Market Models for Inflation

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A thesis submitted for the degree of

Msc. in Mathematical and Computational Finance

Trinity 2008

This work is dedicated to my family,
for all their love and care...

Acknowledgements

Praise is to the One, the Almighty, the merciful and the beneficent. Allah, who is the source of all knowledge and wisdom, taught us what we knew not. I also send Darood to the Holy Prophet Muhammad (saw), who is the Allah's messenger to guide me in affairs of life by his examples and percepts.

I express my heartiest gratitude to my course and thesis Supervisor, Dr. Ben Hambly for his guidance, support and motivation. His valuable suggestions and encouragement during my work made it possible.

I would also like to mention my funding body Higher Education Commission, Pakistan for their great funding scheme that provided so many scholars of Pakistan a chance of higher studies in reputable universities of the world.

In the end I would like to pay thanks to my family and friends for their continuous encouragement and moral support.

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Chapter 1

Introduction

Inflation is currently a big financial issue. Not only it have an affect on a country's economy but it also changes the thinking of an average person. In particular, people start with drawing their money from bank accounts and try to buy a stronger currency or want to invest their money in products that guarantee the value of their money over the time of inflation. It affects people on fixed income, such as pensioners and retirees, the most because a healthy amount of savings today may not have the same purchasing power over time. Simple questions come into one's mind such as how to protect against inflation? How to mantain your standard of living given uncertainity about future inflation?

The answer to these questions is inflation indexed securities. Products in the inflation indexed market are tied to inflation. The idea is to invest your money, for example in the bond market, in such a way that gives you a real return even though the value of your principal amount and the coupons may change. The United Kingdom (UK) introduced the first inflation indexed bonds in 1981 after the remarkable 1970's high inflation. The inflation touched double digits in the 1970's in the UK and to protect again such spikes in the inflation rate the government issued inflation securities. The inflation-indexed market has grown steadily over the past ten years. The government and different corporations are now establishing a wide variety of inflation-indexed securities.

The main objective of this thesis is to study and discuss some of the models for inflation that have been developed. Inspite of the fact that some work has already been done in this field, this still remains an active area of research these days. The need to protect against inflation gives rise to inflation-indexed derivatives and their pricing in an arbitrage free way. We start by looking at inflation, its effects and causes in chapter 2. This chapter includes some basic definitions such as nominal and real rates, provides some details on the inflation derivatives traded in financial markets and inflation-indexed securities.

The third chapter consists of the mathematical background that is needed in chapters 4 and 5. We start by looking at martingale theory and present some key tools of mathematical finance. A small section is devoted to reviewing the change of numeraire technique. We then provide some common and useful definitions that we must know for our work. We also include the HJM framework and extended Vasicek model to help the reader.

Our main work is presented in chapters 4 and 5, in which we discuss some models of inflation. Chapter 4 discusses the Jarrow and Yildirim model of inflation in full detail. This model is considered as the first model of inflation and is based on an analogy with exchange rate models. We also price call options on the inflation index using their technique. In the end, the model developed by Beldgrade, Benhamou and Koehlar in [5] is also included along with the models of Mercurio in [3] for the sake of completeness.

Chapter 5 presents the market models of Mercurio [3] in complete detail. We price the ZCIIS and YYIIS by using Mercurio's first and second market model and by the Jarrow and Yildirim model. One can find the extensions of these models in [4,11,12].

Chapter 2

Inflation

In this chapter, we present some theories and definitions related to inflation. Inflation has its effects on everyone from an average person to the economy of a country and hence is a big area of today's research. One might think that everyone would be equally affected by inflation but thats not the case. Inflation effects you badly when you plan your retirement period because your savings will buy less and less over time and you don't have any source of income. Of course you then think to protect yourself against inflation and hence there is a need for inflation indexed securities. These are the products that guarentee you the same purchasing power from your savings. This chapter starts with the definition of inflation and its effects. The difference between the nominal and real rates is highlighted with a small discussion on inflation-indexed securities and derivatives traded.

2.1 Inflation

Prices of different goods and services do not never same. A candy bar costs more than it used to be 20 years ago. Your bills do not remain the same even if you purchase the same groceries. Inflation is the term to describe when the same goods and services cost more over time. It means you can't buy the same thing for the same price as you could a few years back. Inflation occurs when prices continue to creep upward.

Inflation is the rate of increase in the general level of prices for the same goods and services. In simple words, the rise in prices of goods when there is no compatible rise in wages cause inflation. An increase in the money in circulation may cause a sudden fall in its value and a sharp increase in prices of goods - hence inflation. Inflation is measured as a percentage increase in some kind of inflation index, which may not be unique as there are many types of inflation indexes. A general fall in price level is called deflation.

Inflation does not effect you much if it stays for a small period of time but history shows that the prices never fell back to their previous values except in 1930's when there was a deflation. Since the prices added up each year, a small rate as 4 % could be huge if it stays for a long period of time. In the past fifty years the developed countries have experienced a decrease in the level of inflation and reduced this rate to about 2.00 - 3.00 %

Inflation can be caused by many reasons. Economists are trying to locate its causes and established many theories regarding this. One of those is the demand and supply, in which if the demand increases more than the supply, the prices of the goods increase also and causes inflation. Macroeconomists also think that the increase in the wages of workers can cause inflation because to make significant profits, the company's owner increases the prices of products. Some think that inflation is an indication of growing economy. There are many groups of thought which give different theories but they all consider it a big issue in today's world.

2.2 Nominal Rates, Real Rates and Expected Inflation

We have two choices to look at when valuing the cashflows, the nominal or real cashflows. When we talk about rates or rates of return we refer to the amount in terms of nominal rates. The price of the unit of money never remains the same rather it decays over time due to inflation. The nominal rates are the ones which indicate the rates in terms of money but not in the real sense of its purchasing power. On the other hand the real rates do consider the effect of inflation also. In other words the nominal rates tell you the growth of your money and not the purchasing power. The real rates tell you what is the purchasing power of your money. It is not possible to tell the real rates at the time of investment or borrowing because no one knows the inflation rate for the coming years.

From an economics perspective it is important to differentiate between the real and nominal rates because the real rates are the ones that matter. The nominal rates tells you about the growth of your money at present purchasing power ignoring the effect of inflation. This makes you think that you will get a big amount of money in return but not the purchasing power of the money at that time. The fact is that the nominal rates are more fanciful and the real rates tell you the true value of your money even though it is a small number as compared to the nominal. We give an example below.

We suppose that we buy a 1 year bond with an interest rate of 8 percent that means if we pay \$ 100 at the start of the year we will get \$ 108 at the end of the year. This rate of interest is nominal as we didn't consider the inflation. Suppose now that the inflation rate is 4 % in the same year. That means if you can buy a basket of goods for \$ 100 today, the same basket will cost \$ 104 at the end of the year. So if we buy a bond with 8 % nominal interest rate for \$ 100, sell it after an year for \$ 108 and buy that basket of goods we left with \$ 4. That means your real interest rate is 4 %.

There is a relationship between the real and nominal rates and the expected inflation called Fisher's equation. The Fisher hypothesis is that the real rate of interest remain constant and the nominal rate of interest changes with inflation. Since one cannot predict the inflation and nominal intersest rates Fisher equation roughly says that

$$\text{Real interest rates} = \text{Nominal interest rate - Inflation.}$$

If the inflation is positive, the real rates would be lower than the nominal rates and if the there is deflation the nominal rates would be lower then the real ones. But because we cannot predict the inflation for the coming year we can only deal with either real or nominal rates.

2.3 Inflation-Indexed Securities

In general investors focus on the nominal rate of return on their investments and do not consider the real returns. The inflation do effect the rate of nominal returns on your investment especially when you invest for a long term. Purchasing a normal bond ensures you a nominal rate of return on your investments at maturity and you do not know the real return or in other words the purchasing power of your money because you can not predict the rate of inflation. For example, the purchasing power of \$ 100 would not be the same 30 years from now. If the inflation is at 1 % on average then \$ 100 would have a purchasing power of \$ 74 after 30 years. So one has to take care of inflation when investing money.

Of course you would be interested if someone tells you about a security that guarantees a real rate of return over inflation with no credit risk. A security that guarantees a return higher than the rate of inflation if it is held to maturity is called an inflation-indexed security. These securities guarantee that the holder is going to have the same purchasing power plus a fixed return on his investment.

The use of indexed securities is not new. It was traced back to 18^{th}-century, when the state of Masschusetts introduced bills linked to the price of silver. During the American Revolution, soldiers were issued with depreciation notes in order to preserve the real value of their wages. Experience showed that inflation based on a single good was not a healthy idea and more complex indexation methods have been developed on different sets of baskets throughout history. Several countries experienced a very high inflation during the last half of the 20^{th}-century and began to issue the indexed bonds as means of maintaining the acceptability of long-term contracts. Some recent issues include UK(1981), Australia(1985), Canada(1991), Sweden(1994), US(1997), France(1998), Greece and Italy(2003), Japan(2004) and Germany(2006). Governments mostly issue the inflation-indexed securities but corporations could also benefit from it. There are still some countries which have decided not to issue these securities and some have stopped issuing these securities, for example the Australian government has stopped issuing the CIBs in 2003.

Inflation-indexed securities are instruments that can protect their buyers from changes in the general level of the economy. These securities could be a means of measuring the markets expectations of inflation. The most common cash flow structures are interest indexed bond, current pay indexed, capital indexed bonds and indexed zero-coupon bonds. These securities protect assets and future income against inflation and provide diversification with other assets.

2.4 Derivatives Traded

Inflation derivatives are not a new concept and have been traded for more than a decade. They are generally a form of contract in which two parties agree to payments which are based on the value of an underlying asset or some other data at some particular time. The purpose of inflation derivatives is to minimise the inflation risk in an efficient way for one party while offering the potential for a high return to another. There are a number of instruments that can be used as inflation derivatives.

We assume that the maturity of our contract is M years with payment dates as $T_1, ...T_M$. A swap where Party A pays the inflation rate over a pre-defined period to Party B, while Party B pays a fixed rate to Party A on each payment date is defined as inflation-indexed swap. The inflation rate will be measured as the percentage return of the Consumer Price Index (CPI) over the interval it applies to and is denoted by $I(t)$.

The benchmark is the Zero-Coupon Inflation Indexed Swap (ZCIIS) and is defined as:

Zero-Coupon Inflation Indexed Swap:
In this contract, Party B agrees to pay a fixed amount

$$N^* \left[(1 + K)^M - 1 \right] \tag{2.1}$$

to Party A at the time of maturity i.e. at $T_M = M$, where K and N^* are the contract fixed rate (strike) and nominal value, respectively. In exchange, Party A pays the floating amount in terms of inflation

$$= N^* \left[\frac{I(T_M)}{I_0} - 1 \right] \tag{2.2}$$

to Party B at the time of maturity T_M.

There is no exchange between parties until the time of maturity in a ZCIIS. These are the most flexible contracts and are the building blocks for other structures.

There is another common structure of inflation derivatives in which there is an annual payment in between parties known as Year-on-Year Inflation Indexed Swaps (YYIIS) and is defined as:

Year-on-Year Inflation Indexed Swap:
Let φ_i denote the year fraction of the fixed leg for the time interval $[T_{i-1}, T_i]$ and ψ_i denote that of the floating leg with $T_0 = 0$ that is we start the contract today. In this contract, Party B agrees to pay a fixed amount

$$N^* \varphi_i K$$

to Party B at each time T_i. In exchange, Party A pays Party B the floating amount

$$N^* \psi_i \left[\frac{I(T_i)}{I(T_{i-1})} - 1 \right] \tag{2.3}$$

at each time T_i.

There are other types of structures in inflation derivatives known as Inflation-Indexed Caplets (IIC) and Inflation-Indexed floorlets (IIF). We will not deal with these types in detail but for the sake of completeness we present the definitions here.

Inflation Indexed Caplets / Floorlets

An inflation-indexed cap is a string of IICs and an IIC, in inflation market, can be regarded as a call option on the inflation rate that is implied by the CPI index. An inflation-indexed floor is a string of IIF and an IIF is a put option on the same inflation rate.

The payoff at time T_i is given as

$$N^* \psi_i \left[\omega \left(\frac{I(T_i)}{I(T_{i-1})} - 1 - K \right) \right]^+, \tag{2.4}$$

where K is the strike of IIC or IIF, ψ_i is the contract year fraction for the interval $[T_{i-1}, T_i]$, as before, and $\omega = 1$ for caplet and $\omega = -1$ for a floorlet.

A cap is used to protect its buyer from the inflation erosion above some rate and a floor is used to protect the investments from downside risks due to inflation. Both the investors and the issuers can hedge their inflation exposures with the help of floors and caps.

Chapter 3

Mathematical Background

In this chapter, we will review some basic definition and theories that we need in our later chapters. We start with some fundamentals of mathematical finance that we need in our later work. In the next section we discuss the change of numeriare technique that we use in pricing a call option in chapter 4. We then move on to the Heath-Jarrow-Morton (HJM) framework and extended Vasicek model. We need these in chapter 4 as the Jarrow and Yildirim inflation model is based on HJM framework and discussed this model by considering the dynamics of an extended Vasicek model.

3.1 Fundamentals of Mathematical Finance

This section deals with basic mathematical finance topics that are needed in the next two chapters. We define our financial market and some theoratical concepts and than present two fundamental theorem of asset pricing. These theorems state the conditions to have a complete and arbitrage free market for option pricing. We also included the Girsanov's theorem. This theorem tells us how to convert a physical measure to a risk neutral measure. One can find the details on these topics in [1,9,10].

We start by considering a financial market consisting of $n+1$ assets $\bar{S} = (S^0, S^1, ..., S^n)$ on a time interval $[0,T]$. We assume that S^0 is a risk free asset and the remaining ones are risky assets. We consider a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, where the filtration $\mathcal{F}_t$ is generated by a n-dimensional Brownian motion.
A trading strategy $\left(\bar{\phi}_t\right)$ is a predictable process where $\left(\bar{\phi}_t\right) = (\phi_t^0, ..., \phi_t^n)$. The value of this trading strategy is

$$V_t\left(\phi\right) = \bar{\phi}.\bar{S} = \Sigma_{i=0}^{n} \phi_t^i S_t^i.$$

A trading strategy is self-financing if

$$\begin{aligned} dV_t(\phi) &= \bar{\phi}.d\bar{S} \\ &= \Sigma_{i=0}^{n}\phi_t^i dS_t^i, \end{aligned}$$

that is there is no injection or removal of money. We call a stategy *admissible* if $V_t(\phi) \geq -\alpha$ for some $\alpha > 0$ and $\forall t \in [0,T]$ almost surely.

3.1 Definition
A measure Q, on a filtered probability space is equivalent to the physical measure P, denoted by $Q \sim P$, if they have the same null sets. This is called a martingale measure if all discounted price processes are martingales over $[0,T]$.. In our case Q is such that $\frac{S_i}{S_0}, \forall i$ are martingales under Q.

3.2 Definition
A financial market is complete if every contingent claim is attainable. A *contingent claim* X_T in our financial market is a payoff at time T which depends on the assets S_i over $[0,T]$ and a contingent claim is *attainable* if there is an admissible self-financing strategy such that $V_T(\phi) = X_T$ almost surely

Theorem 3.1 (**The First Fundamental Theorem of Asset Pricing**). Our financial market is arbitrage free if and only if there exists an equivalent martingale measure $Q \sim P$.

Theorem 3.2 (**The Second Fundamental Theorem of Asset Pricing**). An arbitrage free financial market is complete if and only if the equivalent martingale measure Q is unique and hence the prices of the contingent claim are also unique.

There are two fundamental theorems of stochastic calculus that allow us to apply martingale theory to arbitrage theory.

- The Girsanov Theorem shows that a change of measure is equal to a change of drift with respect to the underlying Brownian motion.
- The Martingale Representation Theorem is an existence theorem that shows the existence of an adapted process such that any martingale adapted to the Brownian filtration can be written as an integral of this adapted process adapted to the Brownian filteration. We do not state this as we do not use the martingale representation theorem explicitly in our work.

Theorem 3.3 (**The Girsanov Theorem**)**.** Consider a probability space $(\Omega, \mathcal{F}, P)$ with a n-dimensional Brownian motion W. Let $\theta_t = (\theta_t^1, ..., \theta_t^n)$ be an adapted n-dimensional process and put

$$z_t = \exp\left(-\int_0^t \theta_u.dW_u - \frac{1}{2}\int_0^t \|\theta_u\|^2 du\right),$$

where $\|\theta_u\|^2 = \Sigma_{i=1}^n \left(\theta_u^i\right)^2$.

Let $\tilde{W}_t = W_t + \int_0^t \theta_u du$ and assume that

$$E\left(\int_0^t \|\theta_u\|^2 z_u^2 du\right) < \infty$$

and put

$$\tilde{P}(A) = \int_A z_T(w) P(dw).$$

Then $\tilde{W}$ is a n-dimensional Brownian motion under $\tilde{P}$.

Theorem 3.4 (**Risk Neutral Pricing**)**.** In a complete, arbitrage free financial market, there exists a unique measure Q such that the discounted asset prices are martingales and the price $V(t, S_t)$ of any contingent claim with payoff X_T is

$$V(t, S_t) = S_t^0 E_t^Q\left(\frac{X_T}{S_T^0}\right)$$

3.2 Change of Numeraire

A numeraire is any strictly positive, non-dividend paying tradeable asset.

If we assume that the risk free asset S_0 is strictly positive then S_0 is a numeriare. We will find that the T-bonds are useful numeraire in fixed income settings. We present a theorem on change of numeraire, other details can be find in Brigo-Mercurio [1].

Theorem 3.5. Let N^1 be a numeriare and Q_{N^1} be the measure equivalent to P, such that the asset prices S/N^1 are martingales under Q_{N^1}. For an arbitrary numeraire U, there exists an equivalent measure Q_U such that any contingent claim X_T has price

$$V(t, S_t) = U_t E_t^{Q_U}[X_T/U_T]$$

and moreover

$$\frac{dQ_U}{dQ_{N^1}}\mid_{\mathcal{F}_t}=\frac{U_T}{N_1^T}\frac{N_t^1}{U_t}$$

and $\frac{S}{U}$ is a martingale under Q_U.

3.3 Interest Rates

In this section, we first present some known definitions and then present some results that we use in our later work. A main assumption in our work is that we are dealing with a bond market of ZCBs. All our definitions and results are presented in terms of the ZCBs.

We present the definition of zero-copon bond as:

3.3 Definition (Zero-Coupon Bond)

A zero-coupon bond is a promis to pay 1 unit at a fixed maturity at time T. We call it a $T-$bond and write $P(t,T)$ for its price at time t.

3.4 Definition (LIBOR Rate)

The LIBOR rate is a simple interest that banks charge each other for loans for the time period $[S,T]$, available at t. Mathematically it is defined as

$$F(t,S,T)=\frac{1}{T-S}\left[1-\frac{P(t,S)}{P(t,T)}\right]$$

The spot LIBOR is $F(S,T)=F(S,S,T)=\frac{1}{T-S}\left[1-\frac{1}{P(S,T)}\right]$.

Before giving the definition of forward rates, we define the continuously compounded rate $R(t,S,T)$.

A **continuously compounded rate** can be defined in terms of ZCBs as:

$$e^{R(t,S,T)(T-S)}=\frac{P(t,S)}{P(t,T)}.$$

Or,

$$R(t,S,T)=-\left(\frac{\log(P(t,T))-\log(P(t,S))}{T-S}\right).$$

A **spot continuously compounded** rate is

$$R(S,T)=-\left(\frac{\log(P(S,T))}{T-S}\right).$$

We now define forward rates in terms of a continuous compounded rate

3.3 Definition (Forward Rate)
The instantaneous forward rate is the rate at t for borrowing over $(T, T+\delta t)$ and can be defined from the prices of a T-bond by

$$f(t,T) = \lim_{S \to T} R(t,S,T) = -\frac{\partial \log P(t,T)}{\partial T}.$$

The **short rate** is the rate for borrowing now for an infinitesimal time and is defined as $r_t = f(t,t)$.

Before going to the Heath Jarrow Morton approach, we state a result that we use in chapter 4 in proving Proposition 1. We consider our forward rate to be modelled as:

$$\begin{aligned} df(t,T) &= \alpha(t,T)\,dt + \sigma(t,T)\,dW \\ f(0,T) &= f^*(0,T) \end{aligned} \tag{3.1}$$

where $\alpha(t,T)$ and $\sigma(t,T)$ are adapted processes, the $*$ denotes the market observed value and W is an n-dimensional Brownian motion under P. We assume that

$$\int_0^T \alpha(t,T)\,dt < \infty, \quad \int_0^T \sigma_i^2(t,T)\,dt < \infty \qquad \forall i \quad P \quad a.s.$$

We assume that for a fixed maturity $T > 0$, the instantaneous forward rates evolve as (3.1) where $W = (W_1, ..., W_n)$ is an n-dimensional Brownian motion, $\sigma(t,T) = (\sigma_1(t,T), ..., \sigma_n(t,T))$ is a vector of adapted processes. The product $\sigma(t,T)\,dW(t)$ is a scalar product of two vectors. The initial condition is the observed forward rate curve.

Under the risk neutral measure Q, we have

$$P(t,T) = E_t^Q\left(e^{-\int_t^T r(u)du}\right)$$

and we also have from Definition 3.3

$$P(t,T) = \exp\left(-\int_t^T f(t,s)\,ds\right).$$

As both formulae must be trued and for the consistency in our model, we have to be careful to ensure that there is no arbitrage in the market. This gives us the following theorem:

Theorem 3.6. Under our forward rate model, the bond prices must satisfy

$$dP(t,T) = P(t,T)\left(r(t) + A(t,T) + \frac{1}{2}\|S(t,T)\|^2\right)dt + P(t,T)\,S(t,T)\,dW$$

under P, where

$$\begin{aligned} A(t,T) &= -\int_t^T \alpha(t,s)\,ds, \\ S(t,T) &= -\int_t^T \sigma(t,s)\,ds. \end{aligned}$$

For proof, see [13].

3.4 Heath-Jarrow-Morton Framework (HJM)

In 1986, Ho and Lee developed an alternative to short-rate models and modelled the entire yield curve in a binomial tree setting, [7]. Heath, Jarrow and Morton (HJM), inspired by this, developed a general framework for the modelling of interest-rate dynamics, [8]. They modelled the instantaneous forward rates directly and showed that there is a relationship between the drift and volatility parameters of the forward rate dynamics in an arbitrage free market.

The forward rate dynamics given in (3.1) do not necessarily lead to arbitrage free bond market. Heath, Jarrow and Morton showed that for a unique martingale measure to exist, the function α must have a specified form. They showed that this must be equal to a quantity depending on the vector volatility σ and on the drift rates in the dynamics of n selected zero-coupon bond prices.

HJM Drift Condition
The processes α and σ must satisfy

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)'\,ds, \tag{3.2}$$

under the martingale measure Q for every $t \leq T$. The forward rates can also be expressed as

$$f(t,T) = f^*(0,T) + \int_0^t \alpha(s,T)\,ds + \int_0^t \sigma(s,T)\,dW(s).$$

For proof, see [13].

3.5 Extended Vasicek Model

The Vasicek model for the short rate of interest has mean reversion. As we indicated above, an HJM model can be described by the instantaneous forward rate volatility. The Vasicek type volatility is given as

$$\sigma(t,T) = \sigma e^{\lambda(T-t)}, \tag{3.3}$$

where σ and $\lambda > 0$ are constants. The short rate process for the extended Vasicek model is

$$dr(t) = [\theta(t) - a(t)\, r(t)]\, dt + \sigma(t)\, dW(t). \tag{3.4}$$

If we assume $a(t)$ and $\sigma(t)$ are positive constants, we recover the extended Vasicek model of Hull and White (1994). We get the dynamics of the short rate as

$$dr(t) = (\theta(t) - ar(t))\, dt + \sigma dW(t). \tag{3.5}$$

The time dependent $\theta(t)$ is chosen to exactly fit the term structure of interest rates currently observed in the market. Let $f^*(0,T)$ and $P^*(0,T)$ denote the market instantaneous forward rates and the market value of T-bonds, respectively, at time 0 for all maturities T, then

$$f^*(0,T) = -\frac{\partial \log P^*(0,T)}{\partial T}. \tag{3.6}$$

One can find that in this case

$$\theta(t) = \frac{\partial f^*(0,T)}{\partial T} + af^*(0,T) + \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right). \tag{3.7}$$

One can use the affine term structure to find the value of a T-bond at time t , that is

$$P(t,T) = A(t,T)\, e^{-B(t,T)r(t)}, \tag{3.8}$$

where

$$\begin{aligned} B(t,T) &= \frac{1}{a}\left[1 - e^{a(T-t)}\right], \\ A(t,T) &= \frac{P^*(0,T)}{P^*(0,t)} \exp\left\{B(t,T)\, f^*(0,t) - \frac{a^2}{4a}\left(1 - e^{-2at}\right) B(t,T)^2\right\}. \end{aligned}$$

The extended Vasicek model has an affine term structure, thus it has an explicit solution for bond prices and it is relatively simple to price other instruments in this model.

Chapter 4

Models for Inflation

In this chapter, we will consider some simple models for pricing inflation index derivatives from the literature. The emphasis is going to be on Jarrow and Yildirim (2003) in which they used the HJM model to develop a three factor model for inflation. We present the full details of this model with all the mathematical set up. For the rest of the models we present only a brief description.

4.1 Jarrow and Yildirim Model (2003)

The main reference on pricing inflation-indexed securities and related derivatives using the HJM model is Jarrow and Yildirim (2003)[2]. They considered the Treasury Inflation Protected Securities (TIPS) which is an inflation indexed bond in which the principal is adjusted constantly for inflation and also modify the semi-annual interest payments accordingly. Under the no-arbitrage assumption, they considered a cross-currency economy where nominal dollars correspond to the domestic currency, real dollars corresponds to the foreign currency and the inflation index corresponds to the spot exchange rate. The fluctuations of the real and nominal interest rates and the inflation rate are correlated in their set up.

4.1.1 The Model

We define the notation as used in [2]:

- r for real, n for nominal.
- $P_n(t,T)$: time t price of a nominal zero-coupon bond maturing at time T in dollars.

- $I(t)$: time t CPI-U inflation indexed that is CPI for all urban consumers.
- $P_r(t,T)$: time t price of a real zero-coupon bond maturing at time T in CPI-U units.
- $f_k(t,T)$: time t forward rates for date T where $k \in \{r,n\}$, i.e.

$$P_k(t,T) = \exp\left\{-\int_t^T f_k(t,u)\,du\right\}. \tag{4.1}$$

- $r_k(t) = f_k(t,t)$: the time t spot rate where $k \in \{r,n\}$.
- $B_k(t) = \exp\left\{\int_t^T r_k(v)\,dv\right\}$: time t money market account value for $k \in \{r,n\}$.
- $\mathfrak{B}_k(0)$: time 0 value of a coupon bearing bond in dollars where C is the coupon payment in dollars per period, T is the maturity and F is the face value in dollars, i.e.

$$\mathfrak{B}_n(0) = \sum_{t=1}^{T} CP_n(0,t) + FP_n(0,T). \tag{4.2}$$

- $\mathfrak{B}_{\mathcal{TIPS}}(0)$: time 0 price of a TIPS coupon bearing bond in dollars issued at time $t_0 \leqslant 0$ with a coupon payment C units of the CPI-U, the maturity is time T, and the face value is F units of the CPI-U,

$$\mathfrak{B}_{\mathcal{TIPS}}(0) = \left\{\sum_{t=1}^{T} CI(0)\,P_n(0,t) + FI(0)\,P_n(0,T)\right\}/I(t_0). \tag{4.3}$$

In [2], Jarrow and Yildirim defined the price in dollars of a real zero-coupon bond without an issue date adjustment that is without multiplying by the ratio $\frac{I(0)}{I(t_0)}$ as

$$P_{\mathcal{TIPS}}(t,T) = I(t)\,P_r(t,T). \tag{4.4}$$

The uncertainty in the economy is characterized by a probability space (Ω, F, P) where Ω is a state space, F is a σ-algebra and P is the probability measure on (Ω, F). Let$\{F_t : t \in [0,T]\}$ be the filtration generated by the three Brownian motions $(W_n(t), W_r(t), W_I(t) : t \in [0,T])$. These are standard Brownian motions initialized at zero with correlations give by $dW_n(t)\,dW_r(t) = \rho_{nr}dt, dW_n(t)\,dW_I(t) = \rho_{nI}dt, dW_r(t)\,dW_I(t) = \rho_{rI}dt$.

Given the initial forward rate curve $f_n(0,T)$, we assume that the nominal $T-$maturity forward rate evolves as:

$$df_n(t,T) = \alpha_n(t,T)\,dt + \sigma_n(t,T)\,dW_n(t) \tag{4.5}$$

where $\alpha_n(t,T)$ is random and $\sigma_n(t,T)$ is a deterministic function. We assume that $\alpha_n(t,T)$ is F_t-adapted and jointly measurable with $\int_0^T |\alpha_n(v,T)|\,dv < \infty$ in $P-$a.s. and $\sigma_n(t,T)$ satisfies $\int_0^T \sigma_n(v,T)^2\,dv < \infty$, $P-$a.s.

In the same way, given the initial forward rate curve $f_r(0,T)$, we assume that the real $T-$maturity forward rate evolves as:

$$df_r(t,T) = \alpha_r(t,T)\,dt + \sigma_r(t,T)\,dW_r(t) \tag{4.6}$$

where $\alpha_r(t,T)$ and $\sigma_r(t,T)$ satisfies the same conditions as forward rates for nominal.

The inflation index's evolution is given by

$$\frac{dI_t}{I_t} = \mu_I(t)\,dt + \sigma_I(t)\,dW_I(t) \tag{4.7}$$

where $\mu_I(t)$ is random and $\sigma_I(t)$ is a deterministic function of time satisfying $\mu_I(t)$is F_t-adapted with $E\left[\int_0^\tau \mu_I(t)^2\,dt\right] < \infty$ and $\sigma_I(t)$ satisfies $\int_0^\tau \sigma_I(t)^2\,dv < \infty$ $P-$ a.s.

These evolutions are arbitrage-free and the market is complete if there exits a unique equivalent probability measure Q such that:

$\frac{P_n(t,T)}{B_n(t)}$, $\frac{I(t)P_r(t,T)}{B_n(t)}$, and $\frac{I(t)B_r(t)}{B_n(t)}$ are $Q-$martingales.

By Girsanov's theorem, given that $(W_n(t), W_r(t), W_I(t) : t \in [0,T])$ is a $P-$Brownian motion and that Q is a probability measure equivalent to P, then there exist market prices of risk $(\lambda_n(t), \lambda_r(t), \lambda_I(t) : t \in [0,T])$ such that

$$\tilde{W}_k(t) = W_k(t) - \int_0^t \lambda_k(s)\,ds \text{ for } k \in \{n, r, I\}$$

are $Q-$Brownian motions. We present a proposition that gives necessary and sufficient conditions for the arbitrage free term structure.

Proposition 1:

$\frac{P_n(t,T)}{B_n(t)}$, $\frac{I(t)P_r(t,T)}{B_n(t)}$, and $\frac{I(t)B_r(t)}{B_n(t)}$ are $Q-$martingales if and only if the following conditions hold:

$$\alpha_n(t,T) = \sigma_n(t,T)\left(\int_t^T \sigma_n(t,s)\,ds - \lambda_n(t)\right) \tag{4.8}$$

$$\alpha_r(t,T) = \sigma_r(t,T)\left(\int_t^T \sigma_r(t,s)\,ds - \sigma_I(t)\,\rho_{rI} - \lambda_r(t)\right) \tag{4.9}$$

$$\mu_I(t) = r_n(t) - r_n(t) - \sigma_I(t)\,\lambda_I(t). \tag{4.10}$$

Proof: We prove the second of the above and the others follow in the same way. Since the market is arbitrage free, all tradables must be martingales under Q. Let $\xi_t = \frac{I_t P_r(t,T)}{B_n(t)}$. Then by Ito's formula

$$d\xi_t = -r_n(t)\,dt + \frac{1}{B_n(t)}\left\{I(t)\,dP_r(t,T) + P_r(t,T)\,dI(t) + dP_r(t,T)\,dI(t)\right\}.$$

By using the dynamics of $P_r(t,T)$ in an arbitrage-free market from chapter 3 and $I(t)$, we get

$$\begin{aligned} &= \xi_t\left[A_r(t,T) + \frac{1}{2}\parallel S_r(t,T)\parallel^2 + \sigma_I(t)\,S_r(t,T)\,\rho_{rI} + \mu_I(t)\right]dt \\ &+ \xi_t\left[S_r(t,T)\,dW_r + \sigma_I(t)\,dW_I\right], \end{aligned}$$

where $A_r(t,T) = -\int_t^T \alpha_r(t,s)\,ds$ and $S_r(t,T) = -\int_t^T \sigma_r(t,s)\,ds$.
Now for ξ_t to be a martingale, we need

$$d\xi_t = \xi_t\left[S_r(t,T)\,d\tilde{W}_r + \sigma_I(t)\,d\tilde{W}_I\right]$$

where $\tilde{W}_k$ is as defined above. That is we need,

$$\begin{aligned} &\sigma_I(t)\,S_r(t,T)\,\rho_{rI} + A_r(t,T) + \frac{1}{2}\parallel S_r(t,T)\parallel^2 + \mu_I(t) \\ &\qquad = -\lambda_r(t)\,S_r(t,T) - \lambda_I(t)\,\sigma_I(t). \end{aligned}$$

Differentiating with respect to T, we get

$$-\sigma_I(t)\,\sigma_r(t,T)\,\rho_{rI}) - \alpha_r(t,T) - \sigma_r(t,T)\,S_r(t,T) = \lambda_r(t)\,\sigma_r(t,T).$$

So we have the result,

$$\alpha_r(t,T) = \sigma_r(t,T)\left[\int_t^T \sigma_r(t,s)\,ds - \sigma_I(t)\,\rho_{rI}) - \lambda_r(t)\,\sigma_r(t,T)\right]. \quad \square$$

The first expression in the proposition is the arbitrage-free forward rate drift restriction as in the original HJM model. The second is the analogous arbitrage-free forward rate drift restriction for the real forward rate. We can see the correlation and volatility of the inflation in this expression. The last expression is the Fisher equation relating the nominal interest rate to the real interest rate and expected inflation rate.

The above proposition and Ito's lemma give the following term structure evolution under the martingale measure.

Proposition 2:

Under the martingale measure Q, we have the following price processes:

$$\begin{aligned}
df_n(t,T) &= \sigma_n(t,T)\int_t^T \sigma_n(t,s)\,ds dt + \sigma_n(t,T)\,d\tilde{W}_n(t) \\
df_r(t,T) &= \sigma_r(t,T)\left[\int_t^T \sigma_r(t,s)\,ds - \rho_{rI}\sigma_I(t)\right]dt + \sigma_r(t,T)\,d\tilde{W}_r(t) \\
\frac{dI(t)}{I(t)} &= [r_n(t) - r_r(t)]\,dt + \sigma_I(t)\,d\tilde{W}_I(t) \\
\frac{dP_n(t,T)}{P_n(t,T)} &= r_n(t)\,dt - \int_t^T \sigma_n(t,s)\,dsd\tilde{W}_n(t) \\
\frac{dP_{\mathfrak{TIPS}}(t,T)}{P_{\mathfrak{TIPS}}(t,T)} &= r_n(t)\,dt + \sigma_I(t)\,d\tilde{W}_I(t) - \int_t^T \sigma_r(t,s)\,dsd\tilde{W}_r(t) \\
\frac{dP_r(t,T)}{P_r(t,T)} &= \left[r_r(t) - \rho_{rI}\sigma_I\int_t^T \sigma_r(t,s)\,ds\right]dt - \int_t^T \sigma_r(t,s)\,dsd\tilde{W}_r(t)
\end{aligned}$$

From these expressions, the real and nominal forward rates are normally distributed and the inflation index follows a geometric Brownian motion.

4.1.2 Extended Vasicek Model and Hedge Ratios.

In this section we present the extended Vasicek model for nominal and real rates and also present the hedge ratios as given in [2].
Assuming an exponentially declining volatility of the form

$$\sigma_k(t,T) = \sigma_k e^{-a_k(T-t)} \tag{4.11}$$

and that σ_k is constant for $k \in \{n, r\}$, one can easily obtain the extended Vasicek model for the short rates

$$\begin{aligned}
dr_n(t) &= [\theta_n(t) - a_n r_n(t)]\, dt + \sigma_n dWn(t), &(4.12)\\
dr_r(t) &= \left[\theta_r(t) - \rho_{rI}\sigma_I\sigma_(r) - a_r r_r(t)\right] dt + \sigma_r dWr(t), &(4.13)\\
\frac{dI(t)}{I(t)} &= [r_n(t) - r_r(t)] + \sigma_I dW_I(t), &(4.14)
\end{aligned}$$

where $\theta_k(t)$ are deterministic functions to be used to fit the term structures for nominal and real rates and are given as:

$$\theta_k(t) = \frac{\partial f_k^*(0,T)}{\partial T} + a_k f_k^*(0,t) + \frac{\sigma_k^2}{2a_k}\left(1 - e^{-2a_k t}\right), k \in \{n, r\}. \qquad (4.15)$$

where $\frac{\partial f_x}{\partial T}$ denotes the partial derivative of f_x with respect to its second argument.

Assumming that both nominal and real rates are normally distributed under their respective risk-neutral measures Jarrow and Yildirim showed that r is an Ornstein-Uhlenbeck process and the inflation index $I(t)$, at each time t, is lognormally distributed under Q_n. So we can write, for each $t < T$,

$$\begin{aligned}
I(T) &= I(t)\exp\left\{\int_t^T [r_n(u) - r_r(u)]\, du - \frac{1}{2}\sigma_I^2(T-t)\right.\\
&+ \left.\sigma_I(W_I(T) - W_I(t))\right\}.
\end{aligned}$$

Using the equation (4.11), we can easily find the *deltas* as in the proposition stated below.

Proposition 3:

$$\begin{aligned}
\frac{\partial P_r(t,T)}{\partial r_r(t)} &= -P_r(t,T)\frac{b_r(t,T)}{\sigma_r}, && (4.16)\\
\frac{\partial [I(t)P_r(t,T)]}{\partial r_n(t)} &= 0, && (4.17)\\
\frac{\partial [I(t)P_r(t,T)]}{\partial I(t)} &= P_r(t,T), && (4.18)\\
\frac{\partial [I(t)P_r(t,T)]}{\partial r_r(t)} &= I(t)\left(\frac{\partial P_r(t,T)}{\partial r_r(t)}\right)\\
&= -I(t)P_r(t,T)b_r(t,T)/\sigma_r, && (4.19)\\
\frac{\partial P_n(t,T)}{\partial I(t)} &= 0, && (4.20)\\
\frac{\partial P_n(t,T)}{\partial r_r(t)} &= 0. && (4.21)
\end{aligned}$$

This indicates that the TIPS zero-coupon bond prices do not depend directly on the nominal spot interest rate and hence the TIPS term structure has only two factors. The last two equations i.e. (4.20) and (4.21) show that the nominal zero-coupon bond prices do not depend on the real spot interest rate or the inflation index. But these factors are correlated across the two term structures.

4.1.3 Pricing Options on the Inflation Index.

In their paper, Jarrow and Yildirim presented the closed form formulas for the valuation of the European call option on the inflation index. They consider a European call option on the inflation index with a strike price of K index units and a maturity date T. Noting that the index is not determined in dollars, but dollars per CPI-U unit, they convert the option payoff to dollars by assuming that each unit of option is written on one CPI-U unit. Thus, the payoff of the option at maturity in dollars is:

$$C_T = \max[I_T - K, 0]. \tag{4.22}$$

Under the risk neutral measure, we get the value of the option as:

$$C_t = E_t^Q\left(\max[I(T) - K, 0]\, e^{-\int_t^T r_n(s)ds}\right), \tag{4.23}$$

where E_t^Q is the expectation based on the martingale measure Q.

This can be solved by using $P_n(t,T)$ as numeraire and using (4.6) for the evolution of the inflation index and given the extended Vasicek model for both real and nominal term structures of interest rates. We proceed as follows:

$$C_t = E_t^Q\left(I(T)\,1_{\{I(T)-K>0\}}e^{-\int_t^T r_n(s)ds}\right) - KE_t^Q\left(1_{\{I(T)-K>0\}}e^{-\int_t^T r_n(s)ds}\right),$$
$$= P_n(t,T)\left[E_t^{Q^n}\left(I(T)\,1_{\{I(T)-K>0\}}\right) - KE_t^{Q^n}\left(1_{\{I(T)-K>0\}}\right)\right], \qquad (4.24)$$

where Q_n is an equivalent measure under which $\xi_t = \frac{I(t)P_r(t,T)}{P_n(t,T)}$ is a martingale. Now using proposition 2, we find the dynamics of ξ_t where we do not give the drift term explicitly.

We get,

$$\begin{aligned} d\xi_t &= \xi_t\left[(...)\,dt + \sigma_I(t)\,d\tilde{W}_I(t) - \int_t^T \sigma_r(t,s)\,dsd\tilde{W}_r(t)\right. \\ &+ \left.\int_t^T \sigma_n(t,s)\,dsd\tilde{W}_n(t)\right], \end{aligned}$$

Then by Girsanov's theorem,

$$= \xi_t\left[\sigma_I(t)\,d\bar{W}_I(t) - \int_t^T \sigma_r(t,s)\,dsd\bar{W}_r(t) + \int_t^T \sigma_n(t,s)\,dsd\bar{W}_n(t)\right],$$

By solving the above SDE, we have

$$\begin{aligned} \xi_T &= \xi_t \exp\left\{\sigma_I(t)\int_t^T d\bar{W}_I(t) - \int_t^T \sigma_r^P(u,T)\,d\bar{W}_r(u)\right. \\ &+ \left.\int_t^T \sigma_n^P(u,T)\,d\bar{W}_n(u) - \frac{1}{2}\eta^2\right\}, \end{aligned}$$

where,

$$\sigma_k^P(t,T) = \int_t^T \sigma_k(t,u)\,du$$

where $k \in \{n, r\}$, and

$$\begin{aligned} \eta^2 &= \sigma_I^2(T-t) + \int_t^T \sigma_r^P(u,T)^2\,du + \int_t^T \sigma_n^P(u,T)^2\,du \\ &+ 2\int_t^T \rho_{nr}\sigma_n^P(u,T)\,\sigma_r^P(u,T)\,du + 2\rho_{nI}\sigma_I(t)\int_t^T \sigma_n^P(u,T)\,du \\ &- 2\rho_{rI}\sigma_I(t)\int_t^T \sigma_r^P(u,T)\,du. \end{aligned}$$

Now we solve the second factor in equation (4.24) above i.e, consider

$$\begin{aligned} E_t^{Q^n}\left(1_{\{I(T)-K>0\}}\right) &= Q^n\left(I\left(T\right)-K>0\right), \\ &= Q^n\left(\xi_T>K\right), \\ &= Q^n\left(Z<\frac{\log\left(\frac{\xi_t}{K}\right)-\frac{1}{2}\eta^2}{\eta}\right), \end{aligned}$$

where $Z\sim N\left(0,1\right)$ letting N be the standard cumulative normal distribution function we have,

$$E_t^{Q^n}\left(1_{\{I(T)-K>0\}}\right)=N\left(\frac{\log\left(\frac{I(t)P_r(t,T)}{P_n(t,T)K}\right)-\frac{1}{2}\eta^2}{\eta}\right).$$

The first factor in equation (4.24) follows similarly by noting that the nominal price of a real zero-coupon bond equals the nominal price of the contract paying off one unit of the CPI index at the bond maturity. In formulas,

$$E^Q\left(I\left(T\right)e^{-\int t^T r_n(s)ds}\right)=I\left(t\right)E^Q\left(I\left(T\right)e^{-\int t^T r_r(s)ds}\right)$$

Repeating the above process to calculate Q^n we get,

$$\begin{aligned} C_t &= I\left(t\right)P_r\left(t,T\right)N\left(\frac{\log\left(\frac{I(t)P_r(t,T)}{P_n(t,T)K}\right)+\frac{1}{2}\eta^2}{\eta}\right) \\ &- KP_n\left(t,T\right)N\left(\frac{\log\left(\frac{I(t)P_r(t,T)}{P_n(t,T)K}\right)-\frac{1}{2}\eta^2}{\eta}\right) \qquad (4.25) \end{aligned}$$

The example of extended Vasicek model shows that there are several shortcomings with this model, the most important is that the parameters are not directly observable in the market. The second drawback is that it does not allow a link between instruments that are traded such as zero-coupon and year-on-year products. This last point should not be considered if we are interested in pricing instruments in markets where there is not a liquid market for these products.

4.2 Mercurio Market Models

In 2004 [3], Mercurio gave two market models for inflation-indexed swaps and options. He also priced these products with the Jarrow and Yildirim model in its equivalent short-rate formulation. In his first market model he used a lognormal LIBOR model for both nominal and real rates and a geometric Brownian motion for the inflation index. In his second method he used the fact that the forward index is a martingale at some particular time. We discuss these and his paper in detail in our next chapter.

In (2005), Mercurio and Moreni [4] extended the results of [3] by introducing stochastic volatility as in the Heston model and derived closed form formulae for Inflation Indexed caps and floors under the market model with stochastic volatility.

4.3 Beldgrade-Benhamou-Koehlar Market Model (2004)

To fill the gape in the Jarrow and Yildirim model, Belgrade, Benhamou and Koehler derived a model that is based on traded and liquid market instruments in [5]. They also derived a no-arbitrage relationship between zero coupon and year-on-year swaps using the concept of market model. They focused on a model that is simple enough to have only a few parameters and robust in the sense that it can replicate the market prices. They assumed a market model in which the forward CPI return is modelled as a diffusion with a deterministic volatility.

They discussed the case of Black and Scholes volatility and the Hull and White framework. In the Black-Scholes case, where the volatility is deterministic and homogeneous, they found the formula for year-on-year volatilities as a function of the zero-coupon volatilities. They also performed a convexity adjustment of the inflation swaps that are derived from the difference of martingale measures between the numerator and denominator. They also suggested some boundary conditions to estimate the implicit correlations from the market data.

Chapter 5

Market Models of Inflation

In this chapter we will consider some market models. Our main reference is Mercurio [3] in which he proposed two market models of inflation to price general Inflation-Indexed Swaps (IIS) and options. He also modelled II swaps and options using the Jarrow and Yildirim method as in [2]. In inflation-indexed swaps the inflation rate is either payed on an annual basis or a single payment at the swap maturity. We start by defining the Zero-Coupon Inflation Indexed Swaps (ZCIIS) and Year-on-Year Inflation Indexed Swaps (YYIIS). We then present the models as proposed by Mercurio [3] for YYIIS. We keep the notation introduced in our last chapter.

5.1 Inflation-Indexed Swaps

We have already defined the ZCIIS and YYIIS in chapter 2. We now look at the pricing of these contracts.

5.1.1 Pricing Zero-Coupon IIS

We apply standard no-arbitrage pricing theory, that we developed in chapter 3, to value the inflation-indexed leg of the ZCIIS at time $t, 0 \leq t \leq T_M$. We get,

$$\mathbf{ZCIIS}\left(t, T_M, I_0, N^*\right) = N^* E_n \left\{ e^{-\int_t^{T_M} r_r(u)du} \left[\frac{I\left(T_M\right)}{I_0} - 1 \right] \mid \mathcal{F}_t \right\}, \quad (5.1)$$

where $\mathcal{F}_t$ denotes the σ-algebra generated by the relevant underlying processes up to time t and $E_n\left\{. \mid \mathcal{F}_t\right\}$ denotes the expectation under the risk neutral measure Q_n as defined in chapter 4 when dealing with the extended

Vasicek model.

The nominal price of a real-zero coupon bond is $I(t)\,P_r(t,T)$ that is by eliminating the effect of inflation from the real price of the bond. So the nominal price of real zero-coupon bond equals the nominal price of the contract paying off one unit of the CPI index at the bond maturity. Mathematically, for each $t < T$:

$$I(t)\,P_r(t,T) = I(t)\,E_r\left\{e^{-\int_t^T r_r(u)du} \mid \mathcal{F}_t\right\} = E_n\left\{e^{-\int_t^T r_n(u)du} I(T) \mid \mathcal{F}_t\right\}. \tag{5.2}$$

We can write quation (5.1) as:

$$\begin{aligned}\mathbf{ZCIIS}\,(t,T_M,I_0,N^*) &= \frac{N^*}{I_0}E_n\left\{e^{-\int_t^{T_M} r_r(u)du} I(T_M) \mid \mathcal{F}_t\right\} \\ &- N^* E_n\left\{e^{-\int_t^{T_M} r_r(u)du} \mid \mathcal{F}_t\right\},\end{aligned}$$

Now by using (5.2) and the definition of nominal zero bond, we get

$$\mathbf{ZCIIS}\,(t,T_M,I_0,N^*) = N^*\left[\frac{I(t)}{I_0}P_r(t,T_M) - P_n(t,T_M)\right]. \tag{5.3}$$

At time $t = 0$, the above equation simplifies to

$$\mathbf{ZCIIS}\,(0,T_M,N^*) = N^*\left[P_r(0,T_M) - P_n(0,T_M)\right]. \tag{5.4}$$

Mercurio in [3] noted that the above two equations give model independent prices which are not based on any specific assumptions on the evolution of the interest rate market but follows from the absence of arbitrage. This enables us to strip real zero-coupon bond prices from the quoted prices of zero-coupon inflation-indexed swaps, as given below.

If the market quotes the values of $K = K(T_M)$ for some given maturities T_M, then we equate the present nominal value of equation (2.1), that is

$$N^* P_n(t,T_M)\left[(1+K)^M - 1\right]$$

with equation (5.4). We get the unknown $P_r(0,T_M)$ in the real economy for maturity T_M as

$$P_r(0,T_M) = P_n(0,T_M)\,(1+K(T_M))^M.$$

5.1.2 Pricing Year-on-Year IIS

Finding the value of YYIIS is not as direct as finding the value of ZCIIS. Using the definition as in chapter 2, the value of the payoff (2.3) at time T_i is

$$\mathbf{YYIIS}\left(t,T_{i-1},T_i,\psi_i,N^*\right)=N^*\psi_i E_n\left\{e^{-\int_t^{T_i} r_n(u)du}\left[\frac{I\left(T_i\right)}{I\left(T_{i-1}\right)}-1\right]\mid \mathcal{F}_t\right\}. \tag{5.5}$$

If $t > T_{i-1}$ equation (5.5) reduces to pricing the floating leg of a ZCIIS as in equation (5.4). When $t < T_{i-1}$, equation (5.5) can be written in terms of iterated expectations as

$$=N^*\psi_i E_n\left\{e^{-\int_t^{T_{i-1}} r_n(u)du}E_n\left[e^{-\int_{T_{i-1}}^{T_i} r_n(u)du}\left[\frac{I\left(T_i\right)}{I\left(T_{i-1}\right)}-1\right]\mid \mathcal{F}_{T_{i-1}}\right]\mid \mathcal{F}_t\right\}. \tag{5.6}$$

The inner expectation is the $\mathbf{ZCIIS}\left(T_{i-1},T_i,I\left(T_{i-1}\right),1\right)$, shown above, so we get the result

$$\begin{aligned}\mathbf{YYIIS}\left(t,T_{i-1},T_i,\psi_i,N^*\right) &= N^*\psi_i E_n\left\{e^{-\int_t^{T_{i-1}} r_n(u)du}\left[P_r\left(T_{i-1},T_i\right)\right.\right.\\ &- \left.\left. P_n\left(T_{i-1},T_i\right)\right]\mid \mathcal{F}_t\right\},\\ &= N^*\psi_i E_n\left\{e^{-\int_t^{T_{i-1}} r_n(u)du}P_r\left(T_{i-1},T_i\right)\mid \mathcal{F}_t\right\}\\ &- N^*\psi_i P_n\left(t,T_i\right). \end{aligned} \tag{5.7}$$

This expectation is the nominal price of a payoff equal to the real zero-coupon bond price $P_r\left(T_{i-1},T_i\right)$ at time T_{i-1} in nominal units. If we assumed that the real rates were deterministic, the above equation gives the present value of the forward price of the real bond and we get

$$\begin{aligned}E_n\left\{e^{-\int_t^{T_{i-1}} r_n(u)du}P_r\left(T_{i-1},T_i\right)\mid \mathcal{F}_t\right\} &= P_r\left(T_{i-1},T_i\right)P_n\left(t,T_{i-1}\right),\\ &= \frac{P_r\left(t,T_i\right)}{P_r\left(t,T_{i-1}\right)}P_n\left(t,T_{i-1}\right).\end{aligned}$$

Since we have stochastic real rates in our model of chapter 4 in equations (4.12) to (4.14), the forward price of a real bond must be corrected by a factor depending on both the real and nominal interest rate volatilities and their respective correlations. We price YYIIS first using the Jarrow-Yildirim model and then the two market models as proposed by Mercurio [3].

5.2 Pricing with the Jarrow Yildirim Model

Here onward, we denote the nominal $T-$ forward measure for maturity T by Q_n^T and the associated expectations by E_n^T. So we can write:

$$\begin{aligned}
\mathbf{YYIIS}\left(t,T_{i-1},T_i,\phi_i,N^*\right) &= N^*\phi_i P_n\left(t,T_{i-1}\right)E_n^{T_{i-1}}\left\{P_r\left(T_{i-1},T_i\right)\mid\mathcal{F}_t\right\} \\
&- N^*\psi_i P_n\left(t,T_i\right). \qquad (5.8)
\end{aligned}$$

We use the same formula for zero-coupon bond prices that were given in (3.8), after changing the notation according to our model. That is, we consider

$$\begin{aligned}
P_r\left(t,T\right) &= A_r\left(t,T\right)e^{-B_r(t,T)r_r(t)}, \\
B_r\left(t,T\right) &= \frac{1}{a_r}\left[1-e^{-a_r(T-t)}\right], \\
A_r\left(t,T\right) &= \frac{P_r^M\left(0,T\right)}{P_r^M\left(0,t\right)}\exp\left\{B_r\left(t,T\right)f_r^M\left(0,t\right)-\frac{\sigma_r^2}{4a_r}\left(1-e^{-2a_rt}\right)B_r\left(t,T\right)^2\right\}.
\end{aligned}$$

Now by using the change-of-numeraire technique developed in section (3.2), we get the real instantaneous rate evolution under $Q_n^{T_{i-1}}$ as follows:

$$dr_r\left(t\right)=\left[\theta_r\left(t\right)-a_r\left(t\right)-\rho_{rI}\sigma_r\sigma_I-\rho_{nr}\sigma_n\sigma_rB_n\left(t,T_{i-1}\right)\right]dt-\sigma_rdW_r^{T_{i-1}}\left(t\right), \qquad (5.9)$$

where $W_r^{T_{i-1}}$ is a $Q_n^{T_{i-1}}$-Brownian motion. Since the real rates are normally distributed under $Q_n^{T_{i-1}}$, our real bond price $P_r\left(T_{i-1},T_i\right)$ is lognormally distributed under the same measure. The real rate has the mean and variance, these derivations are similar to that done in [1], p 73, given as follows:

$$\begin{aligned}
E\left[r_r\left(T_{i-1}\right)\mid\mathcal{F}_t\right] &= r_r\left(t\right)e^{-a_r(T_{i-1}-t)}+\beta\left(T_{i-1}\right)-\beta\left(t\right)e^{-a_r(T_{i-1}-t)} \\
&- \rho_{rI}\sigma_r\sigma_IB_r\left(t,T_{i-1}\right)-\frac{\rho_{nr}\sigma_n\sigma_r}{a_r+a_n}\left[B_r\left(t,T_{i-1}\right)-B_n\left(t,T_{i-1}\right)\right. \\
&+ \left.a_rB_r\left(t,T_{i-1}\right)B_n\left(t,T_{i-1}\right)\right]. \\
Var\left[r_r\left(T_{i-1}\right)\mid\mathcal{F}_t\right] &= -\frac{\sigma_r^2}{2a_r}\left[e^{-2a_r(T_{i-1}-t)}-1\right],
\end{aligned}$$

where,

$$\beta\left(t\right)=f_r^*\left(0,t\right)+\frac{\sigma_r^2}{2a_r}\left(e^{-a_rt}-1\right)^2.$$

Now as the real bond price is lognormally distributed under $Q_n^{T_{i-1}}$, we get the closed form formulae for YYIIS as in Mercurio [3]

$$\begin{aligned}
\mathbf{YYIIS}\left(t,T_{i-1},T_i,\psi_i,N^*\right) &= N^*\psi_iP_n\left(t,T_{i-1}\right)\frac{P_r\left(t,T_i\right)}{P_r\left(t,T_{i-1}\right)}e^{C(t,T_{i-1},T_i)} \\
&- N^*\psi_iP_n\left(t,T_i\right), \qquad (5.10)
\end{aligned}$$

where

$$\begin{aligned} C\left(t,T_{i-1},T_i\right) &= \sigma_r B_r\left(t,T_{i-1}\right)\Bigg[B_r\left(t,T_{i-1}\right)\Bigg(\rho_{rI}\sigma_I - \frac{1}{2}\sigma_r B_r\left(t,T_{i-1}\right) \\ &+ \frac{\rho_{nr}\sigma_n}{a_n+a_r}\left(1+a_r B_n\left(t,T_{i-1}\right)\right)\Bigg) - \frac{\rho_{nr}\sigma_n}{a_n+a_r}B_n\left(t,T_{i-1}\right)\Bigg]. \end{aligned}$$

This shows that in the Jarrow-Yildirim model the expectation of a real zero-coupon bond is equal to the current forward price of the zero-coupon bond multiplied by a correction factor where the expectation is taken under a nominal forward measure. The correction factor depends on the volatility of the nominal rate, real rate and the CPI as well as the correlation between the nominal and real rates and the real rates and the CPI, where these volatilities and correlations are instantaneous. This correction factor is due to the fact that the real rates are stochastic in this model and it vanishes for $\sigma_r = 0$.

The value of the inflation-indexed leg of the swap at time t is the sum of all floating payment values at time t. So we get,

$$\begin{aligned} \mathbf{YYIIS}\left(t,\mathcal{T},\Psi,N^*\right) &= N^*\psi_{\iota(t)}\left[\frac{I\left(t\right)}{I\left(T_{\iota(t)-1}\right)}P_r\left(t,T_{\iota(t)}\right) - P_n\left(t,T_{\iota(t)}\right)\right] \\ &+ N^*\Sigma_{i=\iota(t)+1}^{M}\psi_i\left[P_n\left(t,T_{i-1}\right)\frac{P_r\left(t,T_i\right)}{P_r\left(t,T_{i-1}\right)}e^{C(t,T_{i-1},T_i)}\right. \\ &- \left. P_n\left(t,T_i\right)\right], \end{aligned} \tag{5.11}$$

where $\mathcal{T} = \{T_1, T_2, ..., T_M\}$, $\Phi = \{\phi_1, \phi_2, ..., \phi_M\}$ and $\iota(t) = \min\{i : T_i > t\}$. The value of YYIIS at time $t = 0$ is

$$\begin{aligned} \mathbf{YYIIS}\left(0,\mathcal{T},\Psi,N^*\right) &= N^*\psi_1\left[P_r\left(0,T_1\right) - P_n\left(0,T_1\right)\right] + N^*\Sigma_{i=2}^{M}\psi_i\left[P_n\left(0,T_{i-1}\right)\right. \\ &\quad \left.\frac{P_r\left(0,T_i\right)}{P_r\left(0,T_{i-1}\right)}e^{C(0,T_{i-1},T_i)} - P_n\left(0,T_i\right)\right] \\ &= N^*\Sigma_{i=1}^{M}\psi_i P_n\left(0,T_i\right)\left[\frac{1+\tau_i F_n\left(0;T_{i-1},T_i\right)}{1+\tau_i F_r\left(0;T_{i-1},T_i\right)}e^{C(0,T_{i-1},T_i)}\right. \\ &- \left.1\right] \end{aligned} \tag{5.12}$$

Modelling using the Hull-White-Vasicek model, which is a Gaussian model, helps us to find the formulae analytically but the possibility of negative rates and the difficulty in estimating the real rate parameters pushed Mercurio to investigate alternative approaches. In the next two section, we present two different market models for valuation of a YYIIS and inflation-indexed options as proposed by Mercurio [3].

5.3 Pricing with a First Market Model

To price the above YYIIS, Mercurio noticed that we can change the measure and can re-write the expectation in (5.5) as follows

$$\begin{aligned} P_n(t,T_{i-1}) E_n^{T_{i-1}} \{P_r(T_{i-1},T_i) \mid \mathcal{F}_t\} &= P_n(t,T_i) E_n^{T_i} \left\{ \frac{P_r(T_{i-1},T_i)}{P_n(T_{i-1},T_i)} \mid \mathcal{F}_t \right\} \\ &= P_n(t,T_i) E_n^{T_i} \left\{ \frac{1+\tau_i F_n(T_{i-1};T_{i-1},T_i)}{1+\tau_i F_r(T_{i-1};T_{i-1},T_i)} \right. \\ & \left. \mid \mathcal{F}_t \right\}, \end{aligned} \quad (5.13)$$

where F_r and F_n are the LIBOR as defined in chapter 3.
This expectation can be calculated when we specify the distribution of forward rates under the nominal T_i-forward measure.

Mercurio suggested a lognormal LIBOR model and postulated a model for the evolution of the compounded forward rates that appear in (5.13)

In the nominal economy, price of an asset is given by $I(t) P_r(t,T_{i-1})$. This is a martingale under the risk neutral measure $Q_n^{T_{i-1}}$ when discounted by $P_n(t,T_{i-1})$, by definintion, and is the forward CPI. Noting this fact, we let

$$\mathcal{I}_i(t) := I(t) \frac{P_r(t,T_i)}{P_n(t,T_i)}.$$

In [3], Mercurio assumed the lognormal dynamics of $\mathcal{I}_i$, with σ_{iI} , a positive constant and W_i^I a Brownian motion under $Q_n^{T_{i-1}}$ that is,

$$d\mathcal{I}_i(t) = \sigma_{Ii} \mathcal{I}_i(t) dW_I^i(t). \quad (5.14)$$

Now we assume that both nominal and real forward rates follow a LIBOR market model, the analogy with cross-currency derivative pricing gives the dynamics of $F_n(.;T_{i-1},T_i)$ and $F_r(.;T_{i-1},T_i)$ under $Q_n^{T_i}$ as observed by Schlögl, 2002 [6]

$$\begin{aligned} dF_n(t;T_{i-1},T_i) &= \sigma_{ni} F_n(t,T_{i-1},T_i) dW_i^n(t), \\ dF_r(t;T_{i-1},T_i) &= F_r(t;T_{i-1},T_i) [-\rho_{I,ri} \sigma_{Ii} \sigma_{ri} dt + \sigma_{ri} dW_i^r(t)], \end{aligned} \quad (5.15)$$

where W_i^n and W_i^r are two Brownian motions with instantaneous correlation ρ_i and σ_{ni} and σ_{ri} are positive constants. $\rho_{I,ri}$ is the instantaneous correlation between $\mathcal{I}_i(.)$ and $F_r(.;T_{i-1},T_i)$, that is $dW_i^I(t) dW_i^r = \rho_{I,ri} dt$.

The expectation in (5.16) can calculated by noting that the pair

$$(X_i, Y_i) = \left(\log \frac{F_n(T_{i-1}; T_{i-1}, T_i)}{F_n(t; T_{i-1}, T_i)}, \log \frac{F_r(T_{i-1}; T_{i-1}, T_i)}{F_r(t; T_{i-1}, T_i)}\right) \tag{5.16}$$

is distributed as a bivariate normal random variable under $Q_n^{T_i}$. Using the dynamics of $F_n(t; T_{i-1}, T_i)$ and $F_r(t; T_{i-1}, T_i)$ as in (5.15), the mean vector and variance-covariance matrix of this bivariate random variable are given as

$$M_{X_i,Y_i} = \begin{bmatrix} \mu_{x,i}(t) \\ \mu_{y,i}(t) \end{bmatrix}, \quad V_{X_i,Y_i} = \begin{bmatrix} \sigma_{x,i}^2(t) & \rho_i \sigma_{x,i}(t) \sigma_{y,i}(t) \\ \rho_i \sigma_{x,i}(t) \sigma_{y,i}(t) & \sigma_{y,i}^2(t) \end{bmatrix}, \tag{5.17}$$

where

$$\begin{aligned} \mu_{x,i}(t) &= -\frac{1}{2}\sigma_{ni}^2 (T_{i-1} - t), \qquad \sigma_{x,i}(t) = \sigma_{ni}\sqrt{T_{i-1} - t}, \\ \mu_{y,i}(t) &= \left[-\frac{1}{2}\sigma_{ri}^2 - \rho_{I,ri}\sigma_{Ii}\sigma_{ri}\right](T_{i-1} - t), \qquad \sigma_{y,i}(t) = \sigma_{ri}\sqrt{T_{i-1} - t}. \end{aligned}$$

We can decompose a density function $f_{X_i,Y_i}(x, y)$ of (X_i, Y_i) as

$$f_{X_i,Y_i}(x, y) = f_{X_i|Y_i}(x, y) f_{Y_i}(y). \tag{5.18}$$

Where in our case, we have

$$\begin{aligned} f_{X_i|Y_i}(x, y) &= \frac{1}{\sigma_{x,i}(t)\sqrt{2\pi}\sqrt{1-\rho_i^2}} \exp\left[-\frac{\frac{x-\mu_{x,i}(t)}{\sigma_{x,i}(t)} - \rho_i \frac{y-\mu_{y,i}(t)}{\sigma_{y,i}(t)}}{2(1-\rho_i^2)}\right], \\ f_{Y_i}(y) &= \frac{1}{\sigma_{y,i}(t)\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{y-\mu_{y,i}(t)}{\sigma_{y,i}(t)}\right)^2\right]. \end{aligned} \tag{5.19}$$

As $x = \log \frac{F_n(T_{i-1}; T_{i-1}, T_i)}{F_n(t; T_{i-1}, T_i)}$, we have

$$1 + \tau_i F_n(T_{i-1}; T_{i-1}, T_i) = 1 + \tau_i F_n(t, T_{i-1}, T_i) e^x$$

Therefore using this and decomposition in (5.18), we get

$$\begin{aligned} E_n^{T_i}\left\{\frac{1 + \tau_i F_n(T_{i-1}; T_{i-1}, T_i)}{1 + \tau_i F_r(T_{i-1}; T_{i-1}, T_i)} \mid \mathcal{F}_t\right\} &= \int_{-\infty}^{+\infty} \frac{1}{1 + \tau_i F_r(t; T_{i-1}, T_i) e^y} \\ &\quad \left[\int_{-\infty}^{+\infty} (1 + \tau_i F_n(t; T_{i-1}, T_i) e^x) f_{X_i|Y_i}(x, y)\, dx\right] \\ &\quad f_{Y_i}(y)\, dy, \end{aligned}$$

which is then equal to

$$= \int_{-\infty}^{+\infty} \frac{1 + \tau_i F_n\left(t; T_{i-1}, T_i\right) e^{\mu_{x,i}(t) + \rho_i \sigma_{x,i}(t) \frac{y - \mu_{y,i}(t)}{\sigma_{y,i}(t)} + \frac{1}{2}\sigma_{x,i}(t)^2\left(1-\rho_i^2\right)}}{1 + \tau_i F_r\left(t; T_{i-1}, T_i\right) e^{y}} \frac{e^{-\frac{1}{2}\left(\frac{y-\mu_{y,i}(t)}{\sigma_{y,i}(t)}\right)^2}}{\sigma_{y,i}\left(t\right)\sqrt{2\pi}} dy$$

Using the transformation $z = \frac{y - \mu_{y,i}(t)}{\sigma_{y,i}(t)}$,

$$= \int_{-\infty}^{+\infty} \frac{1 + \tau_i F_n\left(t; T_{i-1}, T_i\right) e^{\rho_i \sigma_{x,i}(t) z - \frac{1}{2}\sigma_{x,i}(t)^2 \rho_i^2}}{1 + \tau_i F_r\left(t; T_{i-1}, T_i\right) e^{\mu_{y,i}(t) + \sigma_{y,i}(t) z}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz,$$

Therefore we get,

$$\begin{aligned} \mathbf{YYIIS}\left(t, T_{i-1}, T_i, \psi_i, N^*\right) &= N^* \psi_i P_n\left(t, T_i\right) \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \\ &\quad \frac{1 + \tau_i F_n\left(t; T_{i-1}, T_i\right) e^{\rho_i \sigma_{x,i}(t) z - \frac{1}{2}\sigma_{x,i}(t)^2 \rho_i^2}}{1 + \tau_i F_r\left(t; T_{i-1}, T_i\right) e^{\mu_{y,i}(t) + \sigma_{y,i}(t) z}} dz \\ &- N^* \psi_i P_n\left(t, T_i\right). \end{aligned} \tag{5.20}$$

We cannot value the whole of the inflation-indexed leg of this swap by summing up values in (5.12), as in the Jarrow Yildirim model. This was noticed by Schlögl [6] that the assumptions of volatilities σ_{Ii}, σ_{ni} and σ_{ri} as positive constants for all i is not right.

We can find the relation between two consecutive forward CPI's and corresponding nominal and real forward rates as follows: By definition of forward CPI

$$\frac{\mathcal{I}_i\left(t\right)}{\mathcal{I}_{i-1}\left(t\right)} = \frac{P_r\left(t, T_i\right)}{P_r\left(t, T_{i-1}\right)} \frac{P_n\left(t, T_{i-1}\right)}{P_n\left(t, T_i\right)},$$

and since

$$\frac{P_k\left(t, T_{i-1}\right)}{P_k\left(t, T_i\right)} = 1 + \tau_i F_k\left(t, T_{i-1}, T_i\right).$$

where $k \in \{r, n\}$. That is we have,

$$\frac{\mathcal{I}_i\left(t\right)}{\mathcal{I}_{i-1}\left(t\right)} = \frac{1 + \tau_i F_n\left(t; T_{i-1}, T_i\right)}{1 + \tau_i F_r\left(t; T_{i-1}, T_i\right)}. \tag{5.21}$$

So if we assume $\sigma_{I,i}, \sigma_{ni}$ and σ_{ri} are positive constant for some i, then its clear from the above equation that $\sigma_{r,i-1}$ is no longer a constant.

To deal with this Mercurio [3] used the technique of freezing the forward rates at their time 0 value in the diffusion coefficient of the right hand side of equation (5.21) and noticed that we can still get approximately constant CPI volatility. Note that this gives an approximated value for the model. He then applied this *Freezing* procedure starting from $\sigma_{I,M}$ for each $i < M$ or equivalently for each $i > 2$ and starting from $\sigma_{I,i}$ are all constants and set to one of their admissible values. Then the value of the inflation-indexed leg of the swap at time t is given by

$$\begin{aligned}
\mathbf{YYIIS}(t, \mathcal{T}, \Psi, N^*) &= N^*\psi_{\iota(t)}\left[\frac{I(t)}{I\left(T_{\iota(t)-1}\right)}P_r\left(t, T_{\iota(t)}\right) - P_n\left(t, T_{\iota(t)}\right)\right] \\
&+ N^*\Sigma_{\iota(t)+1}^{M}\psi_i P_n(t, T_i)\left[\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\right. \\
&\quad \frac{1 + \tau_i F_n(t; T_{i-1}, T_i)\, e^{\rho_i\sigma_{x,i}(t)z - \frac{1}{2}\sigma_{x,i}(t)^2\rho_i^2}}{1 + \tau_i F_r(t; T_{i-1}, T_i)\, e^{\mu_{y,i}(t) + \sigma_{y,i}(t)z}}dz \\
&- \left. 1\right]. \qquad (5.22)
\end{aligned}$$

In particular for $t = 0$,

$$\begin{aligned}
\mathbf{YYIIS}(0, \mathcal{T}, \Psi, N^*) &= N^*\psi_1\left[P_r(0, T_1) - P_n(0, T_1)\right] + N^*\Sigma_{i=2}^{M}\psi_i P_n(0, T_i) \\
&\quad \left[\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\right. \\
&\quad \left.\frac{1 + \tau_i F_n(0; T_{i-1}, T_i)\, e^{\rho_i\sigma_{x,i}(0)z - \frac{1}{2}\sigma_{x,i}(0)^2\rho_i^2}}{1 + \tau_i F_r(0; T_{i-1}, T_i)\, e^{\mu_{y,i}(0) + \sigma_{y,i}(0)z}}dz - 1\right] \\
&= N^*\Sigma_{i=1}^{M}\psi_i P_n(0, T_i)\left[\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\right. \\
&\quad \frac{1 + \tau_i F_n(0; T_{i-1}, T_i)\, e^{\rho_i\sigma_{x,i}(0)z - \frac{1}{2}\sigma_{x,i}(0)^2\rho_i^2}}{1 + \tau_i F_r(0; T_{i-1}, T_i)\, e^{\mu_{y,i}(0) + \sigma_{y,i}(0)z}}dz \\
&- \left. 1\right]. \qquad (5.23)
\end{aligned}$$

Hence for this case YYIIS depends on the instantaneous volatilities of nominal and real forward rates and their correlations, and the instantaneous volatilities of forward inflation indices and their correlation with real forward rates, for each payment time $T_i, \quad i = 2, ..., M$.

If we compare Mercurio's first market model with (5.12), (5.22) looks more complicated in terms of the calculations and in terms of input parameters. The input parameters can be determined more easily than those coming from the previous short-rate approach. This is a strong feature of market models due to absence of arbitrage. In this respect (5.22) is preferable to (5.12).

Estimating the volatilities of real rates is a drawback of the Jarrow Yildirim model when valuing a YYIIS with a LIBOR market model. To overcome this problem Mercurio [3] proposed a second market model approach, which we discuss in the next section.

5.4 Pricing with a Second Market Model

In his second market model, Mercurio [3] used the definition of forward CPI and the fact that $\mathcal{I}_i$ is a martingale under $Q_n^{T_i}$. Using this for $t < T_{i-1}$,

$$\begin{aligned}\mathbf{YYIIS}\left(t, T_{i-1}, T_i, \psi_i, N^*\right) &= N^* \psi_i P_n\left(t, T_i\right) E_n^{T_i}\left\{\frac{I\left(T_i\right)}{I\left(T_{i-1}\right)} - 1 \mid \mathcal{F}_t\right\} \\ &= N^* \psi_i P_n\left(t, T_i\right) E_n^{T_i}\left\{\frac{\mathcal{I}_i\left(T_i\right)}{\mathcal{I}_{i-1}\left(T_{i-1}\right)}\right. \\ &- \left. 1 \mid \mathcal{F}_t\right\} \qquad (5.24) \\ &= N^* \psi_i P_n\left(t, T_i\right) E_n^{T_i}\left\{\frac{\mathcal{I}_i\left(T_{i-1}\right)}{\mathcal{I}_{i-1}\left(T_{i-1}\right)} - 1 \mid \mathcal{F}_t\right\}.\end{aligned}$$

Since the purpose of this technique is to overcome the short comings of the Jarrow-Yildirim model in Section 5.2, Mercurio considered the dynamics of $\mathcal{I}_i$ under $Q_n^{T_i}$ as in (5.14) and a similar one for $\mathcal{I}_{i-1}$ under $Q_n^{T_{i-1}}$. Mercurio solved the question regarding the dynamics of $\mathcal{I}_{i-1}$ under $Q_n^{T_i}$ by the change-of-numeraire technique, that is

$$d\mathcal{I}_{i-1}\left(t\right) = -\mathcal{I}_{i-1}\left(t\right)\sigma_{I,i-1}\frac{\tau_i \sigma_{ni} F_n\left(t; T_{i-1}, T_i\right)}{1 + \tau_i F_n\left(t; T_{i-1}, T_i\right)}\rho_{I,ni}dt + \sigma_{I,i-1}\mathcal{I}_{i-1}\left(t\right) dW_{i-1}^I\left(t\right), \qquad (5.25)$$

where $\sigma_{I,i-1}$ is a positive constant, W_{i-1}^I is a $Q_n^{T_i}$ Brownian motion with $dW_{i-1}^I\left(t\right) dW_i^I\left(t\right) = \rho_{i,I}dt$ and $\rho_{I,ni}$ is the instantaneous correlation between $\mathcal{I}_{i-1}\left(.\right)$ and $F_n\left(.; T_{i-1}, T_i\right)$.

The calculations of (5.24) involve the evolution of $\mathcal{I}_{i-1}$ under $Q_n^{T_i}$ that depends on the nominal rate $F_n\left(.; T_{i-1}, T_i\right)$.

This could cause complications like those induced by higher dimensional integrations. To avoid this, Mercurio again used the freezing technique for the drift part of equation (5.25) at its current time t. This makes the distribution of $\mathcal{I}_{i-1}(T_{i-1})$ conditional on $\mathcal{F}_t$ lognormal under $Q_n^{T_i}$ and hence

$$E_n^{T_i}\left\{\frac{\mathcal{I}_i(T_{i-1})}{\mathcal{I}_{i-1}(T_{i-1})} \mid \mathcal{F}_t\right\} = \frac{\mathcal{I}_i(t)}{\mathcal{I}_{i-1}(t)} e^{D_i(t)},$$

where

$$D_i(t) = \sigma_{I,i-1}\left[\frac{\tau_i \sigma_{ni} F_n(t; T_{i-1}, T_i)}{1+\tau_i F_n(t; T_{i-1}, T_i)} \rho_{I,ni} - \rho_{Ii}\sigma_{Ii} + \sigma_{I,i-1}\right](T_{i-1} - t),$$

so that

$$\mathbf{YYIIS}(t, T_{i-1}, T_i, N^*) = N^* \psi_i P_n(t, T_i)\left[\frac{P_n(t, T_{i-1}) P_r(t, T_i)}{P_n(t, T_i) P_r(t, T_{i-1})} e^{D_i(t)} - 1\right]. \tag{5.26}$$

Finally, the value at time t of the inflation-indexed leg of the swap is given by

$$\begin{aligned}\mathbf{YYIIS}(t, \mathcal{T}, \Psi, N^*) &= N^* \psi_{\iota(t)}\left[\frac{I(t)}{I\left(T_{\iota(t)-1}\right)} P_r\left(t, T_{\iota(t)}\right) - P_n\left(t, T_{\iota(t)}\right)\right] \\ &+ N^* \Sigma_{i=\iota(t)+1}^{M} \psi_i \left[P_n(t, T_{i-1}) \frac{P_r(t, T_i)}{P_r(t, T_{i-1})} e^{D_i(t)}\right. \\ &- \left. P_n(t, T_i)\right]. \end{aligned} \tag{5.27}$$

In particular at $t = 0$,

$$\begin{aligned}\mathbf{YYIIS}(0, \mathcal{T}, \Psi, N^*) &= N^* \psi_1 \left[P_r(0, T_1) - P_n(0, T_i)\right] + N^* \Sigma_{i=2}^{M} \psi_i \left[\right. \\ &\quad \left. P_n(0, T_{i-1}) \frac{P_r(0, T_i)}{P_r(0, T_{i-1})} e^{D_i(0)} - P_n(0, T_i)\right] \qquad (5.28) \\ &= N^* \Sigma_{i=1}^{M} \psi_i P_n(0, T_i)\left[\frac{1+\tau_i F_n(0; T_{i-1}, T_i)}{1+\tau_i F_r(0; T_{i-1}, T_i)} e^{D_i(0)} - 1\right]. \end{aligned}$$

The above formula for YYIIS depends on instantaneous volatilities of nominal forward rates; the instantaneous volatilities of forward inflation indices and their correlations; the instantaneous correlations between forward inflation indices and nominal forward rates.

This expression combines the advantage of a fully-analytical formula with the market-model approach and may be preferred to (5.23) for this reason.

The correction term in this formula does not depend on the volatility of real rates as in equation (5.22).

There is a drawback of this formula in that the approximation it is based on can be rough for long maturities T_i. In fact this formula is exact when the correlations between $\mathcal{I}_{i-1}(.)$ and $F_n(.; T_{i-1}, T_i)$ that is $\rho_{I,ni}$ are set to zero and the terms D_i simplified accordingly. This is a major problem as many purchases of inflation products, such as a pension funds, want very long dated instruments. However, in general, such correlations can have a non-negligible impact on the D_i.

In his paper [3], Mercurio then introduced the inflation-indexed caplets and floorlets and priced them using the methods described above. His second market model gives a formula that has the same advantages and drawbacks as the swap price, in terms of input parameters, because the caplet price depends on the instantaneous volatilities of nominal forward rates and the instantaneous correlations between the nominal forward rates and the forward inflation index. The formula is analogous to the Black-Scholes formula and hence is more practical support for the modelling of forward CPIs as geometric Brownian motion.

Mercurio and Moreni then give an extension to the above models of pricing the caplets and floorlets. They noticed that the Black-Scholes like formula is good for pricing II-caps with different maturities but the same strike. They assume that the nominal forward rates are lognormally distributed with constant velocities and that forward CPI's have a Heston evolution with common volatility but stochastic this time. As the payoff that depends on the ratio of two different assets at two different time they use Fourier transforms. They got some partial differential equation that does not seem to solve explicitly and then restrict to some particular cases.

They found closed form formulae when assuming the correlation between forward rates and forward CPI's and between the forward rates and volatility are all zero. They noticed that using freezing drift techniques, it is possible to derive an efficient approximation to the price. They also found that introducing the stochastic volatility gives better results and is a much better fit than in the deterministic case. There is always some opening to find more efficient formulae and one can use a different stochastic volatility process for each forward CPI.

Chapter 6

Conclusion

To understand and price the inflation indexed derivatives one needs to have an understanding of interest rate modelling as this is a building block for the theories on inflation-indexed derivatives. The dynamics of real and nominal interest rates as well as the inflation index is an important assumption in pricing these derivatives. These procducts was initiated for pention holders but now growing rapidly and is a need of today's world.

In our work we discussed the first model of inflation developed by Jarrow and Yildirim in full details. Noting the weaknesses of this model, Mercurio developed two market models that are given in chapter 5. We priced ZCIIS and YYIIS using these models as developed by the authors. We can also price other derivatives inflation indexed caps and floors using these models. In our work we discussed the models developed up till 2005 and did not look at the further extensions.

Hinnerich in 2006 extended the HJM approach proposed by Jarrow and Yildirim [2] to multifactor HJM model and allowed the possibility of jumps in the economy [11]. He proposed the explicit closed form solutions of the derivatives. In 2007, Zhu extended the standard LIBOR market models by introducing mean reversion and by using all stochastic volatilities that are correlated to the LIBOR individually, [12]. There are many other extensions to the models we have presented here as well as other models that one can look at for futher considerations.

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