Simultaneous equal sums of three powers

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Abstract

Using a result of Salberger [9] we show that the number of non-trivial positive integer solutions $x_0, \ldots, x_5 \leqslant B$ to the simultaneous equations

$$x_0^c + x_1^c + x_2^c = x_3^c + x_4^c + x_5^c, \quad x_0^d + x_1^d + x_2^d = x_3^d + x_4^d + x_5^d,$$

is $o(B^3)$ whenever $d > \max\{2, c\}$.

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1 Introduction

The purpose of this short note is to apply recent work of Salberger to an old problem in analytic number theory. More precisely we shall see how Salberger's new results [9] concerning the distribution of rational points on projective algebraic varieties can be used to study the number of positive integer solutions to the simultaneous equations

$$x_0^c + x_1^c + x_2^c = x_3^c + x_4^c + x_5^c, \quad x_0^d + x_1^d + x_2^d = x_3^d + x_4^d + x_5^d, \tag{1.1}$$

in a region $\max x_i \leqslant B$, for fixed positive integers $c < d$. There are clearly $6B^3 + O(B^2)$ trivial solutions in which x_3, x_4, x_5 are a permutation of x_0, x_1, x_2. We write $\mathcal{N}_{c,d}(B)$ for the number of non-trivial solutions, and our primary goal is to estimate this quantity.

When $c = 1$ and $d = 2$ it is rather easy to show that

$$\mathcal{N}_{1,2}(B) = CB^3 \log B(1 + o(1)),$$

for an appropriate constant $C > 0$, so that the non-trivial solutions dominate the trivial ones. In all other cases we would like to know that the trivial solutions dominate the non-trivial ones. This has only been established when $c = 1$, or when $c = 2$ and $d = 3$ or 4. Specifically, it has been shown by Greaves [4] that

$$\mathcal{N}_{1,d}(B) \ll_{d,\varepsilon} B^{\frac{17}{6}+\varepsilon}, \tag{1.2}$$

and by Skinner and Wooley [10] that

$$\mathcal{N}_{1,d}(B) \ll_{d,\varepsilon} B^{\frac{8}{3}+\frac{1}{d-1}+\varepsilon}. \tag{1.3}$$

This latter result reproduces Greaves' result for $d = 7$, and improves upon it for $d \geqslant 8$. Moreover, work of Wooley [12] shows that

$$\mathcal{N}_{2,3}(B) \ll_\varepsilon B^{\frac{7}{3}+\varepsilon}, \tag{1.4}$$

and Tsui and Wooley [11] have shown that

$$\mathcal{N}_{2,4}(B) \ll_\varepsilon B^{\frac{36}{13}+\varepsilon}.$$

We are now ready to record the contribution that we have been able to make to this subject.

Theorem. *Let $\varepsilon > 0$, and suppose that c, d are positive integers such that $c < d$ and $d \geqslant 4$. Then we have*

$$\mathcal{N}_{c,d}(B) \ll_{c,d,\varepsilon} B^{\frac{11}{4}+\varepsilon} + B^{\frac{5}{2}+\frac{5}{3cd}+\varepsilon}.$$

The implied constant in this estimate is allowed to depend at most upon c, d and the choice of ε. When $c = 1$, it is easy to see that our result improves upon (1.2) for $d \geqslant 6$, and upon (1.3) for $d \leqslant 12$. Moreover, it retrieves (1.2) for $d = 5$. When $c < d$ are arbitrary positive integers such that $d \geqslant 4$, it follows from the theorem that

$$\mathcal{N}_{c,d}(B) = o(B^3).$$

An application of Greaves' bound (1.2) shows that the same is true when $c = 1$ and $d = 3$. Similarly, the bound (1.4) of Wooley handles the case $c = 2$ and $d = 3$. This therefore confirms the paucity of non-trivial solutions to the pair of equations (1.1) for all positive integers c, d such that $d > \max\{2, c\}$.

A crucial aspect of our work involves working in projective space. It is clearly natural to talk about rational points on varieties, rather than integral solutions to systems of equations. Whereas a single integer solution to (1.1) can be used to generate infinitely many others by scalar multiplication, all of these will actually correspond to the same projective rational point on the variety defined by (1.1). The transition between estimating $\mathcal{N}_{c,d}(B)$ and counting non-trivial projective rational points of bounded height will be made precise in the following section.

For distinct positive integers c, d the pair of equations (1.1) defines a projective algebraic variety $X_{c,d} \subset \mathbb{P}^5$ of dimension 3. The trivial solutions then correspond to rational points lying on certain planes contained in $X_{c,d}$. Our proof of the theorem makes crucial use of a rather general result due to Salberger [9], that provides a good upper bound for the number of rational points of bounded height that lie on the Zariski open subset formed by deleting all of the planes from an arbitrary threefold in $\mathbb{P}^5$. In order to apply this result effectively we shall need to study the intrinsic geometry of $X_{c,d}$. This will be carried out separately in the final section of this paper.

Our primary goal in this paper was merely to obtain exponents strictly less than 3. However, in private communications with the authors, Salberger has indicated how the upper bound in our theorem can be substantially sharpened. This improvement relies in part on our own Lemma 1, and will appear in print shortly. The authors are grateful to Professor Salberger for this observation, and a number of useful comments that he made about an earlier version of this paper.

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2 Proof of the theorem

Let $X_{c,d} \subset \mathbb{P}^5$ denote the threefold defined by the pair of equations (1.1), for positive integers $c < d$ such that $d \geqslant 4$. Our first task is to consider the possible planes contained in $X_{c,d}$, for which it will clearly suffice to consider the planes contained in F_d, where $F_k \subset \mathbb{P}^5$ denotes the non-singular Fermat hypersurface

$$F_k : \quad x_0^k + x_1^k + x_2^k + x_3^k + x_4^k + x_5^k = 0, \tag{2.1}$$

for any $k \in \mathbb{N}$. Now there are certain obvious planes contained in F_k, which we refer to as "standard". These planes are obtained by first partitioning the indices $\{0, \ldots, 5\}$ into three distinct pairs. For each such pair $\{i, j\}$, one then associates a vector $\mathbf{x}_{i,j} = (x_i, x_j) \in (\overline{\mathbb{Q}} \setminus \{0\})^2$ such that

$$x_i^k + x_j^k = 0.$$

Then if J_0, J_1, J_2 are the sets of indices so formed, and $[\mathbf{x}_{J_0}, \mathbf{x}_{J_1}, \mathbf{x}_{J_2}] \in \mathbb{P}^5$ denotes the corresponding point in F_k, we thereby obtain the plane

$$\{[\lambda_0 \mathbf{x}_{J_0}, \lambda_1 \mathbf{x}_{J_1}, \lambda_2 \mathbf{x}_{J_2}] : \ [\lambda_0, \lambda_1, \lambda_2] \in \mathbb{P}^2\} \subset F_k.$$

It is not hard to see that this procedure produces exactly $15k^3$ standard planes contained in F_k. The following result shows that these are the only planes contained in F_k, if k is at least 4.

Lemma 1. *Let $k \geqslant 4$. Then any plane contained in F_k is standard.*

It ought to be remarked that for $k \geqslant 5$ the classification of planes in F_k is a straightforward consequence of the well-known classification of lines in F_k, as discussed by Debarre [2, §2.5], for example. Thus the chief novelty of Lemma 1 is that we are also able to handle the case $k = 4$. Lemma 1 will be established in the next section. An important consequence of this result is that any plane in the threefold $X_{c,d}$ must correspond to a standard plane in F_d, since we have assumed that $d \geqslant 4$. Among the planes contained in $X_{c,d}$ are the six "trivial" planes

$$x_0 = x_i, \quad x_1 = x_j, \quad x_2 = x_k,$$

where $\{i, j, k\}$ is a permutation of the set $\{3, 4, 5\}$.

We are now ready to complete the proof of the theorem. In estimating $\mathcal{N}_{c,d}(B)$, it is clearly enough to count primitive vectors $\mathbf{x} = (x_0, \ldots, x_5) \in \mathbb{N}^6$, where $\mathbf{x}$ is said to be "primitive" if h.c.f.$(x_0, \ldots, x_5) = 1$. In general, if we know that there are $O(B^\delta)$ primitive vectors that make a contribution to $\mathcal{N}_{c,d}(B)$, for some constant $\delta > 1$, then there will be

$$\ll \sum_{m \leqslant B} (B/m)^\delta \ll B^\delta$$

vectors in total. For any rational point $x = [\mathbf{x}] \in \mathbb{P}^5(\mathbb{Q})$ such that $\mathbf{x} \in \mathbb{Z}^6$ is primitive, we shall write

$$H(x) = \max_{0 \leqslant i \leqslant 5} |x_i|$$

for its height. For any $B \geqslant 1$ and any Zariski open subset $U \subseteq X_{c,d}$ we shall set

$$N_U(B) = \#\{x \in U \cap \mathbb{P}^5(\mathbb{Q}) : \ H(x) \leqslant B\}.$$

A little thought reveals that the trivial planes are the only planes contained in $X_{c,d}$ that can possibly contain rational points $[x_0, \dots, x_5] \in \mathbb{P}^5(\mathbb{Q})$ in which $x_0, \dots, x_5$ share the same sign. Since we are only interested in counting non-trivial solutions to the pair of equations (1.1), it will therefore suffice to estimate $N_U(B)$ in the case that $U \subset X_{c,d}$ is the Zariski open subset formed by deleting all of the planes from $X_{c,d}$.

It is now time to record the result that forms the backbone of our estimate for $N_U(B)$. The following upper bound is due to Salberger [9, Theorem 8.5].

Lemma 2. *Let $\varepsilon > 0$, and suppose that $Z \subset \mathbb{P}^5$ is a geometrically integral threefold defined over $\overline{\mathbb{Q}}$, of degree D. Let Y be the complement of the union of all planes contained in Z. Then we have*

$$\#\{x \in Y \cap \mathbb{P}^5(\mathbb{Q}) : \ H(x) \leqslant B\} \ll_{D,\varepsilon} B^{\frac{11}{4}+\varepsilon} + B^{\frac{5}{2}+\frac{5}{3D}+\varepsilon}.$$

The proof of Lemma 2 follows from applying a birational projection argument to a corresponding bound for threefolds in $\mathbb{P}^4$. This latter result is provided by combining work of the authors' [1, Theorem 3] with the proof of Salberger's earlier estimate [8, Theorem 3.4] for hypersurfaces in $\mathbb{P}^4$.

Before applying Lemma 2 to our situation it is clear that we shall need to say something about the integrality and degree of the threefold $X_{c,d}$, as given by (1.1). In fact we shall show in Lemma 3, in the next section, that $X_{c,d}$ is geometrically integral and has degree cd. Subject to the verification of these facts below, we may therefore conclude from Lemma 2 that

$$N_U(B) \ll_{c,d,\varepsilon} B^{\frac{11}{4}+\varepsilon} + B^{\frac{5}{2}+\frac{5}{3cd}+\varepsilon},$$

for any $\varepsilon > 0$. This completes the proof of the theorem.

3 Geometry of Fermat varieties

Throughout this section let c, d be positive integers such that $c < d$, and let $X_{c,d} \subset \mathbb{P}^5$ denote the threefold (1.1). Our goal is to study the geometry of $X_{c,d}$, with a view to establishing the facts that were employed in the previous section.

Let us begin by considering the singular locus of $X_{c,d}$. Now if $\boldsymbol{\xi} = [\xi_0, \dots, \xi_5]$ is a singular point of $X_{c,d}$ with at least two non-zero coordinates ξ_i, ξ_j, then it follows from the Jacobian criterion that $\xi_i^{c-1}\xi_j^{d-1} = \delta \xi_i^{d-1}\xi_j^{c-1}$, for some $\delta \in \{-1, +1\}$. Hence the $2(d-c)$-th powers of any two non-zero coordinates of $\boldsymbol{\xi}$ must coincide, and so we may conclude that the singular locus of $X_{c,d}$ is finite. We are now ready to establish the following result.

Lemma 3. *$X_{c,d}$ is a geometrically integral variety of degree cd.*

The first part of Lemma 3 may be compared with work of Kontogeorgis [7, Theorem 1.2], who has established that the variety $\sum_{i=0}^n x_i^c = \sum_{i=0}^n x_i^d = 0$ is geometrically reduced and irreducible if $n \geqslant 5$.

In order to prove Lemma 3, we define

$$G_k : \quad x_0^k + x_1^k + x_2^k - x_3^k - x_4^k - x_5^k = 0,$$

in analogy to (2.1). Then G_k is a non-singular hypersurface of dimension 4, and it is clear that $X_{c,d} = G_c \cap G_d$ is a complete intersection in $\mathbb{P}^5$. Hence it follows from [6, Proposition II.8.23] that $X_{c,d}$ is Cohen-Macaulay as a subscheme of G_d. Since the singular locus of $X_{c,d}$ is finite, and so has codimension 3 in $X_{c,d}$, we may therefore apply [3, Theorem 18.15] to deduce that $X_{c,d}$ is geometrically reduced and irreducible. Turning to the degree of $X_{c,d}$, it is not hard to check that G_c and G_d intersect transversely at a generic point of $X_{c,d}$, again using the fact that $X_{c,d}$ has finite singular locus. But then an application of Bézout's theorem, in the form [5, Theorem 18.3], immediately reveals that

$$\deg X_{c,d} = \deg G_c . \deg G_d = cd.$$

This completes the proof of Lemma 3.

Our final task in this section is to establish Lemma 1. Let $k \geqslant 4$, and suppose that we are given a plane Π contained in F_k, as given by (2.1). Then Π must be generated by three non-collinear points $e_0, e_1, e_2 \in F_k$. We may assume after a linear change of variables that

$$e_0 = [1, 0, 0, \mathbf{e}_0], \quad e_1 = [0, 1, 0, \mathbf{e}_1], \quad e_2 = [0, 0, 1, \mathbf{e}_2],$$

for certain $\mathbf{e}_i = (e_{i,3}, e_{i,4}, e_{i,5}) \in \overline{\mathbb{Q}}^3$. But then there exist linear forms of the shape

$$L_0 = u_0, \quad L_1 = u_1, \quad L_2 = u_2, \quad L_i = a_i u_0 + b_i u_1 + c_i u_2,$$

for $i = 3, 4, 5$, that are defined over $\overline{\mathbb{Q}}$ and satisfy

$$L_0(\mathbf{u})^k + L_1(\mathbf{u})^k + L_2(\mathbf{u})^k + L_3(\mathbf{u})^k + L_4(\mathbf{u})^k + L_5(\mathbf{u})^k = 0, \tag{3.1}$$

identically in $\mathbf{u} = (u_0, u_1, u_2)$. In particular we may henceforth assume that none of the forms L_3, L_4, L_5 are identically zero, since it is well-known that a non-singular hypersurface of dimension 3 contains no planes. We proceed to differentiate the identity (3.1) with respect to u_0, giving

$$u_0^{k-1} + a_3 L_3(\mathbf{u})^{k-1} + a_4 L_4(\mathbf{u})^{k-1} + a_5 L_5(\mathbf{u})^{k-1} = 0. \tag{3.2}$$

On writing $M_i(\mathbf{u}) = a_i^{\frac{1}{k-1}} L_i(\mathbf{u})$ for $i = 3, 4, 5$, we must therefore investigate the possibility that

$$u_0^{k-1} + M_3(\mathbf{u})^{k-1} + M_4(\mathbf{u})^{k-1} + M_5(\mathbf{u})^{k-1} = 0, \tag{3.3}$$

identically in $\mathbf{u}$. Now either there exist $\lambda_3, \lambda_4, \lambda_5 \in \overline{\mathbb{Q}}$, not all zero, such that $M_i(\mathbf{u}) = \lambda_i u_0$ for $i = 3, 4, 5$, or we have produced a projective linear space of positive dimension that is contained in the surface $y_0^{k-1} + y_1^{k-1} + y_2^{k-1} + y_3^{k-1} = 0$. Since $k \geqslant 4$, the only such spaces contained in this surface are the obvious lines.

Thus we may assume, without loss of generality, that in either case there exist constants $\lambda_3, \lambda_4 \in \overline{\mathbb{Q}}$ such that

$$M_3(\mathbf{u}) = \lambda_3 u_0, \quad M_4(\mathbf{u}) = \lambda_4 M_5(\mathbf{u}),$$

with $1 + \lambda_3^{k-1} = 0$ and $(1 + \lambda_4^{k-1})M_5(\mathbf{u}) = 0$ identically. We claim that the only possibility here is that $a_4 = a_5 = 0$ in the definition of M_4, M_5. To see this it clearly suffices to suppose for a contradiction that $a_4 a_5 \neq 0$, since L_4, L_5 are non-zero in (3.2). But then, on considering the identity (3.1) satisfied by the original forms $L_0, \ldots, L_5 \in \overline{\mathbb{Q}}[\mathbf{u}]$, we may deduce that there are constants $\alpha, \beta \in \overline{\mathbb{Q}}$ such that

$$\alpha u_0^k + u_1^k + u_2^k = \beta L_5(\mathbf{u})^k,$$

identically in $\mathbf{u}$. This is clearly impossible for $k \geqslant 4$, and so establishes the claim.

In terms of the original linear forms $L_0, \ldots, L_5$ satisfying (3.1), our consideration of (3.3) has therefore led to the conclusion

$$L_3(\mathbf{u}) = \mu_3 u_0, \quad a_4 = a_5 = 0,$$

for μ_3 a k-th root of -1. It remains to consider the possibility that there exist $b_4, b_5, c_4, c_5 \in \overline{\mathbb{Q}}$ such that

$$u_1^k + u_2^k + (b_4 u_1 + c_4 u_2)^k + (b_5 u_1 + c_5 u_2)^k = 0,$$

identically in u_1, u_2. But this corresponds to a line on the Fermat surface of degree k, and so leads to the conclusion that $L_4 = \mu_4 L_i, L_5 = \mu_5 L_j$ for some permutation $\{i, j\}$ of $\{1, 2\}$, where μ_4, μ_5 are appropriate k-th roots of -1. This completes the proof that any plane Π contained in F_k must be standard if $k \geqslant 4$, as claimed in Lemma 1.

Let $m \geqslant 1$ and $k \geqslant 3$ be integers. It is interesting to remark that the argument just presented can be easily generalised to treat possible m-dimensional linear spaces that are contained in the non-singular hypersurface

$$y_0^k + \cdots + y_{2m+1}^k = 0,$$

in $\mathbb{P}^{2m+1}$. Thus it can be shown that all such linear spaces are the obvious ones, and that there are precisely $c_m k^{m+1}$ of them, where

$$c_m = (2m + 1) \cdot (2m - 1) \cdots 3 \cdot 1.$$

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