Marginal utility-based hedging of claims on non-traded assets with partial information

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Abstract

We examine optimal hedging of a claim on a non-traded asset, using a correlated traded asset, when one does not know with certainty the values of the asset price drifts. In this partial information setting, the uncertain parameters are considered as random variables. We filter the drifts from price observations, updating a chosen prior distribution. The result is an effective full information model with random drift parameters. Using a dual approach, we derive representations for the indifference price and optimal hedging strategy, with exponential utility. Using the marginal utility-based price as an approximation to the indifference price, analytic formulae for the optimal hedge are possible, and this allows a simulation study of the optimal hedging program to be carried out. The results indicate improved hedging performance relative to a Black-Scholes strategy which takes the correlation as perfect, and also relative to a utility-based hedging program which does not incorporate learning.

1 Introduction

This paper examines the optimal hedging of a contingent claim on a non-tradeable asset Y in an incomplete market context, using a traded stock S, correlated with Y, when the hedger is restricted to trading strategies in S that are adapted to the observation filtration $\widehat{\mathbb{F}}$ generated by the asset prices. We refer to this as a *partial information* scenario. With continuous observations of prices and correlated log-Brownian motions for S, Y, we therefore assume the agent does not know the values of the assets' expected returns, but does have knowledge of the volatilities and the correlation, $\rho \in [-1, 1]$. It is well known that accurate estimation of drift parameters in this setting is practically impossible, but that reasonable estimates of variances and covariances is possible, and this is the motivation for our assumption.

Examples of underlying assets that are either not traded (or are difficult to trade) include weather indices or baskets of many stocks. In these cases the market becomes incomplete, as there is no tradeable asset which can be used to perfectly replicate the claim payoff. Traders may resort to using a correlated traded asset to hedge the claim, where the correlation is presumed to be close to 1, in effect taking the traded asset as a perfect proxy for the non-traded one. A typical case is the hedging of a basket option using a futures contract on a stock index, where the composition of the basket and the index are not identical.

A number of authors (Ankirchner et al [1], Davis [9], Henderson [13], Hulley and McWalter [14], Monoyios [23, 24], Musiela and Zariphopoulou [25] to name but a few) have studied such *basis risk* models, in which S and Y are typically modelled as correlated geometric Brownian motions with known drifts, volatilities and correlation. We refer to this as a *full information* scenario. If $|\rho| = 1$, the model is complete and a Black-Scholes (henceforth, BS) style perfect hedge is possible. When $|\rho| \neq 1$, many papers resort to exponential indifference valuation and hedging (though Hulley and McWalter [14] use a mean-variance approach, which is an approximation to the exponential hedge, as shown by Kramkov and Sirbu [18]). Explicit pricing and hedging formulae have been derived, and Monoyios [23, 24] developed accurate analytic approximations for optimal hedging strategies, which were used to generate terminal hedging error distributions via simulation methods. This demonstrated that optimal strategies are superior to the BS-style hedge, even for high correlation values.

A caveat to the above conclusions, and the motivation for this work, is that the exponential hedge requires the agent to know the true asset price drifts. Equivalently, the agent needs to be able to observe the underlying Brownian motions driving the asset prices, which is an unrealistic assumption. Since it is then impossible to estimate drift parameters with any degree of accuracy (see Rogers [30] or Monoyios [24]), the utility-based hedging scheme may no longer be feasible in practice. Given the huge attention that this pricing method has received, it is important to examine ways of making the technique more robust to drift parameter uncertainty.

In the context of the classical Merton [21, 22] consumption-investment problems, Rogers [30] has shown how drift parameter uncertainty far outweighs the risk from discrete portfolio rebalancing. Monoyios [24] has shown that drift parameter uncertainty in the lognormal basis risk model is also severe, typically leading to destructive hedging losses or to uncompetitive pricing bounds for the claim. In the face of such uncertainty, we wish to examine whether utility-based hedging, when coupled with learning based on filtering the asset drifts, can outperform the BS-style hedge. The latter requires estimation of the asset volatilities only, which is a less ambitious requirement.

To the above end, we derive an explicit solution to the partial information optimal hedging problem. Using a Bayesian perspective, the uncertainty in the drifts of S, Y is acknowledged by taking their Sharpe ratios (that is, the drift divided by volatility) λ, θ to be *random variables* with a Gaussian prior distribution. Our choice of prior is based on the distribution of the classical estimator of the Sharpe ratios, which we suppose is inferred from asset price observations leading up to the start of the hedging period $[0, T]$. The prior distribution is then updated with the information from subsequent observations by applying a two-dimensional Kalman-Bucy filter. This leads to our first result, Proposition 1, that the partial information model may be converted to a model with Sharpe ratios $(\hat{\lambda}, \hat{\theta}) = (\hat{\lambda}_t, \hat{\theta}_t)_{0 \le t \le T}$ that are random processes adapted to the observation filtration $\hat{\mathbb{F}}$, and which may then be treated as a full information model.

Moreover, under the assumption that the drifts of S and Y have equal prior variances, corresponding to the (not unrealistic) case that the agent uses past observations of equal time length for both asset prices to fix the prior distribution, we are able to derive explicit formulae for the stochastic Sharpe ratios $\hat{\lambda}_t, \hat{\theta}_t$ of each asset in terms of the relevant current asset price.

We treat the optimal hedging problem with random drifts by analysing the dual to the primal control problem. Using a dual representation for indifference prices we derive our main result, Theorem 1, which is a representation in terms of derivatives of the indifference price with respect to the stock and non-traded asset price, for the optimal hedging strategy associated with the utility-based price under partial information. This contains an additional term compared with the classical (full information) basis risk hedging formula, reflecting the additional risk induced by drift parameter uncertainty.

We then derive an analytic approximation for the indifference price (and hence for the associated hedging strategy), valid for small positions in the claim, in the form of the marginal utility-based price of Davis [8]. This is given by a BS-type formula since, pleasingly (and perhaps surprisingly), we show in Proposition 2 that Y is lognormal under the minimal martingale measure Q^M, even though its Q^M-drift, $\hat{\theta}_t - \rho\hat{\lambda}_t$, is stochastic.

These results allow us to simulate the hedging of the claim over many asset price histories, and hence compute the terminal hedging error distribution resulting from the utility-based hedge. This is compared to that from the BS-style hedge, to assess if Bayesian learning can render the optimal hedge superior to the BS-style strategy. We also compute the utility-based hedging error distribution in the absence of learning via filtering, to assess the benefits of the updating procedure. The numerical simulations indicate that the optimal hedging method with learning does indeed out-perform the BS-hedge and the hedge without learning.

Partial information problems under various scenarios have been studied by a number of authors, usually in the context of optimal investment. Notable examples include Rogers [30], Lakner [19, 20], Brennan [6], Brendle [5], and Björk, Davis and Landén [3]. Nagai and Peng

[26] treat risk-sensitive control, and Pham [28], Xia [32] and Xiong and Zhou [33] study mean-variance optimisation problems. Utility-based hedging of claims under partial information has received less attention, though a specialised model of commodity derivatives with partially observed convenience yield has been studied by Carmona and Ludkovski [7], and some mention of partial information pricing was made in Dufresne and Hugonnier [11]. To the best of our knowledge this is the first paper to attempt to analyse the effectiveness of optimal hedging under partial information.

The rest of the paper is organised as follows. In the next section we introduce the basis risk model, in its full information and partial information forms, and reduce the partial information case to a full information model with random drifts using a Kalman-Bucy filter. In Section 3 we solve the optimal hedging problem via convex duality, derive a representation for the optimal hedging strategy under partial information, and obtain an analytic approximation for the indifference price and hedge, using the marginal utility-based price. In Section 4 we conduct a simulation based study of the performance of the optimal strategy with learning. This shows that the strategy does indeed outperform the BS hedge and the hedge without learning.

2 Basis risk model

2.1 Full information case

In a full information model, the setting is a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{0\leq t\leq T}, P)$, where the filtration $\mathbb{F}$ is the P-augmentation of that generated by a two-dimensional Brownian motion $(B, B^{\perp})$. A traded stock price $S := (S_t)_{0\leq t\leq T}$ follows a log-Brownian process given by

$$dS_t = \sigma S_t(\lambda dt + dB_t) =: \sigma S_t d\xi_t, \tag{1}$$

where $\sigma > 0$ and λ are known constants. For simplicity, the interest rate is taken to be 0. The process ξ in (1) defined by $d\xi_t := \lambda dt + dB_t$ will play a role later in the paper as one component of an "observation process" in a partial information model, when λ will be treated as a random variable rather than as a known constant.

A non-traded asset price $Y := (Y_t)_{0\leq t\leq T}$ follows the correlated log-Brownian motion

$$dY_t = \beta Y_t(\theta dt + dW_t) =: \beta Y_t d\zeta_t, \tag{2}$$

with $\beta > 0$ and θ known constants. The Brownian motion W is correlated with B according to

$$d[B, W]_t = \rho dt, \quad W = \rho B + \sqrt{1-\rho^2}B^{\perp}, \quad \rho \in [-1, 1],$$

and the process ζ, given by $d\zeta_t := \theta dt + dW_t$, will act as the second component of an observation process in a partial information model, when θ will be considered a random variable. We shall henceforth refer to the Sharpe ratios λ (respectively, θ) as the drift of S (respectively, Y), for brevity.

A European contingent claim pays the non-negative random variable $h(Y_T)$ at time T. In what follows we shall consider utility maximisation problems with the additional random terminal endowment $nh(Y_T)$, for $n \in \mathbb{R}$, and we make the following assumption on the random endowment.

Assumption 1 The random endowment $nh(Y_T)$ is continuous and bounded below, with finite expectation under any martingale measure.

This assumption ensures that our primal utility maximisation problems will be well-defined. Included in permissible random endowments are long and short put positions as well as long call positions. As is well-known, the case of short calls is problematic in lognormal basis risk models with exponential utility (see, for example, Davis [9]). Some progress on utility-based valuation of unbounded short positions in incomplete markets has been made by Owen and Žitković [27], who consider unbounded random endowments which are nevertheless super-hedgeable, though this is not the case for a claim on a non-traded asset. One way to proceed in our model may be to define feasible prices for short call positions via the dual problem, since the dual representation of the price is well-defined. We plan to address this topic in future research.

As stated in the Introduction, this market is incomplete for $|\rho| \neq 1$. If the correlation is perfect, however, the market becomes complete and perfect hedging is possible (see, for example, Monoyios [24]). No-arbitrage requires that $\theta = \lambda$, and a position in n claims is hedged by $\Delta_t^{(\mathrm{BS})}$ units of S at $t \in [0, T]$, where

$$\Delta_t^{(\mathrm{BS})} = -n \frac{\beta}{\sigma} \frac{Y_t}{S_t} \frac{\partial}{\partial y} \mathrm{BS}(t, Y_t; \beta), \tag{3}$$

and where $\mathrm{BS}(t, y; \beta)$ denotes the BS formula at time t for underlying asset price y and volatility β. From our perspective, the salient feature of (3) is that the perfect hedge does not require knowledge of the values of the drifts λ, θ.

In the incomplete case, $|\rho| \neq 1$, for exponential utility, $U(x) = -e^{-\alpha x}$, with risk aversion parameter $\alpha > 0$, a utility-based price and hedge are computable in analytic form via a perturbation expansion in powers of the dimensionless parameter $a := -\alpha(1-\rho^2)n$ (see Monoyios [23, 24]). In this case a position in n claims is hedged by $\Delta_t^{(\mathrm{I})}$ units of S at $t \in [0, T]$, given by

$$\begin{aligned} \Delta_t^{(\mathrm{I})} &= -n\rho \frac{\beta}{\sigma} \frac{Y_t}{S_t} \frac{\partial p}{\partial y}(t, Y_t), \qquad (4) \\ p(t, y) &= -\frac{1}{\alpha n (1-\rho^2)} \log E^{Q^M} \left[\exp\left(-\alpha(1-\rho^2) n h(Y_T)\right) \middle| Y_t = y \right], \end{aligned}$$

where Q^M is the minimal martingale measure for the model, under which $(S_t)_{0 \le t \le T}$ is a local martingale and under which Y follows

$$dY_t = \beta Y_t \left[(\theta - \rho\lambda) dt + dW_t^{Q^M} \right], \tag{5}$$

for a Q^M-Brownian motion W^{Q^M}.

In [23, 24] the hedging strategy in (4) is shown to be superior to the BS-style hedge (3), in terms of the terminal hedging error distribution produced by selling the claim at the appropriate price (the indifference price or the BS price) and investing the proceeds in the corresponding hedging portfolio. But from (5) we see that the exponential hedge requires knowledge of λ, θ, which are impossible to estimate accurately (see Rogers [30] or Monoyios [24]). This can ruin the effectiveness of indifference hedging, as shown in [24]. It is therefore dubious to draw any meaningful conclusions on the effectiveness of utility-based hedging in this model without relaxing the assumption that the agent knows the true values of the drifts.

2.2 Partial information case

Now we assume the hedger does not know the values of the return parameters λ, θ, so these are considered to be random variables. Equivalently, the agent cannot observe the Brownian motions B, W driving the asset prices, so is required to use strategies adapted to the observation filtration $\hat{\mathbb{F}}$ generated by asset returns.

2.2.1 Choice of prior

We take the the two-dimensional random variable

$$U := \begin{pmatrix} \lambda \\ \theta \end{pmatrix}$$

to have a Gaussian distribution which will be updated as the agent attempts to filter the values of the drifts from asset observations during the hedging interval $[0, T]$.

The choice of Gaussian prior is motivated by the idea that the agent has some past observations of S, Y before time 0, uses these to obtain classical point estimates of the drifts, and the joint distribution of the estimators is used as the prior in a Bayesian framework. Ultimately, in order to obtain explicit solutions, we shall assume that the agent uses observations before time 0 of equal length for both assets. In setting the prior this way, we make the approximation that the asset price observations are continuous, so that σ, β, ρ are known from the quadratic variation and co-variation of S, Y. This is because our goal here is to focus on the severest problem of drift parameter uncertainty.

So, consider, for the moment, an observer with data for S over a time interval of length t_S, and for Y over a window of length t_Y, who considers λ and θ as *constants*, and records the returns dS_t/S_t and dY_t/Y_t in order to estimate the values of the drifts. The best estimator of λ is $\bar{\lambda}(t_S)$ given by

$$\begin{aligned}\bar{\lambda}(t_S) &= \frac{1}{t_S}\int_{t_0}^{t_0+t_S}\frac{dS_u}{\sigma S_u} \\ &= \lambda + \frac{B_{t_0+t_S}}{t_S} \\ &\sim \mathrm{N}\left(\lambda, \frac{1}{t_S}\right),\end{aligned}$$

where $\mathrm{N}(\mu, \Sigma)$ denotes the normal probability law with mean μ and variance Σ. The estimator of λ is normally distributed, with a similar computation for the estimator of θ. The estimator, $(\bar{\lambda}, \bar{\theta})$, of the (supposed constant) vector (λ, θ) is bivariate normal. Defining $\mathrm{v}_0 := 1/t_S$ and $\mathrm{w}_0 := 1/t_Y$ it is easily checked that

$$\begin{pmatrix}\bar{\lambda} \\ \bar{\theta}\end{pmatrix} \sim \mathrm{N}(M, C_0),$$

where the mean vector M and covariance matrix C_0 are given by

$$M = \begin{pmatrix}\lambda \\ \theta\end{pmatrix}, \qquad C_0 = \begin{pmatrix}\mathrm{v}_0 & \rho\min(\mathrm{v}_0, \mathrm{w}_0) \\ \rho\min(\mathrm{v}_0, \mathrm{w}_0) & \mathrm{w}_0\end{pmatrix}. \tag{6}$$

With this in mind, we shall suppose that (λ, θ), now considered as a *random variable*, is bivariate normal according to

$$\lambda \sim \mathrm{N}(\lambda_0, \mathrm{v}_0), \quad \theta \sim \mathrm{N}(\theta_0, \mathrm{w}_0), \quad \mathrm{cov}(\lambda, \theta) = \mathrm{c}_0 := \rho\min(\mathrm{v}_0, \mathrm{w}_0),$$

for some chosen values λ_0, θ_0, typically obtained from past data prior to time 0. This distribution will be updated via subsequent observations of

$$\xi_t := \frac{1}{\sigma}\int_0^t \frac{dS_u}{S_u} = \lambda t + B_t, \qquad \zeta_t := \frac{1}{\beta}\int_0^t \frac{dY_u}{Y_u} = \theta t + W_t,$$

over the hedging interval $[0, T]$.

2.2.2 Kalman-Bucy filter

We are firmly within the realm of a two-dimensional Kalman filtering problem, which we treat as follows. Define the observation filtration by

$$\hat{\mathbb{F}} := (\hat{\mathcal{F}}_t)_{0\le t\le T}, \qquad \hat{\mathcal{F}}_t = \sigma(\xi_s, \zeta_s; 0 \le s \le t).$$

The *observation process*, O, and unobservable *signal process*, U, are defined by

$$O := \begin{pmatrix}\xi_t \\ \zeta_t\end{pmatrix}_{0\le t\le T}, \qquad U := \begin{pmatrix}\lambda \\ \theta\end{pmatrix},$$

satisfying the stochastic differential equations

$$dO_t = U dt + D d\mathbf{B}_t, \qquad dU = \begin{pmatrix}0 \\ 0\end{pmatrix},$$

where

$$D = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix}, \qquad \mathbf{B}_t = \begin{pmatrix} B_t \\ B_t^{\perp} \end{pmatrix}.$$

The optimal filter is $\hat{U}_t := E[U|\hat{\mathcal{F}}_t], 0 \leq t \leq T$, a two-dimensional process defining the best estimates of λ and θ given observations up to time $t \in [0,T]$:

$$\hat{U}_t \equiv \begin{pmatrix} \hat{\lambda}_t \\ \hat{\theta}_t \end{pmatrix} := \begin{pmatrix} E[\lambda|\hat{\mathcal{F}}_t] \\ E[\theta|\hat{\mathcal{F}}_t] \end{pmatrix}, \qquad \begin{pmatrix} \hat{\lambda}_0 \\ \hat{\theta}_0 \end{pmatrix} = \begin{pmatrix} \lambda_0 \\ \theta_0 \end{pmatrix}. \tag{7}$$

The solution to this filtering problem converts the partial information model to a full information model with random drifts, given in the following proposition. To avoid a proliferation of symbols, we abuse notation and write $\hat{\lambda}_t \equiv \hat{\lambda}(t, S_t)$ and $\hat{\theta} \equiv \hat{\theta}(t, Y_t)$ for processes $\hat{\lambda}, \hat{\theta}$ that will turn out to be functions of time and current asset price.

Proposition 1 *The partial information model is equivalent to a full information model in which the asset price dynamics in the observation filtration* $\hat{\mathbb{F}}$ *are*

$$\begin{aligned} dS_t &= \sigma S_t(\hat{\lambda}_t dt + d\hat{B}_t), \qquad (8)\\ dY_t &= \beta Y_t(\hat{\theta}_t dt + d\hat{W}_t), \qquad (9) \end{aligned}$$

where $\hat{B}, \hat{W}$ *are* $\hat{\mathbb{F}}$*-Brownian motions with correlation* ρ*, and the random drifts* $\hat{\lambda}, \hat{\theta}$ *are* $\hat{\mathbb{F}}$*-adpated processes.*

If λ *and* θ *have common initial variance* v_0*, then* $\hat{\lambda}, \hat{\theta}$ *are given by*

$$\begin{pmatrix} \hat{\lambda}_t \\ \hat{\theta}_t \end{pmatrix} = \begin{pmatrix} \lambda_0 \\ \theta_0 \end{pmatrix} + \int_0^t \mathrm{v}_u \begin{pmatrix} d\hat{B}_u \\ d\hat{W}_u \end{pmatrix}, \quad 0 \leq t \leq T, \tag{10}$$

where $(\mathrm{v}_t)_{0\leq t\leq T}$ *is the deterministic function*

$$\mathrm{v}_t := \frac{\mathrm{v}_0}{1+\mathrm{v}_0 t}.$$

Equivalently, $\hat{\lambda}, \hat{\theta}$ *are given as functions of time and current asset price by*

$$\hat{\lambda}_t = \hat{\lambda}(t, S_t) = \frac{\lambda_0 + \mathrm{v}_0\xi_t}{1+\mathrm{v}_0 t}, \qquad \hat{\theta}_t = \hat{\theta}(t, Y_t) = \frac{\theta_0 + \mathrm{v}_0\zeta_t}{1+\mathrm{v}_0 t}, \tag{11}$$

with

$$\xi_t = \frac{1}{\sigma}\log\left(\frac{S_t}{S_0}\right) + \frac{1}{2}\sigma t, \qquad \zeta_t = \frac{1}{\beta}\log\left(\frac{Y_t}{Y_0}\right) + \frac{1}{2}\beta t. \tag{12}$$

Proof By the Kalman-Bucy filter, for example Theorem V.9.2 in Fleming and Rishel [12], $\hat{U}$ satisfies the stochastic differential equation

$$d\hat{U}_t = \ C_t\left(DD^T\right)^{-1}(dO_t - \hat{U}_t dt) \ =: C_t\left(DD^T\right)^{-1} dN_t, \tag{13}$$

where $(N_t)_{0\leq t\leq T}$ is the innovations process, defined by

$$\begin{aligned} N_t &:= O_t - \int_0^t \hat{U}_s ds \\ &= \begin{pmatrix} \xi_t - \int_0^t \hat{\lambda}_s ds \\ \zeta_t - \int_0^t \hat{\theta}_s ds \end{pmatrix} \\ &=: \begin{pmatrix} \hat{B}_t \\ \hat{W}_t \end{pmatrix}, \qquad (14) \end{aligned}$$

and $\hat{B}, \hat{W}$ are $\hat{\mathbb{F}}$-Brownian motions with correlation ρ. The deterministic matrix function C_t is the conditional variance-covariance matrix defined by

$$C_t := E\left[(U-\hat{U}_t)(U-\hat{U}_t)^{\mathrm{T}}\middle|\hat{\mathcal{F}}_t\right] = E\left[(U-\hat{U}_t)(U-\hat{U}_t)^{\mathrm{T}}\right],$$

(T denoting transpose) where the last equality follows because the error $U-\hat{U}_t$ is independent of $\hat{\mathcal{F}}_t$ (Theorem V.9.2 in [12] again).

Using (14), and writing dS_t in terms of $d\xi_t$, as in (1), gives the dynamics (8) of S in the observation filtration; (9) is established similarly.

The matrix $C = (C_t)_{0\le t\le T}$ satisfies the Riccati equation

$$\frac{dC_t}{dt} = -C_t\left(DD^T\right)^{-1}C_t,$$

with C_0 given in (6). Then $R_t := C_t^{-1}$ satisfies the Lyapunov equation

$$\frac{dR_t}{dt} = \left(DD^T\right)^{-1}.$$

Define the elements of the conditional covariance matrix by

$$C_t =: \begin{pmatrix} \mathrm{v}_t & \mathrm{c}_t \\ \mathrm{c}_t & \mathrm{w}_t \end{pmatrix}.$$

Then the filtering equation (13) is a pair of coupled stochastic differential equations:

$$\begin{aligned}\begin{pmatrix} d\hat{\lambda}_t \\ d\hat{\theta}_t \end{pmatrix} &= \frac{1}{1-\rho^2}\begin{pmatrix} \mathrm{v}_t-\rho\mathrm{c}_t & \mathrm{c}_t-\rho\mathrm{v}_t \\ \mathrm{c}_t-\rho\mathrm{w}_t & \mathrm{w}_t-\rho\mathrm{c}_t \end{pmatrix}\begin{pmatrix} d\xi_t-\hat{\lambda}_t dt \\ d\zeta_t-\hat{\theta}_t dt \end{pmatrix} \\ &= \frac{1}{1-\rho^2}\begin{pmatrix} \mathrm{v}_t-\rho\mathrm{c}_t & \mathrm{c}_t-\rho\mathrm{v}_t \\ \mathrm{c}_t-\rho\mathrm{w}_t & \mathrm{w}_t-\rho\mathrm{c}_t \end{pmatrix}\begin{pmatrix} d\hat{B}_t \\ d\hat{W}_t \end{pmatrix}.\end{aligned}$$

Solving the Lyapunov equation yields 3 equations for $\mathrm{v}_t, \mathrm{w}_t, \mathrm{c}_t$:

$$\begin{aligned}\frac{\mathrm{v}_t}{\mathrm{v}_t\mathrm{w}_t-\mathrm{c}_t^2} - \frac{\mathrm{v}_0}{\mathrm{v}_0\mathrm{w}_0-\mathrm{c}_0^2} &= \frac{t}{1-\rho^2}, \\ \frac{\mathrm{w}_t}{\mathrm{v}_t\mathrm{w}_t-\mathrm{c}_t^2} - \frac{\mathrm{w}_0}{\mathrm{v}_0\mathrm{w}_0-\mathrm{c}_0^2} &= \frac{t}{1-\rho^2}, \\ \frac{\mathrm{c}_t}{\mathrm{v}_t\mathrm{w}_t-\mathrm{c}_t^2} - \frac{\mathrm{c}_0}{\mathrm{v}_0\mathrm{w}_0-\mathrm{c}_0^2} &= \frac{\rho t}{1-\rho^2},\end{aligned} \tag{15}$$

where we have written $\mathrm{c}_0 \equiv \rho\min(\mathrm{v}_0, \mathrm{w}_0)$ for brevity.

Now make the simplification $\mathrm{w}_0 = \mathrm{v}_0$. From the discussion in Section 2.2.1, we see that this corresponds to using past observations over the same length of time, $t_S = t_Y$, for both S and Y in fixing the prior. Then $\mathrm{c}_0 = \rho\mathrm{v}_0$, and the solution to the system of equations (15) gives the entries of the matrix C_t as

$$\mathrm{v}_t = \frac{\mathrm{v}_0}{1+\mathrm{v}_0 t}, \quad \mathrm{w}_t = \mathrm{v}_t, \quad \mathrm{c}_t = \rho\mathrm{v}_t.$$

With this simplification, the equation for the optimal filter simplifies to

$$\begin{pmatrix} d\hat{\lambda}_t \\ d\hat{\theta}_t \end{pmatrix} = \mathrm{v}_t\begin{pmatrix} d\xi_t-\hat{\lambda}_t dt \\ d\zeta_t-\hat{\theta}_t dt \end{pmatrix} = \mathrm{v}_t\begin{pmatrix} d\hat{B}_t \\ d\hat{W}_t \end{pmatrix}, \tag{16}$$

which, along with the initial condition in (7), yields (10) and (11).

Finally, the expressions in (12) for ξ_t, ζ_t follow directly from the solutions of (1) and (2) for S and Y.

□

Armed with Proposition 1 we may now treat the model as a full information model with random drift parameters $(\hat{\lambda}_t, \hat{\theta}_t)$, and this is done in the next section. We make the following observations:

- The formulae $\hat{\lambda}(t, S_t), \hat{\theta}(t, Y_t)$ for the random drifts, in terms of current asset price, allow the model to be expressed in Markovian form with only one extra state variable (the stock price S) in the stochastic control problems for the optimal price and hedge compared with the full information case with constant drifts, as we shall see.
- The simplification $\mathrm{w}_0 = \mathrm{v}_0$ is made to allow for a analytic solution to the optimal hedging problem. One can proceed without this assumption, but full computations in analytic form are not possible.

3 Optimal hedging with random drifts

On the stochastic basis $(\Omega, \hat{\mathcal{F}}, \hat{\mathbb{F}}, P)$, the wealth process associated with trading strategy $\pi := (\pi_t)_{0\leq t\leq T}$, an $\hat{\mathbb{F}}$-adapted process satisfying the integrability condition $\int_0^T \pi_t^2 dt < \infty$ a.s., is $X^\pi \equiv X := (X_t)_{0\leq t\leq T}$, satisfying

$$dX_t = \sigma\pi_t(\hat{\lambda}_t dt + d\hat{B}_t). \tag{17}$$

The class $\mathcal{M}$ of local martingale measures for this model consists of measures Q with density processes defined by

$$Z_t := \left.\frac{dQ}{dP}\right|_{\hat{\mathcal{F}}_t} = \mathcal{E}(-\hat{\lambda}\cdot\hat{B} - \psi\cdot\hat{B}^\perp)_t, \quad 0 \leq t \leq T, \tag{18}$$

for integrands ψ satisfying $\int_0^t \psi_u^2 du < \infty$ a.s., for all $t \in [0, T]$ (it is not hard to show that $\int_0^t \hat{\lambda}_u^2 dt < \infty, 0 \leq t \leq T$). For $\psi = 0$ we obtain the minimal martingale measure Q^M.

Under $Q \in \mathcal{M}$, $(\hat{B}^Q, \hat{B}^{\perp,Q})$ is two-dimensional Brownian motion, where

$$d\hat{B}_t^Q := d\hat{B}_t^Q + \hat{\lambda}_t dt, \qquad d\hat{B}_t^{\perp,Q} := d\hat{B}_t^\perp + \psi_t dt,$$

and the asset prices and random drifts satisfy

$$\begin{aligned}
dS_t &= \sigma S_t d\hat{B}_t^Q, \\
dY_t &= \beta Y_t[(\hat{\theta}_t - \rho\hat{\lambda}_t - \sqrt{1-\rho^2}\psi_t)dt + d\hat{W}_t^Q], \\
d\hat{\lambda}_t &= \mathrm{v}_t[-\hat{\lambda}_t dt + d\hat{B}_t^Q], \\
d\hat{\theta}_t &= \mathrm{v}_t[-(\rho\hat{\lambda}_t + \sqrt{1-\rho^2}\psi_t)dt + d\hat{W}_t^Q],
\end{aligned}$$

where $\hat{W}^Q = \rho\hat{B}^Q + \sqrt{1-\rho^2}\hat{B}^{\perp,Q}$.

The relative entropy between $Q \in \mathcal{M}$ and P is defined by

$$\begin{aligned}
\mathcal{H}(Q, P) &:= E\left[\frac{dQ}{dP}\log\frac{dQ}{dP}\right] \\
&= E^Q\left[-\int_0^T \hat{\lambda}_t d\hat{B}_t^Q - \int_0^T \psi_t d\hat{B}_t^{\perp,Q} + \frac{1}{2}\int_0^T \left(\hat{\lambda}_t^2 + \psi_t^2\right) dt\right].
\end{aligned}$$

Using the Q-dynamics of $\hat{\lambda}_t$ it is straightforward to establish that $E^Q \int_0^t \hat{\lambda}_u^2 du < \infty$ for all $t \in [0, T]$. If, in addition, we have the integrability condition

$$E^Q \int_0^t \psi_u^2 du < \infty, \quad 0 \leq t \leq T, \tag{19}$$

then

$$\mathcal{H}(Q,P) = E^{Q}\left[\frac{1}{2}\int_0^T \left(\hat{\lambda}_t^2 + \psi_t^2\right) dt\right] < \infty. \tag{20}$$

In this case we write $Q \in \mathcal{M}_f$, where $\mathcal{M}_f$ denotes the set of martingale measures Q with finite relative entropy with respect to P, and we define $\mathcal{H}(Q,P) := \infty$ otherwise. From (20) we note that the minimal entropy measure Q^E is given by

$$\mathcal{H}(Q^E,P) = E^{Q}\left[\frac{1}{2}\int_0^T \hat{\lambda}_t^2 dt\right],$$

corresponding to $\psi \equiv 0$ in (20). This means that the minimal martingale measure and the minimal entropy measure in this model coincide: $Q^E = Q^M$.

For an initial time $t \in [0,T]$, we define the conditional entropy between $Q \in \mathcal{M}$ and P by

$$H_t(Q,P) := E\left[\left.\frac{Z_T}{Z_t}\log\left(\frac{Z_T}{Z_t}\right)\right|\hat{\mathcal{F}}_t\right], 0 \le t \le T, \tag{21}$$

satisfying $H_0(Q,P) \equiv \mathcal{H}(Q,P)$. Provided the integrability condition (19) is satisfied, then

$$H_t(Q,P) = E^{Q}\left[\left.\frac{1}{2}\int_t^T \left(\hat{\lambda}_u^2 + \psi_u^2\right) du\right|\hat{\mathcal{F}}_t\right],$$

and we define $H_t(Q,P) := \infty$ otherwise. In particular, therefore, recalling that $\hat{\lambda}_t \equiv \hat{\lambda}(t,S_t)$ is a smooth and Lipschitz function of time and current stock price, and that the Q-dynamics of $\hat{\lambda}_t$ do not depend on ψ_t for any $Q \in \mathcal{M}$, the minimal conditional entropy $(H_t(Q^E,P))_{0\le t\le T}$ will be a deterministic function of time and stock price, given by $H_t(Q^E,P) \equiv H^E(t,S_t)$ for a $C^{1,2}([0,T]\times\mathbb{R}^+)$ function H^E defined by

$$H^E(t,s) := E^{Q^E}\left[\left.\frac{1}{2}\int_t^T \hat{\lambda}^2(u,S_u)du\right| S_t = s\right]. \tag{22}$$

3.1 The primal problem

We use an exponential utility function, $U(x) = -\exp(-\alpha x)$, $x \in \mathbb{R}$, $\alpha > 0$. The primal value function $u \equiv u^{(n)}$ is defined as the maximum expected utility of wealth at T from trading S and receiving n units of the claim on Y, when starting at time $t \in [0,T]$:

$$u^{(n)}(t,x,s,y) := \sup_{\pi\in\mathcal{A}} E[U(X_T + nh(Y_T))|X_t = x, S_t = s, Y_t = y], \tag{23}$$

where $\mathcal{A}$ denotes the set of admissible trading strategies. The dynamics of the state variables X, S, Y are given by (17) and (8,9). For starting time 0 we write $u^{(n)}(x) \equiv u^{(n)}(0,x,\cdot,\cdot)$.

The set of admissible strategies is defined as follows. Denote by $\Delta := \pi/S$ be the adapted S-integrable process for the number of shares held. We follow Becherer [2] and take the space of permitted strategies as

$$\mathcal{A} = \{\Delta : (\Delta\cdot S) \text{ is a } (Q,\hat{\mathbb{F}})\text{-martingale for all } Q \in \mathcal{M}_f\},$$

where $(\Delta\cdot S)_t = \int_0^t \Delta_u dS_u$ is the gain from trading over $[0,t]$, $t \in [0,T]$. Other choices for $\mathcal{A}$ are possible, see for example Schachermayer [31] or Delbaen et al [10]. However, these all lead to the same solution for the dual problem (see [10, 15, 31]), and hence to the same primal solution for the utility maximisation problem.

Note that we could also write the value function in (23) as a function of wealth, the non-traded asset price, and the current values of the random drifts: $u(t,x,y;\hat{\lambda}_t,\hat{\theta}_t)$, with the dynamics of the drifts given by (16). We do not pursue this as the dimension of the resulting HJB equation is initially higher than in our formulation, and could be reduced by changing variables using the formulae for the random drifts in (11).

Denote the optimal trading strategy by $\pi^* \equiv \pi^{*,n}$, and the optimal wealth process by $X^* \equiv X^{*,n}$. The utility-based price and hedge for a position in n claims are defined in the now classical manner. The indifference price per claim at $t \in [0,T]$, given $X_t = x, S_t = s, Y_t = y$, is $p^{(n)}$ given by

$$u^{(n)}(t, x - np^{(n)}(t,x,s,y), s, y) = u^{(0)}(t,x,s).$$

The optimal hedging strategy is to hold $(\Delta_t^H)_{0\le t\le T}$ shares of stock at time t, where $\Delta_t^H S_t =: \pi_t^H S_t$, and $\pi^H := \left(\pi_t^H\right)_{0\le t\le T}$, is defined by

$$\pi_t^H := \pi_t^{*,n} - \pi_t^{*,0}, \quad 0 \le t \le T. \tag{24}$$

It is well known that with exponential utility the indifference price is independent of the initial cash wealth x, so we shall write $p^{(n)}(t,x,s,y) \equiv p^{(n)}(t,s,y)$ from now on.

For small positions in the claim (or, equivalently, for small risk aversion), we shall later approximate the indifference price by the marginal utility-based price introduced by Davis [8]. This is the indifference price for infinitesimal diversions of funds into the purchase or sale of claims, and is equivalent (as is well-known) to the limit of the indifference price as $n \to 0$.

Definition 1 (Marginal price) The marginal utility-based price of the claim at $t \in [0,T]$ is $\hat{p}(t,s,y)$ defined by

$$\hat{p}(t,s,y) := \lim_{n\to 0} p^{(n)}(t,s,y).$$

It is well known that with exponential utility the marginal price is also equivalent to the limit of the indifference price as risk aversion goes to zero. Under appropriate conditions (satisfied in this model) it is given by the expectation of the payoff under the optimal measure of the dual problem without the claim. For exponential utility this measure is the minimal entropy measure Q^E and, as we have already seen, in our model $Q^E = Q^M$, giving the representation $\hat{p}(t,s,y) = E^{Q^M}[h(Y_T)|S_t = s, Y_t = y]$, as we shall see in the next section.

3.2 Dual problem and optimal hedge

We attack the primal utility maximisation problem (23) using well-known duality results that are now a classical tool for incomplete market optimisation problems (see the seminal papers by Kramkov and Schachermayer [17] and Karatzas et al [16]). For a problem with the random terminal endowment of a European claim, and with exponential utility, as in this paper, Delbaen et al [10] establish the required duality relations between the primal and dual problems in a semimartingale setting. We shall use these results below to establish a simple algebraic relation (Lemma 1) between the primal value function and the indifference price, which we shall then exploit to derive the representation for the optimal hedging strategy.

The dual problem with starting time 0 has value function defined by

$$v^{(n)}(\eta) := \inf_{Q\in\mathcal{M}} E\left[V(\eta Z_T) + \eta Z_T n h(Y_T)\right],$$

where Z is the density process in (18) and V is the convex conjugate of the utility function:

$$V(\eta) := \sup_{x\in\mathbb{R}} [U(x) - x\eta].$$

For exponential utility V is given by

$$V(\eta) = \frac{\eta}{\alpha}\left[\log\left(\frac{\eta}{\alpha}\right) - 1\right].$$

Hence the dual value function has the well-known entropic representation

$$v^{(n)}(\eta) = V(\eta) + \frac{\eta}{\alpha}\inf_{Q\in\mathcal{M}}\left[\mathcal{H}(Q,P) + \alpha n E^Q h(Y_T)\right].$$

Denoting the dual minimiser that attains the above infimum by $Q^{*,n}$, we observe that $Q^{*,n} \in \mathcal{M}_f$.

For a starting time $t \in [0,T]$ the dual value function is defined by

$$v^{(n)}(t,\eta,s,y) := \inf_{Q\in\mathcal{M}} E\left[V\left(\eta\frac{Z_T}{Z_t}\right) + \eta\frac{Z_T}{Z_t}nh(Y_T)\middle| S_t = s, Y_t = y\right], \tag{25}$$

and we write $v^{(n)}(\eta) \equiv v^{(n)}(0,\eta,\cdot,\cdot)$.

Lemma 1 *The primal value function and indifference price are related by*

$$u^{(n)}(t,x,s,y) = u^{(0)}(t,x,s)\exp\left(-\alpha np^{(n)}(t,s,y)\right), \tag{26}$$

where the value function without the claim is given by

$$u^{(0)}(t,x,s) = -\exp\left(-\alpha x - H^E(t,s)\right), \tag{27}$$

and $H^E(t,s)$ is the conditional minimal entropy function defined in (22).

Proof For brevity, we give the proof for $t = 0$. The proof for a general starting time follows similar lines, and we make some comments on how to adapt the following argument for that case at the end of the proof.

The fundamental duality linking the primal and dual problems in Delbaen et al [10] implies that the value functions $u^{(n)}(x)$ and $v^{(n)}(\eta)$ are conjugate:

$$v^{(n)}(\eta) = \sup_{x\in\mathbb{R}}[u^{(n)}(x) - x\eta], \qquad u^{(n)}(x) = \inf_{\eta>0}[v^{(n)}(\eta) + x\eta].$$

The value of η attaining the above infimum is η^*, given by $v^{(n)}_\eta(\eta^*) = -x$, so that

$$u^{(n)}(x) = v^{(n)}(\eta^*) + x\eta^*,$$

which translates to

$$u^{(n)}(x) = -\exp\left(-\alpha x - \inf_{Q\in\mathcal{M}}\left[\mathcal{H}(Q,P) + \alpha nE^Q h(Y_T)\right]\right). \tag{28}$$

So, in particular,

$$u^{(0)}(x) = -\exp\left[-\alpha x - \mathcal{H}(Q^E,P)\right], \tag{29}$$

where Q^E is the minimal entropy measure: $Q^E = Q^{*,0}$

Combining the dual representations (28) and (29) for the primal problems with and without the claim, with the definition of the indifference price, gives the dual representation for the utility-based price in the form

$$p^{(n)} = \frac{1}{\alpha n}\left[\inf_{Q\in\mathcal{M}}\left[\mathcal{H}(Q,P) + \alpha nE^Q h(Y_T)\right] - \mathcal{H}(Q^E,P)\right], \tag{30}$$

which is the representation found in Delbaen et al [10], modified slightly as we have a random endowment of n claims ([10] considered the case $n = -1$).

In particular, for $n \to 0$ or $\alpha \to 0$, we obtain the marginal price of Davis [8]:

$$\hat{p} := \lim_{n\to 0} p^{(n)} = E^{Q^E}h(Y_T) = E^{Q^M}h(Y_T), \tag{31}$$

the last inequality following from the equality of Q^M and Q^E, as implied by (20).

From (28)–(30), the relation between the primal value functions and indifference price then follows immediately, as

$$\begin{aligned} u^{(n)}(x) &= -\exp\left(-\alpha x - \mathcal{H}(Q^E,P) - \alpha np^{(n)}\right) \\ &= u^{(0)}(x)\exp\left(-\alpha np^{(n)}\right). \end{aligned}$$

Similarly, a corresponding relation for a starting time $t \in [0,T]$ may also be derived. This is achieved using the definition (25) of the dual value function for an initial time $t \in [0,T]$, the conjugacy of $u^{(n)}(t,x,s,y)$ and $v^{(n)}(t,\eta,s,y)$ and the definitions (21) and (22) of the conditional entropy and conditional minimal entropy.

□

Using Lemma 1 we obtain the following representation for the optimal hedging strategy associated with the indifference price. In what follows we assume that the indifference price is a suitably smooth function of (t,s,y), so that (given Lemma 1) we may assume the primal value function is smooth enough to be a classical solution of the associated Hamilton-Jacobi-Bellman (HJB) equation. This smoothness will be confirmed in the next subsection when we view the indifference price as the solution of a stochastic control problem.

Theorem 1 *The optimal hedge for a position in n claims is to hold Δ_t^H units of S at $t \in [0,T]$, where*

$$\Delta_t^H = -n\left(p_s^{(n)}(t,S_t,Y_t) + \rho\frac{\beta}{\sigma}\frac{Y_t}{S_t}p_y^{(n)}(t,S_t,Y_t)\right).$$

Remark 1 We note the extra term in the hedging formula compared with the corresponding full information result (4). The drift parameter uncertainty results in additional risk, manifested as dependence of the indifference price on the stock price, and hence the derivative with respect to the stock price appears in the theorem.

Proof The HJB equation associated with the primal the value function is

$$u_t^{(n)} + \max_{\pi}\mathcal{A}_{X,S,Y}u^{(n)} = 0,$$

where $\mathcal{A}_{X,S,Y}$ is the generator of (X,S,Y) under P. Performing the maximisation over π yields the optimal Markov control as $\pi_t^{*,n} = \pi^{*,n}(t,X_t^{*,n},S_t,Y_t)$, where

$$\pi^{*,n}(t,x,s,y) = -\left(\frac{\hat{\lambda}u_x^{(n)} + \sigma s u_{xs}^{(n)} + \rho\beta y u_{xy}^{(n)}}{\sigma u_{xx}^{(n)}}\right),$$

and where the arguments of the functions on the right-hand-side are omitted for brevity. For the case $n=0$ there is no dependence on y in the value function $u^{(0)}$, and we have $\pi_t^{*,0} = \pi^{*,0}(t,X_t^{*,0},S_t)$, where

$$\pi^{*,0}(t,x,s) = -\left(\frac{\hat{\lambda}u_x^{(0)} + \sigma s u_{xs}^{(0)}}{\sigma u_{xx}^{(0)}}\right).$$

Applying the definition (24) of the optimal hedging strategy along with the representations (26) and (27) from Lemma 1 for the value functions, gives the result.

□

3.3 Stochastic control representation of the indifference price

In this section we discuss the required smoothness of the indifference price required for the validity of Theorem 1. We briefly show how smoothness can be established. Later, we shall in any case approximate the indifference price by the marginal price, for which we will derive an explicit formula that is indeed smooth enough to apply the hedging theorem.

The dual representation (30) of $p^{(n)}$ gives the price of the claim at time 0 as the value function of a control problem:

$$p^{(n)} = \inf_{\psi} E^Q\left[\frac{1}{2\alpha n}\int_0^T \psi_t^2 dt + h(Y_T)\right],$$

to be minimised over control processes $(\psi_t)_{0\le t\le T}$, such that $Q\in\mathcal{M}_f$, and with dynamics for S,Y given by

$$\begin{aligned} dS_t &= \sigma S_t d\hat{B}_t^Q, \\ dY_t &= \beta Y_t[(\hat{\theta}(t,Y_t)-\rho\hat{\lambda}(t,S_t)-\sqrt{1-\rho^2}\psi_t)dt+d\hat{W}_t^Q]. \end{aligned}$$

For a starting time $t\in[0,T]$ we have

$$p^{(n)}(t,s,y)=\inf_{\psi} E^Q\left[\frac{1}{2\alpha n}\int_t^T \psi_u^2 du + h(Y_T)\middle| S_t=s, Y_t=y\right].$$

The HJB dynamic programming PDE associated with $p^{(n)}(t,s,y)$ is

$$\begin{gathered} p_t^{(n)}+\mathcal{A}_{S,Y}^{Q^M}p^{(n)}+\inf_{\psi}\left[\tfrac{1}{2\alpha n}\psi^2-\beta\sqrt{1-\rho^2}\psi y p_y^{(n)}\right]=0, \\ p(T,s,y)=h(y), \end{gathered}$$

where $\mathcal{A}_{S,Y}^{Q^M}$ is generator of (S,Y) under minimal measure:

$$\mathcal{A}_{S,Y}^{Q^M}f(t,s,y)=\beta(\hat{\theta}(t,y)-\rho\hat{\lambda}(t,s))yf_y+\frac{1}{2}s^s f_{ss}+\frac{1}{2}\beta^2y^2f_{yy}+\rho\sigma\beta syf_{sy}.$$

The optimal Markov control is $\psi_t^{*,n}\equiv\psi^{*,n}(t,S_t,Y_t)$, where

$$\psi^{*,n}(t,s,y)=\alpha n\sqrt{1-\rho^2}\beta y p_y^{(n)}(t,s,y),$$

and note that $\psi^{*,0}=0$. Substituting back into the HJB equation, we find that $p^{(n)}$ is expected to solve the semi-linear PDE

$$\begin{gathered} p_t^{(n)}+\mathcal{A}_{S,Y}^{Q^M}p^{(n)}-\tfrac{1}{2}\alpha n(1-\rho^2)\beta^2y^2\left(p_y^{(n)}\right)^2=0, \\ p^{(n)}(T,s,y)=h(y). \end{gathered} \tag{32}$$

We note that for $n=0$ this becomes a linear PDE for the marginal price $\hat{p}$, so that by the Feynman-Kac Theorem we have

$$\hat{p}(t,s,y)=E_{t,s,y}^{Q^M}h(Y_T), \tag{33}$$

consistent with the general result (31). In particular, we shall see that in this case the marginal price is given by a BS-type formula, and is a sufficiently smooth function for Theorem 1 to be applicable with $\hat{p}$ as an approximation to $p^{(n)}$.

In the case $n\neq 0$, the existence of sufficiently smooth solutions to semi-linear PDEs of the type (32) has been considered by Pham [29] and Benth and Karlsen [4], and similar techniques could in principle be used to establish that $p^{(n)}$ is indeed a classical solution to (32). We do not pursue this here, but instead follow Davis [9] and make the transformations

$$\begin{aligned} G_t &:= \frac{1}{\sigma}\log S_t, \\ L_t &:= \frac{1}{\beta}\log Y_t, \\ f(g) &:= h(e^{\beta g}). \end{aligned}$$

The indifference pricing function expressed in the new variables is $J(t,g,\ell)$ defined by the stochastic control problem

$$J(t,g,\ell):=\inf_{\psi}E^Q\left[\frac{1}{2\alpha n}\int_t^T\psi_u^2du+f(L_T)\middle| G_t=g, L_t=\ell\right], \tag{34}$$

subject to state dynamics

$$\begin{aligned} dG_t &= -\frac{1}{2}\sigma dt + d\hat{B}^Q, \\ dL_t &= (a(t, G_t, L_t) - \sqrt{1-\rho^2}\psi_t)dt + d\hat{W}_t^Q, \end{aligned}$$

where, using (11), the function $a(t, g, \ell)$ is given by

$$a(t, g, \ell) = \mathrm{v}_t \left[\theta_0 - \rho\lambda_0 - (L_0 - \rho G_0) + \frac{1}{2}(\beta - \rho\sigma)t + \ell - \rho g\right] - \frac{1}{2}\beta,$$

which is Lipschitz in g, ℓ for all $t \in [0, T]$. The point of making this transformation is that (34) is a standard form of stochastic control problem, whose solution $f^{(n)}$ can be shown to be given by a classical solution of the HJB equation

$$\begin{gathered} J_t + a(t, g, \ell)J_\ell - \tfrac{1}{2}\sigma J_g + \tfrac{1}{2}J_{gg} + \rho J_{g\ell} + \tfrac{1}{2}J_{\ell\ell} + \min_{\psi\in\mathbb{R}}\left[\tfrac{1}{2\alpha n}\psi^2 - \sqrt{1-\rho^2}\psi J_\ell\right] = 0, \\ J(T, g, \ell) = f(\ell). \end{gathered} \tag{35}$$

The salient feature of this equation is that it is uniformly elliptic or parabolic[1] for $\rho^2 \leq 1$, and the proof of Lemma 1 and Theorem 2 in Davis [9] can be adapted to show that the value function J is the unique classical solution of (35), and hence that the indifference pricing function $p^{(n)}(t, s, y)$ is a classical solution of (32), and thus the primal value function $u^{(n)}$ is smooth enough for the proof of Theorem 1 to be valid. We do not give the details here, as we shall in any case use a smooth approximation to the indifference price in rest of our analysis.

3.4 Analytic approximation for the indifference price

To obtain analytic results and hence conduct a simulation study of the effectiveness of the optimal hedging strategy, we approximate the indifference price by the marginal price in (33). The marginal price (and hence the associated trading strategy) can be computed in analytic form since, under Q^M, $\log Y_T$ is Gaussian. We have the following result.

Proposition 2 *Under Q^M, conditional on $S_t = s, Y_t = y$, $\log Y_T \sim \mathrm{N}(m, \Sigma^2)$, where $m \equiv m(t, s, y)$ and $\Sigma^2 \equiv \Sigma^2(t)$ are given by*

$$\begin{aligned} m(t, s, y) &= \log y + \beta\left(\hat{\theta}(t, y) - \rho\hat{\lambda}(t, s) - \frac{1}{2}\beta\right)(T - t) \\ \Sigma^2(t) &= \left[1 + (1-\rho^2)\mathrm{v}_t(T-t)\right]\beta^2(T-t) \end{aligned}$$

Proof This is established by computing the SDEs for Y and for $\hat{\theta}_t - \rho\hat{\lambda}_t$ under Q^M. Indeed, applying the Itô formula to $\log Y_t$ under Q^M, we obtain, for $t < T$,

$$\log Y_T = \log Y_t + \beta\int_t^T \left(\hat{\theta}_u - \rho\hat{\lambda}_u\right)du - \frac{1}{2}\beta^2(T-t) + \beta\int_t^T d\hat{W}_u^{Q^M}, \tag{36}$$

where $\hat{W}^{Q^M}$ is a Brownian motion under Q^M. The dynamics of $\hat{\theta}_t - \rho\hat{\lambda}_t$ under Q^M are

$$d(\hat{\theta}_t - \rho\hat{\lambda}_t) = \sqrt{1-\rho^2}\mathrm{v}_t d\hat{B}_t^{\perp, Q^M},$$

where $\hat{B}^{\perp,Q^M}$ is a Q^M-Brownian motion perpendicular to that driving the stock, related to $\hat{W}^{Q^M}$ by $\hat{W}^{Q^M} = \rho\hat{B}^{Q^M} + \sqrt{1-\rho^2}\hat{B}^{\perp,Q^M}$, and where $\hat{B}^{Q^M}$ is the Brownian motion driving S. Hence, for $u > t$, after changing the order of integration in a double integral, we obtain

$$\int_t^T \left(\hat{\theta}_u - \rho\hat{\lambda}_u\right)du = \left(\hat{\theta}_t - \rho\hat{\lambda}_t\right)(T-t) + \sqrt{1-\rho^2}\int_t^T \mathrm{v}_u(T-u)d\hat{B}_u^{\perp,Q^M}.$$

This can be inserted into (36) to yield the desired result.

[1]In other words, writing the second order terms as $a_{11}J_{gg} + 2a_{12}J_{g\ell} + a_{22}J_{\ell\ell}$, we have $a_{12}^2 \leq a_{11}a_{22}$.

□

We are thus able to obtain BS-style formulae for the price and hedge. For a put option of strike K we easily obtain the following explicit formulae for the marginal price and the associated optimal hedging strategy, where Φ denotes the standard cumulative normal distribution function.

Corollary 1 *With m and Σ as in Proposition 2, define $b \equiv b(t,s,y)$ by*

$$m = \log y + b - \frac{1}{2}\Sigma^2.$$

Then the marginal price at time $t \in [0,T]$ of a put option with payoff $(K - Y_T)^+$ is $\hat{p}(t, S_t, Y_t)$, where

$$\begin{aligned} \hat{p}(t,s,y) &= K\Phi(-d_1 + \Sigma) - ye^b\Phi(-d_1), \\ d_1 &= \frac{1}{\Sigma}\left[\log\left(\frac{y}{K}\right) + b + \frac{1}{2}\Sigma^2\right]. \end{aligned}$$

The optimal hedging strategy given by Theorem 1 with $\hat{p}$ as an approximation to the indifference price is $\hat{\Delta}_t \equiv \hat{\Delta}(t, S_t, Y_t)$, where

$$\hat{\Delta}(t,s,y) = n\rho\frac{\beta}{\sigma}\frac{y}{s}e^b\Phi(-d_1). \tag{37}$$

4 Performance of the optimal strategy

To assess the performance of the optimal strategy based on the marginal price, we conducted the following simulation experiment.

Using chosen values for the "true" drifts λ, θ, we generated asset price paths S, Y over a time frame $[-t_0, T]$, with some small time interval δt (typically 1 trading day or less) between successive prices. We used the data over $[-t_0, 0]$ to estimate the asset drifts, used these estimates to set the means λ_0, θ_0 of the prior distribution, and we set $\mathrm{v}_0 = -1/t_0$. We then update this prior over the hedging timeframe $[0,T]$, and hedge the claim using the strategy in (37).

We suppose a put option of strike K is sold at time 0 for $\hat{p}(0, S_0, Y_0)$ and optimally hedged over $[0,T]$ incorporating updating from filtering, and using rebalancing at intervals of δt. In this way, we generate a terminal hedging error. We repeat this procedure, of setting a prior using past data and then hedging over $[0,T]$, over many paths, in order to generate a terminal hedging error distribution.

We repeated the hedging error computation over the same asset price histories using the BS-style hedge (3), and also with a hedge in the absence of filtering, where we used the initial estimates of the asset drifts to compute the hedge throughout the hedging timeframe. This uses the hedge in (4) and the approximation formulae from Monoyios [24]. Figure 1 shows a typical simulation.

The result of carrying out such a procedure over 10,000 asset price histories is shown in Table 1. The parameters are the same as those used in Figure 1. The results clearly show that the optimal hedge with learning produces a hedging error distribution with higher mean, lower standard deviation and a higher median (all as percentages of the initial option premium), than either the BS hedge or the hedge without learning. Thus, the frequency of profits over losses is increased by the optimal hedging program incorporating learning.

Increasing the correlation to 0.9 gave the results in Table 2. The results are still favourable, even when one sells the option (on average) for a lower value than the BS price, showing that the improvement in hedging performance is not due to starting with a higher wealth in the initial hedging portfolio.

Qualitatively similar results were obtained when varying other parameters. The conclusion is that optimal hedging combined with a filtering algorithm to deal with drift parameter uncertainty can indeed give improved hedging performance over methods which take S as a perfect proxy for Y, and over methods which do not incorporate learning via filtering.

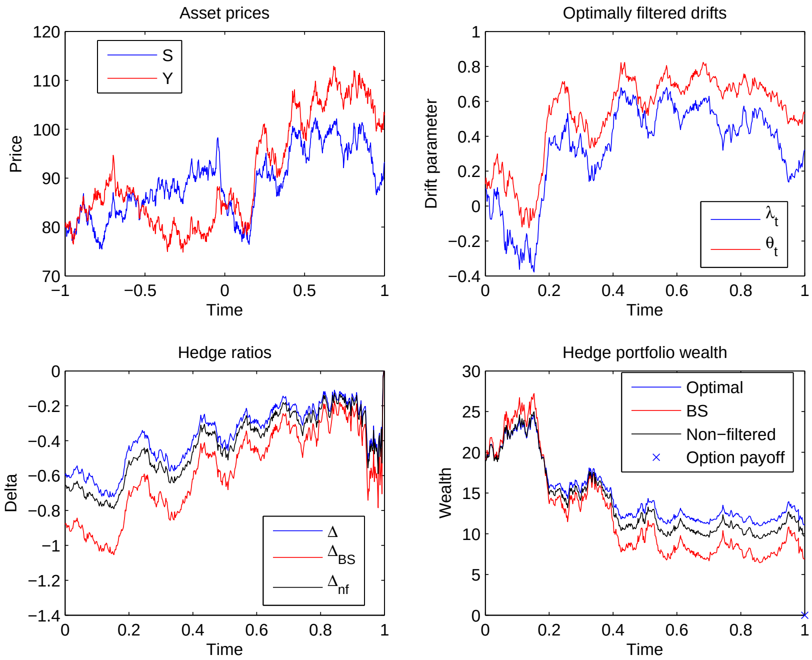

Figure 1: A simulation of the estimation and hedging program over a typical asset price history. The data over $[-t_0, 0]$ is used to find classical estimates of the drifts λ, θ, and the values of the estimates are used to set the means λ_0, θ_0 of the prior distribution, with the prior variance set to $1/t_0$. The parameters used were $t_0 = 1$, $T = 1$, $S_{-t_0} = Y_{-t_0} = 80$, $\lambda = 0.3$, $\sigma = 0.2$, $\theta = 0.4$, $\beta = 0.25$, $\rho = 0.75$, $K = 100$. $\alpha = 0.01$, $n = -1$, $\delta t = 1/504$. The time 0 prices and hedge ratios were $\hat{p}_0 = 19.20$, $\hat{\Delta}_0 = -0.6002$ (optimal hedge with filtering), $p_0^{\mathrm{BS}} = 19.16$, $\Delta_0^{\mathrm{BS}} = -0.8801$ (BS hedge), and $p_0^{\mathrm{NF}} = 18.94$, $\Delta_0^{\mathrm{NF}} = -0.6601$ (optimal hedge without filtering).

Table 1: Hedging error statistics as a fraction of the initial option premium. The parameters are the same as those used in Figure 1. The average initial asset prices at time 0 were $\bar{S}_0 = 84.88$, $\bar{Y}_0 = 86.25$. The average time 0 prices and hedge ratios were $\langle \hat{p}_0 \rangle = 19.98$, $\langle \hat{\Delta}_0 \rangle = -0.5885$ (optimal hedge with filtering), $\langle p_0^{\mathrm{BS}} \rangle = 19.96$, $\langle \Delta_0^{\mathrm{BS}} \rangle = -0.8397$ (BS hedge), and $\langle p_0^{\mathrm{NF}} \rangle = 19.75$, $\langle \Delta_0^{\mathrm{NF}} \rangle = -0.6284$ (optimal hedge without filtering).

	Mean	SD	Median
Optimal Hedge	0.1948	0.5141	0.1834
BS Hedge	0.1143	0.6674	0.0873
Unfiltered Hedge	0.1613	0.5623	0.1567

Table 2: Hedging error statistics as a fraction of the initial option premium. The parameters are the same as those used in Table 1, except that the correlation is now $\rho = 0.9$. The average initial asset prices at time 0 were $\bar{S}_0 = 84.90$, $\bar{Y}_0 = 86.31$. The average time 0 prices and hedge ratios were $\langle \hat{p}_0 \rangle = 19.75$, $\langle \hat{\Delta}_0 \rangle = -0.7325$ (optimal hedge with filtering), $\langle p_0^{\mathrm{BS}} \rangle = 19.91$, $\langle \Delta_0^{\mathrm{BS}} \rangle = -0.8414$ (BS hedge), and $\langle p_0^{\mathrm{NF}} \rangle = 19.64$, $\langle \Delta_0^{\mathrm{NF}} \rangle = -0.7547$ (optimal hedge without filtering).

	Mean	SD	Median
Optimal Hedge	0.1416	0.3948	0.1014
BS Hedge	0.1116	0.4413	0.0678
Unfiltered Hedge	0.1226	0.4004	0.0846

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