

Portfolio Selection in Incomplete Markets with Utility Maximisation



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Abstract

The problem of maximizing the expected utility is well understood in the context of a complete financial market. This dissertation studies the same problem in an arbitrage-free yet incomplete market. Jin and Zhou have characterized the set of the terminal wealths that can be replicated by admissible portfolios. The problem is then transformed into a static optimization problem. It is proved that the terminal wealth is attainable for all utility functions when the market parameters are deterministic. The optimal portfolio is obtained explicitly when the utility function is logarithmic even if the market parameters follow stochastic processes. However we do not succeed in extending this result to the power utility function.

Key words. Utility function, arbitrage-free, incomplete market, Lagrange multiplier, replication, attainable set, market parameters, logarithmic utility, power utility.

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Chapter 1

Introduction

The work of Markowitz on single-period mean-variance portfolio selection [11] laid down the foundation of the financial portfolio theory. Subsequently, in continuous-time portfolio selection models in literature, the agent uses the mean-variance approach but also seeks to maximize expected utility. This utility approach was theoretically justified by von Neumann and Morgenstern [12].

Two main methods are used to solve the mean-variance and the expected utility problems. The first one, attributed to Merton, is called the “forward approach” because the portfolio is obtained dynamically. Duffie and Jackson [2] treated this method, using the Bellman equation in a finite-horizon, incomplete, Markov model. The second approach, used in this dissertation, is called the “backward approach”. An optimal terminal wealth is first identified by solving a static optimization problem and then an efficient portfolio is obtained by replicating the optimal terminal wealth. This method, built around the ideas of equivalent martingale measure, began with Harrison and Kreps [4] and was further developed by Harrison and Pliska [5]. In all these papers, it assumed that the dimension of the underlying Brownian motion is the same as the number of the stocks. This induces a complete market.

A first step toward a martingale analysis of incomplete markets was taken by Pagès [13], who considered a Brownian model in which the number of stocks was strictly less than the dimension of the driving Brownian motion. However, the coefficients of the bond and stock prices in this model were allowed to depend on the underlying Brownian motion model only through the bond and stock prices themselves. Thus, the vector of bond and stock prices formed a Markov process. This specialization created an essentially complete market, and thus it avoided the more interesting case of a market with genuinely unhedgeable risk. A more substantial step was taken by He and Pearson [6] in a discrete-time, finite probability space model, where the authors proposed finding the optimal intermediate consumption and terminal

wealth corresponding to each of the equivalent martingale measures, and then searching over those policies to find a pair yielding the minimum expected total utility.

Karatzas, Lehoczky, Shreve and Xu [9] also studied utility maximization in an incomplete market. For them the incompleteness arises when the number of stocks is strictly smaller than the dimension of the underlying Brownian motion. They showed there is a way to complete the market by introducing additional “fictitious” stocks so that the optimal portfolio for the thus completed market coincides with the optimal portfolio for the original incomplete market.

The main difficulty of the study of utility maximization in incomplete markets is the characterization of the attainable terminal wealth set. Jin and Zhou [7] have succeeded in characterizing it in a paper where they study the mean-variance problem. They also obtained explicit forms of the optimal portfolios when the market parameters are deterministic.

In this paper we use this result to study the utility maximization. The remainder of the paper is organised as follows: In chapter 2 we present the market. In chapter 3 we introduce technical results on the pricing kernels which are vital for our analysis. Chapter 4 presents the utility maximization problem. In chapter 5 we present the result obtained by Jin and Zhou about the characterization of the attainable terminal wealth set. In chapter 6 we prove that the terminal wealth is attainable when the market parameters are deterministic. Chapter 7 is devoted to the logarithmic utility function. We find an explicit form of the optimal portfolio even when the market parameters follow stochastic processes and compare our result to the one obtained by Karatzas, Lehoczky, Shreve and Xu in [9]. Chapter 8 is devoted to the power utility function. Finally chapter 9 concludes the paper.

Chapter 2

Market

2.1 Financial Market

In this paper, T is a fixed terminal time, $(\Omega, F, P, \{F_t\}_{t \geq 0})$ is a fixed complete probability space where $W(t) = (W^1(t), \dots, W^n(t))$ is a standard n -dimensional Brownian motion with $W(0) = 0$ and $F_t = \sigma\{W(s) : 0 \leq s \leq t\}$ is a filtration adapted to $W(t)$.

We denote by $L_F^2(0, T, \mathbf{R}^d)$ the set of all \mathbf{R}^d -valued, F_t measurable stochastic processes $f(\cdot) = \{f(t) : 0 \leq t \leq T\}$ such that

$$\|f(\cdot)\|_{L_F^2(0, T, \mathbf{R}^d)} = \left(E \int_0^T |f(t)|^2 dt \right)^{\frac{1}{2}} < +\infty,$$

by $L_F^\infty(0, T, \mathbf{R}^d)$ the set of all \mathbf{R}^d -valued, F_t measurable stochastic processes $f(\cdot)$ such that

$$\|f(\cdot)\|_{L_F^\infty(0, T, \mathbf{R}^d)} = \text{ess sup}_{(t, \omega) \in [0, T] \times \Omega} |f(t, \omega)| < +\infty,$$

and by $L_{F_T}^2(\Omega, \mathbf{R}^d)$ the set of all \mathbf{R}^d -valued, F_T measurable random variables η such that

$$\|\eta\|_{L_{F_T}^2(\Omega, \mathbf{R}^d)} = \left(E |\eta|^2 \right)^{\frac{1}{2}} < +\infty.$$

Throughout this paper, a (t, ω) -null set is a null-set with respect to the product of the Lebesgue measure on $[0, T]$ and P on Ω .

Notation. We use the following additional notation:

M' : the transpose of any vector or matrix M

$$|M| := \sqrt{\sum_{i,j} m_{i,j}^2} \text{ for any matrix or vector } M = (m_{i,j})$$

The market we consider has a bank account whose price process $S_0(t)$ follows the differential equation:

$$\begin{cases} dS_0(t) = r(t)S_0(t)dt, t \in [0, T], \\ S_0(0) = s_0 > 0, \end{cases} \quad (1)$$

where the interest rate process $r(\cdot) \in L_F^\infty(0, T, \mathbf{R})$.

The m other assets are stocks whose price processes $S_i(t), i = 1, \dots, m$, satisfy the differential equation:

$$\begin{cases} dS_i(t) = S_i(t)[\mu_i(t)dt + \sum_{j=1}^n \sigma_{i,j}(t)dW^j(t)], t \in [0, T], \\ S_i(0) = s_i > 0, \end{cases} \quad (2)$$

where $\mu_i(\cdot) \in L_F^\infty(0, T, \mathbf{R})$ and $\sigma_{ji}(\cdot) \in L_F^\infty(0, T, \mathbf{R})$ are the processes of appreciation and volatility rates respectively.

The total wealth of an agent at time t is denoted by $x(t)$ and the amount invested in the stock i is denoted $\pi_i(t)$.

$\pi(t) = (\pi_1(t), \dots, \pi_m(t))'$ is called a (self-financing) portfolio.

We denote $N_i(t)$ the number of shares of the i -th asset, $i = 0..m$

Then

$$x(t) = \sum_{i=0}^m \pi_i(t) = \sum_{i=0}^m N_i(t)S_i(t). \quad (3)$$

If the agent's strategy is self-financing then

$$\begin{aligned}
dx(t) &= \sum_{i=0}^m N_i(t) dS_i(t), \\
dx(t) &= N_0(t)r(t)S_0(t)dt + \sum_{i=1}^m N_i(t)S_i(t)[\mu_i(t) + \sum_{j=1}^n \sigma_{i,j}(t)dW^j(t)] \quad , \\
dx(t) &= r(t)(x(t) - \sum_{i=1}^m \pi_i(t))dt + \sum_{i=1}^m \mu_i(t)\pi_i(t) + \sum_{i=1}^m \sum_{j=1}^n \pi_i(t)\sigma_{i,j}(t)dW^j(t), \\
dx(t) &= [r(t)x(t) + \sum_{i=1}^m (\mu_i(t) - r(t))\pi_i(t)]dt + \sum_{i=1}^m \sum_{j=1}^n \sigma_{i,j}(t)\pi_i(t)dW^j(t). \quad (4)
\end{aligned}$$

We denote $\sigma(t) = (\sigma_{i,j}(t))_{m \times n}$ the volatility process

and $B(t) = (b_1(t), \dots, b_m(t))' = (\mu_1(t) - r(t), \dots, \mu_m(t) - r(t))'$ the equity premium process.

Then the wealth process $x(t)$ satisfies

$$dx(t) = [r(t)x(t) + B(t)' \pi(t)]dt + \pi(t)' \sigma(t)dW(t), \quad (5)$$

with $x(0) = x_0$, the initial wealth of the agent.

2.2 No Arbitrage and Market Completeness

Basic Assumption (A): There exists $\theta \in L_F^\infty(0, T, \mathbf{R}^n)$ such that $\sigma(t)\theta(t) = B(t)$, a.s., a.e.t $\in [0, T]$.

Remark 2.1 If $\sigma(t)' \sigma(t)$ is uniformly positive definite, the above assumption is satisfied and the market is complete. In this case there exists a unique such θ . In general, however, the process θ , if it exists, may not be unique.

Definition 2.1 A portfolio $\pi(\cdot)$ is said to be admissible if it is self-financing, admissible, F_T -adapted and $\sigma(\cdot)' \pi(\cdot) \in L_F^2(0, T, \mathbf{R}^n)$. The set of all admissible portfolios is denoted by Π . A pair $(x(\cdot), \pi(\cdot))$ is called an admissible wealth-portfolio pair if $(x(\cdot), \pi(\cdot))$ satisfies (5).

For any $\theta \in L_F^\infty(0, T, \mathbf{R}^n)$ define

$$H_\theta(t) = \exp \left\{ - \int_0^t \left[r(s) + \frac{1}{2} |\theta(s)|^2 \right] ds - \int_0^t \theta(s)' dW(s) \right\}. \quad (6)$$

Define

$$\Theta = \left\{ \theta \in L_F^\infty(0, T, \mathbf{R}^n) : \sigma(t)\theta(t) = B(t), a.s., a.e.t \in [0, T] \right\}. \quad (7)$$

The following lemma has been proved in [3].

Lemma 2.1 Let $\theta \in \Theta$, and $(x(\cdot), u(\cdot))$ be an admissible wealth-portfolio pair. Then

$$x(t) = H_\theta(t)^{-1} E(x(T) H_\theta(T) | F_t). \quad (8)$$

Definition 2.2 An admissible portfolio $\pi(\cdot)$ is called an arbitrage opportunity on $[0, T]$ if there exists an initial $x_0 \leq 0$ and a time $t \in [0, T]$, so that the corresponding wealth process $x(\cdot)$ satisfies

$$P(x(t) \geq 0) = 1 \text{ and } P(x(t) > 0) > 0.$$

Moreover, a market is called arbitrage free on $[0, T]$ if there exists no arbitrage opportunity on $[0, T]$.

Proposition 2.1 The market is arbitrage free under Assumption (A).

Proof. Let $\theta \in \Theta$. For the wealth process $x(\cdot)$ under an admissible portfolio $\pi(\cdot)$, it follows from Lemma (2.1) that $x(0) = E[x(T)H_\theta(T)]$. Since $H_\theta(T) > 0$, we must have $x(0) > 0$, a.s., if $x(T) \geq 0$ a.s. and $P(x(T) > 0) > 0$

Definition 2.3 A contingent claim $\xi \in L_{F_T}^2(\Omega, \mathbf{R})$ is said to be replicable if there exists an initial wealth x and an admissible wealth-portfolio pair $(x(\cdot), \pi(\cdot))$ satisfying (5) with $x(T) = \xi$. The market is called complete if any contingent claim $\xi \in L_{F_T}^2(\Omega, \mathbf{R})$ is replicable.

Proposition 2.2 Under Assumption (A), the market is complete if and only if $\text{rank}(\sigma(t)) = m$, a.s., a.e. $t \in [0, T]$.

Proof. We consider the backward stochastic differential equation with a given $\xi \in L_{F_T}^2(\Omega, \mathbf{R})$:

$$dx(t) = [r(t)x(t) + \theta(t)' z(t)]dt + z(t)' dW(t), \quad x(T) = \xi \quad (9)$$

This equation admits a unique solution pair $(x(\cdot), z(\cdot)) \in L_F^2(0, T, \mathbf{R}) \times L_F^2(0, T, \mathbf{R}^m)$. If $\text{rank}(\sigma(t)) = m$, a.s., a.e. $t \in [0, T]$, then there exists $\pi(\cdot)$ such that $\sigma(t)' \pi(t) = z(t)$, a.s., a.e. $t \in [0, T]$. By lemma A.3, we may assume that the process $\pi(\cdot)$ is F_T -measurable. If we substitute $z(t)$ by $\sigma(t)' \pi(t)$ in (9) we conclude that $(x(\cdot), \pi(\cdot))$ is an admissible wealth-portfolio pair with $x(T) = \xi$; hence ξ is replicable.

Conversely, we assume that the market is complete. Then for any $z \in \mathbf{R}^m$, let $y(\cdot)$ solves the following stochastic differential equation

$$dy(t) = [r(t)y(t) + \theta(t)' z]dt + z' dW(t), \quad y(0) = 0.$$

Since $y(T)$ is replicable, there exists $(x(\cdot), \pi(\cdot)) \in L_F^2(0, T, \mathbf{R}) \times L_F^2(0, T, \mathbf{R}^n)$ with $\sigma(\cdot)' \pi(\cdot) \in L_F^2(0, T, \mathbf{R}^m)$ so that

$$dx(t) = [r(t)x(t) + \theta(t)' \sigma(t)' \pi(t)]dt + \pi(t)' \sigma(t)dW(t), \quad x(T) = y(T).$$

If we compare the two preceding equations and use the uniqueness of the Backward Stochastic Differential Equation solution we can conclude that $\sigma(t)' \pi(t) = z$ and this yields $\text{rank}(\sigma(t)) = m$ as $z \in R^m$ is arbitrary.

Remark 2.2 In general, the number of stocks, m , is different from the dimension of the underlying Brownian motion, n , and $\text{rank}(\sigma(t)) = m$ may not hold. Hence, the market is in general incomplete in our setup.

Chapter 3

Pricing Kernels

In this section, we introduce a very important process for our analysis

We define the following process

$$\theta^*(t) = \arg \min_{\theta \in \{\theta \in \mathbf{R}^n : \sigma(t)\theta = B(t)\}} |\theta|^2. \quad (10)$$

Lemma 3.1 *We have the following result:*

$$\theta^*(t)' \theta(t) = |\theta^*(t)|^2, \text{ a.s., a.e. } t \in [0, T].$$

Proof. Pointwisely in (t, ω) , $\theta^*(t)$ minimizes $|\theta|^2$ subject to $\sigma(t)\theta = B(t)$. Hence by the Lagrange approach there is $u \in \mathbf{R}^m$ so that $\theta^*(t)$ minimizes $|\theta|^2 - 2[\sigma(t)\theta - B(t)]'u$ over $\theta \in \mathbf{R}^n$. The zero-derivative condition then gives $\sigma(t)'u(t) = \theta^*(t)$.

This implies that $\{u \in \mathbf{R}^m : \sigma(t)'u = \theta^*(t)\} \neq \emptyset$.

If we define

$$u(t) = \arg \min_{u \in \{u \in \mathbf{R}^m : \sigma(t)'u = \theta^*(t)\}} |u|^2,$$

by virtue of Lemma A.3, we find $\sigma(t)'u(t) = \theta^*(t)$, a.s., a.e. $t \in [0, T]$.

Then for any $\theta \in \Theta$, we have $\theta^*(t)' \theta(t) = u(t)' \sigma(t)\theta(t) = u(t)' B(t) = u(t)' \sigma(t)\theta^*(t) = |\theta^*(t)|^2$.

Remark 3.1 When $\sigma(t)'\sigma(t)$ is uniformly positive definite, the market is complete and the process θ^* is the only θ that satisfies $\sigma(t)\theta(t) = B(t)$

Chapter 4

Portfolio Selection Problem with Utility Maximization

4.1 Utility Functions

In addition to our market described previously, we have a utility function $U(x)$ modelling the utility of an agent's wealth at the terminal time T .

We assume that $U : R \rightarrow R \cup \{-\infty\}$ is increasing on R , continuous on $\{U > -\infty\}$, and satisfies Imada's conditions:

$$\begin{cases} U(0) = 0, \\ U'(0) = \lim_{x \downarrow 0} U'(x) = \infty, \\ U'(\infty) = \lim_{x \rightarrow \infty} U'(x) = 0. \end{cases}$$

Some frequently used utility-functions are:

- The logarithmic utility $U(x) = \log x$
- The power utility $U(x) = \frac{x^\gamma}{\gamma}$
- The exponential utility $U(x) = -\exp(-\alpha x)$

4.2 Portfolio Selection Problem

Fix an initial wealth x_0 . The portfolio selection problem with utility maximization is formulated as

$$\begin{cases} \max E[U(X)], \\ \text{subject to} \begin{cases} (x(\cdot), \pi, \cdot) \in L_{F_t}^2(0, T, \mathbf{R}) \times \Pi, \\ (x(\cdot), \pi, \cdot) \text{ satisfies equation (5) with } x(0) = x_0. \end{cases} \end{cases} \quad (11)$$

We define the following attainable terminal wealth set:

$$A = \{ X \in L_{F_t}^2(\Omega, \mathbf{R}) : \text{there exists } x \in \mathbf{R} \text{ and } \pi(\cdot) \in \Pi \text{ such that } (x(\cdot), \pi(\cdot)) \text{ satisfies (5) with } x(0) = x_0, x(T) = X \}. \quad (12)$$

If we use the pricing formula $x(t) = H_\theta(t)^{-1} E(x(T)H_\theta(T)|F_t)$, we find in particular that there exists $\theta_0 \in \Theta$ such that $E[X(T)H_{\theta_0}(T)] = x_0$. This represents the budget constraint.

Then to solve problem (11), we first solve the static optimization problem:

$$\begin{cases} \max E[U(X)], \\ \text{subject to} \begin{cases} \text{there exists } \theta_0 \in \Theta \text{ such that } E[XH_{\theta_0}(T)] = x_0, \\ X \in A. \end{cases} \end{cases} \quad (13)$$

Remark 4.1 Note that θ_0 is the one which maximizes $E[XH_{\theta_0}(T)]$ among $\theta \in \Theta$.

The problem is to locate the optimal attainable wealth $X^* \in A$. Once this is solved, an optimal portfolio for (11) can be obtained by replicating X^* (which is possible by the definition of A along with the second constraint in (13)).

Compared with the case of a complete market [1], the main difficulty in the present incomplete market situation is to characterize the attainable set A .

Theorem 4.1 *It is straightforward by the definition of A that if $(x^*(\cdot), \pi^*(\cdot))$ is optimal for (11), then $x^*(T)$ is optimal for (13). And conversely, if $X^* \in A$ is optimal for (13), then any wealth-portfolio pair $(x^*(\cdot), \pi^*(\cdot))$ satisfying (13) with $(x^*(\cdot), \pi^*(\cdot)) \in L_{F_t}^2(0, T, \mathbf{R}) \times \Pi$ and $x^*(T) = X^*$ is optimal for (11).*

Theorem 4.2 *If problem (13) admits a solution X^* , then there exists a scalar λ such that X^* is also the optimal solution for the following problem:*

$$\begin{cases} \max E[U(X) - \lambda H_{\theta_0}(t)X + x_0], \\ \text{subject to } X \in A. \end{cases} \quad (14)$$

Conversely, if there is a scalar λ such that the optimal solution X^ of (14) satisfies $E[X^* H_{\theta_0}(T)] = x_0$, then X^* must be an optimal solution of (13).*

Proof. It is easy to see that A is a convex set. Hence the theorem is proved by applying Lagrange multiplier approach (the same fashion as [1]).

But the X^* which maximizes $E[U(X) - \lambda H_{\theta_0}(t)X + x_0]$ is:

$$X^* = (U')^{-1}(\lambda H_{\theta_0}(T)).$$

Then if we have a $\theta_0 \in \Theta$ such that $(U')^{-1}(\lambda H_{\theta_0}(T)) \in A$, then X^* is attainable and $X^* = (U')^{-1}(\lambda H_{\theta_0}(T))$.

After we have found the terminal wealth, we have just to use the replication technique in order to find the optimal portfolio and the wealth process.

Chapter 5

Characterization of the Attainable Terminal Wealth

The next thing to do is to find a characterization of the attainable set A .

Jin and Zhou have found a characterization of the attainable terminal set A in [] when the portfolios are unconstrained. We give below the result and the proof of it.

Theorem 5.1 Given $X \in L_{F_T}^2(\Omega, R)$. The following assertions are equivalent:

- i. $X \in A$
- ii. $E[XH_\theta(T)]$ is independent of $\theta \in \Theta$
- iii. $E[XH_\theta(T)]$ is independent of $\theta \in \Theta_1$ where

$$\Theta_1 = \left\{ \theta \in \Theta : \left\| \theta - \theta^* \right\|_{L_{F_T}^\infty(0, T, \mathbf{R}^n)} \leq 1 \right\} \quad (15)$$

Proof. If $X \in A$, then there is $x \in R$ and a portfolio $\pi(\cdot) \in \Pi$ such that

$$\begin{cases} dx(t) = [r(t)x(t) + B(t)' \pi(t)]dt + \pi(t)' \sigma(t)dW(t), \\ x(0) = x, \quad x(T) = X. \end{cases}$$

Now, for any $\theta \in \Theta$,

$$\begin{aligned} dx(t) &= [r(t)x(t) + B(t)' \pi(t)]dt + \pi(t)' \sigma(t)dW(t) \\ &= [r(t)x(t) + \theta(t)' \sigma(t)' \pi(t)]dt + \pi(t)' \sigma(t)dW(t). \end{aligned}$$

Then $x(0) = x_0 = E[x(T)H_\theta(T)] = E[XH_\theta(T)]$.

Therefore $E[XH_\theta(T)]$ is independent of the choice of $\theta \in \Theta$. This proves that (i) implies (ii).

The implication from (ii) to (iii) is trivial. That's why we have just to prove that (iii) implies (i). We assume that $E[XH_\theta(T)]$ is independent of $\theta \in \Theta_1$.

Then the equation

$$\begin{cases} dX(t) = [r(t)X(t) + \theta(t)' Z(t)]dt + Z(t)' dW(t) \\ x(T) = X \end{cases} \quad (16)$$

admits a unique solution pair $(X_\theta(\cdot), Z_\theta(\cdot))$ with $X_\theta(0) = E[XH_\theta(T)]$. So by the assumption $X_\theta(0)$, $\theta \in \Theta_1$, are all the same and we denote it x_0 .

Let $(X_{\theta^*}(\cdot), Z_{\theta^*}(\cdot))$ solves the equation (16) with $\theta = \theta^*$. We are to prove that there exists a portfolio $\pi_0(\cdot) \in \Pi$ such that

$$Z_{\theta^*}(t) = \sigma(t)' \pi_0(t) \text{ a.s., a.e.t} \in [0, T]. \quad (17)$$

Indeed, define

$$\pi_0(t) = \arg \min_{\pi \in \arg \min_{\pi \in \mathbb{R}^m} |\sigma(t)' \pi - Z_{\theta^*}(t)|^2} |\pi|^2.$$

By the Frank-Wolfe theorem (Lemma A.1), we notice that the set $\arg \min_{\pi \in \mathbb{R}^m} |\sigma(t)' \pi - Z_{\theta^*}(t)|^2$ is nonempty. Moreover $\pi \in \arg \min_{\pi \in \mathbb{R}^m} |\sigma(t)' \pi - Z_{\theta^*}(t)|^2$ if and only if $\sigma(t)\sigma(t)' \pi - \sigma(t)Z_{\theta^*}(t) = 0$. Thus by the Frank-Wolfe theorem $\pi_0(t)$ is well-defined. Furthermore, we can apply Lemma A.3 to conclude that $\pi_0(\cdot)$ is a F_T measurable stochastic process.

We set $\bar{\rho}(t) = \sigma(t)' \pi_0(t) - Z_{\theta^*}(t)$ and

$$\rho(t) = \begin{cases} \bar{\rho}(t)/|\bar{\rho}(t)| & \text{if } \bar{\rho}(t) \neq 0, \\ 0 & \text{if } \bar{\rho}(t) = 0. \end{cases}$$

Then $\rho(\cdot) \in L_F^\infty(0, T, \mathbf{R}^n)$. Moreover, $\sigma(t)\bar{\rho}(t) = \sigma(t)\sigma(t)' \pi_0(t) - \sigma(t)Z_{\theta^*}(T) = 0$ owing to the fact that $\pi_0(t)$ minimizes $|\sigma(t)' \pi - Z_{\theta^*}(T)|^2$. This implies that $\sigma(t)\rho(t) = 0$ and hence

$$\theta^* + \rho \in \Theta_1. \quad (18)$$

On the other hand, $Z_{\theta^*}(t)' \bar{\rho}(t) = [\pi_0(t)' \sigma(t)\bar{\rho}(t) - \bar{\rho}(t)' \bar{\rho}(t)] = -|\bar{\rho}(t)|^2$; thus $Z_{\theta^*}(t)' \rho(t) = -|\bar{\rho}(t)|^2$.

We define $\hat{X}(\cdot)$ to be the solution of the following SDE:

$$\begin{cases} d\hat{X}(t) = [r(t)\hat{X}(t) + (\theta^*(t) + \rho(t))' Z_{\theta^*}(t)]dt + Z_{\theta^*}(t)' dW(t), \\ \hat{X}(0) = x_0. \end{cases}$$

$$\text{Then } E[\hat{X}(T)H_{\theta^*+\rho}(T)] = \hat{X}(0) = x_0 = E[X_{\theta^*}(T)H_{\theta^*+\rho}(T)], \quad (19)$$

where the last equality stems from (18). However,

$$d[\hat{X}(t) - X_{\theta^*}(t)] = r(t)[\hat{X}(t) - X_{\theta^*}(t)]dt + Z_{\theta^*}(t)' \rho(t)dt \text{ with } \hat{X}(0) - X_{\theta^*}(0) = 0;$$

$$\text{hence } \hat{X}(T) - X_{\theta^*}(T) = \int_0^T e^{\int_t^T r(s)ds} Z_{\theta^*}(t)' \rho(t)dt = -\int_0^T e^{\int_t^T r(s)ds} |\bar{\rho}(t)|^2 dt.$$

If we compare that with (19) we conclude that $\bar{\rho}(t) = 0$, a.s., a.e.t $\in [0, T]$ which leads to (17). Since $\sigma(\cdot)\pi_0(\cdot) = Z_{\theta^*}(\cdot) \in L_F^2(0, T, \mathbf{R}^n)$, it follows that $\pi_0(\cdot) \in \Pi$. Now the BSDE (16) that $(X_{\theta^*}(\cdot), Z_{\theta^*}(\cdot))$ satisfies can be rewritten as

$$\begin{cases} dX_{\theta^*}(t) = [r(t)X_{\theta^*}(t) + B(t)' \pi_0(t)]dt + \pi_0(t)' \sigma(t)dW(t), \\ X_{\theta^*}(T) = X, \end{cases}$$

which means that X is attained by the portfolio $\pi_0(\cdot)$.

Therefore if we use this result in our setup, we find that the two following assumptions are equivalent

(i) $X = (U')^{-1}(\lambda H_{\theta_0}(T)) \in A$, for $\theta_0 \in \Theta$

(ii) $E[(U')^{-1}(\lambda H_{\theta_0}(T))H_{\theta}(T)]$ is independent of $\theta \in \Theta$, for $\theta_0 \in \Theta$

Remark 5.1 θ_0 is the one among $\theta \in \Theta$ which maximizes $E[(U')^{-1}(\lambda H_{\theta_0}(T))H_{\theta}(T)]$ for any $\theta \in \Theta$.

Chapter 6

Deterministic case for all Utility Functions

We first prove that if $r(\cdot)$, $\mu(\cdot)$ and $\sigma(\cdot)$ are deterministic processes then the terminal wealth is attainable. Indeed in this case $\theta^*(\cdot)$ is also deterministic. We will prove that $(U')^{-1}(\lambda H_{\theta^*}(T))$ is attainable.

We call P the current martingale measure.

Set an increasing, continuous utility function $U(x)$ which satisfies Imada's conditions.

Set $\theta \in \Theta$. Then

$$\begin{aligned} & E_P[(U')^{-1}(\lambda H_{\theta^*}(T))H_{\theta}(T)] \\ &= E_P \left[(U')^{-1} \left(\lambda \exp \left\{ - \int_0^T \left(r(s) + \frac{|\theta^*(s)|^2}{2} \right) ds - \int_0^T \theta^*(s) dW(s) \right\} \right) \right. \\ & \quad \left. \cdot \exp \left\{ - \int_0^T \left(r(s) ds + \frac{|\theta(s)|^2}{2} \right) ds - \int_0^T \theta'(s) dW(s) \right\} \right] \end{aligned}$$

Define $\tilde{W}(t) = W(t) + \int_0^t \theta(s) ds$ and define a new martingale measure \tilde{P} such that

$$\frac{d\tilde{P}}{dP} = Z(T) \text{ where } Z(t) = \exp \left\{ - \int_0^t \theta(u) dW(u) - \frac{1}{2} \int_0^t \theta(u)^2 du \right\}$$

Then by Girsanov's theorem, \tilde{W} is a Brownian motion under \tilde{P} . Then

$$\begin{aligned}
& E_P[(U')^{-1}(\lambda H_{\theta^*}(T))H_{\theta}(T)] \\
&= E_{\tilde{P}} \left[(U')^{-1} \left(\lambda \exp \left\{ -\int_0^T r(s)ds + \frac{|\theta^*(s)|^2}{2} ds - \int_0^T \theta^*(s)(\theta(s)ds + dW(s)) \right\} \exp \left\{ \int_0^T \theta^*(s)\theta(s)ds \right\} \right) \right. \\
&\quad \left. \cdot \exp \left\{ -\int_0^T r(s)ds \right\} \right] \\
&= E_{\tilde{P}} \left[(U')^{-1} \left(\lambda \exp \left\{ -\int_0^T r(s)ds + \frac{|\theta^*(s)|^2}{2} ds - \int_0^T \theta^*(s)d\tilde{W}(s) \right\} \exp \left\{ \int_0^T \theta^*(s)^2 \right\} \right) \right. \\
&\quad \left. \cdot \exp \left\{ -\int_0^T r(s)ds \right\} \right]
\end{aligned}$$

This expression is independent of $\theta \in \Theta$.

Therefore we can conclude, using the characterization proved before, that $(U')^{-1}(\lambda H_{\theta^*}(T)) \in A$.

We find λ using the budget constraint

$$x_0 = E[(U')^{-1}(\lambda H_{\theta^*}(T))H_{\theta^*}(T)].$$

Therefore when the market parameters $r(\cdot)$, $\mu(\cdot)$ and $\sigma(\cdot)$, are all deterministic processes, we can find the terminal wealth; the optimal portfolio and the wealth process can be found by the replication technique.

Chapter 7

Logarithmic Utility – General Stochastic case

7.1 The terminal wealth is attainable

The focus in this section will be on the logarithmic utility.

Now, the market parameters $r(\cdot)$, $\mu(\cdot)$ and $\sigma(\cdot)$ can follow stochastic processes. Therefore we are here in the most general case.

First, we prove that if θ_0 maximizes $E[(U')^{-1}(\lambda H_{\theta_0}(T))H_{\theta_0}(T)]$ for any $\theta \in \Theta$, θ_0 must be equal to θ^* .

We are working with the current martingale measure P but to simplify the notation, we don't write it.

Suppose $U(x) = \log x$.

Then $U'(x) = \frac{1}{x}$ and $(U')^{-1}(y) = \frac{1}{y}$.

Set θ_0 the maximizer of $E[(U')^{-1}(\lambda H_{\theta_0}(T))H_{\theta_0}(T)]$ for any $\theta \in \Theta$.

$$E[(U')^{-1}(\lambda H_{\theta_0}(T))H_{\theta_0}(T)] = \frac{1}{\lambda} E\left[\frac{H_{\theta_0}(T)}{H_{\theta_0}(T)}\right].$$

Set $\Delta = \theta - \theta_0$ and $\theta_k = \theta_0 + k\Delta$ for k such that $\theta_k \in \Theta$. Note that we have $\sigma\Delta = 0$.

We set

$$f_k = E \left[\frac{H_\theta(T)}{H_{\theta_k}(T)} \right] = E \left[H_\theta(T) \exp \left\{ \int_0^T r(t) dt \right\} \exp \left\{ \int_0^T \theta'_k(t) dW(t) + \frac{1}{2} \int_0^T |\theta_k(t)|^2 dt \right\} \right].$$

Then

$$f_k = E \left[H_\theta(T) \exp \left\{ \int_0^T r(t) dt \right\} \exp \left\{ \int_0^T (\theta_0 + k\Delta) dW(t) + \frac{1}{2} \int_0^T (\theta_0^2 + 2k\Delta\theta_0 + k^2\Delta^2) dt \right\} \right].$$

Because θ_0 is the maximizer of $E[(U')^{-1}(\lambda H_\theta(T))H_\theta(T)]$ for any $\theta \in \Theta$, f_k must be maximal for $\theta_k = \theta_0$, i.e. for $k = 0$.

Then we must have

$$\frac{\partial f_k}{\partial k} \Big|_{k=0} = 0.$$

We have

$$\frac{\partial f_k}{\partial k} = E \left[\frac{H_\theta}{H_{\theta_k}} \left(\int_0^T \Delta dW(t) + \int_0^T (\theta_0\Delta + k\Delta^2) dt \right) \right].$$

Because $\frac{\partial f_k}{\partial k} \Big|_{k=0} = 0$, we have clearly $\theta_0\Delta = 0$.

That means that we have

$$\theta_0 = \theta^*.$$

Therefore we have found a necessary condition for θ_0 .

Thanks to that, our analysis is simplified: if the terminal wealth is attainable, it must be equal to $(U')^{-1}(\lambda H_{\theta^*}(T))$.

We will prove that, even if the market parameters follow stochastic processes, the terminal wealth is attainable when the utility function is logarithmic.

Set $\lambda \in R$ and $\theta \in \Theta$.

$$\begin{aligned} E[(U')^{-1}(\lambda H_{\theta^*}(T))H_{\theta}(T)] &= E\left[\frac{H_{\theta}(T)}{\lambda H_{\theta^*}(T)}\right] \\ &= \frac{1}{\lambda} E\left[\exp\left\{-\int_0^T (\theta(s)' - \theta^*(s)') dW(s) - \frac{1}{2} \int_0^T (\|\theta(s)\|^2 - \|\theta^*(s)\|^2) ds\right\}\right]. \end{aligned}$$

We set a new process

$$\Delta(s) = \theta(s) - \theta_0(s).$$

Then

$$\Delta(s)^2 = \theta(s)^2 + \theta^*(s)^2 - 2\theta(s)' \theta^*(s).$$

Because $\theta(s)' \theta^*(s) = \theta^*(s)^2$, we find that

$$\Delta(s)^2 = \theta(s)^2 + \theta^*(s)^2 - 2\theta^*(s)^2 = \theta(s)^2 - \theta^*(s)^2.$$

Then

$$E[(U')^{-1}(\lambda H_{\theta^*}(T))H_{\theta}(T)] = \frac{1}{\lambda} E\left[\exp\left\{-\int_0^T \Delta(s) dW(s) - \frac{1}{2} \int_0^T \Delta(s)^2 ds\right\}\right].$$

$\left(\int_0^t -\Delta(s) dW(s)\right)$ is a martingale, then

$$E\left[\exp\left\{-\int_0^t \Delta(s) dW(s) - \frac{1}{2} \int_0^t \Delta(s)^2 ds\right\}\right] \quad (20)$$

is the stochastic exponential of the martingale $\left(\int_0^t -\Delta(s) dW(s)\right)$ and is therefore a local martingale.

But $\theta(s)$ and $\theta^*(s)$ $\theta \in L_F^\infty(0, T, R^n)$ then $\int_0^T \|\theta(s)\|^2$ and $\int_0^T \|\theta^*(s)\|^2$ are bounded.

Therefore

$$E \left[\exp \left\{ \frac{1}{2} \int_0^T \Delta(s)^2 ds \right\} \right]$$

is also bounded.

Therefore by Novikov's theorem, the stochastic exponential (20) is also a martingale.

Then

$$E \left[\exp \left\{ - \int_0^T \Delta(s) dW(s) - \frac{1}{2} \int_0^T \Delta(s)^2 ds \right\} \right] = 1$$

$$\text{and } E[(U')^{-1}(\lambda H_{\theta^*}(T)) H_{\theta}(T)] = \frac{1}{\lambda}.$$

This does not depend on $\theta \in \Theta$. This proves that

$$E[(U')^{-1}(\lambda H_{\theta^*}(T)) H_{\theta}(T)] \in A$$

and is attainable.

7.2 Determination of the terminal wealth and the optimal portfolio

The budget constraint gives us:

$$\frac{1}{\lambda} = x_0. \tag{21}$$

Therefore the terminal wealth is

$$X^* = x^*(T) = x_0 H_{\theta^*}(T)^{-1} = x_0 \exp \left\{ \int_0^T \left(r(t) + \frac{|\theta^*(t)|^2}{2} \right) dt + \int_0^T \theta^*(t)' dW(t) \right\} \quad (22)$$

The replicating portfolio is determined by the equation

$$dx^*(t) = \left(r(t)x^*(t) + \theta^*(t)' q^*(t) \right) dt + q^*(t) dW(t) \quad (23)$$

and the final wealth given above.

We conjecture that

$$x^*(t) = x_0 H_{\theta^*}(t)^{-1}.$$

Then

$$\begin{aligned} dx^*(t) &= x_0 \left[H_{\theta^*}(t)^{-1} \left(r(t)dt + \theta^{*'}(t) dW(t) \right) + \frac{1}{2} \times 2 H_{\theta^*}(t)^{-1} |\theta^*(t)|^2 dt \right] \\ &= x^*(t) \left(r(t)dt + \theta^{*'}(t) dW(t) \right) + x^*(t) |\theta^*(t)|^2 dt \\ &= \left(r(t)x^*(t) + \theta^{*'}(t) \theta^*(t) x^*(t) \right) dt + \theta^{*'}(t) x^*(t) dW(t). \end{aligned} \quad (24)$$

That's why we must have, by comparing (22) and (23),

$$q^*(t) = \theta(t) x^*(t)$$

If we assume that $(\sigma(t)\sigma'(t))$ is invertible, then the optimal portfolio is

$$\Pi^*(t) = (\sigma(t)\sigma'(t))^{-1} \sigma(t) \theta^*(t) x^*(t),$$

which, because $\sigma(t)\theta^*(t) = B(t)$, can be rewritten

$$\Pi^*(t) = (\sigma(t)\sigma'(t))^{-1} B(t) x^*(t). \quad (25)$$

And the value function is

$$u(x) = \sup E[U(x)] = \log x_0 + E \int_0^T \left(r(t) + \frac{1}{2} \|\theta^*(t)\|^2 \right) dt ,$$

which is finite ([8]).

7.3 Comparison with the result of Karatzas, Lehoczky, Shreve and Xu

We will compare this result to the result proved by Karatzas, Lehoczky, Shreve and Xu in [9].

First we present their result.

They consider that in addition to the bank account, there exist m stocks and a d -dimensional Brownian motion such that $d \geq m$. Then the number of sources of uncertainty is as large as the number of stocks available for investment.

The stocks and the bank account follow the same stochastic differential equations as those described in the chapter 2. Therefore the financial market is the same as the one described in the section (2.1).

But they suppose a standing assumption in their model:

Standing Assumption 7.1 *The matrix $\sigma(t)$ has a rank equal to m for every t .*

Therefore the matrix $(\sigma(t)\sigma'(t))$ is invertible.

In this setup there is only one process such that $\sigma(t)\theta(t) = B(t)$ because $\theta(t)$ is defined as

$$\theta(t) \equiv \sigma'(t)(\sigma(t)\sigma'(t))^{-1} B(t) .$$

They give another definition of completeness

Definition 7.1 *A financial market is complete if $m = d$, and incomplete if $m < d$.*

The utility maximization problem for an incomplete market ($m < d$) is studied by the method of “fictitious completion”. They introduce $d - m$ additional stocks driven by the d -dimensional Brownian motion W , thus create a fictitious complete market in which the utility maximization problem is solved as in the complete market. The appreciation rates for these additional stocks are then determined so that the optimal portfolio in the resulting complete market does not invest in the additional stocks.

According to these assumptions and given that the market parameters follow stochastic processes, they prove the same results as us: they find the same terminal wealth, the same optimal portfolio and the same value function.

We can see that they find a similar result as us by a different way. Their result is based on the fact that the matrix $\sigma(t)$ has a rank equal to the number of stocks for every t and then on the uniqueness of the process $\theta(t)$ whereas in our case we do not specify it and $\theta \in \Theta$. Our result generalises their idea.

Chapter 8

Power Utility

8.1 Deterministic case

Now we will study the power utility function.

We have

$$U(x) = \frac{x^\gamma}{\gamma} \text{ with } 0 < \gamma < 1.$$

Then $U'(x) = x^{\gamma-1}$ and $(U')^{-1}(y) = y^{\frac{1}{\gamma-1}}$.

We first study the deterministic case, i.e. when $r(\cdot)$, $\mu(\cdot)$, $\sigma(\cdot)$ and then $\theta^*(\cdot)$ are deterministic processes.

We have proved in the chapter 6 that $(U')^{-1}(\lambda H_{\theta^*}(T))$ is attainable. We will use this fact in order to derive the terminal wealth and the optimal portfolio.

$(U')^{-1}(\lambda H_{\theta^*}(T))$ is attainable then is the terminal wealth.

Set $\theta \in \Theta$ and $\lambda \in R$.

The budget constraint gives us

$$E[(U')^{-1}(\lambda H_{\theta^*}(T))H_\theta(T)] = x_0.$$

$$E[(U')^{-1}(\lambda H_{\theta^*}(T))H_\theta(T)]$$

$$\begin{aligned}
&= \lambda^{\frac{1}{\gamma-1}} E \left[\exp \left\{ \int_0^T \frac{1}{\gamma-1} \left(r(s) + \frac{1}{2} |\theta^*(s)|^2 \right) ds + \int_0^T \frac{1}{\gamma-1} \theta^*(s)' dW(s) \right. \right. \\
&\quad \left. \left. - \int_0^T \left(r(s) + \frac{1}{2} |\theta(s)|^2 \right) ds - \int_0^T \theta(s)' dW(s) \right\} \right] \\
&= \lambda^{\frac{1}{\gamma-1}} \exp \left\{ \int_0^T \frac{1}{\gamma-1} \left(r(s) + \frac{1}{2} |\theta^*(s)|^2 \right) ds - \int_0^T \left(r(s) + \frac{1}{2} |\theta(s)|^2 \right) ds \right\} \\
&\quad \cdot E \left[\exp \left\{ \int_0^T \left(\frac{1}{\gamma-1} \theta^*(s)' - \theta(s)' \right) dW(s) \right\} \right] \\
&= \lambda^{\frac{1}{\gamma-1}} \exp \left\{ \int_0^T \left(\frac{\gamma}{1-\gamma} r(s) + \frac{1}{2(1-\gamma)} |\theta^*(s)|^2 - \frac{1}{2} |\theta(s)|^2 \right) ds \right\} \cdot \exp \left\{ \int_0^T \frac{1}{2} \left(\frac{1}{\gamma-1} \theta^*(s)' - \theta(s)' \right)^2 ds \right\} \\
&= \lambda^{\frac{1}{\gamma-1}} \exp \left\{ \int_0^T \left[\frac{\gamma}{1-\gamma} r(s) + \frac{1}{2(1-\gamma)} |\theta^*(s)|^2 - \frac{1}{2} |\theta(s)|^2 \right. \right. \\
&\quad \left. \left. + \frac{1}{2(1-\gamma)^2} |\theta^*(s)|^2 + \frac{1}{2} |\theta(s)|^2 - \frac{\overbrace{|\theta^*(s)|^2}}{\theta^*(s)' \theta(s)} \frac{1}{1-\gamma} \right] ds \right\} \\
&= \lambda^{\frac{1}{\gamma-1}} \exp \left\{ \frac{\gamma}{1-\gamma} \left[\int_0^T \left(r(s) + \frac{|\theta^*(s)|^2}{2(1-\gamma)} \right) ds \right] \right\}.
\end{aligned}$$

We can check that this does not depend on $\theta \in \Theta$. This proves again that $(U')^{-1}(\lambda H_{\theta^*}(T))$ is replicable when the utility function is a power utility function.

Therefore

$$x_0 = \lambda^{\frac{1}{\gamma-1}} \exp \left\{ \frac{\gamma}{1-\gamma} \left[\int_0^T \left(r(s) + \frac{|\theta^*(s)|^2}{2(1-\gamma)} \right) ds \right] \right\} \quad (26)$$

$$\text{and } \lambda = x_0^{\gamma-1} \exp \left\{ \gamma \left[\int_0^T \left(r(s) + \frac{|\theta^*(s)|^2}{2(1-\gamma)} \right) ds \right] \right\}.$$

The terminal wealth is therefore given by

$$X^* = x^*(T) = (U')^{-1}(\lambda H_{\theta^*}(T)) = x_0 m(T) \quad (27)$$

$$\text{with } m(t) = \exp \left\{ \int_0^t \left[r(s) + \frac{(1-2\gamma)|\theta^*(s)|^2}{2(1-\gamma)^2} \right] ds + \int_0^t \frac{\theta^*(s)' dW(s)}{1-\gamma} \right\} \quad (28)$$

The replicating portfolio is determined by the equation

$$dx^*(t) = [r(t)x^*(t) + \theta^*(t)' q(t)] dt + q^*(t) dW(t) \quad (29)$$

We conjecture that

$$x^*(t) = x_0 m(t)$$

Then

$$dx^*(t) = \left[r(t)x^*(t) + \frac{|\theta^*(t)|^2}{1-\gamma} x^*(t) \right] dt + \frac{x^*(t)\theta^*(t)'}{1-\gamma} dW(t) \quad (30)$$

Then we must have, by comparing (28) and (29),

$$q^*(t) = \frac{\theta^*(t)x^*(t)}{1-\gamma} \quad (31)$$

If we assume that $(\sigma(t)\sigma'(t))$ is invertible, then the optimal portfolio is

$$\pi^*(t) = \frac{(\sigma(t)\sigma'(t))^{-1} \sigma(t)\theta^*(t)}{1-\gamma} x^*(t)$$

and because $\sigma(t)\theta^*(t) = B(t)$, it can be rewritten

$$\pi^*(t) = \frac{(\sigma(t)\sigma'(t))^{-1} B(t)}{1-\gamma} x^*(t). \quad (32)$$

And the value function is

$$u(x) = E\left[\frac{x^{*\gamma}}{\gamma}\right] = \frac{x_0^\gamma}{\gamma} E\left[\exp\left\{\int_0^T \left[r(s) + \frac{(1-2\gamma)|\theta^*(s)|^2}{2(1-\gamma)^2}\right] ds\right\}\right]. \quad (33)$$

8.2 General Stochastic case

First, as we did it for the logarithmic case, we prove that if θ_0 maximizes $E[(U')^{-1}(\lambda H_{\theta_0}(T))H_{\theta_0}(T)]$ for any $\theta \in \Theta$, θ_0 must be equal to θ^* .

Set θ_0 the maximizer of $E[(U')^{-1}(\lambda H_{\theta_0}(T))H_{\theta_0}(T)]$ for any $\theta \in \Theta$.

$$\begin{aligned} & E[(U')^{-1}(\lambda H_{\theta_0}(T))H_{\theta_0}(T)] \\ &= \lambda^{\frac{1}{\gamma-1}} E\left[\exp\left\{\int_0^T \frac{1}{\gamma-1} \left(r(s) + \frac{1}{2}|\theta_0(s)|^2\right) ds + \int_0^T \frac{1}{\gamma-1} \theta_0(s)' dW(s) \right. \right. \\ & \quad \left. \left. - \int_0^T \left(r(s) + \frac{1}{2}|\theta(s)|^2\right) ds - \int_0^T \theta(s)' dW(s)\right\}\right]. \end{aligned}$$

Set $\Delta = \theta - \theta_0$ and $\theta_k = \theta_0 + k\Delta$ for k such that $\theta_k \in \Theta$. Note that we have $\sigma\Delta = 0$.

We set

$$f_k = E[(U')^{-1}(\lambda H_{\theta_k}(T))H_{\theta}(T)].$$

Then

$$\begin{aligned} f_k &= \lambda^{\frac{1}{\gamma-1}} E \left[\exp \left\{ \int_0^T \frac{1}{\gamma-1} \left(r(s) + \frac{1}{2} |\theta_k(s)|^2 \right) ds + \int_0^T \frac{1}{\gamma-1} \theta_k(s)' dW(s) \right. \right. \\ &\quad \left. \left. - \int_0^T \left(r(s) + \frac{1}{2} |\theta(s)|^2 \right) ds - \int_0^T \theta(s)' dW(s) \right\} \right] \\ &= \lambda^{\frac{1}{\gamma-1}} E \left[\exp \left\{ \int_0^T \frac{1}{\gamma-1} \left(r(s) + \frac{1}{2} |\theta_0 + k\Delta|^2 \right) ds + \int_0^T \frac{1}{\gamma-1} (\theta_0 + k\Delta)' dW(s) \right. \right. \\ &\quad \left. \left. - \int_0^T \left(r(s) + \frac{1}{2} |\theta(s)|^2 \right) ds - \int_0^T \theta(s)' dW(s) \right\} \right]. \end{aligned}$$

Because θ_0 is the maximizer of $E[(U')^{-1}(\lambda H_{\theta_0}(T))H_{\theta}(T)]$ for any $\theta \in \Theta$,

f_k must be maximal for $\theta_k = \theta_0$, i.e. for $k = 0$.

Then we must have

$$\frac{\partial f_k}{\partial k} \Big|_{k=0} = 0.$$

We have

$$\frac{\partial f_k}{\partial k} = E[(U')^{-1}(\lambda H_{\theta_k}(T))H_{\theta}(T) \left(\int_0^T \frac{1}{1-\gamma} \Delta dW(s) + \frac{1}{1-\gamma} \int_0^T (k\Delta^2 + \theta_0\Delta) dt \right)].$$

Because $\frac{\partial f_k}{\partial k} \Big|_{k=0} = 0$, we have clearly $\theta_0\Delta = 0$.

That means that we have

$$\theta_0 = \theta^*.$$

Therefore we have found a necessary condition for θ_0 .

Thanks to that, our analysis is simplified: if the terminal wealth is attainable, it must be equal to $(U')^{-1}(\lambda H_{\theta^*}(T))$.

Nevertheless, I have not succeeded in proving that $(U')^{-1}(\lambda H_{\theta^*}(T))$ is attainable. I have tried the same method as for the logarithmic case. We can see that this method does not work in this case. I develop it below.

We suppose that $r(\cdot)$, $\mu(\cdot)$, $\sigma(\cdot)$ and then $\theta^*(\cdot)$ follow stochastic processes.

Set $\theta \in \Theta$ and $\lambda \in R$.

$$E[(U')^{-1}(\lambda H_{\theta^*}(T))H_{\theta}(T)] \\ = \lambda^{\frac{1}{1-\gamma}} \exp\left\{\int_0^T \frac{\gamma}{1-\gamma} r(s) ds\right\} E\left[\exp\left\{\int_0^T \frac{1}{2} \left(\frac{1}{1-\gamma} |\theta^*(s)|^2 - |\theta(s)|^2\right) ds + \int_0^T \left(\frac{1}{1-\gamma} \theta^*(s)' - \theta(s)'\right) dW(s)\right\}\right]$$

In order to use the same method as used for the logarithmic case, we would like

$$\exp\left\{\int_0^T \frac{1}{2} \left(\frac{1}{1-\gamma} |\theta^*(s)|^2 - |\theta(s)|^2\right) ds + \int_0^T \left(\frac{1}{1-\gamma} \theta^*(s)' - \theta(s)'\right) dW(s)\right\} \quad (34)$$

to be an exponential martingale. This would be a sufficient condition but not a necessary condition.

If we set

$$\Delta = \theta(s) - \frac{1}{1-\gamma} \theta^*(s),$$

we find

$$\Delta^2 = |\theta(s)|^2 + \left(\frac{1}{1-\gamma}\right)^2 |\theta^*(s)|^2 - \frac{2}{1-\gamma} \theta(s)' \theta^*(s)$$

and, because $\theta(s)' \theta^*(s) = \theta^*(s)^2$, we have

$$\Delta^2 = |\theta(s)|^2 + \left[\left(\frac{1}{1-\gamma}\right)^2 - \frac{2}{1-\gamma} \right] |\theta^*(s)|^2 = |\theta(s)|^2 + \frac{-1+2\gamma}{(1-\gamma)^2} |\theta^*(s)|^2.$$

For (34) to be an exponential martingale, we would like that

$$\Delta^2 = |\theta(s)|^2 - \frac{1}{1-\gamma} |\theta^*(s)|^2.$$

Therefore we should have

$$-\frac{1}{1-\gamma} = \frac{-1+2\gamma}{(1-\gamma)^2},$$

which implies that $\gamma = 0$ which is impossible! But we must notice that

when $\gamma \rightarrow 0$, $\frac{x^\gamma}{\gamma} \approx \ln(x)$, and we find the logarithmic case.

Remark 8.1 So far, I haven't succeeded in showing that the expression (34) is a martingale.

Remark 8.2 This does not prove that the terminal wealth is not attainable. This just shows why we cannot use the method used for the logarithmic case.

In their paper, Karatzas, Lehoczky, Shreve and Xu have found the terminal wealth, the optimal portfolio and the final value when the market parameters $r(\cdot)$, $\mu(\cdot)$, $\sigma(\cdot)$ are deterministic processes but they did not manage to extend these results to a more general stochastic case.

Conjecture 8.1 The logarithmic function is the only one for which the terminal wealth is attainable and for which we can obtain explicitly the optimal portfolio in incomplete markets.

Chapter 9

Conclusions

In this paper, we have studied the portfolio selection in a continuous-time incomplete market by maximizing utility. In their paper about the mean-variance problem, Zhou and Jin have completely characterized the attainable terminal wealth set. However this result is totally independent of the portfolio selection problem and therefore it can be used in our framework. The original problem has then been transferred into a static optimization problem with some terminal conditions.

Using this, we have proved that the terminal wealth is attainable when the market parameters are deterministic. Moreover in the chapter 7 we have extended this result to obtain an explicit form of the optimal portfolio for the logarithmic utility function even if the market parameters follow stochastic processes. We did not, however, succeed in doing the same thing for some other utility functions.

Therefore we can conjecture that the logarithmic function is the only one for which we can derive an optimal portfolio explicitly. However, this has not been proved and this is the main remaining part of the subject. Further, the general stochastic case is also still remaining for the mean-variance problem because Jin and Zhou have already done the deterministic case.

Appendix A

Some Lemmas

In this section we give some useful technical lemmas. We do not give the proofs of these lemmas but these can be found in [].

Lemma A.1 *If a quadratic function $f : \mathbf{R}^d \rightarrow \mathbf{R}$ is bounded below on a nonempty polyhedron S , then f attains its infimum on S .*

Lemma A.2 *Given $a \in \mathbf{R}^n$ and $A \in \mathbf{R}^{m \times n}$. If $a \notin \{A'u : u \in \mathbf{R}_+^m\}$, then there exists $v \in \mathbf{R}^n \setminus \{0\}$ such that $a'v = -1$ and $Av \geq 0$.*

Lemma A.3 *Let $X \equiv \{X(t) : 0 \leq t \leq T\}$ be a given n -dimensional, F_t -measurable stochastic process. Assume that $S(t, \omega) := \{y \in \mathbf{R}^m : f(X(t, \omega), y) \leq 0\} \neq \emptyset$ for any $(t, \omega) \in [0, T] \times \Omega$, where $f : \mathbf{R}^n \times \mathbf{R}^m \rightarrow \mathbf{R}^k$ is jointly measurable in both variables and continuous in the second variable. Then the process $\alpha \equiv \{\alpha(t) : 0 \leq t \leq T\}$ defined as $\alpha(t, \omega) := \arg \min_{y \in S(t, \omega)} |y|^2$ is also F_t -measurable.*

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