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Matched Asymptotic Expansions for Valuing Spread Options

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Abstract

Spread Options are crucial in the energy, currency and fixed income, and commodity markets. The problem with spread options is that there are no closed-form formulae to price or hedge them. In this paper, we use matched asymptotic expansions in order to price spread options. We use both one-factor and two-factor models. In the one-factor models we assume the spread follows one of the following processes: Geometric Brownian Motion, Ornstein-Uhlenbeck and Arithmetic Brownian Motion. In the two-factor models, we assume the assets follow one of these processes.

Chapter 1

Introduction

The main goal of this paper is to apply matched asymptotic expansions in order to price spread options. Valuing these options is a big challenge as there are no closed formulae to price or hedge them. Many pricing and hedging schemes have been developed based on the work done by Black, Scholes and Merton. Despite the importance of these theories, they were not able to price or hedge many of the widely used options. One of which is spread options which we will discuss in this paper.

As defined in [1], spread options are options written on the difference of two assets whose time t values are denoted by $S_1(t)$ and $S_2(t)$. The payoff of this option at maturity time T is $\max(S_1(T) - S_2(T) - K, 0)$ where K is the strike. The spread $S(T)$ of the option is defined as $S(T) = S_1(T) - S_2(T)$.

Spread options are crucial in the energy, currency and fixed income, and commodity markets. In the commodity market, there are spreads on the difference between prices of the same commodity at different times (calendar spreads), at different locations (location spreads) and with different grades (quality spreads)[1]. In the currency and fixed income market, the spread is usually on the difference between two interest or swap rates or on the difference between two yields,... For instance, as stated in [1], in the U.S. fixed income market, there are spreads between maturities such as the NOB spreads (notes-bonds) and between quality levels such as the TED spreads (treasury bills-Eurodollars). Lastly, in the energy market, spread options are widely used. Two main spread options are used by NYMEX (New York Mercantile Exchange) namely: spread options on heating oil/crude oil and gasoline/crude oil spread options. In the energy market, there are different kinds of spread options such as temporal spreads which are based on the difference between the prices of the same commodity at different times and the locational spreads which are based on the difference between the prices of the same commodities at different locations, but the most widely used spread options in the energy market, are spread options on two different commodities[1]. These spread options are crucial in the energy market as they quantify the cost of production of the end products from the raw material used to produce them. The wide use of spread options in many markets has motivated financial mathematicians to work on pricing or hedging them.

In this paper, we try to value spread options by considering one and two-factor models. The former are discussed in chapters two, three and four, while

the latter are discussed in chapters five and six. For the one-factor models, we use three different models for the spread; namely Geometric Brownian Motion, Ornstein Uhlenbeck and Arithmetic Brownian Motion. In chapter 2, we apply matched asymptotic expansion to the transition density function of each of the three processes. The results of this chapter are used in chapter three in order to price spread options using risk-neutral pricing. In chapter 4, on the other hand, we directly apply matched asymptotic expansions on the value of the option in each of the three cases. We then plot graphs and monitor the difference between different models.

We then proceed to the two-factor models where we assume that the assets both follow one of the three processes mentioned above. We start off in chapter five by applying matched asymptotic expansion to the transition density function of each of the three processes. Finally, in chapter six, we directly apply matched asymptotic expansions on the value of the option in each of the three cases. A brief summary of the paper and some concluding remarks are provided in chapter seven.

Chapter 2

Transition Density of Spread Options Based on One Factor Models

2.1 Geometric Brownian Motion

We consider spread options where the Spread S_t follows a Geometric Brownian Motion and thus satisfies the following stochastic differential equation

$$dS_t = rS_t dt + \sigma S_t dW_t$$

where dW_t is the increment of a standard Brownian Motion, r is the risk-free interest rate and σ is the volatility. Note that both r and σ are assumed to be constant. Let P be the transition density function of this process. By Kolmogorov's forward equation (Fokker-Planck equation)

$$\frac{\partial P}{\partial t} = \frac{1}{2} \frac{\partial^2 (\sigma^2 S^2 P)}{\partial S^2} - \frac{\partial (rSP)}{\partial S} \quad (1)$$

with $P(S, 0) = \delta(S - S_0)$. Assume a short period of time i.e. the time is small, so that

$$\sigma^2 t = \epsilon^2 \tau$$

where $0 < \epsilon \ll 1$. Equation (1) now becomes

$$\frac{1}{\epsilon^2} \frac{\partial P}{\partial \tau} = \frac{1}{2} S^2 \frac{\partial^2 P}{\partial S^2} + (2 - \alpha) S \frac{\partial P}{\partial S} + (1 - \alpha) P$$

where $\alpha = r/\sigma^2$. Expanding in ϵ , we write

$$P(S, \tau) \sim P_0(S, \tau) + \epsilon^2 P_1(S, \tau) + \dots$$

Collecting together the terms of $O(\frac{1}{\epsilon^2})$, we get

$$\frac{\partial P_0}{\partial \tau} = 0 \text{ with } P_0(S, 0) = \delta(S - S_0)$$

The solution to this problem can be easily found to be

$$P_0(S, \tau) = \delta(S - S_0)$$

To see the behavior around the initial value S_0 , we rescale near S_0 by introducing an inner variable

$$S = S_0(1 + \epsilon x)$$

Hence, (1) becomes

$$\frac{\partial P^i}{\partial \tau} = \frac{1}{2}(1 + \epsilon x)^2 \frac{\partial^2 P^i}{\partial x^2} + (2 - \alpha)\epsilon(1 + \epsilon x) \frac{\partial P^i}{\partial x} + (1 - \alpha)\epsilon^2 P^i$$

with

$$P^i(x, 0) = \delta(S_0 \epsilon x) = \frac{1}{S_0 \epsilon} \delta(x)$$

We now expand

$$P^i(x, \tau; \epsilon) \sim \frac{1}{\epsilon} P_0^i(x, \tau) + P_1^i(x, \tau) + \dots$$

Collecting together the terms of $O(\frac{1}{\epsilon})$, we get

$$\frac{\partial P_0^i}{\partial \tau} = \frac{1}{2} \frac{\partial^2 P_0^i}{\partial x^2}$$

with

$$P_0^i(x, 0) = \frac{1}{S_0} \delta(x)$$

This problem has a similarity solution of the form

$$P_0^i(x, \tau) = \sqrt{\tau} f\left(\frac{x}{\sqrt{\tau}}\right)$$

If we let $\xi = x/\sqrt{\tau}$, we get

$$f'' + \xi f' - f = 0$$

If we differentiate both sides of this equation, we get

$$f''' + \xi f'' = 0$$

We deduce that $f'' = e^{-\frac{\xi^2}{2}}$ Hence, we get

$$P_0^i(x, \tau) = \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{x^2}{2\tau}}$$

We now proceed further and find P_1^i . Comparing the $O(1)$ terms, we get

$$\frac{\partial P_1^i}{\partial \tau} = \frac{1}{2} \frac{\partial^2 P_1^i}{\partial x^2} + x \frac{\partial^2 P_0^i}{\partial x^2} + (2 - \alpha) \frac{\partial P_0^i}{\partial x}$$

with

$$P_1^i(x, 0) = 0$$

Using the fact that if $u_\tau = \frac{1}{2}u_{xx} + v$ where $v_\tau = \frac{1}{2}v_{xx}$, then $u = \tau v$ is a particular solution, we get that $P_1^i = (2 - \alpha)\tau P_{0x}^i$ is a particular solution. Also, if $u_\tau = \frac{1}{2}u_{xx} + vx$ where $v_\tau = \frac{1}{2}v_{xx}$, then $u = x\tau v + \frac{1}{2}\tau^2 v_x$ is a particular solution. Hence, $P_1^i(x, \tau) = x\tau P_{0xx}^i + \frac{1}{2}\tau^2 P_{0xxx}^i$ is a particular solution. Hence,

$$P_1^i = (2 - \alpha)\tau \frac{\partial P_0^i}{\partial x} + x\tau \frac{\partial^2 P_0^i}{\partial x^2} + \frac{1}{2}\tau^2 \frac{\partial^3 P_0^i}{\partial x^3}$$

Then, by finding the first, second and third derivatives of P_0^i with respect to x , we get

$$P_1^i(x, \tau) = \frac{e^{-\frac{x^2}{2\tau}}}{\sqrt{2\pi\tau}} \left((\alpha - \frac{3}{2})x + \frac{1}{2} \frac{x^3}{\tau} \right)$$

Hence,

$$P \sim \frac{e^{-\frac{x^2}{2\tau}}}{\sqrt{2\pi\tau}} \left(\frac{1}{\epsilon} + x(\alpha - \frac{3}{2}) + \frac{1}{2} \frac{x^3}{\tau} + \dots \right)$$

2.2 Ornstein-Uhlenbeck

We consider spread options where the Spread Y_t satisfies the following stochastic differential equation

$$dY_t = rY_t dt + \sigma S_0 dW_t$$

where S_0 is a constant, dW_t is the increment of a standard Brownian Motion, r is the risk-free interest rate and σ is the volatility. Note that both r and σ are assumed to be constant. Let P be the transition density function of this process. By Kolmogorov's forward equation (Fokker-Planck equation)

$$\frac{\partial P}{\partial t} = \frac{\partial^2 (\frac{1}{2} S_0^2 \sigma^2 P)}{\partial Y^2} - \frac{\partial (rY P)}{\partial Y} \quad (2)$$

with $P(Y, 0) = \delta(Y - S_0)$. Assume a short period of time i.e. the time is small, so that

$$\sigma^2 t = \epsilon^2 \tau$$

where $0 < \epsilon \ll 1$. Equation (2) now becomes

$$\frac{1}{\epsilon^2} \frac{\partial P}{\partial \tau} = \frac{S_0^2}{2} \frac{\partial^2 P}{\partial Y^2} - \alpha Y \frac{\partial P}{\partial Y} - \alpha P$$

Expanding in ϵ , we write

$$P(Y, \tau) \sim P_0(Y, \tau) + \epsilon^2 P_1(Y, \tau) + \dots$$

Collecting together the terms of $O(\frac{1}{\epsilon^2})$, we get

$$\frac{\partial P_0}{\partial \tau} = 0 \text{ with } P_0(Y, 0) = \delta(Y - S_0)$$

The solution to this problem is easily found to be

$$P_0(Y, \tau) = \delta(Y - S_0)$$

To see the details of the behavior of Y around the initial value S_0 , we rescale near S_0 , by introducing an inner variable

$$Y = S_0(1 + \epsilon x)$$

Hence, equation (2) becomes

$$\frac{\partial P^i}{\partial \tau} = \frac{1}{2} \frac{\partial^2 P_0^i}{\partial x^2} - \alpha \epsilon (1 + \epsilon x) \frac{\partial P^i}{\partial x} - \alpha \epsilon^2 P^i$$

with

$$P^i(x, 0) = \frac{1}{\epsilon S_0} \delta(x)$$

By inner expansion,

$$P(x, \tau; \epsilon) \sim \frac{1}{\epsilon} P_0^i(x, \tau) + P_1^i(x, \tau) + \dots$$

Collecting together the terms of $O(\frac{1}{\epsilon})$,

$$\frac{\partial P_0^i}{\partial \tau} = \frac{1}{2} \frac{\partial^2 P_0^i}{\partial x^2}$$

with

$$P_0^i(x, 0) = \frac{1}{S_0} \delta(x)$$

This problem has a similarity solution of the form

$$P_0^i(x, \tau) = \sqrt{\tau} f\left(\frac{x}{\sqrt{\tau}}\right)$$

If we let $\xi = x/\sqrt{\tau}$, we get

$$f'' + \xi f' - f = 0$$

If we differentiate both sides of this equation, we get

$$f''' + \xi f'' = 0$$

Hence, we get

$$P_0^i(x, \tau) = \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{x^2}{2\tau}}$$

We proceed further to find P_1^i . Comparing the $O(1)$ terms, we get

$$\frac{\partial P_1^i}{\partial \tau} = \frac{1}{2} \frac{\partial^2 P_1^i}{\partial x^2} - \alpha \frac{\partial P_0^i}{\partial x}$$

with

$$P_1^i(x, 0) = 0$$

Using the fact that if $u_\tau = \frac{1}{2}u_{xx} + v$ where $v_\tau = \frac{1}{2}v_{xx}$, then $u = \tau v$ is a particular solution, we get

$$P_1^i(x, \tau) = -\alpha\tau \frac{\partial P_0^i}{\partial x}$$

Then, by finding the first derivative of P_0^i with respect to x , we get

$$P_1^i(x, \tau) = \frac{1}{\sqrt{2\pi\tau}} x \alpha e^{-\frac{x^2}{2\tau}}$$

Hence,

$$P \sim \frac{e^{-\frac{x^2}{2\tau}}}{\sqrt{2\pi\tau}} \left(\frac{1}{\epsilon} + \alpha x + \dots \right)$$

2.3 Arithmetic Brownian Motion

We consider spread options where the Spread X_t follows Arithmetic Brownian Motion and thus satisfies the following stochastic differential equation

$$dX_t = rS_0 dt + \sigma S_0 dW_t$$

where dW_t is the increment of a standard Brownian Motion, r is the risk-free interest rate and σ is the volatility. Note that both r and σ are assumed to be constant. Let P be the transition density function of this process. By Kolmogorov's forward equation (FokkerPlanck equation)

$$\frac{\partial P}{\partial t} = \frac{1}{2} \frac{\partial^2 (\sigma^2 S_0^2 P)}{\partial X^2} - \frac{\partial (rS_0 P)}{\partial X} \quad (3)$$

with $P(X, 0) = \delta(X - S_0)$. Assume a short period of time i.e. the time is small, so that

$$\sigma^2 t = \epsilon^2 \tau$$

where $0 < \epsilon \ll 1$. Equation (3) now becomes

$$\frac{1}{\epsilon^2} \frac{\partial P}{\partial \tau} = \frac{1}{2} S_0^2 \frac{\partial^2 P}{\partial X^2} - \alpha S_0 \frac{\partial P}{\partial X}$$

Expanding in ϵ , we write

$$P(X, \tau) \sim P_0(X, \tau) + \epsilon^2 P_1(X, \tau) + \dots$$

Collecting together the terms of $O(\frac{1}{\epsilon^2})$, we get

$$\frac{\partial P_0}{\partial \tau} = 0 \text{ with } P_0(X, 0) = \delta(X - S_0)$$

The solution to this problem can be easily found to be

$$P_0(X, \tau) = \delta(X - S_0)$$

To see the behavior around the initial value S_0 , we rescale near S_0 introducing an inner variable

$$X = S_0(1 + \epsilon x)$$

Hence, (3) becomes

$$\frac{\partial P^i}{\partial \tau} = \frac{1}{2} \frac{\partial^2 P^i}{\partial x^2} - \alpha \epsilon \frac{\partial P^i}{\partial x}$$

with

$$P(x, 0) = \delta(S_0 \epsilon x) = \frac{1}{S_0 \epsilon} \delta(x)$$

We now expand

$$P^i(x, \tau; \epsilon) \sim \frac{1}{\epsilon} P_0^i(x, \tau) + P_1^i(x, \tau) + \dots$$

Collecting together the terms of $O(\frac{1}{\epsilon})$, we get

$$\frac{\partial P_0^i}{\partial \tau} = \frac{1}{2} \frac{\partial^2 P_0^i}{\partial x^2}$$

with

$$P_0^i(x, 0) = \frac{1}{S_0} \delta(x)$$

This problem has a similarity solution of the form

$$P_0^i(x, \tau) = \sqrt{\tau} f\left(\frac{x}{\sqrt{\tau}}\right)$$

If we let $\xi = x/\sqrt{\tau}$, we get

$$f'' + \xi f' - f = 0$$

If we differentiate both sides of this equation, we get

$$f''' + \xi f'' = 0$$

Hence, we get

$$P_0^i(x, \tau) = \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{x^2}{2\tau}}$$

We now proceed further and find P_1^i . Comparing the $O(1)$ terms, we get

$$\frac{\partial P_1^i}{\partial \tau} = \frac{1}{2} \frac{\partial^2 P_1^i}{\partial x^2} - \alpha \frac{\partial P_0^i}{\partial x}$$

with

$$P_1^i(x, 0) = 0$$

Using the fact that if $u_\tau = \frac{1}{2}u_{xx} + v$ where $v_\tau = \frac{1}{2}v_{xx}$, then $u = \tau v$ is a particular solution, we get

$$P_1^i = -\alpha\tau \frac{\partial P_0^i}{\partial x}$$

Then, by finding the first derivative of P_0^i with respect to x , we get

$$P_1^i(x, \tau) = \alpha x \frac{e^{-\frac{x^2}{2\tau}}}{\sqrt{2\pi\tau}}$$

Hence,

$$P \sim \frac{e^{-\frac{x^2}{2\tau}}}{\sqrt{2\pi\tau}} \left(\frac{1}{\epsilon} + \alpha x + \dots \right)$$

Chapter 3

Option Pricing by integrating the transition density obtained from one-factor models

Using the results obtained in the last chapter, we can obtain approximations for the value of the spread option. This is done by integrating the product of the approximation of the transition density function (obtained in chapter 2) and the payoff.

3.1 Geometric Brownian Motion

From the results of section 1 of the previous chapter, we have

$$P \sim \frac{e^{-\frac{x^2}{2\tau}}}{\sqrt{2\pi\tau}} \left(\frac{1}{\epsilon} + x\left(\alpha - \frac{3}{2}\right) + \frac{1}{2} \frac{x^3}{\tau} + \dots \right)$$

Hence, the value of the spread option at time $t = 0$ is

$$V = \int_K^\infty p(x', 0, x, T)(x-K)dx = \int_K^\infty \frac{e^{-\frac{x^2}{2\tau}}}{\sqrt{2\pi\tau}} \left(\frac{1}{\epsilon} + x\left(\alpha - \frac{3}{2}\right) + \frac{1}{2} \frac{x^3}{\tau} + \dots \right) (x-K)dx$$

Evaluating this integral, we get

$$V = \frac{1}{\sqrt{2\pi\tau}} \left(\frac{\tau}{\epsilon} e^{-\frac{K^2}{2\tau}} - \frac{K}{\epsilon} \sqrt{2\pi\tau} (1 - N(\frac{K}{\sqrt{\tau}})) \right) + \left(\alpha - \frac{3}{2} \right) \tau \sqrt{2\pi\tau} (1 - N(\frac{K}{\sqrt{\tau}})) + O(\epsilon)$$

3.2 Ornstein-Uhlenbeck

From the results of section 2 of the previous chapter, we have

$$P \sim \frac{e^{-\frac{x^2}{2\tau}}}{\sqrt{2\pi\tau}} \left(\frac{1}{\epsilon} + \alpha x + \dots \right)$$

Hence, the value of the spread option at time $t = 0$ is

$$V = \int_K^\infty p(x', 0, x, T)(x - K)dx = \int_K^\infty \frac{e^{-\frac{x^2}{2\tau}}}{\sqrt{2\pi\tau}} \left(\frac{1}{\epsilon} + \alpha x + \dots \right) (x - K) dx$$

Evaluating this integral, we get

$$V = \frac{1}{\sqrt{2\pi\tau}} \left(\frac{\tau}{\epsilon} e^{-\frac{K^2}{2\tau}} - \frac{K}{\epsilon} \sqrt{2\pi\tau} (1 - N(\frac{K}{\sqrt{\tau}})) \right) + \alpha \tau \sqrt{2\pi\tau} (1 - N(\frac{K}{\sqrt{\tau}})) + O(\epsilon)$$

3.3 Arithmetic Brownian Motion

From the results of section 3 of the previous chapter, we have

$$P \sim \frac{e^{-\frac{x^2}{2\tau}}}{\sqrt{2\pi}} \left(\frac{1}{\epsilon\sqrt{\tau}} + \alpha \frac{x}{\sqrt{\tau}} + \dots \right)$$

Hence, the value of the spread option at time $t = 0$ is

$$V = \int_K^\infty p(x', 0, x, T)(x - K)dx = \int_K^\infty \frac{e^{-\frac{x^2}{2\tau}}}{\sqrt{2\pi}} \left(\frac{1}{\epsilon\sqrt{\tau}} + \alpha \frac{x}{\sqrt{\tau}} + \dots \right) (x - K) dx$$

Evaluating this integral, we get

$$V = \frac{1}{\sqrt{2\pi\tau}} \left(\frac{\tau}{\epsilon} e^{-\frac{K^2}{2\tau}} - \frac{K}{\epsilon} \sqrt{2\pi\tau} (1 - N(\frac{K}{\sqrt{\tau}})) \right) + \alpha \tau \sqrt{2\pi\tau} (1 - N(\frac{K}{\sqrt{\tau}})) + O(\epsilon)$$

Chapter 4

Option Pricing by finding the value directly for One Factor Models

4.1 Geometric Brownian Motion

This section is based on the work done in [2]. In this section, we consider spread options where the Spread S_t satisfies the following stochastic differential equation

$$dS_t = rS_t dt + \sigma S_t dW_t$$

where dW_t is the increment of a standard Brownian Motion, r is the risk-free interest rate and σ is the volatility. Note that both r and σ are assumed to be constant. Let V be the value of the spread option; then V satisfies the Black-Scholes Partial Differential Equation (BSPDE)

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0 \quad (4)$$

If we measure the time backwards from expiry and scale it with σ^2 , writing $t = T - t'/\sigma^2$, then (4) becomes

$$\frac{\partial V}{\partial t'} = \frac{1}{2}S^2 \frac{\partial^2 V}{\partial S^2} + \alpha S \frac{\partial V}{\partial S} - \alpha V$$

where $\alpha = r/\sigma^2$. Suppose the scaling is so small, α so that

$$t' = \epsilon^2 \tau$$

where $0 < \epsilon \ll 1$. Hence, the equation now becomes

$$\frac{1}{\epsilon^2} \frac{\partial V}{\partial \tau} = \frac{1}{2} S^2 \frac{\partial^2 V}{\partial S^2} + \alpha S \frac{\partial V}{\partial S} - \alpha V$$

We expand in ϵ , writing

$$V(S, \tau) \sim V_0(S, \tau) + \epsilon^2 V_1(S, \tau) + \dots$$

By comparing the coefficients of $\frac{1}{\epsilon^2}, \dots$, we get

$$\frac{\partial V_0}{\partial \tau} = 0$$

and

$$\frac{\partial V_1}{\partial \tau} = \alpha \left(S \frac{\partial V_0}{\partial S} - V_0 \right)$$

with the final condition

$$V_0 + \epsilon^2 V_1 = \begin{cases} S - K(1 - \epsilon^2 \alpha \tau) & \text{if } S - K \gg \epsilon K, \text{ far above the strike} \\ 0 & \text{if } K - S \gg \epsilon K, \text{ far below the strike} \end{cases}$$

By rescaling near the strike K , introducing an inner variable, we write

$$S = K(1 + \epsilon x)$$

If we also rescale

$$V(S, \tau) = \epsilon K v(x, \tau),$$

the equation becomes

$$\frac{1}{\epsilon^2} \frac{\partial v}{\partial \tau} = \frac{1}{2\epsilon^2} (1 + \epsilon x)^2 \frac{\partial^2 v}{\partial x^2} + \frac{\alpha}{\epsilon} (1 + \epsilon x) \frac{\partial v}{\partial x} - \alpha v,$$

and the payoff is

$$v(x, 0) = \max(x, 0)$$

By expanding,

$$v(x, \tau; \epsilon) \sim v_0(x, \tau) + \epsilon v_1(x, \tau) + \dots$$

Collecting the terms of $O(1)$ together, we get

$$\frac{\partial v_0}{\partial \tau} = \frac{1}{2} \frac{\partial^2 v_0}{\partial x^2}$$

with $v_0(x, 0) = \max(x, 0)$. As the conditions at $x = \pm\infty$ are

$$v(x, \tau) \sim x \quad \text{as } x \rightarrow +\infty$$

and

$$v(x, \tau) \rightarrow 0 \quad \text{as } x \rightarrow -\infty,$$

this problem has a similarity solution of the form

$$v_0(x, \tau) = \sqrt{\tau} f(x/\sqrt{\tau})$$

so,

$$f'' + \xi f' - f = 0,$$

with $f \rightarrow 0$ as $\xi \rightarrow -\infty$ and $f \sim \xi$ as $\xi \rightarrow \infty$ where $\xi = x/\sqrt{\tau}$. By differentiating this equation, we obtain

$$v_0(x, \tau) = xN(x/\sqrt{\tau}) + \sqrt{\tau}n(x/\sqrt{\tau})$$

where $N(\cdot)$ is the cumulative normal and $n(\cdot)$ is its derivative. Using the original variables, this expression becomes

$$V(S, t) \sim (S - K)N\left(\frac{S/K - 1}{\sigma\sqrt{T-t}}\right) + \sigma\sqrt{T-t}Kn\left(\frac{S/K - 1}{\sigma\sqrt{T-t}}\right).$$

We, then proceed further in order to find the two-term inner expansion $v_0 + \epsilon v_1$. By comparing the term of $O(\epsilon)$, we get

$$\frac{\partial v_1}{\partial \tau} = \frac{1}{2} \frac{\partial^2 v_1}{\partial x^2} + x \frac{\partial^2 v_0}{\partial x^2} + \alpha \frac{\partial v_0}{\partial x}$$

with $v_1(x, 0) = 0$. Using the fact that if $u_\tau = \frac{1}{2}u_{xx} + v$ where $v_\tau = \frac{1}{2}v_{xx}$, then $u = \tau v$ is a particular solution, and that if $u_\tau = \frac{1}{2}u_{xx} + vx$ where $v_\tau = \frac{1}{2}v_{xx}$, then $u = x\tau v + \frac{1}{2}\tau^2 v_x$, we get

$$v_1(x, \tau) = x\tau \frac{\partial^2 v_0}{\partial x^2} + \frac{1}{2}\tau^2 \frac{\partial^3 v_0}{\partial x^3} + \alpha\tau \frac{\partial v_0}{\partial x} = \frac{1}{2}x\sqrt{\tau}n(x/\sqrt{\tau}) + \alpha\tau N(x/\sqrt{\tau}).$$

Hence, the two-term inner expansion $v_0 + \epsilon v_1$ can be easily found. Using the original variables, this expression becomes

$$V(S, t) \sim (S - K + r(T - t))N\left(\frac{S/K - 1}{\sigma\sqrt{T - t}}\right) + \sigma\sqrt{T - t}(S + K)n\left(\frac{S/K - 1}{\sigma\sqrt{T - t}}\right)$$

4.2 Ornstein-Uhlenbeck

We consider spread options where the Spread Y_t satisfies the following stochastic differential equation

$$dY_t = rY_t dt + \sigma S_0 dW_t$$

where dW_t is the increment of a standard Brownian Motion, r is the risk-free interest rate and σ is the volatility. Note that both r and σ are assumed to be constant. Let V be the value of the spread option, then V satisfies the following Partial Differential Equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_0^2 \frac{\partial^2 V}{\partial Y^2} + rY \frac{\partial V}{\partial Y} - rV = 0 \quad (5)$$

If we measure the time backwards from expiry and scale it with σ^2 , writing $t = T - t'/\sigma^2$, then (5) becomes

$$\frac{\partial V}{\partial t'} = \frac{1}{2}S_0^2 \frac{\partial^2 V}{\partial Y^2} + \alpha Y \frac{\partial V}{\partial Y} - \alpha V$$

where $\alpha = r/\sigma^2$. Suppose the scaling is so small, so that

$$t' = \epsilon^2 \tau$$

where $0 < \epsilon \ll 1$. Hence, the equation now becomes

$$\frac{1}{\epsilon^2} \frac{\partial V}{\partial \tau} = \frac{1}{2}S_0^2 \frac{\partial^2 V}{\partial Y^2} + \alpha Y \frac{\partial V}{\partial Y} - \alpha V$$

We expand in ϵ writing

$$V(Y, \tau) \sim V_0(Y, \tau) + \epsilon^2 V_1(Y, \tau) + \dots$$

By comparing the coefficients of $\frac{1}{\epsilon^2}, 1, \epsilon^2, \dots$, we get

$$\frac{\partial V_0}{\partial \tau} = 0$$

and

$$\frac{\partial V_1}{\partial \tau} = \alpha(Y \frac{\partial V_0}{\partial Y} - V_0)$$

with the final condition

$$V_0 + \epsilon^2 V_1 = \begin{cases} Y - K(1 - \epsilon^2 \alpha \tau) & \text{if } Y - K \gg \epsilon K, \text{ far above the strike} \\ 0 & \text{if } K - Y \gg \epsilon K, \text{ far below the strike} \end{cases}$$

By rescaling near the strike K , introducing an inner variable, we write

$$Y = K(1 + \epsilon x),$$

If we also rescale

$$V(Y, \tau) = \epsilon K v(x, \tau),$$

the equation becomes

$$\frac{1}{\epsilon^2} \frac{\partial v}{\partial \tau} = \frac{S_0^2}{2\epsilon^2 K^2} \frac{\partial^2 v}{\partial x^2} + \frac{\alpha}{\epsilon} (1 + \epsilon x) \frac{\partial v}{\partial x} - \alpha v,$$

and the payoff is

$$v(x, 0) = \max(x, 0).$$

By expanding,

$$v(x, \tau; \epsilon) \sim v_0(x, \tau) + \epsilon v_1(x, \tau) + \dots$$

Collecting the terms of $O(1)$ together, we get

$$\frac{\partial v_0}{\partial \tau} = \frac{1}{2} \frac{S_0^2}{K^2} \frac{\partial^2 v_0}{\partial x^2}$$

with $v_0(x, 0) = \max(x, 0)$. As the conditions at $x = \pm\infty$ are

$$v(x, \tau) \sim x \quad \text{as } x \rightarrow +\infty$$

and

$$v(x, \tau) \rightarrow 0 \quad \text{as } x \rightarrow -\infty,$$

this problem has a similarity solution of the form

$$v_0(x, \tau) = \sqrt{\tau} f(x/\sqrt{\tau})$$

so,

$$\frac{S_0^2}{K^2} f'' + \xi f' - f = 0,$$

with $f \rightarrow 0$ as $\xi \rightarrow -\infty$ and $f \sim \xi$ as $\xi \rightarrow \infty$ where $\xi = x/\sqrt{\tau}$. By differentiating this equation, we obtain

$$v_0(x, \tau) = xN\left(\frac{x}{\sqrt{\tau}} \frac{K}{S_0}\right) + \frac{S_0}{K} \sqrt{\tau} n\left(\frac{x}{\sqrt{\tau}} \frac{K}{S_0}\right)$$

where $N(\cdot)$ is the cumulative normal and $n(\cdot)$ is its derivative. Using the original variables, this expression becomes

$$V(Y, t) \sim (Y - K)N\left(\frac{Y - K}{S_0 \sigma \sqrt{T - t}}\right) + \sigma \sqrt{T - t} S_0 n\left(\frac{Y - K}{S_0 \sigma \sqrt{T - t}}\right)$$

We, then proceed further in order to find the two-term inner expansion $v_0 + \epsilon v_1$. By comparing the term of $O(\epsilon)$, we get

$$\frac{\partial v_1}{\partial \tau} = \frac{1}{2} \frac{S_0^2}{K^2} \frac{\partial^2 v_1}{\partial x^2} + \alpha \frac{\partial v_0}{\partial x}$$

with $v_1(x, 0) = 0$. Using the fact that if $u_\tau = \frac{1}{2} \frac{S_0^2}{K^2} u_{xx} + v$ where $v_\tau = \frac{1}{2} \frac{S_0^2}{K^2} v_{xx}$, then $u = \tau v$ is a particular solution, we get

$$v_1(x, \tau) = \alpha \tau \frac{\partial v_0}{\partial x} = \alpha \tau N\left(\frac{x}{\sqrt{\tau}} \frac{K}{S_0}\right).$$

Hence, the two-term inner expansion $v_0 + \epsilon v_1$ can be easily found. Using the original variables, this expression becomes

$$V(Y, t) \sim (Y - K(1 - r(T - t)))N\left(\frac{Y - K}{S_0 \sigma \sqrt{T - t}}\right) + S_0 \sigma \sqrt{T - t} n\left(\frac{Y - K}{S_0 \sigma \sqrt{T - t}}\right)$$

4.3 Arithmetic Brownian Motion

We consider spread options where the Spread X_t satisfies the following stochastic differential equation

$$dX_t = rS_0dt + \sigma S_0dW_t$$

where dW_t is the increment of a standard Brownian Motion, r is the risk-free interest rate and σ is the volatility. Note that both r and σ are assumed to be constant. Let V be the value of the spread option, then V satisfies the following Partial Differential Equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_0^2 \frac{\partial^2 V}{\partial X^2} + rS_0 \frac{\partial V}{\partial X} - rV = 0 \quad (6)$$

If we measure the time backwards from expiry and scale it with σ^2 , writing $t = T - t'/\sigma^2$, then (6) becomes

$$\frac{\partial V}{\partial t'} = \frac{1}{2}S_0^2 \frac{\partial^2 V}{\partial X^2} + \alpha S_0 \frac{\partial V}{\partial X} - \alpha V$$

where $\alpha = r/\sigma^2$. Suppose the scaling is so small, so that

$$t' = \epsilon^2 \tau$$

where $0 < \epsilon \ll 1$. Hence, the equation now becomes

$$\frac{1}{\epsilon^2} \frac{\partial V}{\partial \tau} = \frac{1}{2}S_0^2 \frac{\partial^2 V}{\partial X^2} + \alpha S_0 \frac{\partial V}{\partial X} - \alpha V$$

We now expand in ϵ writing

$$V(X, \tau) \sim V_0(X, \tau) + \epsilon^2 V_1(X, \tau) + \dots$$

By comparing the coefficients of $\frac{1}{\epsilon^2}, 1, \epsilon^2, \dots$, we get

$$\frac{\partial V_0}{\partial \tau} = 0$$

and

$$\frac{\partial V_1}{\partial \tau} = \alpha(S_0 \frac{\partial V_0}{\partial X} - V_0)$$

with the final condition

$$V_0 + \epsilon^2 V_1 = \begin{cases} X - K(1 - \epsilon^2 \alpha \tau) & \text{if } X - K \gg \epsilon K, \text{ far above the strike} \\ 0 & \text{if } K - X \gg \epsilon K, \text{ far below the strike} \end{cases}$$

If we rescale near the strike K , introducing an inner variable, we get

$$X = K(1 + \epsilon x),$$

If we also rescale

$$V(X, \tau) = \epsilon K v(x, \tau),$$

the equation becomes

$$\frac{1}{\epsilon^2} \frac{\partial v}{\partial \tau} = \frac{S_0^2}{2\epsilon^2 K^2} \frac{\partial^2 v}{\partial x^2} + \frac{\alpha S_0}{\epsilon K} \frac{\partial v}{\partial x} - \alpha v,$$

and the payoff is

$$v(x, 0) = \max(x, 0).$$

By expanding,

$$v(x, \tau; \epsilon) \sim v_0(x, \tau) + \epsilon v_1(x, \tau) + \dots$$

Collecting the terms of $O(1)$ together, we get

$$\frac{\partial v_0}{\partial \tau} = \frac{1}{2} \frac{S_0^2}{K^2} \frac{\partial^2 v_0}{\partial x^2}$$

with $v_0(x, 0) = \max(x, 0)$. As the conditions at $x = \pm\infty$ are

$$v(x, \tau) \sim x \quad \text{as } x \rightarrow +\infty$$

and

$$v(x, \tau) \rightarrow 0 \quad \text{as } x \rightarrow -\infty,$$

This problem has a similarity solution of the form

$$v_0(x, \tau) = \sqrt{\tau} f(x/\sqrt{\tau})$$

so,

$$\frac{S_0^2}{K^2} f'' + \xi f' - f = 0,$$

with $f \rightarrow 0$ as $\xi \rightarrow -\infty$ and $f \sim \xi$ as $\xi \rightarrow \infty$ where $\xi = x/\sqrt{\tau}$. By differentiating this equation, we obtain

$$v_0(x, \tau) = xN\left(\frac{x}{\sqrt{\tau}} \frac{K}{S_0}\right) + \frac{S_0}{K} \sqrt{\tau} n\left(\frac{x}{\sqrt{\tau}} \frac{K}{S_0}\right)$$

where $N(\cdot)$ is the cumulative normal and $n(\cdot)$ is its derivative. Using the original variables, this expression becomes

$$V(X, t) \sim (X - K)N\left(\frac{X - K}{S_0 \sigma \sqrt{T - t}}\right) + \sigma \sqrt{T - t} S_0 n\left(\frac{X - K}{S_0 \sigma \sqrt{T - t}}\right)$$

We, then proceed further in order to find the two-term inner expansion $v_0 + \epsilon v_1$. By comparing the term of $O(\epsilon)$, we get

$$\frac{\partial v_1}{\partial \tau} = \frac{1}{2} \frac{S_0^2}{K^2} \frac{\partial^2 v_1}{\partial x^2} + \alpha \frac{S_0}{K} \frac{\partial v_0}{\partial x}$$

with $v_1(x, 0) = 0$. Using the fact that if $u_\tau = \frac{1}{2} \frac{S_0^2}{K^2} u_{xx} + v$ where $v_\tau = \frac{1}{2} \frac{S_0^2}{K^2} v_{xx}$, then $u = \tau v$ is a particular solution, we get

$$v_1(x, \tau) = \alpha \tau \frac{S_0}{K} \frac{\partial v_0}{\partial x} = \alpha \tau \frac{S_0}{K} N\left(\frac{x}{\sqrt{\tau}} \frac{K}{S_0}\right).$$

Hence, the two-term inner expansion $v_0 + \epsilon v_1$ can be easily found. Using the original variables, this expression becomes

$$V(X, t) \sim (X - K + r(T - t)S_0)N\left(\frac{X - K}{S_0 \sigma \sqrt{T - t}}\right) + S_0 \sigma \sqrt{T - t} n\left(\frac{X - K}{S_0 \sigma \sqrt{T - t}}\right)$$

4.4 Empirical Results

We plot the approximations of the value of the spread option obtained by two-term inner expansions when the spread follows each of the three processes: Geometric Brownian Motion, Ornstein-Uhlenbeck and Arithmetic Brownian Motion. Examples are shown in figures 4.1, 4.2 and 4.3. In these examples, we set the time of expiry $T=1$, the volatility $\sigma = 0.3$ (thus $\sigma^2 T = 0.09$), the risk neutral interest rate $r = 0.05$ and the strike $K = 100$.

In fact, we notice from these graphs that there are slight differences among the three approximations. However, in figure 4.1, we notice that as x increases, the value increases more drastically than in figures 4.2 and 4.3. This is due to the fact that in figure 4.1, the spread follows the Geometric Brownian Motion and thus the volatility, σS , depends on S and thus on x . Hence, as x increases, the volatility increases leading to a slightly higher value. As for the cases when the spread follows Ornstein-Uhlenbeck process or Arithmetic Brownian Motion, the volatility is σS_0 which is independent of S and thus independent of x so the increase or decrease in x does not affect the volatility and thus the values obtained in these cases are slightly lower than those obtained for when the spread follows Geometric Brownian Motion. Note that the approximations obtained in figures 4.2 and 4.3 are identical. This is due to the fact that we only considered the two-term inner expansion and ignored the higher order terms; the difference between these two approximations can be seen in the terms of $O(\epsilon^2)$. Finally, note that in all three cases, when $x = 0$, the value starts increasing more strongly compared to when $x < 0$. The reason for that is the fact that $x = 0$ implies that the spread is equal to the strike. When $x < 0$, the spread is less than the strike and when $x > 0$, the spread is greater than the strike. It is when the spread is greater than the strike that the spread option has a larger value as the payoff of the spread option is $\max(S - K, 0)$ where S is the spread.

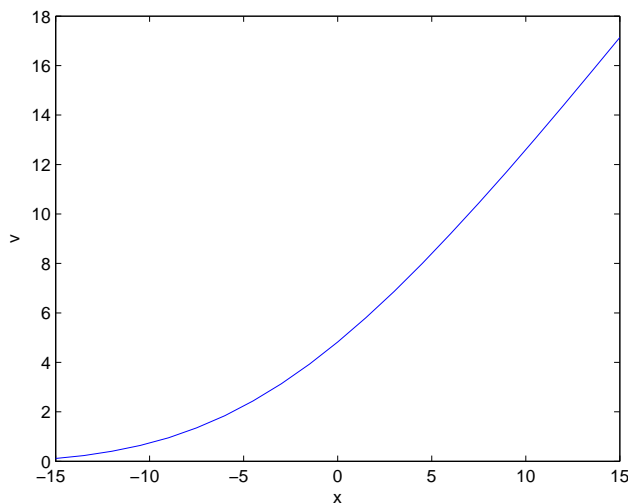


Figure 4.1: Approximate value of the spread option, using two-term inner expansion when the spread follows Geometric Brownian Motion, as a function of x . Time to expiry is one year; the strike is $K = 100$; the volatility is $\sigma = 0.3$, and the risk-free interest rate is $r = 0.05$.

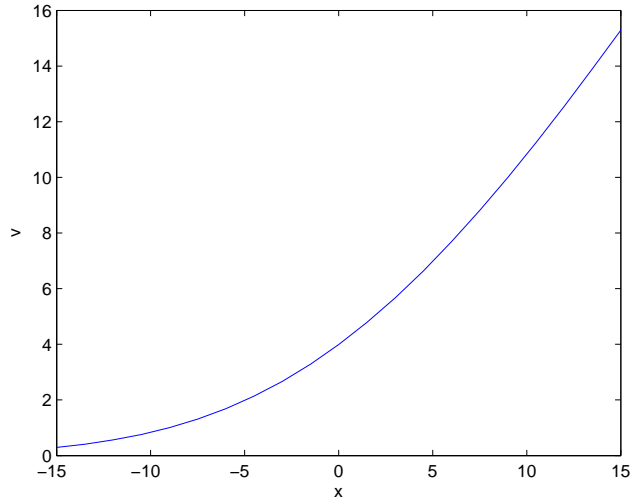


Figure 4.2: Approximate value of the spread option, using two-term inner expansion when the spread follows Ornstein-Uhlenbeck, as a function of x . Time to expiry is one year; the strike is $K = 100$; volatility is $\sigma = 0.3$, and the risk-free interest rate is $r = 0.05$.

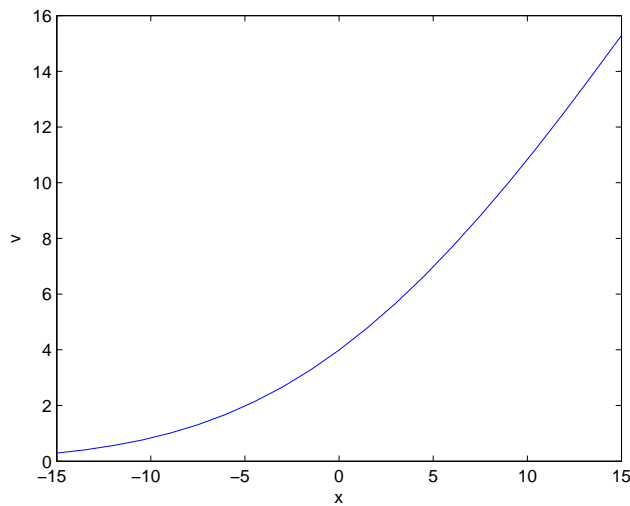


Figure 4.3: Approximate value of the spread option, using two-term inner expansion when the spread follows Arithmetic Brownian Motion, as a function of x . Time to expiry is one year; the strike is $K = 100$; volatility is $\sigma = 0.3$, and the risk-free interest rate is $r = 0.05$.

Chapter 5

Finding the Transition Density Function using Two-Factor Models

5.1 Geometric Brownian Motion

In this section, we will assume that the assets S_1 and S_2 follow the geometric Brownian Motion so that

$$dS_1 = rS_1dt + \sigma_1S_1dW_1$$

$$dS_2 = rS_2dt + \sigma_2S_2dW_2$$

where W_1 and W_2 are standard Brownian Motions with correlation ρ , r is the risk-free interest rate and σ_1 and σ_2 are the volatilities of S_1 and S_2 respectively. Note that r , σ_1 and σ_2 are assumed to be constant. Let P be the transition density function of this option. P satisfies the two-dimensional Kolmogorov Forward Equation

$$\frac{\partial P}{\partial t} = -\frac{\partial(rS_1P)}{\partial S_1} - \frac{\partial(rS_2P)}{\partial S_2} + \frac{1}{2} \frac{\partial^2(\sigma_1^2 S_1^2 P)}{\partial S_1^2} + \frac{1}{2} \frac{\partial^2(\sigma_2^2 S_2^2 P)}{\partial S_2^2} + \frac{\partial^2(\rho\sigma_1\sigma_2 S_1 S_2 P)}{\partial S_1 \partial S_2}$$

Assume a short period of time i.e. the time is small, so that

$$t = \epsilon^2 \tau$$

where $0 < \epsilon \ll 1$. After simplifying the derivatives, the expression reads

$$\begin{aligned} \frac{1}{\epsilon^2} \frac{\partial P}{\partial \tau} &= \frac{1}{2} (\sigma_1^2 S_1^2 \frac{\partial^2 P}{\partial S_1^2} + 2\rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 P}{\partial S_1 \partial S_2} + \sigma_2^2 S_2^2 \frac{\partial^2 P}{\partial S_2^2}) + (2\sigma_1^2 + \sigma_1\sigma_2\rho - r) S_1 \frac{\partial P}{\partial S_1} \\ &+ (2\sigma_2^2 + \sigma_1\sigma_2\rho - r) S_2 \frac{\partial P}{\partial S_2} + (\sigma_2^2 + \sigma_1^2 + \sigma_1\sigma_2\rho - 2r) P \end{aligned} \quad (7)$$

Expanding in ϵ , we write,

$$P(S_1, S_2, \tau) \sim P_0(S_1, S_2, \tau) + \epsilon^2 P_1(S_1, S_2, \tau) + \dots$$

Collecting together the terms of $O(\frac{1}{\epsilon^2})$, we get

$$\frac{\partial P_0}{\partial \tau} = 0$$

with

$$P_0(S_1, S_2, 0) = \delta(S_1 - S_2 - S_0)$$

This problem can be easily solved to obtain

$$P_0(S_1, S_2, 0) = \delta(S_1 - S_2 - S_0)$$

To observe the behavior around the initial value S_0 , we now rescale near S_0 , introducing two inner variables

$$S_1 = S_0(1 + \epsilon x_1)$$

$$S_2 = S_0(1 + \epsilon x_2)$$

Equation (7) now read

$$\begin{aligned} \frac{\partial P^i}{\partial \tau} &= \frac{1}{2} ((1 + \epsilon x_1)^2 \sigma_1^2 \frac{\partial^2 P^i}{\partial x_1^2} + 2\rho\sigma_1\sigma_2(1 + \epsilon x_1)(1 + \epsilon x_2) \frac{\partial^2 P^i}{\partial x_1 \partial x_2} + (1 + \epsilon x_2)^2 \sigma_2^2 \frac{\partial^2 P^i}{\partial x_2^2}) \\ &+ (2\sigma_1^2 + \sigma_1\sigma_2\rho - r)\epsilon(1 + \epsilon x_1) \frac{\partial P^i}{\partial x_1} + (2\sigma_2^2 + \sigma_1\sigma_2\rho - r)\epsilon(1 + \epsilon x_2) \frac{\partial P^i}{\partial x_2} + \epsilon^2 (\sigma_2^2 + \sigma_1^2 + \sigma_1\sigma_2\rho - 2r) P_0^i \end{aligned}$$

with

$$P^i(x_1, x_2, 0) = \frac{1}{S_0\epsilon} \delta(x_1 - x_2)$$

If we expand as follows

$$P^i \sim \frac{1}{\epsilon} P_0^i + P_1^i + \dots$$

and compare the terms of the same order, we get

$$\frac{\partial P_0^i}{\partial \tau} = \frac{1}{2} (\sigma_1^2 \frac{\partial^2 P_0^i}{\partial x_1^2} + 2\rho\sigma_1\sigma_2 \frac{\partial P_0^i}{\partial x_1 \partial x_2} + \sigma_2^2 \frac{\partial^2 P_0^i}{\partial x_2^2})$$

with

$$P_0^i(x_1, x_2, 0) = \frac{1}{S_0} \delta(x_1 - x_2)$$

The solution can be found to be

$$P_0^i(x_1, x_2, \tau) = \frac{1}{2\pi\tau\sqrt{\sigma_1\sigma_2}} e^{-\left(\frac{x_1^2}{2\sigma_1^2\tau} + \frac{2\rho x_1 x_2}{2\sigma_1\sigma_2\tau} + \frac{x_2^2}{2\sigma_2^2\tau}\right)}$$

We proceed further to find P_1^i

$$\begin{aligned} \frac{\partial P_1^i}{\partial \tau} = & \frac{1}{2} (\sigma_1^2 \frac{\partial^2 P_1^i}{\partial x_1^2} + 2\rho\sigma_1\sigma_2 \frac{\partial P_1^i}{\partial x_1 \partial x_2} + \sigma_2^2 \frac{\partial^2 P_1^i}{\partial x_2^2}) + \sigma_1^2 x_1 \frac{\partial^2 P_0^i}{\partial x_1^2} + 2\rho\sigma_1\sigma_2(x_1+x_2) \frac{\partial^2 P_0^i}{\partial x_1 \partial x_2} + \sigma_2^2 x_2 \frac{\partial^2 P_0^i}{\partial x_2^2} \\ & + (2\sigma_1^2 + \rho\sigma_1\sigma_2 - r) \frac{\partial P_0^i}{\partial x_1} + (2\sigma_2^2 + \rho\sigma_1\sigma_2 - r) \frac{\partial P_0^i}{\partial x_2} \end{aligned}$$

with

$$P_1^0(x_1, x_2, 0) = 0.$$

By performing tedious calculations, one can find the solution to this PDE. The solution is of the form

$$Q(x, y) e^{-\left(\frac{x_1^2}{2\sigma_1^2\tau} + \frac{2\rho x_1 x_2}{2\sigma_1\sigma_2\tau} + \frac{x_2^2}{2\sigma_2^2\tau}\right)} \quad (5.1)$$

where Q is a polynomial in x and y .

5.2 Ornstein-Uhlenbeck

In this section, we will assume that the assets Y_1 and Y_2 follow the following stochastic differential equations

$$dY_1 = rY_1 dt + \sigma_1 S_0 dW_1$$

$$dY_2 = rY_2dt + \sigma_2S_0dW_2$$

where W_1 and W_2 are standard Brownian Motions with correlation ρ , r is the risk-free interest rate and σ_1 and σ_2 are the volatilities of Y_1 and Y_2 respectively. Note that r , S_0 , σ_1 and σ_2 are assumed to be constant. Let P be the transition density function of this process. P satisfies the two-dimensional Kolmogorov Forward Equation

$$\frac{\partial P}{\partial t} = -\frac{\partial(rY_1P)}{\partial Y_1} - \frac{\partial(rY_2P)}{\partial Y_2} + \frac{1}{2}\frac{\partial^2(\sigma_1^2S_0^2P)}{\partial Y_1^2} + \frac{1}{2}\frac{\partial^2(\sigma_2^2S_0^2P)}{\partial Y_2^2} + \frac{\partial^2(\rho\sigma_1\sigma_2S_0^2P)}{\partial Y_1\partial Y_2}$$

Assume a short period of time i.e. the time is small, so that

$$t = \epsilon^2\tau$$

where $0 < \epsilon \ll 1$. After simplifying the derivatives, the expression reads

$$\frac{1}{\epsilon^2}\frac{\partial P}{\partial \tau} = \frac{S_0^2}{2}\left(\sigma_1^2\frac{\partial^2 P}{\partial Y_1^2} + 2\rho\sigma_1\sigma_2\frac{\partial^2 P}{\partial Y_1\partial Y_2} + \sigma_2^2\frac{\partial^2 P}{\partial Y_2^2}\right) - rY_1\frac{\partial P}{\partial Y_1} - rY_2\frac{\partial P}{\partial Y_2} - 2rP \quad (8)$$

We now expand in ϵ writing,

$$P(Y_1, Y_2, \tau) \sim P_0(Y_1, Y_2, \tau) + \epsilon^2(Y_1, Y_2, \tau) + \dots$$

Comparing the terms of $O(\frac{1}{\epsilon^2})$, we get

$$\frac{\partial P_0}{\partial \tau} = 0$$

with

$$P_0(Y_1, Y_2, 0) = \delta(Y_1 - Y_2 - S_0)$$

The solution is easily found to be

$$P_0(Y_1, Y_2, \tau) = \delta(Y_1 - Y_2 - S_0)$$

To observe the behavior around S_0 , we rescale near the initial value S_0 , introducing two inner variables

$$Y_1 = S_0(1 + \epsilon x_1)$$

$$Y_2 = S_0(1 + \epsilon x_2)$$

Equation (8) now reads

$$\frac{1}{\epsilon^2} \frac{\partial P^i}{\partial \tau} = \frac{1}{2\epsilon^2} (\sigma_1^2 \frac{\partial^2 P^i}{\partial x_1^2} + 2\rho\sigma_1\sigma_2 \frac{\partial^2 P^i}{\partial x_1 \partial x_2} + \sigma_2^2 \frac{\partial^2 P^i}{\partial x_2^2}) - \frac{r}{\epsilon} (1 + \epsilon x_1) \frac{\partial P^i}{\partial x_1} - \frac{r}{\epsilon} (1 + \epsilon x_2) \frac{\partial P^i}{\partial x_2} - 2rP^i$$

If we expand as follows

$$P^i(x_1, x_2, \tau; \epsilon) \sim \frac{1}{\epsilon} P_0^i(x_1, x_2, \tau) + P_1^i(x_1, x_2, \tau) + \dots$$

and compare the terms of the same order, we get

$$\frac{\partial P_0^i}{\partial \tau} = \frac{1}{2} (\sigma_1^2 \frac{\partial^2 P_0^i}{\partial x_1^2} + 2\rho\sigma_1\sigma_2 \frac{\partial^2 P_0^i}{\partial x_1 \partial x_2} + \sigma_2^2 \frac{\partial^2 P_0^i}{\partial x_2^2})$$

with

$$P_0^i(x_1, x_2, 0) = \frac{1}{S_0} \delta(x_1 - x_2)$$

Hence, the solution is found to be

$$P_0^i(x_1, x_2, \tau) = \frac{1}{2\pi\tau\sqrt{\sigma_1\sigma_2}} e^{-\left(\frac{x_1^2}{2\sigma_1^2\tau} + \frac{2\rho x_1 x_2}{2\sigma_1\sigma_2\tau} + \frac{x_2^2}{2\sigma_2^2\tau}\right)}$$

We now proceed to find P_1^i

$$\frac{\partial P_1^i}{\partial \tau} = \frac{1}{2} (\sigma_1^2 \frac{\partial^2 P_1^i}{\partial x_1^2} + 2\rho\sigma_1\sigma_2 \frac{\partial^2 P_1^i}{\partial x_1 \partial x_2} + \sigma_2^2 \frac{\partial^2 P_1^i}{\partial x_2^2}) - r \frac{\partial P_0^i}{\partial x_1} - r \frac{\partial P_0^i}{\partial x_2}$$

with

$$P_1^i(x_1, x_2, 0) = 0$$

Using the fact that if $u_\tau = \frac{1}{2}(\sigma_1^2 u_{x_1 x_1} + 2\rho\sigma_1\sigma_2 u_{x_1 x_2} + \sigma_2^2 u_{x_2 x_2}) + v$ where v satisfies $v_\tau = \frac{1}{2}(\sigma_1^2 v_{x_1 x_1} + 2\rho\sigma_1\sigma_2 v_{x_1 x_2} + \sigma_2^2 v_{x_2 x_2})$, then $u = \tau v$ is a particular solution, we deduce that

$$P_1^i(x_1, x_2, \tau) = -r\tau \left(\frac{\partial P_0^i}{\partial x_1} + \frac{\partial P_0^i}{\partial x_2} \right) = \frac{r}{2\pi\sqrt{\sigma_1\sigma_2}} e^{-\left(\frac{x_1^2}{2\sigma_1^2\tau} + \frac{\rho x_1 x_2}{\sigma_1\sigma_2\tau} + \frac{x_2^2}{2\sigma_2^2\tau}\right)} \left(\frac{x_1}{\sigma_1^2\tau} + \frac{\rho(x_1+x_2)}{\sigma_1\sigma_2\tau} + \frac{x_2}{\sigma_2^2\tau} \right)$$

Finally, we find that

$$P(x_1, x_2, \tau) \sim \frac{1}{2\pi\tau\sqrt{\sigma_1\sigma_2}} e^{-\left(\frac{x_1^2}{2\sigma_1^2\tau} + \frac{2\rho x_1 x_2}{2\sigma_1\sigma_2\tau} + \frac{x_2^2}{2\sigma_2^2\tau}\right)} \left(\frac{1}{\epsilon} + \frac{rx_1}{\sigma_1^2} + \frac{r\rho(x_1+x_2)}{\sigma_1\sigma_2} + \frac{rx_2}{\sigma_2^2} + O(\epsilon) \right)$$

5.3 Arithmetic Brownian Motion

In this section, we will assume that the assets X_1 and X_2 follow the arithmetic Brownian Motion so that

$$dX_1 = rS_0 dt + \sigma_1 S_0 dW_1$$

$$dX_2 = rS_0 dt + \sigma_2 S_0 dW_2$$

where W_1 and W_2 are standard Brownian Motions with correlation ρ , r is the risk-free interest rate and σ_1 and σ_2 are the volatilities of X_1 and X_2 respectively. Note that r , σ_1 and σ_2 are assumed to be constant. Let P be the transition density function of this process. P satisfies the two-dimensional Kolmogorov Forward Equation

$$\frac{\partial P}{\partial t} = -\frac{\partial(rS_0 P)}{\partial X_1} - \frac{\partial(rS_0 P)}{\partial X_2} + \frac{1}{2} \frac{\partial^2(\sigma_1^2 S_0^2 P)}{\partial X_1^2} + \frac{1}{2} \frac{\partial^2(\sigma_2^2 S_0^2 P)}{\partial X_2^2} + \frac{\partial^2(\rho\sigma_1\sigma_2 S_0^2 P)}{\partial X_1 \partial X_2}$$

Assume a short period of time i.e. the time is small, so that

$$t = \epsilon^2 \tau$$

where $0 < \epsilon \ll 1$. After simplifying the derivatives, the expression reads

$$\frac{1}{\epsilon^2} \frac{\partial P}{\partial \tau} = \frac{S_0^2}{2} \left(\sigma_1^2 \frac{\partial^2 P}{\partial X_1^2} + 2\rho\sigma_1\sigma_2 \frac{\partial^2 P}{\partial X_1 \partial X_2} + \sigma_2^2 \frac{\partial^2 P}{\partial X_2^2} \right) - rS_0 \left(\frac{\partial P}{\partial X_1} + \frac{\partial P}{\partial X_2} \right) \quad (9)$$

We now expand in ϵ , writing

$$P(X_1, X_2, \tau) \sim P_0(X_1, X_2, \tau) + \epsilon^2 P_1(X_1, X_2, \tau) + \dots$$

Comparing the terms of $O(\frac{1}{\epsilon^2})$, we get

$$\frac{\partial P_0}{\partial \tau} = 0$$

with

$$P_0(X_1, X_2, 0) = \delta(X_1 - X_2 - S_0)$$

The solution is easily found to be

$$P_0(X_1, X_2, \tau) = \delta(X_1 - X_2 - S_0)$$

To see the behavior near the initial values S_0 , we rescale around S_0 introducing two inner variables

$$X_1 = S_0(1 + \epsilon x_1)$$

$$X_2 = S_0(1 + \epsilon x_2)$$

Equation (9) now reads

$$\frac{1}{\epsilon^2} \frac{\partial P^i}{\partial \tau} = \frac{1}{2\epsilon^2} (\sigma_1^2 \frac{\partial^2 P^i}{\partial x_1^2} + 2\rho\sigma_1\sigma_2 \frac{\partial^2 P^i}{\partial x_1 \partial x_2} + \sigma_2^2 \frac{\partial^2 P^i}{\partial x_2^2}) - \frac{r}{\epsilon} (\frac{\partial P^i}{\partial x_1} + \frac{\partial P^i}{\partial x_2})$$

If we expand as follows

$$P(x_1, x_2, \tau; \epsilon) \sim \frac{1}{\epsilon} P_0(x_1, x_2, \tau) + P_1(x_1, x_2, \tau) + \dots$$

and compare the terms of the same order, we get

$$\frac{\partial P_0^i}{\partial \tau} = \frac{1}{2} (\sigma_1^2 \frac{\partial^2 P_0^i}{\partial x_1^2} + 2\rho\sigma_1\sigma_2 \frac{\partial^2 P_0^i}{\partial x_1 \partial x_2} + \sigma_2^2 \frac{\partial^2 P_0^i}{\partial x_2^2})$$

with

$$P_0^i(x_1, x_2, 0) = \frac{1}{S_0} \delta(x_1 - x_2)$$

Hence, the solution is found to be

$$P_0^i(x_1, x_2, \tau) = \frac{1}{2\pi\tau\sqrt{\sigma_1\sigma_2}} e^{-\left(\frac{x_1^2}{2\sigma_1^2\tau} + \frac{2\rho x_1 x_2}{2\sigma_1\sigma_2\tau} + \frac{x_2^2}{2\sigma_2^2\tau}\right)}$$

We now proceed to find P_1^i

$$\frac{\partial P_1^i}{\partial \tau} = \frac{1}{2}(\sigma_1^2 \frac{\partial^2 P_1^i}{\partial x_1^2} + 2\rho\sigma_1\sigma_2 \frac{\partial^2 P_1^i}{\partial x_1\partial x_2} + \sigma_2^2 \frac{\partial^2 P_1^i}{\partial x_2^2}) - r\left(\frac{\partial P_0^i}{\partial x_1} + \frac{\partial P_0^i}{\partial x_2}\right)$$

with

$$P_1^i(x_1, x_2, 0) = 0$$

Using the fact that if $u_\tau = \frac{1}{2}(\sigma_1^2 u_{x_1 x_1} + 2\rho\sigma_1\sigma_2 u_{x_1 x_2} + \sigma_2^2 u_{x_2 x_2}) + v$ where v satisfies $v_\tau = \frac{1}{2}(\sigma_1^2 v_{x_1 x_1} + 2\rho\sigma_1\sigma_2 v_{x_1 x_2} + \sigma_2^2 v_{x_2 x_2})$, then $u = \tau v$ is a solution, we deduce that

$$P_1^i(x_1, x_2, \tau) = -r\tau\left(\frac{\partial P_0^i}{\partial x_1} + \frac{\partial P_0^i}{\partial x_2}\right) = \frac{r}{2\pi\sqrt{\sigma_1\sigma_2}} e^{-\left(\frac{x_1^2}{2\sigma_1^2\tau} + \frac{\rho x_1 x_2}{\sigma_1\sigma_2\tau} + \frac{x_2^2}{2\sigma_2^2\tau}\right)} \left(\frac{x_1}{\sigma_1^2\tau} + \frac{\rho(x_1 + x_2)}{\sigma_1\sigma_2\tau} + \frac{x_2}{\sigma_2^2\tau}\right)$$

Finally, we find that

$$P(x_1, x_2, \tau) \sim \frac{1}{2\pi\tau\sqrt{\sigma_1\sigma_2}} e^{-\left(\frac{x_1^2}{2\sigma_1^2\tau} + \frac{2\rho x_1 x_2}{2\sigma_1\sigma_2\tau} + \frac{x_2^2}{2\sigma_2^2\tau}\right)} \left(\frac{1}{\epsilon} + \frac{rx_1}{\sigma_1^2} + \frac{r\rho(x_1 + x_2)}{\sigma_1\sigma_2} + \frac{rx_2}{\sigma_2^2} + O(\epsilon)\right)$$

Chapter 6

Option Pricing by Finding the Value for Two-Factor Models

6.1 Geometric Brownian Motion

In this section, we will assume that the assets S_1 and S_2 follow the geometric Brownian Motion so that

$$dS_1 = rS_1 dt + \sigma_1 S_1 dW_1$$

$$dS_2 = rS_2 dt + \sigma_2 S_2 dW_2$$

where W_1 and W_2 are standard Brownian Motions with correlation ρ , r is the risk-free interest rate and σ_1 and σ_2 are the volatilities of S_1 and S_2 respectively. Note that r , σ_1 and σ_2 are assumed to be constant. Let V be the value of the option and thus V satisfies the following Stochastic Differential Equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}(\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + 2\rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} + \sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2}) + rS_1 \frac{\partial V}{\partial S_1} + rS_2 \frac{\partial V}{\partial S_2} - rV = 0 \quad (10)$$

Let

$$\zeta = S_1 - S_2$$

and

$$\eta = S_1 + S_2$$

Writing V as a function of ζ and η , equation (10) becomes

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{1}{8}(\sigma_1^2(\zeta + \eta)^2(\frac{\partial^2 V}{\partial \zeta^2} + 2\frac{\partial^2 V}{\partial \zeta \partial \eta} + \frac{\partial^2 V}{\partial \eta^2}) + 2\rho\sigma_1\sigma_2(\eta^2 - \zeta^2)(\frac{\partial^2 V}{\partial \eta^2} - \frac{\partial^2 V}{\partial \zeta^2}) \\ + \sigma_2^2(\eta - \zeta)^2(\frac{\partial^2 V}{\partial \zeta^2} - 2\frac{\partial^2 V}{\partial \zeta \partial \eta} + \frac{\partial^2 V}{\partial \eta^2})) + r\zeta\frac{\partial V}{\partial \zeta} + r\eta\frac{\partial V}{\partial \eta} - rV = 0 \end{aligned}$$

If we measure the time backwards from expiry, writing $t = T - t'$, and assume small time scaling such that

$$t' = \epsilon^2 \tau$$

where $0 < \epsilon \ll 1$, we get

$$\begin{aligned} \frac{1}{\epsilon^2} \frac{\partial V}{\partial \tau} = \frac{1}{8}(\sigma_1^2(\zeta + \eta)^2(\frac{\partial^2 V}{\partial \zeta^2} + 2\frac{\partial^2 V}{\partial \zeta \partial \eta} + \frac{\partial^2 V}{\partial \eta^2}) + 2\rho\sigma_1\sigma_2(\eta^2 - \zeta^2)(\frac{\partial^2 V}{\partial \eta^2} - \frac{\partial^2 V}{\partial \zeta^2}) \\ + \sigma_2^2(\eta - \zeta)^2(\frac{\partial^2 V}{\partial \zeta^2} - 2\frac{\partial^2 V}{\partial \zeta \partial \eta} + \frac{\partial^2 V}{\partial \eta^2})) + r\zeta\frac{\partial V}{\partial \zeta} + r\eta\frac{\partial V}{\partial \eta} - rV \end{aligned}$$

By expanding in ϵ writing

$$V(S_1, S_2, \tau; \epsilon) \sim V_0(S_1, S_2, \tau) + \epsilon V_1(S_1, S_2, \tau) + \dots$$

Comparing the terms of the same order in ϵ we see that

$$\frac{\partial V_0}{\partial \tau} = 0$$

with

$$V_0(S_1, S_2, 0) = \max(S_1 - S_2 - K, 0)$$

To see the behavior around the strike, we rescale near K , introducing a new inner variable

$$\zeta = K(1 + \epsilon x)$$

and

$$\eta = Ky.$$

If we rescale at the same time

$$V(\zeta, \eta) = \epsilon K v(x, y),$$

the equation becomes

$$\begin{aligned} \frac{1}{\epsilon^2} \frac{\partial v}{\partial \tau} = & \frac{1}{8\epsilon^2} ((\sigma_1^2(1+\epsilon x+y)^2 - 2\rho\sigma_1\sigma_2(y^2 - (1+\epsilon x)^2) + \sigma_2^2(y-1-\epsilon x)^2) \frac{\partial^2 v}{\partial x^2} + \frac{2}{\epsilon} (\sigma_1^2(1+\epsilon x+y)^2) \frac{\partial^2 v}{\partial x \partial y} \\ & - \sigma_2^2(y-1-\epsilon x)^2 + (\sigma_1^2(1+\epsilon x+y)^2 + 2\rho\sigma_1\sigma_2(y^2 - (1+\epsilon x)^2) + \sigma_2^2(y-1-\epsilon x)^2) \frac{\partial^2 v}{\partial y^2}) + \frac{r}{\epsilon} (1+\epsilon x) \frac{\partial v}{\partial x} + r y \frac{\partial v}{\partial y} - r v. \end{aligned}$$

with $v(x, 0) = \max(x, 0)$ We now expand

$$v(x, \tau; \epsilon; y) \sim v_0(x, \tau) + \epsilon v_1(x, \tau) + \dots$$

Collecting together the terms of $O(\frac{1}{\epsilon^2})$, we get

$$\frac{\partial v_0}{\partial \tau} = \frac{1}{8} (\sigma_1^2(1+y)^2 - 2\rho\sigma_1\sigma_2(y^2 - 1) + \sigma_2^2(y-1)^2) \frac{\partial^2 v_0}{\partial x^2}$$

with $v_0(x, 0) = \max(x, 0)$. Let $a = \frac{1}{4} (\sigma_1^2(1+y)^2 - 2\rho\sigma_1\sigma_2(y^2 - 1) + \sigma_2^2(y-1)^2)$ (Note that a is independent of x). The above equation becomes

$$\frac{\partial v_0}{\partial \tau} = \frac{1}{2} a \frac{\partial^2 v_0}{\partial x^2}$$

As the conditions at $x = \pm\infty$ are

$$v(x, \tau) \sim x \quad \text{as } x \rightarrow +\infty$$

and

$$v(x, \tau) \rightarrow 0 \quad \text{as } x \rightarrow -\infty,$$

this problem has a similarity solution of the form

$$v_0(x, \tau) = \sqrt{\tau} f\left(\frac{x}{\sqrt{\tau}}\right)$$

Hence,

$$a f'' + \xi f' - f = 0$$

with $f \rightarrow 0$ as $\xi \rightarrow -\infty$ and $f \sim \xi$ as $\xi \rightarrow \infty$ where $\xi = x/\sqrt{\tau}$. By differentiating this equation, it is easily found that

$$v_0(x, \tau, y) = xN\left(\frac{x}{\sqrt{a\tau}}\right) + \sqrt{a\tau}n\left(\frac{x}{\sqrt{a\tau}}\right)$$

In the original variables, we have

$$V(S_1, S_2, t) \sim (S_1 - S_2 - K)N\left(\frac{S_1 - S_2 - K}{K\sqrt{a(T-t)}}\right) + K\sqrt{a(T-t)}n\left(\frac{S_1 - S_2 - K}{K\sqrt{a(T-t)}}\right)$$

where

$$a = \frac{1}{4K^2}(\sigma_1^2(K + S_1 + S_2)^2 - 2\rho\sigma_1\sigma_2((S_1 + S_2)^2 - K^2) + \sigma_2^2(S_1 + S_2 - K)^2)$$

We now proceed to find v_1 by comparing the terms of $O(\frac{1}{\epsilon})$

$$\frac{\partial v_1}{\partial \tau} = \frac{a}{2} \frac{\partial^2 v_1}{\partial x^2} + \frac{x}{4} \frac{\partial^2 v_0}{\partial x^2} (\sigma_1^2(1+y) + 2\rho\sigma_1\sigma_2 - \sigma_2^2(y-1)) + 2 \frac{\partial^2 v_0}{\partial x \partial y} (\sigma_1^2(1+y)^2 - \sigma_2^2(y-1)^2) + r \frac{\partial v_0}{\partial x}$$

Using the fact that if $u_\tau = \frac{a}{2}u_{xx} + v$ where v satisfies $v_\tau = \frac{a}{2}v_{xx}$, then $u = \tau v$ is a particular solution, and the fact that if $u_\tau = \frac{a}{2}u_{xx} + xv$ where v satisfies $v_\tau = \frac{a}{2}v_{xx}$, then $u = x\tau v + \frac{1}{2}\tau^2 v_x$, we deduce that

$$\begin{aligned} v_1(x, \tau; y) &= 2\tau \frac{\partial^2 v_0}{\partial x \partial y} (\sigma_1^2(1+y)^2 - \sigma_2^2(y-1)^2) + r\tau \frac{\partial v_0}{\partial x} + (\sigma_1^2(1+y) + 2\rho\sigma_1\sigma_2 - \sigma_2^2(y-1)) \left(\frac{x\tau}{4} \frac{\partial^2 v_0}{\partial x^2} + \frac{\tau^2}{8} \frac{\partial^3 v_0}{\partial x^3} \right) \\ &= -\frac{x\sqrt{\tau}}{2a\sqrt{a}} (\sigma_1^2(1+y)^2 - \sigma_2^2(y-1)^2) (\sigma_1^2(1+y) - 2\rho\sigma_1\sigma_2 y + \sigma_2^2(y-1)) n\left(\frac{x}{\sqrt{a\tau}}\right) + r\tau N\left(\frac{x}{\sqrt{a\tau}}\right) + \left(\frac{x\sqrt{\tau}}{4\sqrt{a}} n\left(\frac{x}{\sqrt{a\tau}}\right) \right. \\ &\quad \left. + \frac{\tau}{8a} n'\left(\frac{x}{\sqrt{a\tau}}\right)\right) (\sigma_1^2(1+y) + 2\rho\sigma_1\sigma_2 + \sigma_2^2(y-1)) \end{aligned}$$

Using the two-term inner expansion, $v_0 + \epsilon v_1$, and replacing in the original values we get

$$\begin{aligned} V(S_1, S_2, t) &\sim (S_1 - S_2 - K)N\left(\frac{S_1 - S_2 - K}{K\sqrt{a(T-t)}}\right) + K\sqrt{a(T-t)}n\left(\frac{S_1 - S_2 - K}{K\sqrt{a(T-t)}}\right) \\ &\quad - \frac{\sqrt{(T-t)}}{K^2} (\sigma_1^2(S_1 + S_2 + K) - \sigma_2^2(S_1 + S_2 - K)) (\sigma_1^2(S_1 + S_2 + K) - 2\rho\sigma_1\sigma_2(S_1 + S_2) \\ &\quad + \sigma_2^2(S_1 + S_2 - K)) \left(\frac{S_1 - S_2 - K}{2Ka\sqrt{a}}\right) n\left(\frac{S_1 - S_2 - K}{K\sqrt{a(T-t)}}\right) + rK(T-t)N\left(\frac{S_1 - S_2 - K}{K\sqrt{a(T-t)}}\right) \\ &\quad + \left(\frac{(S_1 - S_2 - K)\sqrt{T-t}}{4\sqrt{a}} n\left(\frac{S_1 - S_2 - K}{K\sqrt{a(T-t)}}\right) + \frac{(T-t)}{8a} n'\left(\frac{S_1 - S_2 - K}{K\sqrt{a(T-t)}}\right)\right) (\sigma_1^2(S_1 + S_2 + K) + 2\rho\sigma_1\sigma_2 K \\ &\quad - \sigma_2^2(S_1 + S_2 - K)) \end{aligned}$$

6.2 Ornstein-Uhlenbeck

In this section, we will assume that the assets Y_1 and Y_2 follow the following stochastic differential equations

$$dY_1 = rY_1 dt + \sigma_1 S_0 dW_1$$

$$dY_2 = rY_2 dt + \sigma_2 S_0 dW_2$$

where W_1 and W_2 are standard Brownian Motions with correlation ρ , r is the risk-free interest rate and σ_1 and σ_2 are the volatilities of Y_1 and Y_2 respectively. Note that r , S_0 , σ_1 and σ_2 are assumed to be constant. Let V be the value of the option and thus V satisfies the following Stochastic Differential Equation

$$\frac{\partial V}{\partial t} + \frac{S_0^2}{2} (\sigma_1^2 \frac{\partial^2 V}{\partial Y_1^2} + 2\rho\sigma_1\sigma_2 \frac{\partial^2 V}{\partial Y_1 \partial Y_2} + \sigma_2^2 \frac{\partial^2 V}{\partial Y_2^2}) + rY_1 \frac{\partial V}{\partial Y_1} + rY_2 \frac{\partial V}{\partial Y_2} - rV = 0 \quad (11)$$

Let

$$\zeta = Y_1 - Y_2$$

and

$$\eta = Y_1 + Y_2$$

Hence, equation (10) now becomes

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{S_0^2}{2} (\sigma_1^2 (\frac{\partial^2 V}{\partial \zeta^2} + 2\frac{\partial^2 V}{\partial \zeta \partial \eta} + \frac{\partial^2 V}{\partial \eta^2}) + 2\rho\sigma_1\sigma_2 (\frac{\partial^2 V}{\partial \eta^2} - \frac{\partial^2 V}{\partial \zeta^2}) + \sigma_2^2 (\frac{\partial^2 V}{\partial \zeta^2} - 2\frac{\partial^2 V}{\partial \zeta \partial \eta} + \frac{\partial^2 V}{\partial \eta^2})) \\ + r\zeta \frac{\partial V}{\partial \zeta} + r\eta \frac{\partial V}{\partial \eta} - rV = 0 \end{aligned}$$

If we measure the time backwards from expiry, writing $t = T - t'$, and assume small time scaling such that

$$t' = \epsilon^2 \tau$$

where $0 < \epsilon \ll 1$, we get

$$\begin{aligned} \frac{1}{\epsilon^2} \frac{\partial V}{\partial \tau} = \frac{S_0^2}{2} (\sigma_1^2 (\frac{\partial^2 V}{\partial \zeta^2} + 2\frac{\partial^2 V}{\partial \zeta \partial \eta} + \frac{\partial^2 V}{\partial \eta^2}) + 2\rho\sigma_1\sigma_2 (\frac{\partial^2 V}{\partial \eta^2} - \frac{\partial^2 V}{\partial \zeta^2}) + \sigma_2^2 (\frac{\partial^2 V}{\partial \zeta^2} - 2\frac{\partial^2 V}{\partial \zeta \partial \eta} + \frac{\partial^2 V}{\partial \eta^2})) \\ + r\zeta \frac{\partial V}{\partial \zeta} + r\eta \frac{\partial V}{\partial \eta} - rV \end{aligned}$$

By expanding in ϵ writing

$$V(\zeta, \eta, \tau; \epsilon) \sim V_0(\zeta, \eta, \tau) + \epsilon V_1(\zeta, \eta, \tau) + \dots$$

Comparing the terms of the same order in ϵ we see that

$$\frac{\partial V_0}{\partial \tau} = 0$$

with

$$V_0(\zeta, \eta, 0) = \max(\zeta - K, 0)$$

Hence,

$$V_0(\zeta, \eta, \tau) = \max(\zeta - K, 0)$$

To see the behavior around the strike, we rescale near K , introducing a new inner variable

$$\zeta = K(1 + \epsilon x)$$

and

$$\eta = Ky.$$

If we rescale at the same time

$$V(\zeta, \eta) = \epsilon K v(x, y),$$

the equation becomes

$$\begin{aligned} \frac{1}{\epsilon^2} \frac{\partial v}{\partial \tau} &= \frac{S_0^2}{2} \left(\frac{1}{\epsilon^2 K^2} (\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2) \frac{\partial^2 v}{\partial x^2} + \frac{2}{\epsilon K^2} \frac{\partial^2 v}{\partial x \partial y} (\sigma_1^2 - \sigma_2^2) + \frac{1}{K^2} \frac{\partial^2 v}{\partial y^2} (\sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2) \right) \\ &+ \frac{r}{\epsilon} (1 + \epsilon x) \frac{\partial v}{\partial x} + ry \frac{\partial v}{\partial y} - rv. \end{aligned}$$

with $v(x, 0) = \max(x, 0)$. We now expand

$$v(x, \tau; \epsilon; y) \sim v_0(x, \tau) + \epsilon v_1(x, \tau) + \dots$$

Collecting together the terms of $O(\frac{1}{\epsilon^2})$, we get

$$\frac{\partial v_0}{\partial \tau} = \frac{S_0^2}{2K^2}(\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2) \frac{\partial^2 v_0}{\partial x^2}$$

with $v_0(x, 0) = \max(x, 0)$. Let $a = \frac{S_0^2}{K^2}(\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2)$ (Note that a is independent of x). Thus the above equation becomes

$$\frac{\partial v_0}{\partial \tau} = \frac{1}{2}a \frac{\partial^2 v_0}{\partial x^2}$$

As the conditions at $x = \pm\infty$ are

$$v(x, \tau) \sim x \quad \text{as } x \rightarrow +\infty$$

and

$$v(x, \tau) \rightarrow 0 \quad \text{as } x \rightarrow -\infty,$$

this problem has a similarity solution of the form

$$v_0(x, \tau) = \sqrt{\tau} f\left(\frac{x}{\sqrt{\tau}}\right)$$

Hence,

$$af'' + \xi f' - f = 0$$

with $f \rightarrow 0$ as $\xi \rightarrow -\infty$ and $f \sim \xi$ as $\xi \rightarrow \infty$ where $\xi = x/\sqrt{\tau}$. By differentiating this equation, it is easily found that

$$v_0(x, \tau, y) = xN\left(\frac{x}{\sqrt{a\tau}}\right) + \sqrt{a\tau}n\left(\frac{x}{\sqrt{a\tau}}\right)$$

In the original variables, we have

$$V(Y_1, Y_2, t) \sim (Y_1 - Y_2 - K)N\left(\frac{Y_1 - Y_2 - K}{K\sqrt{a(T-t)}}\right) + K\sqrt{a(T-t)}n\left(\frac{Y_1 - Y_2 - K}{K\sqrt{a(T-t)}}\right)$$

We now proceed to find v_1 by comparing the terms of $O(\frac{1}{\epsilon})$

$$\frac{\partial v_1}{\partial \tau} = \frac{a}{2} \frac{\partial^2 v_1}{\partial x^2} + 2 \frac{\partial^2 v_0}{\partial x \partial y} \frac{\sigma_1^2 - \sigma_2^2}{K^2} + r \frac{\partial v_0}{\partial x}$$

Using the fact that if $u_\tau = \frac{a}{2}u_{xx} + v$ where v satisfies $v_\tau = \frac{a}{2}v_{xx}$, then $u = \tau v$ is a particular solution, we deduce that

$$v_1(x, \tau; y) = 2 \frac{\tau}{K^2} (\sigma_1^2 - \sigma_2^2) \frac{\partial^2 v_0}{\partial x \partial y} + r\tau \frac{\partial v_0}{\partial x} = 2 \frac{\tau}{K^2} (\sigma_1^2 - \sigma_2^2) \frac{\partial N(\frac{x}{\sqrt{a\tau}})}{\partial y} + r\tau N(\frac{x}{\sqrt{a\tau}}) = r\tau N(\frac{x}{\sqrt{a\tau}})$$

Using the two-term inner expansion, $v_0 + \epsilon v_1$, and replacing in the original values we get

$$V(Y_1, Y_2, t) \sim (Y_1 - Y_2 - K(1 - r(T - t)))N\left(\frac{Y_1 - Y_2 - K}{K\sqrt{a(T - t)}}\right) + K\sqrt{a(T - t)}n\left(\frac{Y_1 - Y_2 - K}{K\sqrt{a(T - t)}}\right)$$

6.3 Arithmetic Brownian Motion

In this section, we will assume that the assets X_1 and X_2 follow the Arithmetic Brownian Motion so that

$$dX_1 = rS_0 dt + \sigma_1 S_0 dW_1$$

$$dX_2 = rS_0 dt + \sigma_2 S_0 dW_2$$

where W_1 and W_2 are standard Brownian Motions with correlation ρ , r is the risk-free interest rate and σ_1 and σ_2 are the volatilities of X_1 and X_2 respectively. Note that r , σ_1 and σ_2 are assumed to be constant. Let P be the transition density function of this process. Let V be the value of the option and thus V satisfies the following Stochastic Differential Equation

$$\frac{\partial V}{\partial t} + \frac{S_0^2}{2} (\sigma_1^2 \frac{\partial^2 V}{\partial X_1^2} + 2\rho\sigma_1\sigma_2 \frac{\partial^2 V}{\partial X_1 \partial X_2} + \sigma_2^2 \frac{\partial^2 V}{\partial X_2^2}) + rS_0 (\frac{\partial V}{\partial X_1} + \frac{\partial V}{\partial X_2}) - rV = 0 \quad (12)$$

Let

$$\zeta = X_1 - X_2$$

and

$$\eta = X_1 + X_2$$

Hence, equation (12) now becomes

$$\frac{\partial V}{\partial t} + \frac{S_0^2}{2} (\sigma_1^2 (\frac{\partial^2 V}{\partial \zeta^2} + 2\frac{\partial^2 V}{\partial \zeta \partial \eta} + \frac{\partial^2 V}{\partial \eta^2}) + 2\rho\sigma_1\sigma_2 (\frac{\partial^2 V}{\partial \eta^2} - \frac{\partial^2 V}{\partial \zeta^2}) + \sigma_2^2 (\frac{\partial^2 V}{\partial \zeta^2} - 2\frac{\partial^2 V}{\partial \zeta \partial \eta} + \frac{\partial^2 V}{\partial \eta^2})) + 2rS_0 \frac{\partial V}{\partial \eta} - rV = 0$$

If we measure the time backwards from expiry, writing $t = T - t'$, and assume small time scaling such that

$$t' = \epsilon^2 \tau$$

where $0 < \epsilon \ll 1$, we get

$$\frac{1}{\epsilon^2} \frac{\partial V}{\partial t} = \frac{S_0^2}{2} (\sigma_1^2 (\frac{\partial^2 V}{\partial \zeta^2} + 2 \frac{\partial^2 V}{\partial \zeta \partial \eta} + \frac{\partial^2 V}{\partial \eta^2}) + 2\rho\sigma_1\sigma_2 (\frac{\partial^2 V}{\partial \eta^2} - \frac{\partial^2 V}{\partial \zeta^2}) + \sigma_2^2 (\frac{\partial^2 V}{\partial \zeta^2} - 2 \frac{\partial^2 V}{\partial \zeta \partial \eta} + \frac{\partial^2 V}{\partial \eta^2})) + 2rS_0 \frac{\partial V}{\partial \eta} - rV$$

By expanding in ϵ writing

$$V(\zeta, \eta, \tau; \epsilon) \sim V_0(\zeta, \eta, \tau) + \epsilon V(\zeta, \eta, \tau) + \dots$$

Comparing the terms of the same order in ϵ we see that

$$\frac{\partial V_0}{\partial \tau} = 0$$

with

$$V_0(\zeta, \eta, 0) = \max(\zeta - K, 0)$$

Hence,

$$V_0(\zeta, \eta, \tau) = \max(\zeta - K, 0)$$

To see the behavior around the strike, we rescale near K , introducing a new inner variable

$$\zeta = K(1 + \epsilon x)$$

and

$$\eta = Ky.$$

If we rescale at the same time

$$V(\zeta, \eta) = \epsilon K v(x, y),$$

the equation becomes

$$\frac{1}{\epsilon^2} \frac{\partial v}{\partial \tau} = \frac{S_0^2}{2} \left(\frac{1}{\epsilon^2 K^2} (\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2) \frac{\partial^2 v}{\partial x^2} + \frac{2}{\epsilon K^2} \frac{\partial^2 v}{\partial x \partial y} (\sigma_1^2 - \sigma_2^2) + \frac{1}{K^2} \frac{\partial^2 v}{\partial y^2} (\sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2) \right) + 2 \frac{rS_0}{K} \frac{\partial v}{\partial y} - rv.$$

with $v(x, 0) = \max(x, 0)$. We now expand

$$v(x, \tau; \epsilon; y) \sim v_0(x, \tau) + \epsilon v_1(x, \tau) + \dots$$

Collecting together the terms of $O(\frac{1}{\epsilon^2})$, we get

$$\frac{\partial v_0}{\partial \tau} = \frac{S_0^2}{2K^2} (\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2) \frac{\partial^2 v_0}{\partial x^2}$$

with $v_0(x, 0) = \max(x, 0)$. Let $a = \frac{S_0^2}{K^2} (\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2)$ (Note that a is independent of x). Thus the above equation becomes

$$\frac{\partial v_0}{\partial \tau} = \frac{1}{2} a \frac{\partial^2 v_0}{\partial x^2}$$

As the conditions at $x = \pm\infty$ are

$$v(x, \tau) \sim x \quad \text{as } x \rightarrow +\infty$$

and

$$v(x, \tau) \rightarrow 0 \quad \text{as } x \rightarrow -\infty,$$

this problem has a similarity solution of the form

$$v_0(x, \tau) = \sqrt{\tau} f\left(\frac{x}{\sqrt{\tau}}\right)$$

Hence,

$$af'' + \xi f' - f = 0$$

with $f \rightarrow 0$ as $\xi \rightarrow -\infty$ and $f \sim \xi$ as $\xi \rightarrow \infty$ where $\xi = x/\sqrt{\tau}$. By differentiating this equation, it is easily found that

$$v_0(x, \tau, y) = xN\left(\frac{x}{\sqrt{a\tau}}\right) + \sqrt{a\tau}n\left(\frac{x}{\sqrt{a\tau}}\right)$$

In the original variables, we have

$$V(X_1, X_2, t) \sim (X_1 - X_2 - K)N\left(\frac{X_1 - X_2 - K}{K\sqrt{a(T-t)}}\right) + K\sqrt{a(T-t)}n\left(\frac{X_1 - X_2 - K}{K\sqrt{a(T-t)}}\right)$$

We now proceed to find v_1 by comparing the terms of $O(\frac{1}{\epsilon})$

$$\frac{\partial v_1}{\partial \tau} = \frac{a}{2} \frac{\partial^2 v_1}{\partial x^2} + 2 \frac{\partial^2 v_0}{\partial x \partial y} \frac{\sigma_1^2 - \sigma_2^2}{K^2}$$

Using the fact that if $u_\tau = \frac{a}{2}u_{xx} + v$ where v satisfies $v_\tau = \frac{a}{2}v_{xx}$, then $u = \tau v$ is a particular solution, we deduce that

$$v_1(x, \tau; y) = 2 \frac{\tau}{K^2} (\sigma_1^2 - \sigma_2^2) \frac{\partial^2 v_0}{\partial x \partial y} = 2 \frac{\tau}{K^2} (\sigma_1^2 - \sigma_2^2) \frac{\partial N(\frac{x}{\sqrt{a\tau}})}{\partial y} = 0$$

Using the two-term inner expansion, $v_0 + \epsilon v_1$, and replacing in the original values we get

$$V(X_1, X_2, t) \sim (X_1 - X_2 - K)N\left(\frac{X_1 - X_2 - K}{K\sqrt{a(T-t)}}\right) + K\sqrt{a(T-t)}n\left(\frac{Y_1 - Y_2 - K}{K\sqrt{a(T-t)}}\right)$$

Chapter 7

Conclusion

In this paper, we found approximations to the value of spread options using matched asymptotic expansions. We were motivated by the importance of spread options and their wide use in the financial markets. Despite the fact that many mathematical tools and approximations for these options have been proposed in the literature, there are still many challenging problems that financial mathematicians face when it comes to pricing and hedging spread options. We have tried to find suitable approximations in this paper using both one-factor and two-factor models. In the one-factor model we assumed the spread followed one of the following: Geometric Brownian Motion, Ornstein-Uhlenbeck and Arithmetic Brownian Motion. In the two-factor model, on the other hand, we assumed that the assets followed one of these processes. We only focused on two-term inner expansions, but this work can be easily extended to obtain more accurate results.

The goal of this paper was to show how one can use matched asymptotic expansions in order to approximate the value of spread options. We shed light on one of the challenges in the field of mathematical finance, pricing and hedging spread options. We are aware that there is much more to be done in this field and hopefully with the increasing computing capabilities and the availability of more data, it will be easier to find more suitable approximations to these options.

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