{\theta}V_T|\mathcal{F}_t) - D_{\theta}V_t \text{ and } D_{\theta}Y_t = D_{\theta}M_t - D_{\theta}V_t$$

and the density process $D_{\theta}Z$ is given by

$$D_{\theta}M_T - D_{\theta}M_t = \int_t^T D_{\theta}Z_s dB_t, \quad 0 \leq \theta \leq t \leq T.$$

It implies that a version of $\{(D_{\theta}Y_t, D_{\theta}Z_t); 0 \leq \theta, t \leq T\}$ is given by

$D_{\theta}Y_t = 0, \; D_{\theta}Z_t = 0, \quad 0 \leq t < \theta \leq T,$

$\mathrm{d}D_\theta Y_t = -\left[\partial_y f(t, Y_t, Z_t) D_\theta Y_t + \partial_z f(t, Y_t, Z_t) D_\theta Z_t + D_\theta f(t, Y_t, Z_t)\right]\mathrm{d}t + D_\theta Z_t \mathrm{d}B_t,$

$\theta \le t \le T.$

It remains to show that for the considered version of the Malliavin derivatives of Y and Z, $D_s Y_s = Z_s$. For $t \le s$,

$$Y_s = Y_t - \int_t^s f(r, Y_r, Z_r)\mathrm{d}r + \int_t^s Z_r \mathrm{d}B_r.$$

Then for $t < \theta \le s$,

$$D_\theta Y_s = Z_\theta - \int_\theta^s \partial_y f(r, Y_r, Z_r) D_\theta Y_r + \partial_z f(r, Y_r, Z_r) D_\theta Z_r + D_\theta f(r, Y_r, Z_r)\mathrm{d}r + \int_\theta^s Z_r \mathrm{d}B_r.$$

By taking $\theta = s$, it implies that

$$D_s Y_s = Z_s \quad \text{a.s.}$$

■

4.3 Application to European Option Pricing in the Constrained Case

In this section, we study some nonlinear backward equations for the pricing of contingent claims with constrains on the wealth or portfolio processes, and demonstrate how the above proposition and comparison theorem apply to two simple examples in finance. In other words, we only focus on the applications of the properties proved instead of the details in pricing the contingent claims. If one is interested in pricing the claims in the following examples, please refer to (Jouini and Kallal, 1995a) [30] and (Cvitanic and Karatzas, 1993) [31].

For simplicity, suppose in a market there 2 assets, where the first one is riskless while the other is risky. The rest hypothesis is the same as section 3.5.

4.3.1 Replicating claims with difference risk premium for long and short positions (Jouini and Kallal, 1995a) [30]

We suppose in the market there is different risk premium for long and short positions. Let $\theta^1 - \theta^2$ be the difference in excess return between long and short positions in the stocks, where θ^1 and θ^2 are predictable and bounded processes. Then by corollary 3.3.2, we have that, given a square-integrable contingent claim ξ, there exists a unique square-integrable replication strategy $(X, \sigma^* \pi_t)$ which satisfies

$$dX_t = [r_t X_t + \pi_t \sigma_t \theta_t^1 + (\pi_t)^- \sigma_t (\theta_t^1 - \theta_t^2)]dt + \pi_t \sigma_t dB_t,$$

$$X_T = \xi,$$

where X_t is the fair price of the contingent claim ξ at time t.

Let $(\widetilde{X}, \sigma^* \widetilde{\pi})$ be the solution of the solution of the LBSDE

$$d\widetilde{X}_t = \left[r_t \widetilde{X}_t + \widetilde{\pi}_t \sigma_t \theta_t^1 + (\widetilde{\pi}_t) \sigma_t (\theta_t^1 - \theta_t^2)\right]dt + \widetilde{\pi}_t \sigma_t dB_t,$$

$$\widetilde{X}_T = \xi.$$

It is interesting to find a sufficient condition which ensures that $X_t = \widetilde{X}_T$. Then using the comparison theorem and the Malliavin calculus, we have the following proposition.

Proposition 4.3.1 Suppose that the coefficients $r_t, \theta_t^1, \theta_t^2, \sigma_t$ are deterministic functions of t and suppose that $\xi \in \mathbb{D}^{1,2}$. If $(\sigma_u^*)^{-1} \mathbb{D}_u \xi \leq 0, d\mathbb{P} \otimes du$ a.s., then the price for ξ is $X = \widetilde{X}$.

Proof:

By proposition 4.2.2, $\left(\widetilde{X}, \widetilde{\pi}\right) \in \mathbb{L}^2\left(0, T; (\mathbb{D}^{1,2}) \times (\mathbb{D}^{1,2})\right)$, and for $1 \leq i \leq m$, a version of $\left\{\left(\mathbb{D}_u^i \widetilde{X}_t, \mathbb{D}_u^i \widetilde{\pi}_t\right); 0 \leq u \leq t \leq T\right\}$ is

$$d\mathbb{D}_u^i \widetilde{X}_t = \left[r_t \mathbb{D}_u^i \widetilde{X}_t + \mathbb{D}_u^i \widetilde{\pi}_t \sigma_t \theta_t^1 + (\mathbb{D}_u^i \widetilde{\pi}_t) \sigma_t (\theta_t^1 - \theta_t^2)\right]dt + \mathbb{D}_u^i \widetilde{\pi}_t \sigma_t dB_t,$$

$$\mathbb{D}_u^i \widetilde{X}_T = \mathbb{D}_u^i \xi.$$

Set $Y_t^u = (\sigma_u^*)^{-1} \mathbb{D}_u \widetilde{X}_t$ and $Z_t^u = (\mathbb{D}_u \widetilde{\pi}_t)(\sigma_u)^{-1}$, for $0 \leq u \leq t \leq T$. We can easily see that $(Y_t^u, Z_t^u, u \leq t \leq T)$ is the solution of the BSDE

$$dY_t^u = -[-r_t Y_t^u - Z_t^u \sigma_t \theta_t^1 - Z_t^u \sigma_t (\theta_t^1 - \theta_t^2)]dt + Z_t^u \sigma_t dB_t,$$

$$Y_T^u = (\sigma_u^*)^{-1} \mathbb{D}_u^i \xi.$$

Then we apply the comparison theorem to (Y^u, Z^u) and $(0,0)$. So if $(\sigma_u^*)^{-1}\mathbb{D}_u^i\xi \leq 0$, then $Y_u^u \leq 0$ i.e. $(\sigma_u^*)^{-1}\mathbb{D}_u\widetilde{X}_u \leq 0$.

Since by proposition 4.2.2, it follows $\sigma_u^t\pi_u$=$\mathbb{D}_u\widetilde{X}_u$. Then we get $\pi_u \leq 0$. Therefore, the price for ξ is $X = \widetilde{X}$. ∎

4.3.2 Replicating claims with high interest rate for borrowing (Cvitanic and Karatzas, 1993) [31]

We suppose that in a market, an investor is allowed to borrow money at time t at an interest rate $R_t > r_t$, where r_t is the bond rate and R_t is a predictable and bounded process. Hence the amount borrowed at time t is equal to $(Y_t - \pi_t)^-$. Then by corollary 3.3.2, we have that, given a square-integrable contingent claim ξ, there exists a unique square-integrable replication strategy $(Y, \sigma^*\pi_t)$ which satisfies

$$dY_t = [r_tY_t + \pi_t\sigma_t\theta_t - (R_t - r_t)(Y_t - \pi_t)^-]dt + \pi_t\sigma_t dB_t,$$

$$Y_T = \xi,$$

where Y_t is the fair price of the contingent claim ξ at time t.

Let $(\widetilde{Y}, \sigma^*\widetilde{\pi})$ be the solution of the solution of the LBSDE

$$d\widetilde{Y}_t = \left[r_t\widetilde{Y}_t + \widetilde{\pi}_t\sigma_t\theta_t - (R_t - r_t)\left(\widetilde{Y}_t - \widetilde{\pi}_t\right)\right]dt + \widetilde{\pi}_t\sigma_t dB_t,$$

$$\widetilde{Y}_T = \xi.$$

Similarly, using the comparison theorem and the Malliavin calculus, we have the following proposition.

Proposition 4.3.2 Suppose that the coefficients $r_t, R_t, \theta_t, \sigma_t$ are deterministic functions of t and suppose that $\xi \in \mathbb{D}^{1,2}$. If $(\sigma_u^*)^{-1}\mathbb{D}_u\xi \geq \xi, d\mathbb{P} \otimes du$ a.s., then the price for ξ is $Y = \widetilde{Y}$.

Proof:

We use exactly the same argument as proposition 4.3.1, but apply the comparison theorem to $\left(\widetilde{X}, \widetilde{\pi}_t\right)$ and (Y^u, Z^u), instead of $(0, 0)$ and (Y^u, Z^u). ∎

Chapter 5

Conclusions

In this paper, we studied the following class of backward stochastic differential equations

$$dY_t^j = -f^j(t, Y_t, L(M)_t)dt + dM_t^j, \quad Y_T = \xi,$$

on a general filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, wher L is a prescribed non-linear mapping which sends M to an adapted process $L(M)$, and M, a correction term, is a martingale to be determined.

Based on Liang, Lyons and Qian's paper, which showed BSDEs may be reformulated as ordinary functional differential equations, the existence and uniqueness of local $\mathbb{L}^p$ solutions of the above BSDE was proved by using a fixed-point theorem. Then the global solution was obtained by subdividing the time interval $[0, T]$. Therefore, the result in Liang, Lyons and Qian's paper is a special case of ours.

Our greatest contribution in the paper is that we successfully established the corresponding comparison theorem of the above BSDE, which had not done by anyone in the literature before.

Furthermore, we studied the solution of

$$dY_t = -f(t, Y_t, Z_t) + Z_t^* dB_t, \quad Y_T = \xi$$

in the Malliavin's sense, and revisited and proved the proposition 4.2.2 (El Karoui, Peng and Quenez, 1997) by a different approach.

By those generalised theorems and propositions stated and proved by us, some standard results in the literature were recalled and showed as the special cases.

Finally, we applied the theory of the above BSDE demonstrated in the paper to European option pricing in both the unconstrained and constrained cases.

In conclusion, we taken advantage of Liang, Lyons and Qian's idea all the way through the paper, and clearly illustrate some important properties of BSDEs and their applications to finance.

However, some theory in the paper is needed further studies. For example, we may improve the comparison theorem to make the condition on the mapping L more nature as the condition put on L seems to be artificial in some sense. Moreover, the proposition 4.2.2 can be generalised under the solution of

$$dY_t^j = -f^j(t, Y_t, L(M)_t)dt + dM_t^j, \quad Y_T = \xi.$$

Actually, we tried this generation. However, according to the comments from some academic supervisors, our proof for the generalised version is not enough rigorous, so we took out the part from the paper. The proof is not that straightforward as it looks like. In fact, it is complicated. Basically, we had some problems about how to define the derivative of L and how to ensure the differentiability of L, etc. We believe there is no barrier to generalise this proposition in the future after doing more research on the Malliavin Calculus and the general type of BSDEs in finance.

In addition, Liang, Lyons and Qian's approach can also apply to the optimal control problems, the utility maximization problems with backward stochastic dynamics and the theory of forward-backward stochastic differential equations. These issues may require further studies.

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