FINANCIAL BENCHMARK TRACKING PROBLEMS UNDER A STOCHASTIC LINEAR QUADRATIC CONTROL FRAMEWORK

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Abstract

In this thesis we analyse the problem of tracking a financial benchmark via trading a portfolio of a small number of assets on a finite time horizon. The development of general stochastic linear quadratic control (SLQ) theory in recent years allows us to study this investment problem using this approach. We formulate the problem under the SLQ control framework and derive an optimal feedback control solution using stochastic Riccati equations and an accompanying equation. We then apply our theory to benchmark problems involving tracking a continuously compounded given growth rate and a stock market index to obtain novel solutions. An outline of how we might implement the model in practice is also given.

1 Introduction

The world's financial markets are governed by uncertainty. This uncertainty could be in the future prices of stocks which inherently transform into uncertainty amongst other markets such as bonds and consumer-sensitive markets such as housing and retail. As a consequence, we are continuously faced with challenges for obtaining optimal results under such conditions. In this thesis, we will study investment-specific problems that are concerned with optimally managing a portfolio of assets to satisfy a certain criteria. Our problems are formulated using systems of stochastic differential equations (SDEs) which build in this uncertainty via a white noise factor which takes into account the many random forces at play. The dynamic nature of our systems means that the decisons we make over time are continuously changing. We must develop an optimal strategy which would ensure that we achieve the best expected result in our investment problems subject to certain constraints enforced upon us. Such constrained continuous-time optimisation problems are known in the literature as stochastic optimal control problems. In particular, this thesis is concerned with a special subset of optimal control problems which are comprised of a quadratic cost functional and a state process (governed by an SDE) that is linear and homogeneous/non-homogeneous in both the state and control variables. These problems are known as **stochastic linear quadratic** (SLQ) control problems and the nice structural properties of these systems can be exploited to obtain useful insights and, in some cases, elegant solutions.

SLQ control theory has developed enormously in the last 10 years due to significant contributions made by a number of papers and books including Chen et al. (1998), Yong and Zhou (1999) and Ait Rami and Zhou (2000). In fact the theory of linear quadratic control itself has evolved dramatically since the pioneering work of Wonham (1960) and Kalman (1968), the former concerning matrix Riccati equations and the latter being a study on deterministic linear quadratic control, of which SLQ control is an extension. In finance, a number of investment problems have been studied under the effective SLQ control framework. For example, Zhou and Li (2000) consider continuous-time mean-variance portfolio selection and make use of the SLQ control model to derive the optimal feedback control and the efficient frontier. We will in fact extract some theory from this paper to help us solve one of our investment problems.

Our investment problems involve tracking a financial benchmark using a small portfolio of stocks and a risk-free bond on a **finite** time horizon, which will be formulated under an SLQ control model. The benchmarks under consideration are a continuously compounded growth rate and a stock market index. We wish to use a model that ensures the tracking performance is independent of the stocks that make up our portfolio. As a result we are not concerned as to how we pick the stocks in the portfolio, rather how we can best trade a **given** set of stocks (and the bond) in order to track our benchmark as closely as possible. Furthermore, as we are trading with a few given assets only, rather than all the available ones in the market, our tracking problems are said to be in an **incomplete** market. In practice, such tracking problems are likely to be of particular interest to asset managers. A fund manager's portfolio performance is typically judged against certain financial benchmarks to determine how well it has performed. For example, many global equity funds typically benchmark the MSCI World Index, bond funds are commonly benchmarked against the Lehman Brothers Aggregate Index and commodity funds usually benchmark the S&P Goldman Sachs Commodity Index. Our focus on trading a **small** set of stocks derives from the impracticality of a fund manager tracking a large market index (e.g. Dow Jones Wilshire 5000) with a small fund/set through holding positions in **all** the stocks.

The motivation for this thesis comes from a similar paper published by Yao et al. (2006) who use SLQ control theory to track the same benchmarks. However, a fundamental difference between both papers is that we formulate our SLQ control problems under a **finite** time horizon while theirs concerns an infinite time horizon. According to the literature, benchmark tracking under a finite time horizon in this context has not been studied and we will attempt to derive new solutions to these problems using SLQ control theory. We will obtain a completely different problem class to Yao et al. (2006), involving stochastic Riccati equations which can, in certain cases, be solved explicitly without the need to resort to computational techniques such as semidefinite programming to obtain the optimal feedback control. Furthermore, in considering our systems under a finite time horizon, stabilisability of the feedback control becomes a non-issue and consequently we do not need to introduce a discount factor in the tracking objective. We will consider two slightly different methods in trying to solve our benchmark problems, primarily for their ease with respect to the problem in question and also as a demonstration of versatility of the SLQ control method in general.

The remainder of this thesis is organised as follows. In section 2, we set up the first of our benchmark problems, namely tracking a growth rate, and then present a derivation of the optimal feedback control which we use to solve the problem, in sections 3 and 4. In section 5, we formulate and study the market index tracking problem and in section 6 we provide an outline of how the model can be implemented in practice for the growth rate tracking case. Finally, a conclusion is presented in section 7.

2 Formulating the growth rate tracking problem

Let us introduce the following notation which will be used throughout this thesis:

- M' is the transpose of any vector or matrix M;
- M_j is the j th entry of any vector M;
- S^n is the space of all $n \times n$ symmetric matrices;
- S^n_+ is the subspace of all non-negative definite matrices of S^n;
- $C([0,T]; X)$ is the space of all $X-$valued continuous functions on $[0,T]$.

We follow the approach of Yao et al. (2006) and set up our market model as follows. Consider a market with $m+1$ traded assets, where the first asset is a riskless bond and the rest are risky stocks which comprise a market index (e.g. FTSE-100). Let $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$ be a filtered probability space with a standard m-dimensional Brownian motion $W(t) = (W_1(t), \ldots.., W_m(t))'$ with $W(0) = 0$ and $t \in [0, T]$. We assume throughout this paper that $\mathcal{F}_t$ is generated by $W(.)$. Let the riskless bond, price S_0, grow at the risk-free, **constant** interest rate r :

$$\begin{cases} dS_0 = rS_0(t)\,dt, \\ S_0 = S_{00}. \end{cases} \tag{1}$$

Suppose the m traded stocks, each with price $S_i(t)$, $i = 1, .., m$, satisfy the following stochastic differential equation (SDE):

$$\begin{cases} dS_i(t) = b_i S_i(t)\,dt + \sum_{j=1}^{m} \sigma_{ij} S_i(t)\,dW_j(t), \\ S_i(0) = S_{i0}, \end{cases} \tag{2}$$

where b_i and σ_{ij} are the appreciation rates and volatilities of the stock respectively and are assumed, in this problem, to be **constant**. We define the **covariance matrix** by

$$\sigma = \begin{bmatrix} \sigma_1 \\ : \\ \sigma_m \end{bmatrix} \equiv (\sigma_{ij})_{m \times m}$$

The tracking problem we are considering is as follows. Suppose we have a set consisting of n out of the m traded stocks in a market index. How we have chosen these stocks is not an issue, we simply assume we are given said set. The aim is to control the investment of an initial wealth x_0 in the stocks and the bond such that its performance closely tracks a given deterministic, continuously compounded growth rate $\mu > 0$, over a finite time horizon. We want to do this by dynamically allocating our wealth between the stocks in the portfolio and the bond. We assume, for this problem, that n is much smaller than m , so that we are working in an incomplete market. Furthermore, we assume, without loss of generality that the first n

of the m stocks are selected for the set.

Let $\pi_i(t)$, $i = 1, ..., n$ be the allocation of the wealth to the i-th stock at time t. In control theory we say $\pi(.)$ is the **control**. Denote the wealth of the investment at time t by $x(t)$. Then, under a self-financing **portfolio** $\pi(.) = (\pi_1(.), ..., \pi_n(.))'$, we have the commonly-known **wealth equation** (see Zhou and Li (2000) for a proof):

$$\begin{cases} dx(t) = \{rx(t) + \sum_{i=1}^{n} [b_i - r] \pi_i(t)\} dt + \sum_{j=1}^{m} \sum_{i=1}^{n} \sigma_{ij} \pi_i(t) dW_j(t), \\ x(0) = x_0. \end{cases} \tag{3}$$

We say $x(.)$ is the state process under the control $\pi(.)$. Let

$$b := \begin{bmatrix} b_1 - r \\ : \\ b_n - r \end{bmatrix} = (b_1 - r, ..., b_n - r)'$$

and let σ_n be the $n \times m$ matrix which is identical to the matrix containing the first n rows of σ. Then (3) becomes

$$\begin{cases} dx(t) = \left\{rx(t) + b'\pi(t)\right\} dt + \pi' \sigma_n dW(t), \\ x(0) = x_0. \end{cases} \tag{4}$$

We say $\pi(.)$ is an **admissible** portfolio (control) if $\pi(.) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^n)$, the space of all $\mathbb{R}^n$-valued, $\mathcal{F}_t$-adapted measurable processes satisfying $E \int_0^T \|\pi(t)\|^2 dt < \infty$. Our tracking problem is therefore formulated as

$$min \qquad E \int_0^T \left[x(t) - x_0 e^{\mu t}\right]^2 dt$$

$$s.t. \begin{cases} dx(t) = \left\{ rx(t) + b'\pi(t) \right\} dt + \pi'\sigma_n dW(t), \\ x(0) = x_0, \end{cases} \tag{5}$$

where "s.t." denotes "subject to" and $x_0 e^{\mu t}$ is the growth trajectory with pre-specified growth factor $\mu > 0$. This problem is known as being **homogeneous** in the state and control variables. We analyse homogeneous SLQ control problems in section 5. Let

$$y(t) = x(t) - x_0 e^{\mu t}.$$

Then our tracking problem becomes, after simplification of the dynamics in the state process,

$$min \quad E \int_0^T [y(t)]^2 dt$$

$$s.t. \begin{cases} dy(t) = \left\{ ry(t) + b'\pi(t) + (r-\mu) x_0 e^{\mu t} \right\} dt + \pi'\sigma_n dW(t), \\ y(0) = 0. \end{cases} \tag{6}$$

Thus our objective is to minimise a quadratic cost functional subject to a linear, **non-homogeneous** system in the state and control variables. The equation is non-homogeneous due to the presence of the $(r-\mu) x_0 e^{\mu t}$ term. Since the dynamics are stochastic, we have a **stochastic linear quadratic** (SLQ) control problem, which we can solve using the theory presented in the next section of this paper.

3 Solving SLQ control problems

In this section we consider the following SLQ problem, found in Zhou and Li (2000), of which (6) is a subset:

$$J(u(.)) = \quad min \quad E\left\{ \int_0^T \frac{1}{2} \left[y(t)' Q(t) y(t) + u(t)' R(t) u(t) \right] dt + \frac{1}{2} y(T)' H y(T) \right\}$$

$$s.t. \begin{cases} dy(t) = [A(t)\,y(t) + B(t)\,u(t) + f(t)]\,dt + \sum_{j=1}^{m} D_j(t)\,u(t)\,dW_j(t)\,, \\ y(0) = y_0. \end{cases} \tag{7}$$

Here, $u(.) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^m)$ is the control and

$$Q \in C\left([0,T]\,; S^n_+\right),$$

$$R \in C\left([0,T]\,; S^m\right), H \in S^n_+,$$

$$A \in C\left([0,T]\,; \mathbb{R}^n\right), B, D_j \in C\left([0,T]\,; \mathbb{R}^{n\times m}\right) \quad ,$$

$$f \in L^2 C\left(0, T; \mathbb{R}^n\right).$$

Let us introduce the following **stochastic Riccati equation** (SRE):

$$\begin{cases} \frac{dP(t)}{dt} & = -P(t)\,A(t) - A(t)'\,P(t) - Q(t) \\ & +P(t)\,B(t)\left[R(t) + \sum_{j=1}^{m} D_j(t)'\,P(t)\,D_j(t)\right]^{-1} B(t)'\,P(t)\,, \\ P(T) & = H, \\ K(t) & \equiv R(t) + \sum_{j=1}^{m} D_j(t)'\,P(t)\,D_j(t) > 0, \end{cases} \tag{8}$$

along with an accompanying equation

$$\begin{cases} \frac{dg(t)}{dt} = -A(t)'\,g(t) + P(t)\,B(t)\left[R(t) + \sum_{j=1}^{m} D_j(t)'\,P(t)\,D_j(t)\right]^{-1} B(t)'\,g(t) - P(t)\,f(t)\,, \\ g(T) = 0. \end{cases} \tag{9}$$

Although being a non-linear deterministic ordinary differential equation (ODE), equation (8) is called stochastic to indicate that it has come about in the context of an SLQ control problem. In fact (8) can be modified to a backward stochastic differential equation in the general case where the coefficient matrices are random (see Chen et al. (1998)). The third constraint in (8) is required for the SLQ problem to have a minimising optimal control (Yong and Zhou (1999)). The derivations of the SRE and its accompanying equation are indeed an interesting exercise in itself and can be obtained via three different approaches: (a) the stochastic maximum principle/Hamiltonian equations, (b) dynamic programming/Hamilton-Jacobi-Bellman (HJB) equations and (c) completion of squares (see Yong and Zhou (1999) for details). Suppose $P \in C\left([0,T];S_{+}^{n}\right)$ and $g \in C\left([0,T];\mathbb{R}^{n}\right)$ are solutions to (8) and (9) respectively. We now proceed to derive an expression for the optimal feedback control to the SLQ problem (7), as found in Zhou and Li (2000). Note our derivation is, in essence, using the approach of (a). By Ito's lemma, suppressing the t term, we have

$$\begin{aligned} d\left(y'Py\right) &= dy'Py + y'dPy + y'Pdy + dy'Pdy \\ &= \left(Ay+Bu+f\right)'Pydt + \left(\sum_{j=1}^{m} D_j u\right)' PydW \\ &+ y'\left(-PA - A'P - Q + PBK^{-1}B'P\right)ydt \\ &+ y'P\left(Ay+Bu+f\right)dt + y'P\left(\sum_{j=1}^{m} D_j u\right)dW \\ &+ \left(\sum_{j=1}^{m} D_j u\right)' P \left(\sum_{j=1}^{m} D_j u\right)dt \\ &= \left(2\left(u'B'Py + y'Pf\right) + y'\left(-Q + PBK^{-1}B'P\right)y + \sum_{j=1}^{m} u'D_j'PD_j u\right)dt \\ &+ \left(....\right)dW. \end{aligned}$$

Now integrate both sides from 0 to T and take expectations to give the following expression:

$$\begin{aligned} E\left(y'\left(T\right)Hy\left(T\right)\right) &= y_0'P\left(0\right)y_0 \\ &+ E\int_0^T \left(2\left(u'B'Py + y'Pf\right) + y'\left(-Q + PBK^{-1}B'P\right)y + \sum_{j=1}^{m} u'D_j'PD_j u\right)dt. \end{aligned} \tag{10}$$

Also apply Ito's formula to the function $y'g$. This gives

$$d(y'g) = \left(u'B'g + y'PBK^{-1}B'g + f'g - y'Pf\right)dt + (....)\,dW.$$

Integrating both sides, as before, from 0 to T and taking expectations gives us the following:

$$E\int_0^T (y'Pf)\,dt = y_0'g(0) + E\int_0^T \left(u'B'g + y'PBK^{-1}B'g + f'g\right)dt, \tag{11}$$

where we have used $g(T) = 0$ from (9). We now substitute (10) and (11) into $J(u(.))$ (7) and simplify to obtain

$$\begin{aligned} J(u(.)) &= \frac{1}{2}E\int_0^T \left(u'Ku + y'PBK^{-1}B'Py + 2u'B'(Py+g) + 2y'PBK^{-1}B'g + 2f'g\right)dt \\ &+ \frac{1}{2}y_0'P(0)\,y_0 + y_0'g(0) \\ &= \frac{1}{2}E\int_0^T \left\{\left(u + K^{-1}B'(Py+g)\right)' K\left(u + K^{-1}B'(Py+g)\right) + 2f'g - gBK^{-1}B'g\right\}dt \\ &+ \frac{1}{2}y_0'P(0)\,y_0 + y_0'g(0) \\ &\geq \frac{1}{2}y_0'P(0)\,y_0 + y_0'g(0). \end{aligned}$$

We achieve equality if and only if

$$u^*(t,y) = -K(t)^{-1}B(t)'(P(t)\,y + g(t)).$$

Replacing $K(t) = R(t) + \sum_{j=1}^m D_j(t)'P(t)D_j(t)$, we therefore have the following expression for the **optimal feedback control** to the SLQ problem (7):

$$u^*(t,y) = -\left[R(t) + \sum_{j=1}^m D_j(t)'P(t)D_j(t)\right]^{-1} B(t)'(P(t)\,y + g(t)). \tag{12}$$

4 Solving the growth rate tracking problem

We now return to the growth rate tracking problem (6) and utilise the theory from section 3. Comparing (6) with problem (7), we have that

$$R(t) = 0, \tag{13}$$

$$H = 0 \tag{14}$$

$$A(t) = r \tag{15}$$

$$B(t) = b' = ((b_1 - r), ..., (b_n - r)) \tag{16}$$

$$f(t) = (r - \mu) x_0 e^{\mu t} \tag{17}$$

$$D_j(t) = \sigma'_n \tag{18}$$

Furthermore our state process $y(t)$ and the unknown function $P(t)$ is one-dimensional, so that $Q(t) = 1$. Because of the one-dimensional property of both these functions we can, in fact, solve our growth tracking problem explicitly to derive a closed-form solution for the optimal feedback control $u^* \equiv \pi^*$. Our SRE (8) reduces to

$$\begin{cases} \frac{dP(t)}{dt} = (\rho - 2r) P(t) - 1, \\ P(T) = 0, \\ P(t)[\sigma\sigma'] > 0, \qquad t \in [0, T] \end{cases} \tag{19}$$

where

$$\rho = B \left[\sum_{j=1}^{m} D'_j D_j \right]^{-1} B' = B [\sigma\sigma']^{-1} B'. \tag{20}$$

As $(\rho - 2r)$ is constant (no t dependence) for our problem, solving (19) gives

$$P(t) = \frac{1}{\rho - 2r} \left(1 - e^{-(\rho - 2r)(T - t)}\right). \tag{21}$$

The accompanying equation (9) therefore becomes

$$\begin{cases} \frac{dg(t)}{dt} = (\rho - r)\, g(t) - \frac{(r-\mu) x_0 e^{\mu t}}{\rho - 2r} \left(1 - e^{-(\rho-2r)(T-t)}\right), \\ g(T) = 0. \end{cases} \tag{22}$$

We can solve this using Mathematica. The solution, after some algebra, is

$$g(t) = \frac{x_0 \left[(\rho - 2r)\, e^{-(\rho-r)(T-t)+\mu T} + (r-\mu)\, e^{\mu t} - (\rho - r - \mu)\, e^{-(\rho-2r)(T-t)+\mu t}\right]}{(\rho - 2r)(\rho - r - \mu)}. \tag{23}$$

Substituting (13), (16), (18), (21) and (23) into equation (12) and simplifying thus gives us the following explicit optimal feedback control solution, in the notation of our problem:

$$\pi^*(t, y) = -\left[\sigma\sigma'\right]^{-1} B' \left(y + \frac{x_0 \left[(\rho - 2r)\, e^{-(\rho-r)(T-t)+\mu T} + (r-\mu)\, e^{\mu t} - (\rho - r - \mu)\, e^{-(\rho-2r)(T-t)+\mu t}\right]}{(\rho - r - \mu)\left(1 - e^{-(\rho-2r)(T-t)}\right)}\right). \tag{24}$$

5 The market index tracking problem

In this section we consider the problem of tracking a market index (e.g. FTSE-100) using the same given portfolio of n stocks and the risk-free bond, as described in section 2. We assume our stock price S_i follows (2) and model the market index as a weighted sum of the constituent stocks (Yao et al. (2006)):

$$\begin{cases} I(t) &= \sum_{j=1}^{m} \alpha_j S_j(t) \\ I(0) &= I_0, \end{cases},$$

where α_j is the weight of the shares of stock j in the index. Our tracking problem is therefore

$$min \quad E \int_0^T (x(t) - I(t))^2\, dt$$

subject to the wealth equation (4). Let us consider formulating this as a **homogeneous** SLQ control problem by writing out the SDEs for the wealth process $x(.)$, market index $I(.)$ and the constituent stocks $S_i(.)$. In doing this, we eliminate the need for an accompanying equation and are just left with an SRE to consider. Thus we have the following:

$$min \quad E\int_0^T \left(x(t) - I(t)\right)^2 \quad dt$$

$$s.t. \begin{cases} dx(t) & = \left\{rx(t) + \sum_{i=1}^n [b_i - r]\,\pi_i(t)\right\} dt + \sum_{j=1}^m \sum_{i=1}^n \sigma_{ij}\pi_i(t)\, dW_j(t)\,, \\ dI(t) & = \sum_{i=1}^m \alpha_i b_i S_i(t)\, dt + \sum_{i=1}^m \sum_{j=1}^m \alpha_i \sigma_{ij} S_i(t)\, dW_j(t)\,, \\ dS_1(t) & = b_1 S_1(t)\, dt + \sigma_{1j} S_1(t)\, dW_j(t)\,, \\ \vdots & \\ dS_m(t) & = b_m S_m(t)\, dt + \sigma_{mj} S_m(t)\, dW_j(t)\,, \\ (x(0), I(0), S_i(0)) & = (x_0, I_0, S_{i0})\,. \end{cases} \tag{25}$$

Our state process is now multi-dimensional (as opposed to one-dimensional in the growth tracking problem) as we have

$$y(t) \quad = \quad \begin{bmatrix} x(t) \\ I(t) \\ S_1(t) \\ \vdots \\ S_m(t) \end{bmatrix}$$

and our control is now the following vector:

$$\pi(t) = \begin{bmatrix} \pi_1(t) \\ \vdots \\ \pi_n(t) \end{bmatrix}.$$

In writing our system in this way we see that it is indeed homogeneous in both the state and control process. Our SLQ problem (25) is now a subset of the following general homogeneous SLQ control system:

$$\begin{aligned} &min \quad E\int_0^T \left[y(t)' Q(t) y(t) + u(t)' R(t) u(t) \right] dt \\ &s.t. \quad \begin{cases} dy(t) &= [A(t) y(t) + B(t) u(t)] dt + \sum_{j=1}^m [C_j(t) y(t) + D_j(t) u(t)] dW_j(t), \\ y(0) &= y_0. \end{cases} \end{aligned} \tag{26}$$

From Yong and Zhou (1999) we have that the SRE for this control problem is given by

$$\begin{cases} \dot{P} &= -PA - A'P - \sum_{j=1}^m C_j' P C_j - Q \\ &\quad + \left(B'P + \sum_{j=1}^m D_j' P C_j \right)' \left(R + \sum_{j=1}^m D_j' P D_j \right)^{-1} \left(B'P + \sum_{j=1}^m D_j' P C_j \right), \\ P(T) &= 0, \\ R &+ \sum_{j=1}^m D_j' P D_j > 0, \end{cases} \tag{27}$$

where $\dot{P} = \frac{dP}{dt}$ and the t term has been suppressed in the equation. Furthermore, the optimal feedback control is given by

$$u^*(t,y) = -\left[R + \sum_{j=1}^m D_j' P D_j \right]^{-1} \left(B'P + \sum_{j=1}^m D_j' P C_j \right) y(t). \tag{28}$$

Comparing coefficients, we therefore have the following:

$$Q = \begin{bmatrix} 1 & -1 & 0 & \dots & 0 \\ -1 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}_{(m+2)\times(m+2)} \qquad R = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}_{n\times n} \tag{29}$$

$$A = \begin{bmatrix} r & 0 & 0 & \dots & 0 \\ 0 & 0 & \alpha_1 b_1 & \dots & \alpha_m b_m \\ 0 & 0 & b_1 & \dots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & b_m \end{bmatrix}_{(m+2)\times(m+2)} \qquad B = \begin{bmatrix} b_1 - r & b_2 - r & \dots & b_n - r \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix}_{(m+2)\times n} \tag{30}$$

$$C_j = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \alpha_1 \sigma_{1j} & \dots & \alpha_m \sigma_{mj} \\ 0 & 0 & \sigma_{1j} & \dots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \sigma_{mj} \end{bmatrix}_{(m+2)\times(m+2)} \qquad D_j = \begin{bmatrix} \sigma_{1j} & \sigma_{2j} & \dots & \sigma_{nj} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}_{(m+2)\times n} \tag{31}$$

where $j = 1, ..., n$. Suppose we consider, for simplicity, the hypothetical case of a market index comprised of 2 stocks i.e. $m = 2$ and we wish to track this index with a portfolio consisting of 1 constituent stock i.e. $n = 1$. Then, our state vector is 4-dimensional:

$$y(t) = \begin{bmatrix} x(t) \\ I(t) \\ S_1(t) \\ S_2(t) \end{bmatrix}$$

and our control vector is $u(t) = \pi_1(t)$. Furthermore our unknown P in the SRE (27) is a 4×4 matrix:

$$P(t) = \begin{bmatrix} P_{11}(t) & P_{12}(t) & P_{13}(t) & P_{14}(t) \\ P_{21}(t) & P_{22}(t) & P_{23}(t) & P_{24}(t) \\ P_{31}(t) & P_{32}(t) & P_{33}(t) & P_{34}(t) \\ P_{41}(t) & P_{42}(t) & P_{43}(t) & P_{44}(t) \end{bmatrix}, \tag{32}$$

with $P(T) = 0$ and coefficient matrices

$$Q = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, R = 0,$$

$$A = \begin{bmatrix} r & 0 & 0 & 0 \\ 0 & 0 & \alpha_1 b_1 & \alpha_2 b_2 \\ 0 & 0 & b_1 & 0 \\ 0 & 0 & 0 & b_2 \end{bmatrix}, B = \begin{bmatrix} b_1 - r \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$C_j = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_1\sigma_{1j} & \alpha_2\sigma_{2j} \\ 0 & 0 & \sigma_{1j} & 0 \\ 0 & 0 & 0 & \sigma_{2j} \end{bmatrix}, D_j = \begin{bmatrix} \sigma_{1j} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad (33)$$

$j = 1, 2.$

SRE (27) is now in a considerably complex matrix form, even in the case of considering just 2 stocks in the market index. We have, suppressing the t term,

$$PA = \begin{bmatrix} rP_{11} & 0 & \alpha_1 b_1 P_{12} + b_1 P_{13} & \alpha_2 b_2 P_{12} + b_2 P_{14} \\ rP_{21} & 0 & \alpha_1 b_1 P_{22} + b_1 P_{23} & \alpha_2 b_2 P_{22} + b_2 P_{24} \\ rP_{31} & 0 & \alpha_1 b_1 P_{32} + b_1 P_{33} & \alpha_2 b_2 P_{32} + b_2 P_{34} \\ rP_{41} & 0 & \alpha_1 b_1 P_{42} + b_1 P_{43} & \alpha_2 b_2 P_{42} + b_2 P_{44} \end{bmatrix},$$

$$A'P = \begin{bmatrix} rP_{11} & rP_{12} & rP_{13} & rP_{14} \\ 0 & 0 & 0 & 0 \\ \alpha_1 b_1 P_{21} + b_1 P_{31} & \alpha_1 b_1 P_{22} + b_1 P_{32} & \alpha_1 b_1 P_{23} + b_1 P_{33} & \alpha_1 b_1 P_{24} + b_1 P_{34} \\ \alpha_2 b_2 P_{21} + b_2 P_{41} & \alpha_2 b_2 P_{22} + b_2 P_{42} & \alpha_2 b_2 P_{23} + b_2 P_{43} & \alpha_2 b_2 P_{24} + b_2 P_{44} \end{bmatrix},$$

$$C_1'PC_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & \gamma & \varepsilon \end{bmatrix},$$

$$C_2'PC_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \zeta & \eta \\ 0 & 0 & \kappa & \lambda \end{bmatrix},$$

$$\left(B'P+\sum_{j=1}^{2}D_j'PC_j\right)' = \begin{bmatrix} (b_1-r)\,P_{11} \\ (b_1-r)\,P_{12} \\ (b_1-r)\,P_{13}+\nu \\ (b_1-r)\,P_{14}+\xi \end{bmatrix},$$

$$\left(\sum_{j=1}^{2}D_j'PD_j\right)^{-1} = \phi,$$

where

$$\alpha = \alpha_1\sigma_{11}\left(\alpha_1\sigma_{11}P_{22}+\sigma_{11}P_{32}\right)+\sigma_{11}\left(\alpha_1\sigma_{11}P_{23}+\sigma_{11}P_{33}\right), \quad (34)$$

$$\beta = \alpha_2\sigma_{21}\left(\alpha_1\sigma_{11}P_{22}+\sigma_{11}P_{32}\right)+\sigma_{21}\left(\alpha_1\sigma_{11}P_{24}+\sigma_{11}P_{34}\right), \quad (35)$$

$$\gamma = \alpha_1\sigma_{11}\left(\alpha_2\sigma_{21}P_{22}+\sigma_{21}P_{42}\right)+\sigma_{11}\left(\alpha_2\sigma_{21}P_{23}+\sigma_{21}P_{43}\right), \quad (36)$$

$$\varepsilon = \alpha_2\sigma_{21}\left(\alpha_2\sigma_{21}P_{22}+\sigma_{21}P_{42}\right)+\sigma_{21}\left(\alpha_2\sigma_{21}P_{24}+\sigma_{21}P_{44}\right), \quad (37)$$

$$\zeta = \alpha_1\sigma_{12}\left(\alpha_1\sigma_{12}P_{22}+\sigma_{12}P_{32}\right)+\sigma_{12}\left(\alpha_1\sigma_{12}P_{23}+\sigma_{12}P_{33}\right), \quad (38)$$

$$\eta = \alpha_2\sigma_{22}\left(\alpha_1\sigma_{12}P_{22}+\sigma_{12}P_{32}\right)+\sigma_{22}\left(\alpha_1\sigma_{12}P_{24}+\sigma_{12}P_{34}\right), \quad (39)$$

$$\kappa = \alpha_1\sigma_{12}\left(\alpha_2\sigma_{22}P_{22}+\sigma_{22}P_{42}\right)+\sigma_{12}\left(\alpha_2\sigma_{22}P_{23}+\sigma_{22}P_{43}\right), \quad (40)$$

$$\lambda = \alpha_2\sigma_{22}\left(\alpha_2\sigma_{22}P_{22}+\sigma_{22}P_{42}\right)+\sigma_{22}\left(\alpha_2\sigma_{22}P_{24}+\sigma_{22}P_{44}\right), \quad (41)$$

$$\nu = \alpha_1\sigma_{11}^2P_{12}+\sigma_{11}^2P_{13}+\alpha_1\sigma_{12}^2P_{12}+\sigma_{12}^2P_{13}, \quad (42)$$

$$\xi = \alpha_2\sigma_{11}\sigma_{21}P_{12}+\sigma_{11}\sigma_{21}P_{14}+\alpha_2\sigma_{12}\sigma_{22}P_{12}+\sigma_{12}\sigma_{22}P_{14}, \quad (43)$$

$$\phi = \left(\left(\sigma_{11}^2+\sigma_{12}^2\right)P_{11}\right)^{-1}. \quad (44)$$

Combining the above terms on the right-hand side of (27) gives us the following matrix expression:

$$\begin{bmatrix} \dot{P_{11}}(t) & \dot{P_{12}}(t) & \dot{P_{13}}(t) & \dot{P_{14}}(t) \\ \dot{P_{21}}(t) & \dot{P_{22}}(t) & \dot{P_{23}}(t) & \dot{P_{24}}(t) \\ \dot{P_{31}}(t) & \dot{P_{32}}(t) & \dot{P_{33}}(t) & \dot{P_{34}}(t) \\ \dot{P_{41}}(t) & \dot{P_{42}}(t) & \dot{P_{43}}(t) & \dot{P_{44}}(t) \end{bmatrix} = \begin{bmatrix} \check{a} & \check{b} & \check{c} & \check{d} \\ \check{e} & \check{f} & \check{g} & \check{h} \\ \check{i} & \check{j} & \check{k} & \check{l} \\ \check{m} & \check{n} & \check{o} & \check{p} \end{bmatrix}, \tag{45}$$

where the entries in the matrix on the right-hand side of (45) are functions of t and are given by

$$\check{a} = -2rP_{11} - 1 + \phi\left(b_1 - r\right)^2 P_{11}^2, \tag{46}$$

$$\check{b} = -rP_{12} + 1 + \phi\left(b_1 - r\right)^2 P_{11}P_{12}, \tag{47}$$

$$\check{c} = -\alpha_1 b_1 P_{12} - b_1 P_{13} - rP_{13} + \phi\nu\left(b_1 - r\right) P_{11}, \tag{48}$$

$$\check{d} = -\alpha_2 b_2 P_{12} - b_2 P_{14} - rP_{14} + \phi\xi\left(b_1 - r\right) P_{11}, \tag{49}$$

$$\check{e} = -rP_{21} + 1 + \phi\left(b_1 - r\right)^2 P_{11}P_{12}, \tag{50}$$

$$\check{f} = -1 + \phi\left(b_1 - r\right)^2 P_{12}^2, \tag{51}$$

$$\check{g} = -\alpha_1 b_1 P_{22} - b_1 P_{23} + \phi\nu\left(b_1 - r\right) P_{12}, \tag{52}$$

$$\check{h} = -\alpha_2 b_2 P_{22} - b_2 P_{24} + \phi\xi\left(b_1 - r\right) P_{12}, \tag{53}$$

$$\check{i} = -rP_{31} - \alpha_1 b_1 P_{21} - b_1 P_{31} + \phi\nu\left(b_1 - r\right) P_{11}, \tag{54}$$

$$\check{j} = -\alpha_1 b_1 P_{22} - b_1 P_{32} + \phi\nu\left(b_1 - r\right) P_{12}, \tag{55}$$

$$\check{k} = -\alpha_1 b_1 P_{32} - 2b_1 P_{33} - \alpha_1 b_1 P_{23} - (\alpha + \zeta) + \phi\nu^2, \tag{56}$$

$$\check{l} = -\alpha_2 b_2 P_{32} - b_2 P_{34} - \alpha_1 b_1 P_{24} - b_1 P_{34} - (\beta + \eta) + \phi\nu\xi, \tag{57}$$

$$\check{m} = -rP_{41} - \alpha_2 b_2 P_{21} - b_2 P_{41} + \phi\xi\left(b_1 - r\right)P_{11}, \tag{58}$$

$$\check{n} = -\alpha_2 b_2 P_{22} - b_2 P_{42} + \phi\xi\left(b_1 - r\right)P_{12}, \tag{59}$$

$$\check{o} = -\alpha_1 b_1 P_{42} - b_1 P_{43} - \alpha_2 b_2 P_{23} - b_2 P_{43} - (\gamma + \kappa) + \phi\nu\xi, \tag{60}$$

$$\check{p} = -\alpha_2 b_2 P_{42} - 2b_2 P_{44} - \alpha_2 b_2 P_{24} - (\varepsilon + \lambda) + \phi\xi^2. \tag{61}$$

Thus, we have 16 ordinary differential equations with 16 final conditions

$$P_{uv}\left(T\right) = 0, u = 1, .., 4, v = 1, ..4. \tag{62}$$

Although it seems like a system of 16 coupled ODEs we discover that we can, in fact, solve these equations one-by-one, beginning from the upper left hand corner of the entries in (45). Let us start with the first ODE in this system, given by

$$\begin{cases} \dot{P_{11}}\left(t\right) & = -2rP_{11}\left(t\right) - 1 + \phi\left(t\right)\left(b_1 - r\right)^2 P_{11}^2\left(t\right), \\ P_{11}\left(T\right) & = 0. \end{cases} \tag{63}$$

This is an ODE only in the P_{11} term as $\phi = \left(\left(\sigma_{11}^2 + \sigma_{12}^2\right)P_{11}\right)^{-1}$. Then (63) becomes

$$\begin{cases} \dot{P_{11}}\left(t\right) & = \left(-2r + \frac{(b_1 - r)^2}{\left(\sigma_{11}^2 + \sigma_{12}^2\right)}\right)P_{11}\left(t\right) - 1, \\ P_{11}\left(T\right) & = 0. \end{cases} \tag{64}$$

which we can easily solve to obtain

$$P_{11}\left(t\right) = \frac{1}{\chi}\left(1 - e^{-\chi(T-t)}\right) \tag{65}$$

where

$$\chi = -2r + \frac{\left(b_1 - r\right)^2}{\left(\sigma_{11}^2 + \sigma_{12}^2\right)}$$

We substitute (44) into equation (47) and solve

$$\begin{cases} \dot{P_{12}}(t) &= \left(-r + \frac{(b_1-r)^2}{\left(\sigma_{11}^2+\sigma_{12}^2\right)}\right) P_{12}(t) + 1, \\ P_{12}(T) &= 0. \end{cases} \tag{66}$$

for P_{12}. We can continue in this way, solving for $P_{13}, P_{14}, ...$ right down to the last coefficient in our unknown matrix, P_{44}, found in the bottom right-hand corner of (45). The nice structure of the matrix equations allows us to do this as each ODE comprises of terms which can be obtained from solving the ODEs that have followed before it. As a further demonstration let us see how we can in fact solve the final ODE for P_{44}. This is given by

$$\begin{cases} \dot{P_{44}}(t) &= -\alpha_2 b_2 P_{42}(t) - 2b_2 P_{44}(t) - \alpha_2 b_2 P_{24}(t) - (\varepsilon(t) + \lambda(t)) + \phi(t)\,\xi(t)^2, \\ P_{44}(T) &= 0. \end{cases} \tag{67}$$

The terms P_{42} and P_{24} are known as we can solve equations (59) and (53). The functions ε and λ contain only the one unknown we are solving for (see (37) and (41)) as the other terms will have been solved in sequence previously. The ϕ term, given by (44), is easily known from using (65). Furthermore, ξ, given by (43), is entirely composed of functions that can be solved from the ODEs that will have come in sequence before (67). Thus we can solve for the final term P_{44} and consequently obtain a complete solution for the matrix P which solves the SRE.

As a result, we can finally obtain an expression for the optimal feedback control for our simplified market index tracking problem by substituting matrix P into (28). Although we have looked at the simpler case of a market index comprised of just 2 stocks, it is easy to see how we can extend the iterative methodology for obtaining P in the general SLQ problem (25) stated at the beginning of this section. We can program an algorithm which can solve these equations efficiently in sequence to generate all the matrix coeffcients and therefore obtain the optimal feedback control.

6 Using the model in practice

Having discovered solutions to both tracking problems, we provide an outline in this section of how we can use the SLQ control model in practice. We consider the growth-rate tracking problem due to the fact that we have an explicit formula for the feedback control (tracking a market index would be considered in a similar way).

6.1 Computing the feedback control

Suppose we are given a portfolio of 10 stocks (n) from the FTSE-100 index (m). We wish to obtain a 10% annual return (μ) on our portfolio via controlling an initial investment of £1000 (x_0) amongst the stocks and our risk free bond. Suppose the risk-free interest rate on our bond is fixed at 4% (r) and our tracking period is a particular quarter of a year ($T = 60$ trading days). Using our SLQ control theory approach we have that our optimal feedback control is given explicitly by (24):

$$\pi^* (t, y) = - [\sigma\sigma']^{-1} B' \left(y + \frac{x_0 \left[(\rho - 2r) e^{-(\rho - r)(T-t) + \mu T} + (r - \mu) e^{\mu t} - (\rho - r - \mu) e^{-(\rho - 2r)(T-t) + \mu t} \right]}{(\rho - r - \mu) \left(1 - e^{-(\rho - 2r)(T-t)}\right)} \right)$$

where

$$\begin{aligned} y(t) &= x(t) - x_0 e^{\mu t} \\ \sigma &= [\sigma_1, ..., \sigma_{100}]' = (\sigma_{ij})_{100\times 100} \\ B' &= [b_1 - r, ..., b_{10} - r]' \\ \rho &= B [\sigma\sigma'] B' \end{aligned}$$

In order to calculate the optimal feedback control we first need to estimate the drift b_k and covariance matrix σ parameters of our stocks. We can do this by a standard statistical method using historical data and maximum likelihood estimates. For a stock satisfying the process $dS(t) = bS(t)\,dt + cS(t)\,dW(t)$ we can write $log\left(\frac{S(t)}{S(0)}\right) = \left(b - \frac{1}{2}c^2\right)t + cW_t = \Upsilon t + cW_t$.

To estimate b (or Υ) and c, the stock process is sampled over some time horizon T_1 prior to our tracking period T at discrete time intervals $t_k = k\delta t = \frac{kT_1}{N}$, for $k = 0, 1, ..., N$, and the log-returns of our stocks

$$X_k = log\left(\frac{S\left(t_{k+1}\right)}{S\left(t_k\right)}\right)$$

are found. The sample time horizon could be, for example, one year ($T_1 = 250$ trading days) and S could be the daily closing price of our stocks. It can be shown (e.g. Luenberger (1998)) that the maximum likelihood estimate for Υ is given by

$$\widehat{\Upsilon} \quad = \quad \frac{1}{N\delta t}\sum_{k=0}^{N-1} X_k = \frac{1}{T_1} log\left(\frac{S\left(T\right)}{S\left(0\right)}\right)$$

and that the maximum likelihood estimate for the covariance matrix σ of the 100 index stocks has coefficients

$$\widehat{\sigma_{ij}} \quad = \quad \frac{1}{T_1 - 1}\sum_{k=0}^{T_1} \left(x_{ik} - \overline{x_i}\right)\left(x_{jk} - \overline{x_j}\right)$$

where x is the observed values of the log-returns of our stocks and $\overline{x}$ is the mean of the observed log-returns of our stocks.

Now that we have estimates for B' and covariance matrix σ we can substitute these into (24), along with the other parameters which we have specified above to generate our optimal feedback control at the beginning of the tracking period. For simplicity in executing this model we can assume that short selling of stocks is permitted and that transaction costs are ignored. Once we have obtained the optimal feedback control π^* we would use this to dynamically adjust our positions accordingly amongst our 10 stocks and the bond which would ensure (in theory at least) that we are tracking our pre-specified growth rate over the desired period. We can also decide how frequently we wish to trade (i.e. adjust) the portfolio during the tracking period based on the feedback control law specified by the model. For example we could update once every trading day or once every week (5 trading days) or even longer.

We can also re-run the model at an interim time during the tracking period if we wish, in which case we would have to update our parameters B' and σ.

6.2 Comments

One of the major considerations in the implementation of the SLQ control model is the usage of **past** data to estimate our parameters B' and σ. It would be important to test how sensitive our model is to these parameters in determining the feedback control. Furthermore, it would be instrumental to experiment with strategies involving how frequently we trade the portfolio and how many times we can update the feedback control to determine which combinations give us favourable tracking performance. This is important as the stocks in our portfolio may behave very differently prior to and during the tracking period. As our parameters are estimated before the tracking horizon it is important that our model takes into account these trends and adjusts the feedback control accordingly. Intuition thus suggests that trading frequently under a feedback control that has updated parameters should give better results than trading frequently under a feedback control which has been computed just once at the beginning of the tracking period. Indeed, Yao et al. (2006), whose paper concerns the tracking problem under an infinite time horizon, show that the former strategy works better, using a portfolio of 5 randomly generated stocks from the S&P 500 index, tracking a growth rate of 50% per annum.

There is also a key practical consideration in the implementation of our model. While we have permitted short selling without any constraints, in practice this may not be feasible. There may arise a situation where heavy borrowing is required (e.g. a large drop in the value of the FTSE-100) to maintain favourable tracking and this may not be possible. Thus, investigating the relationship between the ratio of total short/long amount and total net wealth amongst the stocks in our portfolio during the tracking period, as done in Yao et al. (2006), would be instructive. Again, we can test with different trading frequencies to see which strategy gives us the most favourable amount of borrowing. This indicates that the SLQ control model will find best use in practice when used as a reference tool for comparing the performance of a portfolio against a benchmark rather than as a trading tool for outperforming the benchmark. For example, a wealth manager can experiment by executing the model on his portfolio with a desired growth rate and then, after observing the feedback control, can decide whether the amount needed to short is viable.

7 Conclusion

This thesis presents an interesting and powerful approach to analysing and solving investment problems involving the tracking of financial benchmarks on a finite time horizon. We have considered two particular tracking problems concerning benchmarks which are widely used in practice: a constant given growth rate and a stock market index. For both problems, our approach was to formulate a stochastic linear quadratic control problem, which was either homogeneous or non-homogeneous in the state and control variables, and to then generate the optimal feedback control using stochastic Riccati equations.

For the growth-rate tracking problem we formulated a non-homogeneous stochastic linear quadratic control problem. The non-homogeneous property of our system allowed us to use a stochastic Riccati equation and an accompanying equation which were easily solved because of the one-dimensionality of our state process. We were able to derive an explicit formula for the optimal feedback control as a result.

The market index problem was more complex due to the stochastic nature of the market index process itself. We formulated a homogeneous SLQ problem in the state and control variables which eliminated the need for an accompanying equation. The resulting SRE was in matrix form and we were able to discover an iterative procedure born out of the nice structure of this matrix which would allow us to solve for the different coefficients in the unknown matrix P. We found this iterative methodology in the context of a market comprising of just 2 stocks and commented that the method can be extended to our general tracking problem involving m stocks and using computer algorithms for efficiency. In fact, this particular aspect of this thesis would be interesting to investigate further. Having discovered a process for obtaining the solution to the SRE we can then proceed to find the optimal feedback control in matrix form.

Implementing an SLQ control model in practice involves first estimating the drift vector and covariance matrices of the constituent stocks from a market index. Once we have found these, we can compute the optimal feedback control which would enable us to adjust our portfolios accordingly to track our benchmarks as close as possible. We can also experiment with the model to test which strategies could give us the best tracking performance. A major drawback in the model's implementation, however, is that we can only update the feedback control at discrete time intervals during the tracking period, leading to suboptimality. Deciding between how frequently to trade and obtaining optimality is an issue which Yao et al. (2006) raise in their paper and further study into this problem, both theoretically and practically, would be important.

This subject has an interesting future direction developing out of situations that may arise in practice. For example, a wealth manager may not be able to short the stocks in his portfolio by the amount required by the optimal feedback control law, or, he may have a constraint imposed on the overall wealth. These restrictions will give rise to a new set of more sophisticated control problems which fall outside the SLQ control theory we have studied in this thesis and would be highly interesting to study both from a theoretical and practical perspective.

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