Structural Models of Credit with Default Contagion

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Abstract

Multi-asset credit derivatives trade in huge volumes, yet no models exist that are capable of properly accounting for the spread behaviour of dependent companies. In this thesis we consider new ways of incorporating a richer and more realistic dependence structure into multi-firm models. We focus on the structural framework in which firm value is modelled as a geometric Brownian motion, with default as the first hitting time of an exponential default threshold. Specification of a dependence structure consisting of a common driving influence and firm-specific inter-company ties allows for both default causality and default asymmetry and we incorporate default contagion in the first passage framework for the first time.

Building on the work by Zhou (2001a), we propose an analytical model for corporate bond yields in the presence of default contagion and two-firm credit default swap baskets. We derive closed-form solutions for credit spreads, and results clearly highlight the importance of dependence assumptions. Extending this framework numerically, we calculate CDS spreads for baskets of three firms with a wide variety of credit dependence specifications. We examine the impact of firm value correlation and credit contagion for symmetric and asymmetric baskets, and incorporate contagion that has a declining impact over time.

Contents

Chapter 1

Introduction

Credit risk, the risk that an obligor does not honour its obligations, impacts all global financial markets. Its influence is pervasive and as an understanding and appreciation of it has grown, the credit derivatives market has flourished. Banks, corporations and investors must all understand and manage credit exposures; the market for them to do so now represents trillions of dollars[1] and liquid indices exist in Europe, Asia and the USA. Single-name products have become standard and as the market has deepened over recent years it has fuelled a dramatic surge in the number and complexity of multi-asset products, from options on collateralised debt obligation (CDO) tranches to CDOs of CDOs, so called CDO-squareds. Valuing these types of derivatives, which depend on portfolios of reference credits, requires the ability to accurately model the underlying credit dependence structure. Unfortunately, in this regard, the models have not kept up with the market. In this thesis we investigate ways of introducing realistic dependence dynamics into models of credit.

1.1 Background

Credit risk is often thought of as default risk, and indeed we will use both terms, often interchangeably. Whilst default is usually associated with bankruptcy, however, this is just one of the credit events we are interested in. More broadly, credit risk is exposure to a possible credit event, which as defined by the ISDA (International Securities and Derivatives Association), include bankruptcy, failure to pay, obligation acceleration,

[1] $26 trillion currently according to The Economist, Sept 23-29th 2006, The dark side of debt. But then, at 'just' $9 trillion six months ago, this does not seem a particularly meaningful number.

obligation default, repudiation/moratorium and restructuring. More generally for some market participants, credit risk can also be considered to include spread risk.

One of the first things to become apparent when modelling credit is the scarcity of data. Credit events do not happen very often. When they do, however, they often involve significant losses, the size and timing of which are completely unknown beforehand. As a result, credit events are hard to characterise, above the fact that they are usually unexpected and unpredictable[2]. Credit curves (a plot of yield against time to maturity) reflect this uncertainty, taking a whole variety of shapes from upward-sloping to humped, and credit spreads are always positive (even for very short maturities), frequently exhibiting jumps in value.

The credit rating agencies, such as Standard and Poors, Moody's, DBRS and Fitch, evaluate the credit risk outlook for individual companies and assign credit ratings. Companies with similar ratings are supposed to be similarly creditworthy and investors pay considerable attention to them; however, the approach is very crude and generally of limited use. Credit ratings are well known to be at best a lagging indicator of credit quality, as the rating agencies are slow to anticipate changes in corporate strength. Ratings tend to react to news and market moves rather than encompass a long-term view of overall corporate health, and in fact the direction a rating is moving in is more indicative of the likelihood of default than the rating itself[3].

Another important point to note when analysing credit exposures is that companies do not operate in isolation and so it is unrealistic to assume that credit events are independent. In reality a whole network of links exists between companies in related businesses, industries and markets and the impact of individual credit events can ripple through the market as a form of contagion. It is thus of fundamental importance when modelling credit, not only to understand the drivers of credit risk at an individual company, but also the dependence structure between related companies. Whether accounting for counterparty risk in the price of a single-name credit derivative, or considering credit risk in a portfolio context, an understanding of credit dependence is essential to accurate risk evaluation and pricing.

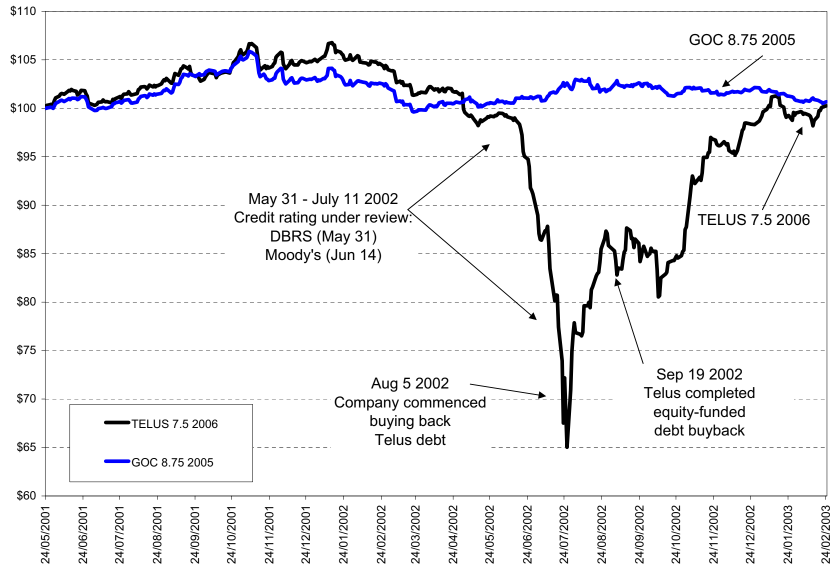
[2]Usually, but arguably, not always. The default of Bethlehem Steel in 2001 is a good case in point. In a declining industry with escalating pension and health-care liabilities, its default became just a matter of time. Whether General Motors now faces a similar future is a topical question.

[3]Darrell Duffie, 2004 Clarendon Lectures in Finance, Oxford University

1.2 Motivation

A good example of credit market and rating agency behaviour is illustrated by the Canadian telecoms company, Telus Corporation, in 2002. Coincident with the high-profile bankruptcy of WorldCom (May – July 2002) and the market obsession with the dire state of the telecoms sector in general (predominately as a result of the excessive leverage stemming from the bidding for 3G mobile licenses) the rating agencies and investors became jittery about Telus. The company was struggling in some areas of its business and had a high level of debt as a result of a foolish acquisition. The whole sector came under pressure globally and investors dumped Telus, preferring to sell first and ask questions later. Short-sellers joined the fray, the company's credit rating slid to sub-investment grade, triggering another round of selling, and liquidity dried up. The fact that the company was a strong ongoing concern generating significant cash flow was overlooked as the market became awash with rumours of potential debt restructuring. Figure 1.1 says it all – the episode was a case of market and rating agency panic, caused by contagion from other players in the industry.

Figure 1.1: Relative Price Performance of Telus vs. Government of Canada



Source: Telus

The bond shown in the graph, Telus 7.5 2006, lost 35% of its value for reasons unrelated to the company's individual circumstances and financial strength. The fact that the decline in value was unwarranted, driven entirely by Telus' relationship with other companies experiencing difficulties is evidenced by the fact that within six months the bond had regained its original value.

In Chapter 3 we provide a full classification of the main default triggers. Broadly speaking, these can be broken down into macro factors impacting many companies, firm-specific situations and inter-company ties. Based on this specification, we are able to characterise the different types of credit dependence structures exhibited in the market. The situation with WorldCom and Telus is just one example of one of the many ways in which the impact of default by one company can be manifest elsewhere due to credit contagion. Another, more recent example is given by the auto parts maker Delphi which filed for Chapter 11 bankruptcy in October 2005. The first major casualty of the struggling US automotive industry, Delphi's bankruptcy sent a shockwave through the markets. This time however, unlike in the case of Telus, the reaction was likely warranted.

1.3 Applications

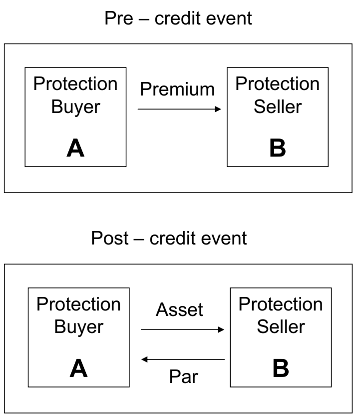
From the example and discussion above, it is clear that credit risk can become extremely involved, particularly in a portfolio setting, and an understanding of it is of great importance to a wide variety of participants in the financial markets, either from a risk management or a pricing perspective. Banks must be able to model the exposure of their loan book, both at the individual and aggregate level. Companies must similarly understand their total business exposure, whilst investors and fund managers need to price credit instruments and model risk exposures for their investments within the context of their entire portfolio. Many global banks suffered massive write-downs in their loans in 2001 and 2002 as a result of poor accounting for exposures on a portfolio basis.

The largest application for credit models is in the structuring, pricing and hedging of credit derivatives and as interest in trading credit risk grows, the credit derivatives market continues to develop in depth, volume and complexity. Pricing credit derivatives provides the motivation for the models developed in this thesis, and so we briefly outline the structures of two of the most important.

1.3.1 Credit Default Swaps

The simplest type of credit derivative is a credit default swap (CDS) and can be considered as a form of insurance. If company C is the reference credit, then the purchaser, A, of the CDS pays a stream of premiums for the life of the swap to a counterparty, B, to buy protection against default by company C. The basic outline is illustrated in Figure 1.2.

Figure 1.2: Credit Default Swap



If the bond $C(t, T)$ is the reference asset for the contract and company C defaults at time $\tau < T'$ where T' is the swap maturity, then assuming w.l.o.g. that $C(t, T)$ has unit par value, A receives $1 - C(\tau, T)$ from B at time τ. If no default occurs before time T', A receives nothing. Assuming that A pays premiums p at times T'_i for $i = 1 \ldots m$ where $T'_m < T'$, the maturity date of the swap, the cash flows from the perspective of the protection buyer A, ignoring counterparty risk, are

$$(1 - C(\tau, T))\mathbb{I}_{\{\tau \leq T'\}}\mathbb{I}_{\tau}(t) - p\sum_{i=1}^{m} \mathbb{I}_{\{T'_i < \tau\}}\mathbb{I}_{T'_i}(t)$$

where $\mathbb{I}_E$ is the indicator function of an event E,

$$\mathbb{I}_{\{t<\tau\}}P = \begin{cases} P & \text{if} \quad t < \tau \\ 0 & \text{if} \quad t \geq \tau \end{cases}$$

and $x\mathbb{I}_s(t)$ denotes a payment x at time $t = s$. If B can be assumed to be free of default risk, this is a reasonable view of A's credit risk. Usually, however, this is not the case and A is also exposed to B's credit risk, requiring knowledge of the credit dependence between B and C.

The size of the overall CDS market has doubled every year since it began trading in earnest back in 1996[4], and the launch of the iTraxx and CDX indices in Europe and the USA in 2004 has dramatically enhanced liquidity. In addition to single-name CDS contracts, market participants can now trade a wide variety of standardised products, ranging from first-to-default baskets to options and tranches on the underlying indices. A k^{th}-to-default swap, for example, is very similar to a single-name CDS, except that there are several reference credits, rather than one, and the protection buyer receives a payment on the occurrence of the k^{th} default. A full understanding of the dependence structure between the reference entities is therefore needed to model this and other types of basket derivatives.

1.3.2 Collateralised Debt Obligations

Originating from products in the mortgage market, collateralised debt obligations (CDOs) create tranched exposure to a portfolio of reference credits. Originally backed by physical bonds or loans, nowadays CDOs are generally synthetic structures and exposure is to a portfolio of single-name credit default swaps. The basic structure is illustrated in Figure 1.3. A special purpose vehicle (SPV) issues multiple tranches of credit-linked notes which are backed by a portfolio of CDSs. Investors absorb losses associated with defaults by the reference assets in order of seniority, starting with the equity investors.

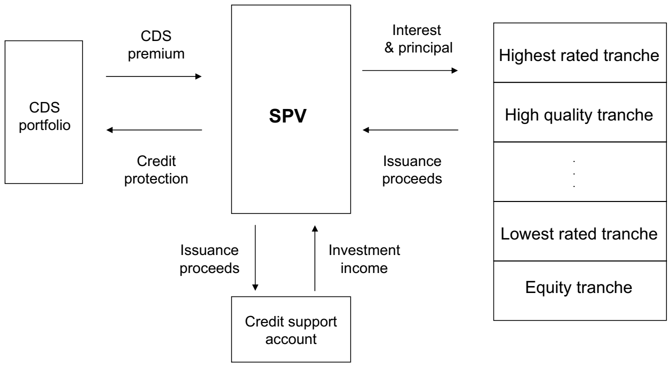
Whilst bespoke CDOs continue to be structured and traded, it is the recent availability of standardised index tranches and their derivatives that is driving liquidity and generating a market in correlation. As increasingly huge sums of money are traded based on these multi-name products, the ability to incorporate a realistic dependence structure into the models becomes ever more important.

1.4 Models

The broad-based need for good credit models has generated a lot of interest from academics and practitioners and two distinct approaches to modelling default have

[4]According to the International Index Company, www.itraxx.com.

Figure 1.3: Collateralised Debt Obligation



evolved in the literature. In the reduced form approach, default is exogenously specified as the first jump time of a Poisson process with some intensity λ. In this framework, also known as the intensity approach, default time τ is exponentially distributed with parameter λ, where λ is interpreted as the conditional default arrival rate. Default is unpredictable in this setup and so implied credit spread properties are quite plausible – short credit spreads are strictly positive for example. Pricing formulæ and default probabilities are also highly tractable, building on existing interest rate theory. For a general stochastic intensity, the probability of default is given by

$$\mathbb{P}\left[\tau \leq T\right] = 1 - \mathbb{E}\left[e^{-\int_t^T \lambda(s)ds}\right],$$

while the value of a defaultable claim X at time T with zero recovery value in the event of default is

$$\mathbb{E}\left[Xe^{-\int_t^T r_f(s)ds}\mathbb{I}_{\{\tau>T\}}\right] = \mathbb{E}\left[Xe^{-\int_t^T (r_f(s)+\lambda(s))ds}\right]$$

where $r_f(s)$ is the risk-free rate. Introducing default into the valuation framework just requires an adjustment of interest rates. In the portfolio context, copulas[5] have become the standard pricing tool enabling the dependence structure to be considered separately from the individual term structures of default risk.

[5]Copulas are tools linking marginal distributions with a joint distribution. For more details see Nelsen (1999) or Cherubini et al. (2004).

Intensity models and copulas are popular because they are easy to implement and straightforward to calibrate to market data. However, since default intensities are exogenously specified, they bear no relation to the underlying cause of default or company structure. The models provide no reason why a company defaults, but, rather, enable default probabilities to be derived from market prices. This is not a very satisfactory basis to build a model on.

Structural models, on the other hand, are driven by firm fundamentals and model the evolution of firm assets, with default occurring when the value of the firm falls below some threshold level. In this framework, debt is a contingent claim on firm assets and firm assets are usually modelled as a geometric Brownian motion. This is attractive since it enables standard Black-Scholes option pricing formulæ to be used; however, the approach has a number of shortcomings. Since firm value is modelled as a diffusion process, default events are predictable, bond prices converge to their default value and short credit spreads tend to zero. This behaviour is not seen in practice, and predicted credit spreads are too low. It is also not always evident how to measure the value and volatility of firm assets. Both are assumed to be known but often this information is not publicly available.

These and calibration issues make structural models harder to implement. Nonetheless, their economic foundation makes them a good starting point for developing more complicated models that are better able to explain market behaviour than the more ad-hoc reduced form models. Incorporating a realistic specification of dependence into a structural framework constitutes the focus of this thesis.

1.5 Thesis Outline

We begin by covering the basics of single-firm structural models in Chapter 2 before discussing in detail the characteristics of default and credit dependence in Chapter 3. These chapters set the stage for the rest of the thesis – they outline exactly what it is we want to model and provide an overview of previous work in the field. In Chapters 4 and 5 we develop a two-dimensional analytical model. We value corporate bonds in the presence of default contagion and two-company CDS baskets, deriving analytical formulæ and illustrating the impact of dependence assumptions on spreads. Chapter 6 extends this framework numerically to incorporate a much richer specification of credit dependence, allowing us to value three-company CDS baskets with asymmetric default contagion. Concluding remarks are in Chapter 7.

We assume throughout that firm values are modelled as geometric Brownian motions (GBMs) with default as the first hitting time of an exponential default barrier. This assumption is necessary for the analysis in Chapters 4 and 5 but is not integral to the methodology of Chapter 6. Incorporating jumps or other types of discontinuities would require significant modifications to the framework, but the extension to a continuous non-GBM world should be relatively straightforward.

Chapter 2

Single-Firm Structural Models

The original structural model dates back to the early seventies and the papers of Black and Scholes (1973) and Merton (1974). Their work seeks to relate credit events to economic fundamentals by modelling the dynamics of the assets of a firm with default occurring if the value of the firm drops below some threshold level. In this chapter, we begin by outlining Merton's original approach. There have been many extensions to the original work but since the methodology is broadly the same throughout, it is instructive to consider the basic model in some detail before outlining the various improvements that have been suggested. We then describe first passage models, and in particular, the formulation that provides the basis for the majority of this thesis. The purpose of this chapter is to provide an overview of the use of the structural approach to single-firm credit modelling, and it forms the foundation for the multiple-firm work considered in subsequent chapters.

2.1 Merton's Model

Merton (1974) considered a firm with the following characteristics:

- Two funding sources – equity and a single homogeneous class of debt
- The debt is considered as a zero coupon bond, par value K, maturity T
- In the event of non-payment of the debt at time T, the bondholders take control of the firm and equityholders receive nothing
- The firm cannot issue any senior claims, pay cash dividends or do a share repurchase prior to the maturity of the debt.

It is further assumed that there are no transaction costs or taxes, short selling is permitted and there are no problems with the divisibility of assets – standard 'perfect market' assumptions. Interest rates are assumed to be constant, such that riskless discount bond prices (unit par value) are $B(t,T) = \exp(-r_f(T-t))$, where r_f is the risk-free rate. If $V(t)$ is defined to be the value of the firm at time t, $C(t,T)$ the value of the debt, $E(t)$ the value of the equity, then $V(t) = E(t) + C(t,T)$ and both equity and debt can be viewed as contingent claims on firm assets. If $V(T) > K$ then the bondholders are repaid and the balance of the firm goes to the equityholders. If $V(T) < K$, default occurs, the bondholders take over the firm and the equityholders receive nothing. In other words

$$E(V,T) = \max(0, V(T) - K) \tag{2.1}$$

$$C(T,T) = \min(K, V(T)) = K - \max(0, K - V(T)). \tag{2.2}$$

Thus the value of the equity can be considered as a European call option on the assets of the firm, strike K, whilst the debt issue can be viewed as a portfolio consisting of a default-free loan, face-value K and a short European put on the assets of the firm with strike K.

The value of the firm is modelled as a geometric Brownian motion on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$,

$$dV(t) = \alpha V(t)dt + \sigma V(t)dW(t) \tag{2.3}$$

where α and σ are constants representing the drift and volatility respectively, and $W(t)$ is a standard Brownian motion. If $Y_1 = F_1(V,t)$, $Y_2 = F_2(V,t)$ are two functions of the value of the firm and time, where $V = V(t)$ in all that follows, then for $i = 1, 2$, using Itô's lemma

$$dY_i = \frac{\partial F_i}{\partial V}dV + \frac{\partial F_i}{\partial t}dt + \frac{1}{2}\sigma^2 V^2 \frac{\partial^2 F_i}{\partial V^2}dt \tag{2.4}$$

since $(dV)^2 = \sigma^2 V^2 dt$. If we consider a portfolio, P, consisting of the entity Y_1 hedged with $-\Delta$ lots of the entity Y_2,

$$P = Y_1 - \Delta Y_2,$$

then from equations (2.3) and (2.4)

$$\begin{aligned} dP &= dY_1 - \Delta dY_2 \\ &= (\frac{1}{2}\sigma^2 V^2 F_{1_{VV}} + \alpha V F_{1_V} + F_{1_t})dt - \Delta(\frac{1}{2}\sigma^2 V^2 F_{2_{VV}} + \alpha V F_{2_V} + F_{2_t})dt \\ &+ (\sigma V F_{1_V} - \Delta \sigma V F_{2_V})dW. \end{aligned}$$

Taking $\Delta = F_{1_V}/F_{2_V}$ ensures that the portfolio is risk-free. Since in the absence of arbitrage it must grow at the risk-free rate,

$$\begin{aligned} dP &= (\frac{1}{2}\sigma^2 V^2 F_{1_{VV}} + F_{1_t})dt - \Delta(\frac{1}{2}\sigma^2 V^2 F_{2_{VV}} + F_{2_t})dt \\ &= r_f P dt = r_f(F_1 - \Delta F_2)dt. \end{aligned}$$

Thus

$$\frac{1}{F_{1_V}}(F_{1_t} + \frac{1}{2}\sigma^2 V^2 F_{1_{VV}} - r_f F_1) = \frac{1}{F_{2_V}}(F_{2_t} + \frac{1}{2}\sigma^2 V^2 F_{2_{VV}} - r_f F_2). \tag{2.5}$$

This holds for any two functions $F_1(V,t)$, $F_2(V,t)$ of V and t, and therefore each side of the equation must be equal to some function $a(V,t)$. Hence for any function $Y = F(V,t)$

$$\frac{1}{2}\sigma^2 V^2 F_{VV} - aF_V - r_f F + F_t = 0.$$

Writing $a(V,t) = (\sigma\lambda - \alpha)V$, $\lambda = \lambda(V,t)$ is the market price of risk and the equation can be written

$$\frac{1}{2}\sigma^2 V^2 F_{VV} + (\alpha - \sigma\lambda)VF_V - r_f F + F_t = 0. \tag{2.6}$$

Merton (1974) assumes that V is a tradable asset, in which case $F = V$ is a solution to this equation, and

$$(\alpha - \sigma\lambda)V - r_f V = 0.$$

Hence $\lambda = \frac{\alpha - r_f}{\sigma}$ is the market price of risk and equation (2.6) reduces to the Black-Scholes equation

$$\frac{1}{2}\sigma^2 V^2 F_{VV} + r_f V F_V - r_f F + F_t = 0. \tag{2.7}$$

The assumption that the value of the firm is tradable is a drawback of the structural approach since in practice it is neither tradable nor often observable. Justification given for the assumption is that $V(t) = E(t) + C(t,T)$ and both debt and equity are tradable. Since the market is complete, V can be replicated by dynamically trading these securities, and therefore can be viewed as the price process of a tradable asset. However, this ignores the fact that whilst equity can usually be assumed to be tradable, company debt takes many forms, only some of which is normally public.

Under the assumption that firm assets are tradable, the framework is attractive since well known Black-Scholes pricing formulæ can be used to price both debt and equity. $E(t)$ is a function of V and t and therefore it satisfies equation (2.7) with appropriate boundary conditions. Since it is equivalent to a European call option on the value of the firm, strike K, from standard theory

$$E(t) = V\Phi(d_1) - Ke^{-r_f(T-t)}\Phi(d_2) \tag{2.8}$$

where Φ is the standard cumulative normal distribution function and

$$\begin{aligned} d_1 &= \frac{\ln(\frac{V}{K}) + (r_f + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= d_1 - \sigma\sqrt{T-t}. \end{aligned} \tag{2.9}$$

Similarly, from the Black-Scholes price for a European put option,

$$C(t,T) = V\Phi(-d_1) + Ke^{-r_f(T-t)}\Phi(d_2). \tag{2.10}$$

Of interest is the credit spread $S(t,T)$, the difference in yield between a risky bond and a risk-free bond. Denoting the yield of $C(t,T)$ by $y(t,T)$, we have

$$y(t,T) = \frac{-1}{T-t}\ln\left(\frac{C(t,T)}{K}\right).$$

Therefore the spread is

$$S(t,T) = \frac{-1}{T-t}\left(\ln\left(\frac{C(t,T)}{K}\right) - \ln B(t,T)\right) = \frac{-1}{T-t}\ln\left(\Phi(d_2) + \frac{1}{d}\Phi(-d_1)\right) \tag{2.11}$$

where $d = \frac{Ke^{-r_f(T-t)}}{V}$ is a measure of leverage.

From equation (2.3), it is also straightforward to calculate default probabilities since

$$V(t) = V_0 e^{(\alpha - \frac{1}{2}\sigma^2)t + \sigma W_t} \tag{2.12}$$

where V_0 is given and $W_t \sim N(0,t)$. If τ denotes the time of default, then at time t the default probability is

$$\begin{aligned} \mathbb{P}(\tau = T|t) &= \mathbb{P}(V(t)e^{(\alpha-\frac{1}{2}\sigma^2)(T-t)+\sigma W_{T-t}} < K) \\ &= \Phi\left(\frac{\ln\frac{K}{V(t)} - (\alpha - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right). \end{aligned}$$

Equation (2.3) and the subsequent delta-hedging arguments can be easily extended to account for dividends, taxes or coupon payments. For example, if the firm pays out an amount $cVdt$ in time dt, and Y_1, Y_2 have continuous payment streams c_1Y_1dt and c_2Y_2dt then

$$\begin{aligned} dV &= (\alpha - c)Vdt + \sigma VdW \qquad \text{and} \\ dP &= dY_1 - \Delta dY_2 + c_1Y_1dt - \Delta c_2Y_2dt. \end{aligned}$$

Hence, as above, if λ is the market price of risk

$$\frac{1}{2}\sigma^2V^2F_{1_{VV}} + (\alpha - \sigma\lambda)VF_{1_V} - r_fF_1 + c_1F_1 + F_{1_t} = 0. \tag{2.13}$$

Under the assumption that V is a tradable asset, $\alpha - \sigma\lambda = r_f - c$, and for any function $Y = F(V, t)$, this reduces to

$$\frac{1}{2}\sigma^2 V^2 F_{VV} + (r_f - c)VF_V - r_f F + F_t + c_1 F = 0 \tag{2.14}$$

where c is the continuous payment stream paid by the firm, and c_1 is the continuous payment stream received by $F(V, t)$. Credit spreads and default probabilities can then be derived as before.

In Merton's framework, it is therefore possible to derive the value of a firm's bonds and equity, its default probability and credit spread. The model is easy to understand, based as it is on existing Black-Scholes option pricing theory, and is intuitively appealing since it relates default directly to firm fundamentals. A number of the assumptions are overly simplistic, for example having just one class of debt, constant interest rates, no coupons and default only at maturity of the debt, but these can easily be relaxed and incorporated into the framework.

There are two big problems with the theory, however, that are not so easy to address. Firstly, everything is based on the value of a firm's assets. Whilst equity prices are easy to observe for publicly traded companies, the value of a firm is not, and even less so its volatility. Secondly, since the firm's value is modelled as a diffusion process, default events are predictable. As a result bond prices do not jump in the event of a default, but converge to their default value, and short credit spreads tend to zero. Neither of these phenomena is seen in practice.

2.1.1 Measuring Model Input Data

The issue here is the problem of measuring the value of the firm assets, V, and its volatility, σ, neither of which is generally observable in the market.

The usual implementation method, as suggested by Jones et al. (1984) involves using the market value of a company's equity and its instantaneous volatility. These are much easier to observe in practice than V and σ. To implement Merton's original model, the amount of debt outstanding must be measured, and then scheduled debt payments mapped into a single payment at a chosen date in some way.

In Merton's model, equity is a call option on the assets of the firm, so as in (2.8),

$$E(t) = V\Phi(d_1) - Ke^{-r_f(T-t)}\Phi(d_2). \tag{2.15}$$

For a given firm, the parameters T and K are known and r_f can be estimated from the market, so equity value is expressed as a function of V and σ. Now applying Itô's lemma to $E(t)$, since

$$\begin{aligned} dE &= E_V dV + E_t dt + \frac{1}{2}\sigma^2 V^2 E_{VV} dt \\ &= (\alpha V E_V + E_t + \frac{1}{2}\sigma^2 V^2 E_{VV})dt + \sigma V E_V dW, \end{aligned}$$

if σ_E is the instantaneous volatility of the company's equity, then

$$E\sigma_E = \sigma V \frac{\partial E}{\partial V}. \tag{2.16}$$

Therefore, given E and σ_E, equations (2.15) and (2.16) can be solved for V and σ.

A second approach suggested by Hull et al. (2004) uses Merton's model within the compound option framework of Geske (1979). Options data are readily quoted and available in the market place and market implied volatilities are easily calculated. The underlying idea is that since equity can be modelled as a European call option, an option on the firm's equity is a compound option. Using the valuation framework for compound options derived by Geske (1979), for a given option maturity it is possible to equate the value of the option on a firm calculated using the Black-Scholes equation (using the market-implied volatility) with the value calculated using the compound option approach and Merton's model. This gives a relationship between the implied volatility, T, d and σ, where $d = \frac{Ke^{-r_f(T-t)}}{V}$ is the measure of leverage defined in (2.11) for the credit spread. For a given maturity T and a given implied volatility, this therefore gives a relationship between d and σ. Hence, by considering the implied volatilities of two options with this same maturity date, it is possible to obtain two such relationships which can be solved for d and σ, and thus from equation (2.11), calculate the credit spread for a zero coupon bond with maturity T.

This method is attractive since the variables needed to calculate credit spreads can be estimated directly from implied volatility data. It is no longer necessary to estimate instantaneous equity volatility or to map a company's liability structure, some of which may be off-balance sheet, to a single zero coupon bond as in the method of Jones et al. (1984).

Hull et al. (2004) test both methods against empirical observations and find their approach to be a slight improvement. The compound option approach also provides some insights into the relationship between option markets and credit markets, namely credit spreads, option volatilities and skews. Care needs to be taken when using implied volatilities, however, since they can introduce instabilities. Out of the money implied volatilities are highly sensitive, and in the case of more exotic and barrier options, multiple implied volatilities are possible[1].

2.2 A Review of the Single-Firm Literature

Since it was first published in the early 1970's, there have been many attempts to extend Merton's basic model in order to address some of its limitations. For example, debt covenants specifying default conditions, multiple debt issues incorporating coupons and a seniority structure, non-constant interest rates and bankruptcy costs.

Black and Cox (1976) proposed the first version of what are now known as first passage models, relaxing the assumption that default can only occur at maturity. Instead, as in a bond indenture agreement, bondholders file for bankruptcy when the value of the firm drops below some time-dependent critical value $K_\tau = Ke^{-r_f(T-t)}$. This problem is modelled as in Merton (1974), for no default before maturity T, with the addition of a barrier option. The method is outlined in detail in Section 2.3 and the first passage framework forms the basis for subsequent chapters. In their paper, Black and Cox also address the valuation of corporate bonds in the case of several different classes of debt.

A number of papers relax the constant term structure of interest rates assumption to consider first-passage models incorporating both default and interest rate risk. Kim et al. (1993) and Cathcart and El-Jahel (1998) use the CIR term structure model of interest rates, whilst Longstaff and Schwartz (1995) and Briys and de Varenne (1997) assume rates evolve according to the Vasicek (1977) model. Longstaff and Schwartz (1995) also allow for payment priority on default to vary. Nielsen et al. (1993) and Saá-Requejo and Santa-Clara (1999) further consider the extension of a stochastic interest rate environment to include a stochastic default barrier.

Hull and White (1995), Hull and White (2001) and Avellaneda and Zhu (2001) propose structural models but rather than modelling the value of the firm, they consider a credit or default index – any process that is a measure of the firm's financial health and can trigger a default. Default is modelled as the first passage of a default barrier

[1]For some examples, see Chapter 1 of Shaw (1998).

where the barrier is initially specified such that model default probabilities are consistent with those in the market. Hull and White consider a discrete time framework which Avellaneda and Zhu extend to continuous time.

Whilst initially the model produces a term structure of credit spreads which is fitted to the market (and therefore short spreads are positive at $t = 0$), the shape of the default barrier is not very realistic since short spreads are being forced to be non-zero in a structural framework. Immediately after $t = 0$, the firm will either have defaulted or it will have moved away from the barrier and short spreads will be zero. Consequently, short spreads are extremely volatile in this framework.

A further approach based on the firm value methodology, considered by Leland (1994), Leland and Toft (1996) and Mella-Barral and Perraudin (1997) looks at the question of optimal capital structure. Bankruptcy is assumed to be an endogenous event triggered by the equityholders to maximise equity value. Due to their foundation in firm fundamentals, structural models lend themselves well to the consideration of issues of corporate finance and capital structure.

In another variant, Francois and Morellec (2004) and Moraux (2003) consider the application of excursion theory to the structural framework. A firm defaults only when certain criteria are met regarding how long firm value is below a default threshold. The length of time a firm is in this 'distressed' state, the severity of the distress and the occurrence of previous distressed states can all be taken into account to define when a default is triggered.

Moody's KMV uses Merton's framework for credit modelling as the motivation of a 'distance-to-default' statistic that is then used in conjunction with Moody's database of historical default information to assess default probabilities. The approach is known as the KMV model and is widely used in the market.

The distance-to-default is a summary statistic reflecting a combination of accounting and market data. Motivated by d_2 in the Black-Scholes formula, (2.9), for unit time, ignoring the drift term, it represents a standardised distance of the value of firm assets from the default barrier[2],

$$\text{Distance-to-default} = \frac{\ln(\text{Market value of assets}) - \ln(\text{Default point})}{\text{Asset volatility}},$$

[2]This is just one common distance-to-default definition in use. Moody's KMV seem to use both this measure and some similar variants in their models, but do not disclose too many details.

where the value of firm assets and their volatility are approximated from equity values and volatilities (see Section 2.1.1), and the default point is set as some function of the face value of the firm's short-term and long-term liabilities. For a given firm, the statistic is then used to give an expected default frequency by calibration with historical default data. For further details, see www.moodyskmv.com.

The biggest problem with the diffusion models outlined above is the fact that credit spreads, in particular those for short-maturity bonds, are far too low. Model bond prices also converge smoothly to their default level rather than drop precipitously at or around the time of the default as tends to happen in practice. Since both characteristics are due to the predictable nature of default in these models, the obvious way to improve results is to introduce an element of unpredictability or uncertainty into the model formulation. Two interesting ways of doing this include introducing an unpredictable jump term, and assuming that information available to bondholders is incomplete.

Originally proposed by Merton (1976) to model stocks and options, jump-diffusion models incorporate both a continuous diffusive term and a discontinuous jump component to model firm dynamics. The jump term is usually modelled as a Poisson process and results in an incomplete market, requiring a further assumption in order to derive a pricing equation. Merton (1976) assumes a Capital Asset Pricing Model (CAPM) framework in which jump risk is diversifiable and therefore not rewarded with any excess return and prices an equity call option in the presence of lognormally distributed jumps. Zhou (1997b) extends this approach to include a constant default barrier to model credit spreads. Zhou (2001b) takes a similar approach, but applies utility arguments rather than assuming the CAPM, and considers a time-dependent default barrier. In both papers, Zhou models recovery on default using a writedown factor which is a function of V/K. In this way, recovery rates depend on the value of the firm at default. Jumps are assumed to be lognormal and the pricing equation is solved in each case by discretising the time interval and using Monte Carlo methods. The analysis is further extended to include the possibility of stochastic interest rates.

Kijima and Suzuki (2001) consider a time-dependent jump intensity which is estimated from market data to allow the model to be consistent with the current term structure of interest rates. The jump term is assumed to be lognormally distributed and default is only permitted at maturity to enable closed form solutions which can then be used to price credit derivatives. The model includes stochastic interest rates and covers the case of multiple classes of debt.

Cathcart and El-Jahel (2006) propose a model integrating the structural and intensity modelling frameworks. With company strength modelled through a signalling variable, default can occur either through the signalling process hitting a lower boundary, or according to a hazard rate which itself depends on the level of interest rates and the signalling variable. In a different approach, Cariboni and Schoutens (2004) model firm value as a Levy process rather than as a geometric Brownian motion and price credit default swaps. Default is triggered by the crossing of a default barrier as before, and the pure-jump nature of the Levy process allows for asymmetries, fat-tails, jumps and instantaneous defaults. Hilberink and Rogers (2002) use Levy processes with one-sided jumps in the context of work by Leland (1994) and Leland and Toft (1996) on optimal capital structure with endogenous default, addressing the problem of zero short spreads in these earlier models. Chen and Kou (2005) extend this to consider a two-sided jump model in which jump sizes have a double exponential distribution. They show that the combination of endogenous default and jump risk can produce a wide variety of shapes for both credit spreads and the equity option implied volatilities.

The addition of a jump term is one way to introduce default unpredictability into the basic structural model. Another is to assume that information regarding firm value and/or the level of the default barrier is incomplete from the perspective of bondholders. This idea was first introduced by Duffie and Lando (2001) who built on the work by Leland and Toft (1996) to assume that whilst management and equityholders have full knowledge of the asset process and act to maximise the value of equity by setting an optimal default threshold, bondholders receive only noisy reports of firm value at discrete times. Kusuoka (1999) and Coculescu et al. (2006) consider extensions to this framework in which bond investors receive the noisy asset reports continuously whilst Cetin et al. (2004) assume that rather than the bondholders seeing management's information set plus noise, managers restrict the information that is available and the market therefore sees a reduction in management's information set. Frey and Schmidt (2006) consider the situation in which firm value is filtered from discretely observed news and in each of these approaches, the framework admits a default intensity that can be used for pricing.

A similar method is the basis for the CreditGrades Model detailed in Finger et al. (2002) in which assets are modelled as a geometric Brownian motion with an uncertain default barrier. The lognormal distribution of the barrier is related to historical recovery rates; once fixed at $t = 0$ it is constant but unknown and parameters are chosen to ensure consistency with market credit spreads. The model is designed to highlight changing spreads or divergent views on spreads, it is not a pricing tool; however, short spreads become very unstable when firm assets are near their minimum value, giving unrealistic yield curve dynamics.

Giesecke (2003), Giesecke and Goldberg (2004) and Giesecke (2006) take a slightly different approach and assume that whilst default is a publicly observable event, either the value of the firm or the level of the default barrier (or both) is unknown. For each possible scenario, Giesecke (2002) considers the relationship between the incomplete information framework and the existence of a default intensity in detail. Schmidt (2006) considers a generalisation that allows for jumps in both the value of the firm and the default barrier. Default is modelled as the first hitting time of a stochastic and unobservable default barrier, admitting a default intensity that can be used for pricing.

Fouque et al. (2006a) take a different approach and incorporate a mean-reverting stochastic volatility in a first passage framework. Building on techniques developed for the equity markets, they analyse the impact of different volatility time scales on the credit curve and find that a fast stochastic volatility time scale is effective for increasing short spreads whilst longer maturity spreads are captured by a slowly varying volatility.

Schönbucher (2003) provides a good summary of empirical research related to structural models, covering the shape and movement of credit spreads, issuer ranking and pricing accuracy. It is well-documented that in general structural models tend to over-price both higher quality bonds and those with shorter maturities. Research by Jones et al. (1984), Ogden (1987), Lyden and Saraniti (2001) and Eom et al. (2004) on a number of different structural models supports this widely-held view. Ericsson et al. (2006) find that whilst bond spreads are systematically underestimated, the same does not apply to CDS spreads. Their work suggests that structural models may be well suited to the valuation of CDSs, but that factors other than default risk need to be taken into account when valuing corporate bonds.

2.3 The First Passage Model

2.3.1 Modelling Framework

We consider now the first passage model proposed by Black and Cox (1976) and outline the framework we use for modelling firm value and default throughout the remainder of this thesis. We assume the same tradability and perfect market assumptions as in Section 2.1 and model firm value as a geometric Brownian motion. In the risk-neutral measure, (2.3) becomes

$$dV(t) = (r_f - q)V(t)dt + \sigma V(t)dW(t) \qquad (2.17)$$

where the risk-free rate, r_f, dividend yield, q, and volatility, σ, are constants and $W(t)$ is a Brownian motion defined as before. In other words,

$$\begin{aligned} V(t) &= V(0)e^{(\mu t + \sigma W(t))} \\ d\ln V(t) &= \mu dt + \sigma dW(t) \end{aligned}$$

where $\mu = r_f - q - \frac{1}{2}\sigma^2$ is the expected rate of growth of the firm.

2.3.2 Barrier Formulation

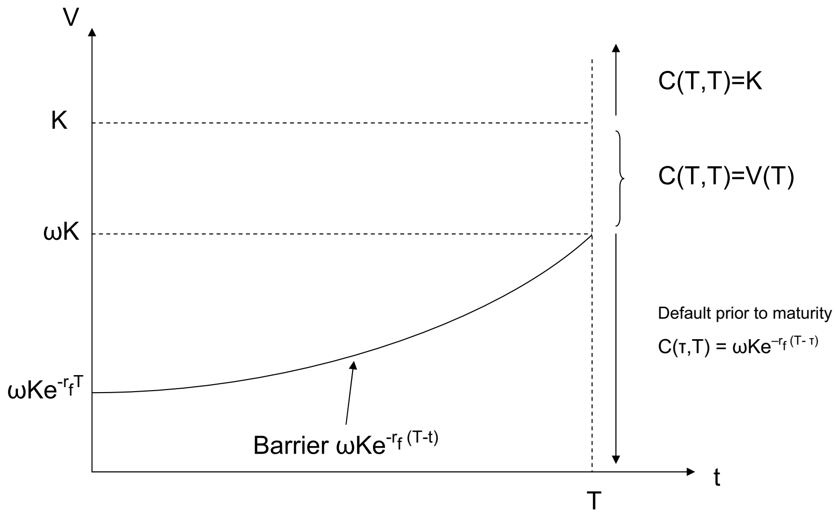
In the original structural model proposed by Merton (1974), default could only occur at debt maturity if the value of the firm was insufficient to pay debtholders. This is not a very realistic scenario, as many issuers default well before their debt matures, and most debt issues contain safety covenants to protect bondholders. These are contractual provisions, the specific details of which vary by bond issue, giving bondholders the right to force the issuer to reorganise or go bankrupt in certain instances. In effect, this means that once a company breaches a debt safety covenant, it defaults on its outstanding debt. Black and Cox (1976) extended Merton's model to a first-passage framework to deal with default prior to maturity. They model the safety covenant by assuming that once firm value falls to some lower barrier, $b(t)$, which may be time-dependent, the company defaults on its debt and bondholders take ownership of the assets.

Black and Cox (1976) assume that the time-dependence is exponential and so the bankruptcy level, or barrier, has the form $b(t) = Ae^{-\gamma(T-t)}$. This makes intuitive sense since the barrier takes the form of the discounted debt value. In practice, it is often taken as a weighted average of short-term and long-term liabilities. Black and Cox (1976) compute bond value and survival probability for $Ae^{-\gamma(T-t)} \leq Ke^{-r_f(T-t)}$, and consider the case $Ae^{-\gamma(T-t)} = \omega Ke^{-r_f(T-t)}$, for $0 \leq \omega \leq 1$, so that the default barrier is a constant fraction of discounted par value.

As shown in Figure 2.1, payments in this formulation are

- A payment equal to barrier level $\omega Ke^{-r_f(T-t)}$ on default prior to maturity.
- Repayment of firm value $V(T)$ for $\omega K \leq V(T) \leq K$ at maturity.
- Repayment of par K for $V(T) \geq K$ at maturity.

Figure 2.1: Black & Cox Barrier Formulation



Black and Cox's formulation of the barrier is good – it does not lead to inconsistency and its level can be varied by changing the factor ω. More generalisation can also be introduced by replacing the risk-free interest rate with a general discount rate, and thereby changing the slope of the barrier. One attribute of the barrier that does not seem so realistic, however, is the fact that once firm value is worth par at bond maturity, bondholders are repaid in full. A firm is unlikely to be able to refinance its whole net worth to repay a bond issue, and there would be costs associated with liquidating everything. It would seem more realistic to have a cushion, so that bondholders are only repaid in full if the firm is worth more than some threshold level.

For such a simple underlying idea, the specification of a default threshold is actually surprisingly strewn with potential pitfalls. Longstaff and Schwartz (1995) have an inconsistency at maturity. In their formulation, the firm can be solvent at maturity, and therefore not in default, but with asset value less than bond face value. Such a company would clearly be unable to repay bondholders and would default, in contrast to the outcome of the model.

It is also important to ensure that in the event of default prior to maturity, bondholders do not receive a greater amount than permitted by the default-value of the firm. This is the situation in Nielsen et al. (1993) where the payoff on default is independent of both the level of the barrier and value of the firm.

Briys and de Varenne (1997) specifically look to address these inconsistencies in a first-passage model with stochastic interest rates. They define the barrier as a discounted fraction of par,

$$b(t) = \omega K B(t, T)$$

where $B(t, T)$ is the value of a default-free zero coupon bond. In the case of stochastic interest rates, this is then a stochastic default barrier. Default payments are defined to be a fraction of remaining asset value.

2.3.2.1 Alternative Barrier Formulation

In a very similar vein to the Black and Cox model, we propose an exponential barrier of the form

$$b(t) = Ke^{-\gamma(T-t)}$$

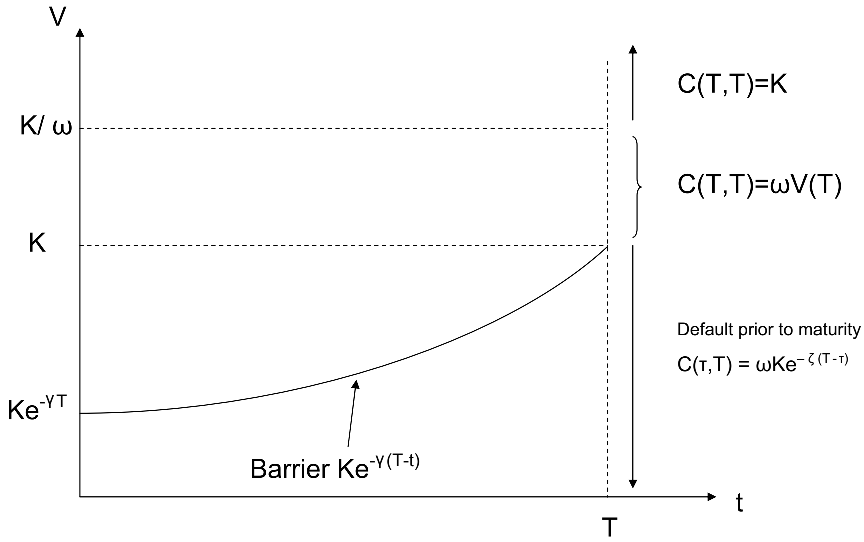
where γ is the barrier growth rate, K is par value and T is bond maturity. This barrier is therefore pinned at maturity – it is always worth par, K, at maturity, T, and as γ increases, the amount that the barrier is initially discounted increases. In other words, for different types of firm with the same debt profile, it is possible to assign barriers that relate to more or less restrictive debt covenants. As γ increases, the default probability should decrease, all other factors remaining constant.

On default we assume that bondholders are paid a fraction of discounted par value. We assume for generality some discount factor ζ and incorporate the write-down factor, $0 \leq \omega \leq 1$, in the payment on default, rather than in the level of the barrier as done by Black and Cox. ω represents the fact that in the event of default or a restructuring, a portion of the defaulting company's value is lost to bondholders. This can be as a result of legal costs, for example, or due to a breach in the absolute priority rule. Payments are illustrated in Figure 2.2 and are

- Payment on default at barrier: $\omega K e^{-\zeta(T-\tau)}$
- Payment at maturity: $\min(\omega V(T), K)$

where τ represents the default time.

Figure 2.2: Alternative Barrier Formulation



In this way, there is no inconsistency in the payment at maturity, and bondholders are repaid in full once $V(T) \geq K/\omega$. In reality, this level is likely to be more like K/η for some $\omega \leq \eta \leq 1$, but for the sake of simplicity and fewer parameters, we use K/ω.

Having the added flexibility of a general barrier growth rate γ means that we can set $\mu = \gamma$ so that it equals the drift in firm value. As we see later, this can considerably simplify many models. Since it is not possible to observe or measure the drift in firm value or the growth rate of the default barrier, this is an attractive simplification, the implication being that leverage is constant.

Finally, to ensure that bondholders are not repaid more than discounted face value in the event of default prior to maturity, we require that

$$\omega K e^{-\zeta(T-t)} \leq K e^{-r_f(T-t)}.$$

As we discuss in more detail in later chapters, Zhou (2001a) looks at default correlations for two firms, modelled using a first passage structural approach. For each firm the default barrier is defined to be $K_i e^{\gamma_i t}$. The paper considers default correlations and not bond valuation, and so is concerned with modelling default probabilities over some time period, and not with the payoff on maturity. Since in this framework the barrier level is pinned at $t = 0$ and not at $t = T$, the value of the barrier at maturity is $K_i e^{\gamma_i t}$, where presumably K_i represents bond par value. Ensuring that maturity payments are consistent across bond maturities is not so obvious using this approach. It would seem to make more sense to pin the level of the barrier at maturity and then to vary the rate at which it is discounted.

2.3.3 Solution Using the Method of Images

With firm value modelled by (2.17) and boundary conditions as specified in Section 2.3.2, the default probability is a simple hitting probability and straightforward to calculate. Black and Cox (1976) use this default probability to value corporate bonds by calculating the expected discounted value of default payments. Bielecki and Rutkowski (2002) provide a thorough overview of Black and Cox's approach and an outline of their proof.

We provide an alternative derivation of bond prices using the well-known method of images[3]. When done in this way, the calculation is very intuitive and straightforward, requiring no prior knowledge of the probability theory used in Black and Cox's proof. In a somewhat related paper, Ericsson and Reneby (1998) provide formulæ for three main building blocks – a down-and-out call option, a down-and-out binary option and a unit down-and-in claim. These they then use to value a variety of corporate securities by writing the contracted payoffs of each security as a sum of the basic building blocks.

We consider the value of a zero coupon bond $C(t, T)$, par value K, maturity T, with default barrier and payment on default specified as in Section 2.3.2.1. Defining the linear operator $\mathcal{L}$ by

$$\mathcal{L}C(V, t, T) = \frac{1}{2}\sigma^2 V^2 C_{VV} + (r_f - q)VC_V - r_f C + C_t, \tag{2.18}$$

by the delta-hedging arguments of Section 2.1, $C(V, t, T)$ then satisfies

$$\begin{aligned} \mathcal{L}C(V,t,T) &= 0 \qquad V(t) > b(t), \quad t < T \\ C(V,T,T) &= \min\{\omega V(T), K\} \\ C(b(t),t,T) &= \omega K e^{-\zeta(T-t)} \\ C(V,t,T) &\to K e^{-r_f(T-t)} \quad \text{as} \quad V \to \infty. \end{aligned} \tag{2.19}$$

[3]See Wilmott et al. (1995) for further details of the method of images.

For ease of notation, we write $C(V,t,T)$ in this section to show the dependence of C on firm value, V, explicitly. To solve (2.19) we consider the maturity payment and barrier payment separately, splitting the problem into two,

$$\textbf{Problem 1:} \quad \begin{aligned} \mathcal{L}C_1(V,t,T) &= 0 \qquad V(t) > b(t), \quad t < T \\ C_1(V,T,T) &= \min\{\omega V(T), K\} \\ C_1(b(t),t,T) &= 0 \\ C_1(V,t,T) &\to Ke^{-r_f(T-t)} \quad \text{as} \quad V \to \infty \end{aligned} \tag{2.20}$$

$$\textbf{Problem 2:} \quad \begin{aligned} \mathcal{L}C_2(V,t,T) &= 0 \qquad V(t) > b(t), \quad t < T \\ C_2(V,T,T) &= 0 \\ C_2(b(t),t,T) &= \omega K e^{-\zeta(T-t)} \\ C_2(V,t,T) &\to 0 \quad \text{as} \quad V \to \infty \end{aligned} \tag{2.21}$$

We then have a regular barrier option problem, (2.20), and a rebate, (2.21), both of which are straightforward to solve by the method of images. By the linearity of the operator $\mathcal{L}$, the value of $C(V,t,T)$ is then the sum of $C_1(V,t,T)$ and $C_2(V,t,T)$, and as shown in Appendix A is

$$\begin{aligned} C(V,t,T) &= \omega V(t) e^{-q(T-t)} \left[\Phi(d_{bq}) - \Phi(d_{1q}) - \left(\frac{b(t)}{V(t)}\right)^{\kappa+2} \left[\Phi(d^i_{bq}) - \Phi(d^i_{1q})\right] \right] \\ &+ Ke^{-r_f(T-t)} \left[\Phi(d_{2q}) - \left(\frac{b(t)}{V(t)}\right)^{\kappa} \Phi(d^i_{2q}) \right] \\ &+ \omega K e^{-\zeta(T-t)} \left(\frac{V(t)}{b(t)}\right)^{c} \left[\Phi(-d_{cq}) + \left(\frac{b(t)}{V(t)}\right)^{\kappa+2c} \Phi(d^i_{cq}) \right] \end{aligned} \tag{2.22}$$

where

$$
\begin{aligned}
d_{1q} &= \frac{\ln(\omega V(t)/K) + (\mu + \sigma^2)(T-t)}{\sigma\sqrt{T-t}} \\
d_{1q}^i &= \frac{\ln(\omega b(t)^2/(KV(t))) + (\mu + \sigma^2)(T-t)}{\sigma\sqrt{T-t}} \\
d_{2q} &= d_{1q} - \sigma\sqrt{T-t} \\
d_{2q}^i &= d_{1q}^i - \sigma\sqrt{T-t} \\
d_{bq} &= \frac{\ln(V(t)/b(t)) + (\mu - \gamma + \sigma^2)(T-t)}{\sigma\sqrt{T-t}} \\
d_{bq}^i &= \frac{\ln(b(t)/V(t)) + (\mu - \gamma + \sigma^2)(T-t)}{\sigma\sqrt{T-t}} \\
d_{cq} &= \frac{\ln(V(t)/b(t)) + (\mu - \gamma + c\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \\
d_{cq}^i &= \frac{\ln(b(t)/V(t)) + (\mu - \gamma + c\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \\
\kappa &= \frac{2}{\sigma^2}(\mu - \gamma) \\
c &= \frac{1}{\sigma^2}\left(\gamma - \mu - \sqrt{(\mu - \gamma)^2 + 2\sigma^2(r_f - \zeta)}\right).
\end{aligned}
\tag{2.23}
$$

This is easily shown to be identical to that in Black and Cox (1976) (correcting for the typographical error in their paper) by setting $\omega = 1$ and $\zeta = \gamma$ to correspond to their formulation of maturity and default payments.

In the special case that payment on default is a fraction of discounted par, $\omega K e^{-r_f(T-t)}$, we can set $\zeta = r_f$ and the formulæ simplify. As discussed in Bakshi et al. (2006), this formulation of the default payment is supported by empirical research and is the framework we use in subsequent chapters. In this case, the value of the bond is

$$
\begin{aligned}
C(V,t,T) = \omega V(t)e^{-q(T-t)}&\left[\Phi(d_{bq}) - \Phi(d_{1q}) - \left(\frac{b(t)}{V(t)}\right)^{\kappa+2}\left[\Phi(d_{bq}^i) - \Phi(d_{1q}^i)\right]\right] \\
+ \quad Ke^{-r_f(T-t)}&\left[\Phi(d_{2q}) + \omega\Phi(d_{cq}^i) + \left(\frac{b(t)}{V(t)}\right)^{\kappa}\left[\omega\Phi(-d_{cq}) - \Phi(d_{2q}^i)\right]\right]
\end{aligned}
\tag{2.24}
$$

where the variables are defined as in (2.23) above, with the exception of

$$
\begin{aligned}
d_{cq} &= \frac{\ln(V(t)/b(t)) + (\gamma - \mu)(T-t)}{\sigma\sqrt{T-t}} \\
d_{cq}^i &= \frac{\ln(b(t)/V(t)) + (\gamma - \mu)(T-t)}{\sigma\sqrt{T-t}} \\
c &= -\kappa.
\end{aligned}
$$

Using (2.24), Figures 2.3 and 2.4 illustrate yield curve sensitivity to initial credit quality (with $\sigma = 0.2$) and firm volatility (with initial credit quality of two), respectively, generating a variety of different yield curve shapes. Initial credit quality is discussed in more detail in Chapter 4 and is equal to initial firm value divided by the initial level of the barrier. Lower credit quality means the firm is initially closer to bankruptcy and therefore more risky. Similarly, a more volatile firm is more risky and so we see yields increase with increasing volatility and declining credit quality.

Figure 2.3: Varying credit quality

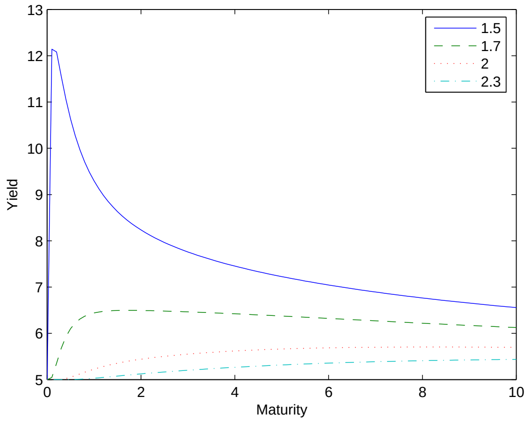
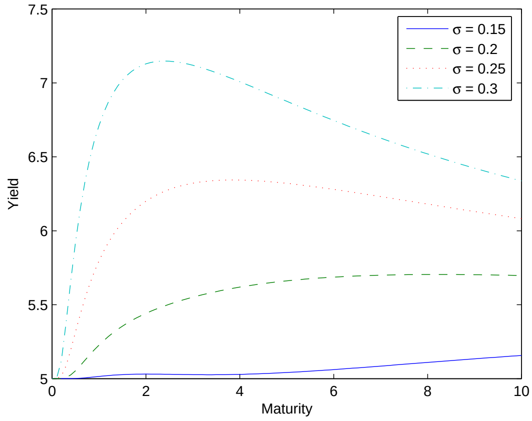


Figure 2.4: Varying firm volatility



$$K = 100, \ r_f = 0.05, \ q = 0, \ \omega = 0.7, \ \gamma = 0.03$$

2.4 Summary

In this chapter, we have provided an introduction to the structural approach to modelling credit and an overview of its applicability in valuing a single company's debt. Starting with Merton's original model, we outlined the mathematical ideas, provided a summary of the relevant literature and considered the most realistic formulation of a default barrier. We concluded by providing an alternative method for calculating bond prices in a first passage setting.

The resulting framework and specification of firm value dynamics form the basis for the rest of the thesis. There have been many attractive developments to the basic structural model, for example stochastic interest rates, that we mention only in passing. Our focus, rather, is the extension of the first passage methodology to the multiple-firm setting, incorporating a realistic dependence structure. We begin in the next chapter by outlining what it is that we wish to model and the extent of other research in this area.

Chapter 3

A Review of Multiple-Firm Models

In modelling the default event for a single firm, we considered the dynamics of the company's economic strength over time. Firms do not operate in isolation, however, and so in a portfolio setting we need to account for the dependence structure between companies in addition to their individual corporate characteristics. This is of fundamental importance across finance, whether to managing a loan book, running an investment portfolio, or in the structuring, pricing and hedging of many derivatives contracts, from first-to-default baskets to collateralised debt obligations (CDOs). Added to which, counterparty risk introduces credit risk and frequently the need to account for credit dependence at the level of the individual firm. In each case, knowledge not only of individual default probabilities is required, but a complete specification of the default dependence structure of the companies being considered.

There are a number of different approaches to including dependence within both the structural and reduced-form frameworks. The purpose of this chapter is to summarise the various methods that have been considered to date and to motivate the model we develop in subsequent chapters. To this end, it is first instructive to consider what it is that actually drives default, how the effects of a company default can spread to another firm or firms as a form of contagion, and hence what characteristics credit dependence structures exhibit in the market. We can then specify what attributes a realistic multi-asset credit model should possess and therefore what we are working towards.

3.1 What Drives Default?

Within the context of credit derivatives, default refers more broadly to the occurrence of a 'credit event' as defined by the ISDA and outlined in Section 1.1. Credit dependence we then take to mean either dependence in the occurrence of credit events or more generally in changes in credit quality (for example, spread widening).

A company default can be triggered in three main ways

1. By factors that directly impact multiple companies, of which there are two types,

   (a) Cyclical – there is a high degree of cyclical default correlation, with more defaults and higher credit spreads in economic recessions. This can be related back to the level of interest rates and various macro-economic factors.

   (b) Market-wide shock – an external factor that directly impacts multiple companies, either in the same industry or related for some other reason. This should be contrasted with the situation when one company impacts others. Examples include

   - Excess capacity in the global steel industry impacting global steel companies
   - SARS and the Gulf War impacting the travel industry
   - Debt incurred through the bidding for 3G licences crippling global telecoms companies
   - IT spending glut and subsequent decline post Y2K impacting chip manufacturers and IT consultancies
   - Severe catastrophes such as an earthquake in Tokyo or the September 11th attacks
   - Liquidity breakdown or systematic meltdown
   - Sovereign risk – default, moratorium on capital outflows, devaluation.

2. By company-specific incidents or situations. There are endless examples, some of which include

   - Weak finances, the inability to refinance, insufficient liquid assets

- Bad management, bad capital expenditure decisions etc.
- Too much competition or dying business
- Business risk – e.g. a satellite fails on launch, a power plant suffers an explosion etc.

3. Due to inter-company ties. These come in many guises and can be physical or purely a matter of perception.
   - Legal e.g. parent-subsidiary relationship
   - Financial e.g. trade-creditor agreement, financial guarantee, loan. For example, fear for the major US investment banks following the collapse of LTCM led to a bailout by the Federal Reserve.
   - Mutual capital holdings. Italian businesses, Japanese banks and Korean Chaebols are three examples of whole systems built on inter-company holdings and ties, generating networks of related interests.
   - Supplier-purchaser or other business orientated relationships and contracts.
   - Perceived link or similarity. For example the spate of (or fear of) accounting fraud at Enron, WorldCom, Tyco, GE and Parmalat in 2002.

Default dependence then occurs primarily through two mechanisms

1. As a direct consequence of a common factor driving default, whether cyclical or a market shock, as in the examples above.
2. Due to inter-company ties, which can be thought of as a form of contagion.

In other words, the dependence structure within a given group of companies is extremely involved. There is a network of links, some of which are due to exposure to common factors, some of which are due to direct or indirect links with other firms. Corporate defaults and spread changes are clearly not independent and the inter-relationship is non-symmetrical and can exhibit both contagion and feedback. A realistic multi-asset credit model needs to have the flexibility to incorporate the types of dependence structures described above, whilst also enabling calibration and ease of computation.

3.2 Dependence Measures

When people talk of measuring the dependence between two random variables, they are often referring to their correlation, and more specifically their linear correlation. However, linear correlation is just one of many measures of stochastic dependence, and in many financial and credit applications, it is not the most appropriate one.

The linear correlation, $\rho(X, Y)$, between two random variables X and Y is

$$\rho(X, Y) = \frac{\text{Cov}[X, Y]}{\sqrt{\text{Var}[X] \cdot \text{Var}[Y]}} = \frac{\mathbb{E}[X \cdot Y] - \mathbb{E}[X] \cdot \mathbb{E}[Y]}{\sqrt{\text{Var}[X] \cdot \text{Var}[Y]}}, \tag{3.1}$$

where $\text{Cov}[X, Y]$ is the covariance between X and Y and Var stands for variance. In n dimensions, pairwise correlations and covariances are represented in symmetric, positive definite $n \times n$ matrices.

Linear correlation is a measure of the linear dependence between random variables and is the appropriate measure of dependence for multivariate normal distributions[1]. However, many financial applications fall outside the Gaussian world, and as a more general dependence measure, linear correlation has a number of shortcomings. It requires finite variance, it is not invariant under non-linear strictly increasing transformations, and zero correlation does not necessarily imply independence (except in the multi-variate normal case).

Rank correlation measures do not suffer from these limitations and are increasingly commonly used in credit applications. If the random variables X and Y have distribution functions F_X and F_Y, and joint distribution function F, then Spearman's rank correlation, ρ_S, is defined by

$$\begin{aligned} \rho_S(X, Y) &= \rho(F_X(X), F_Y(Y)) \\ &= 12\,\mathbb{E}[F_X(X) \cdot F_Y(Y)] - 3 \end{aligned} \tag{3.2}$$

where $\rho(X, Y)$ is the linear correlation. (3.2) is a direct result of the fact that $F_X(X)$ and $F_Y(Y)$ are standard uniform distributions. If (X_1, Y_1) and (X_2, Y_2) are two independent pairs of random variables taken from F, then Kendall's rank correlation, ρ_τ, is

$$\rho_\tau(X, Y) = \mathbb{P}\left[(X_1 - X_2)(Y_1 - Y_2) > 0\right] - \mathbb{P}\left[(X_1 - X_2)(Y_1 - Y_2) < 0\right]$$

[1] More generally, linear correlation is the canonical dependence measure for spherical and elliptical distributions.

For further details on the various dependence measures, their characteristics and suitability, see Embrechts et al. (2002) or Cherubini et al. (2004). In this thesis we distinguish between dependence, which refers to the general concept of association, and correlation, which refers to linear correlation since we are working purely with processes driven by Brownian motions.

3.3 The Literature

There are, very broadly speaking, two general approaches to modelling defaults in a portfolio context – static and dynamic credit models. In the static approach, the total number of defaults, or the probability of a given number of defaults, is calculated for a fixed time period. Examples of this approach include Moody's Binomial Expansion method, factor models and the CreditMetrics modelling framework. Davis and Lo (2001a), Schönbucher (2003) and Gupton et al. (1997), respectively, provide more details of these approaches. From a risk perspective, knowing the probable number of defaults within a portfolio over a certain period of time is very useful, however, in order to price basket credit derivatives, we are interested in the timing and identity of the defaults as well as the number and so here we consider just dynamic credit risk models.

In the dynamic approach, the default processes of the individual obligors within a portfolio are modelled in much the same way as in the case of a single firm. Overlaid on this is the default dependence structure which derives from both the specification of the individual default processes and their inter-relationship. As for the single-firm case, there are two main types of dynamic credit risk models – structural, or firm value, and intensity, or reduced form. As will become apparent, whilst there is a lot of interest in this field, there remains a lot to be done to achieve a realistic and applicable model of credit dependence.

The structural models remain our primary focus, however more work has been done in a portfolio context using other approaches. Since many of the ideas these use are of interest to the further development of multi-asset structural models, we summarise the structural approaches and then provide a brief outline of a few of the most relevant alternatives.

3.3.1 Structural Models

Paralleling the single firm case, structural approaches to modelling credit in a portfolio context take the form of multi-dimensional diffusive processes, jump-diffusion processes, Levy processes and systems with incomplete information.

Extending Merton's diffusion model to two dimensions, for two firms with firm asset values $V_1(t)$ and $V_2(t)$ respectively, it is possible to model a range of dependent defaults by considering

$$dV_1(t) = \mu_1 V_1(t)dt + \sigma_1 V_1(t)dW_1(t) \qquad dV_2(t) = \mu_2 V_2(t)dt + \sigma_2 V_2(t)dW_2(t)$$

where the Brownian motions $W_1(t)$ and $W_2(t)$ are correlated with correlation coefficient ρ,

$$\text{cov}\left(W_1(t), W_2(t)\right) = \rho t.$$

Other notation is as in Section 2.1 and default is at maturity T if the value of firm assets is less than some default threshold for each firm. Variation of ρ enables the whole range of possible default correlations and the model can be extended to multiple firms by introducing additional correlated Brownian motions.

Within this set-up, if the value and volatility of firm assets are derived from observed share price data as in Section 2.1.1, it follows that the local correlation between share prices is equal to the local correlation between firm value processes. This is not true in practice. Share price correlations are significantly higher. Allowing default to occur only at maturity is also unrealistic and provides very little useful information about the timing of defaults. Zhou (2001a) addresses this issue, extending the Black & Cox framework to model default of company i as the first hitting time of a time-dependent barrier $C_i(t) = e^{\lambda_i t} K_i$. Firm assets are related through correlated Brownian motions as above and λ_i represents the growth rate of the firm's liabilities.

Zhou models the default status of a firm by the default indicator process, $N_i(t)$, which is zero until default, at which time it jumps to 1. Default correlation between two firms is then

$$\text{Corr}[N_1(t), N_2(t)] = \frac{\mathbb{E}[N_1(t) \cdot N_2(t)] - \mathbb{E}[N_1(t)] \cdot \mathbb{E}[N_2(t)]}{\sqrt{\text{Var}[N_1(t)] \cdot \text{Var}[N_2(t)]}}. \tag{3.3}$$

Since $N_i(t)$ are Bernoulli binomial random variables,

$$\mathbb{E}[N_i(t)] = \mathbb{P}(N_i(t) = 1) \tag{3.4}$$

$$\text{Var}[N_i(t)] = \mathbb{P}(N_i(t) = 1) \cdot [1 - \mathbb{P}(N_i(t) = 1)], \tag{3.5}$$

and

$$\mathbb{E}[N_1(t)\cdot N_2(t)] = \mathbb{E}[N_1(t)] + \mathbb{E}[N_2(t)] - \mathbb{P}(N_1(t) = 1 \,\text{or}\, N_2(t) = 1). \tag{3.6}$$

Specification of $\mathbb{P}(N_i(t) = 1)$ and $\mathbb{P}(N_1(t) = 1 \,\text{or}\, N_2(t) = 1)$ is therefore sufficient to calculate the default correlation. For two firms, under the assumption that $\lambda_i = \mu_i$, Zhou (2001a) shows that

$$\mathbb{P}(N_i(t) = 1) = 2\Phi\left(-\frac{\ln(V_i(0)/K_i)}{\sigma_i\sqrt{t}}\right),$$

where Φ is the standard normal cumulative probability distribution function, derives a formula for $\mathbb{P}(N_1(t) = 1 \,\text{or}\, N_2(t) = 1)$ and calculates default correlations using (3.3). The assumption $\lambda_i = \mu_i$ means that assets and liabilities grow at the same rate and hence leverage is constant. This assumption considerably simplifies the solution and is shown by Zhou (2001a) to have little impact on results.

Hull and White (2001) retain Zhou's framework but calibrate the time-dependent default barriers to a term structure of default hazard rates. They do this in a discrete time setting so that initially all term structures are fitted by specification of the default barriers. However, problems arise due to the predictable nature of the structural framework, and the fact that in a purely diffusion setting, short spreads converge to zero. Forcing the barrier to generate non-zero short spreads results in a very unrealistically-shaped barrier. It also goes wrong immediately after $t = 0$ since it is then known whether or not default has occurred. If it has not, the value of the firm will have moved away from the barrier, causing short spreads to instantly decline to zero. Short spreads are consequently very unstable in this framework.

Hull et al. (2005) extend the first-passage framework to an n-dimensional setting, assuming that the value of company i's assets evolves according to

$$d\ln V_i = \mu_i dt + \alpha_i(t)\sigma_i dF(t) + \sqrt{1-\alpha_i(t)^2}\sigma_i dU_i(t)$$

where $F(t)$ and $U_i(t)$ are Brownian motions. $F(t)$ is common to all companies, representing the default environment, whilst $U_i(t)$ is firm-specific. The correlation between asset processes is then driven by the $\alpha_i(t)s$, which may be either stochastic or time-dependent. Using Monte Carlo simulation, Hull et al. (2005) price CDO tranches with a piecewise constant default barrier calculated for each time period to ensure consistency with individual market CDS prices. The framework can be extended to incorporate multiple common Brownian motions, $F_j(t)$, with weightings $\alpha_{ij}(t)$.

The approaches of Zhou (2001a), Hull and White (2001) and Hull et al. (2005) model defaults in a dynamic setting, capturing default timing, and the first passage framework goes some way to rectifying the under-estimation of default probabilities that is characteristic of Merton's model. However, default dependence is purely a secondary effect of the way the model is specified. If one firm defaults, and another firm's assets are correlated to the defaulted firm's assets, then that firm is more likely to default. The dependence structure derives purely from the similar processes driving asset values and therefore the framework captures the cyclical aspect of default dependence described in Section 3.1, but does not account for any causality of default events or allow for contagion. The criticisms of the diffusion model from Chapter 2 also apply – since the asset processes are continuous, defaults are always predictable and short spreads are zero. In this setting it is therefore not possible to have an unexpected move in spreads (or default) at one company that sparks resulting moves at other companies.

In a recent paper, Fouque et al. (2006b) extend the first passage stochastic volatility framework proposed in Fouque et al. (2006a) to the multi-firm setting with a default dependence structure deriving from two influences – the correlation between asset processes driving individual names and common stochastic volatility factors. Approximations are derived for the joint survival probability and the distribution of defaults, but the analysis is extremely involved and whilst there are two components to the dependence structure, both act similarly and again there is no mechanism for contagion.

As in the single-firm case, introducing jumps or incomplete information into the multi-firm setting and specifying an appropriate dependence structure could lead to more realistic default dynamics. Very little work has been done in either area, however. Cathcart and El-Jahel (2002) allow default to occur either through a firm hitting its default barrier, or according to a jump-event. Dependence between default events then occurs through correlated asset processes and related hazard rates. Spreads and default correlations are calculated for two related companies and whilst it is an attractive framework, it is extremely numerically intensive, particularly when looking to extend the model to higher dimensions. Chen and Kou (2005) discuss the desirability of extending their jump-diffusion approach to credit to more dimensions, and Huang and Kou (2004) provide solutions for two-dimensional barrier options, but integrating the two remains an area for further research. Collin-Dufresne et al. (2002) and Giesecke (2003) consider dependence modelling in a portfolio context with incomplete information. Collin-Dufresne et al. (2002) model firm assets as geometric Brownian motions and assume that investors see some lagged firm value, where the lag is not perfectly known. To simplify the problem there are just two lagged states and all firms are assumed to be simultaneously in one or the other. In this scenario, each firm's default time is an unpredictable stopping time with a default intensity. Investors define a prior distribution on the lag and update it as defaults are observed, leading to jumps in default intensities at the time of a default. The more unexpected a default, the more likely that the prior distribution was wrong, and the larger the resultant move for related companies.

Giesecke (2003) considers the case in which only default events are observable to investors. The value of firm assets and the level of the default threshold are assumed to be unknown. Cyclical default dependence can be introduced through correlated Brownian motions driving firm values, whilst contagion or inter-firm dependence occurs by making the default thresholds dependent. Once a firm defaults, investors update their information on related firms and consequently spreads and default probabilities for these firms jump.

Whilst the incomplete information setting makes intuitive sense and has many attractive attributes – e.g. default clustering and non-zero short spreads, much work remains to be done to create a model with a non-symmetrical, dynamic dependence structure that can be calibrated and related to market prices.

In another approach, Luciano and Schoutens (2005), Moosbrucker (2006), Baxter (2006a) and Baxter (2006b) assume that firm values are driven by Levy processes rather than geometric Brownian motions. Luciano and Schoutens (2005) and Moosbrucker (2006) assume that firm values are modelled as geometric Brownian motions time-changed by a common Gamma business time, leading to Variance Gamma processes. The dependence structure is introduced through the common stochastic time-change and the model incorporates jumps, asymmetries and fat-tails. Moosbrucker (2006) extends the work of Luciano and Schoutens (2005) to include dependency between the Brownian motions and breaks the stochastic time change into systematic and idiosyncratic components. Baxter (2006a) extends this framework to consider firm value modelled as a combination of a continuous Brownian term and a discontinuous Variance-Gamma process. Each term is further decomposed into a global factor and an idiosyncratic factor, leading to two correlations – one between the Brownian components, the other between jump terms. Baxter (2006b) considers the relative merits of a variety of different Levy models centred around the Gamma model and their ability to price CDO tranches.

3.3.2 Other Models

Intensity models (also known as reduced-form models and first proposed by Jarrow and Turnbull (1995)) have received considerably more attention in the multi-firm setting than structural models, particularly amongst practitioners. In this framework, the i^{th} obligor defaults when a point process $N_i(t)$ with intensity $\lambda_i(t)$ jumps. $N_i(t)$ is usually a Poisson process in which case default is exponentially distributed with parameter $\lambda_i(t)$. One way to introduce dependence is through correlated intensity processes, with the rationale that default intensities are driven by common macroeconomic variables (see, for example, Duffie and Singleton (1999)). From a modelling perspective, this is attractive since the processes $N_i(t)$ become independent conditional on the realisation of the default intensities. However, in this setup default correlation is usually far too low – at most it is of the same order of magnitude as default probabilities.

Davis and Lo (2001b) assume the existence of two risk states – normal and enhanced, the latter triggered by a default and giving occasional periods of wider credit spreads and default clusters. All obligors in a portfolio are treated equally; default by one obligor causes the default intensity of the remaining entities to jump to an elevated level for an exponentially distributed length of time. Since all bonds are treated the same, the model considers systematic risk rather than inter-firm linkages.

Jarrow and Yu (2001) link default intensities at the individual obligor level, rather than looking at the whole portfolio as in Davis and Lo (2001b). Default is modelled as the jump in a point process where the default intensity is based on economic state variables and also contains counterparty-specific jump terms capturing inter-firm linkages. Two classes of firm are introduced – primary and secondary. The former are generally larger companies and their default intensities depend purely on macro-economic factors, introducing a cyclical or market level dependence structure at this level. Conditional on the economic variables, defaults of the primary firms are independent. Secondary companies are influenced by events at the primary companies and their default intensities depend on both economic variables and the default processes of the primary firms. This introduces a much richer dependence structure since default by a primary firm causes a jump in default probabilities of related secondary firms, and default clustering. This work is extended by Yu (2005) to allow for an arbitrary dependence structure, relaxing the need for primary and secondary firms.

Related to the intensity models, by far the most popular multi-asset models used by the market in recent years have been those using copulas. Copula functions enable the distributions of marginal default times to be specified separately from the dependence structure, enabling easy implementation and calibration. Initiating from the paper by Li (2000), there has been a huge amount of work done applying copulas to finance. Nelsen (1999), Schönbucher (2003) and Cherubini et al. (2004) provide a full and rigorous overview of the mathematics of copula functions and their application to financial modelling. Due to their ease of use, copula methods have rapidly become the market standard for modelling portfolios of credits, however, they suffer considerable limitations[2]. As basically static models, copulas are unable to model the dynamics of the credit process, their dependence structure is arbitrary and overly simplistic, and one of the ways in which they are commonly implemented, using base correlations, admits arbitrage – hardly the basis for a good model! As the market is all too aware, there is an urgent need for something better.

One further area that has seen some interest of late is the application of network theory. By representing the dependence structure between companies as a network, it is possible to investigate how changes in the network influence the default properties of the portfolio. Egloff et al. (2004) use weighted graphs to incorporate both common macroeconomic factors and business links and consider the loss distribution of the portfolio as the interdependence changes. Hatchett and Kühn (2006) extend this work to include more general types of business links in a large portfolio setting, confirming the finding by Egloff et al. (2004) that the dependence structure has a noticeable impact on the tails of the loss distribution. Giesecke and Weber (2004) use a similar approach, associating default contagion with the local dependence of firms on their business partners. By representing firms as nodes on a lattice, contagion is incorporated by making each company's financial health dependent on the state of connected companies. Buttle (2004) incorporates an intensity-based default mechanism into a network structure, considers the impact of the network specification on default events and shows how to price simple multi-name products.

[2]Mikosch (2006) provides a critical discussion of the widespread usage of copula methods; countering arguments are given by Genest and Remillard (2006).

3.4 Model Motivation

The form that a good multi-firm credit model should take is clearly the subject of great debate, but broadly speaking it seems reasonable to require that it should

1. **Be intuitive.** The model should be based on economic fundamentals and interested parties should be able to understand how the model relates to the event of company default. This is the primary advantage of the structural model over most other approaches.

2. **Capture real-world default dynamics.** Default events should be dynamic and driven by macroeconomic and market-wide influences as well as idiosyncratic factors. Defaults should be at times predictable, but more often, unexpected.

3. **Incorporate a realistic dependence structure.** Both positive and negative credit dependence should be possible, and it should be asymmetric[3]. The model should allow default propagation in the form of a type of credit contagion, with the possibility of default clustering triggered by either cyclical or common factors.

4. **Be implementable.** The model should not be so complicated and incomprehensible that it is impossible to understand or use, either from an analytical or a numerical standpoint.

5. **Enable calibration.** Models which highlight the sensitivity of prices to the dependence specification and default dynamics are good, but at the end of the day, the aim is to price multi-name credit derivatives. The model should be consistent with market CDS prices and enable the valuation of k^{th}-to-default baskets and CDO tranches.

Taken together, the above represent the holy grail of credit modelling and the route to riches. As a starting point, the aim of this thesis is to extend the first passage structural model to account for a more realistic dependence structure, one that is dynamic, asymmetric with respect to default risk, incorporates both macro driving influences and contagion and can result in periods of default clustering. There are many further attributes that would be attractive, for example removing the predictable nature of

[3]Symmetrical credit dependence is a frequent modelling assumption, but is unrealistic. The bankruptcy of Microsoft, for example, would have a huge impact on a local computer supplier but Microsoft would be unlikely to notice if the latter went out of business.

defaults at the individual level. However, these are for the future. By necessity, there is a balance to be struck between extending beyond a naive asset correlation structure and obtaining an easily calibratable model. Our focus is on the former, and so we leave questions of calibration to one side. By proposing new ways of incorporating a realistic dependence structure, our aim is to highlight the importance of credit dependence assumptions and to develop a framework that has the potential to form the basis for an implementable approach to multi-asset credit modelling.

In Chapters 4 and 5, we consider an analytical approach, providing formulæ and results for the valuation of bonds in the presence of default contagion and two-firm CDS baskets. Prohibited from extending this to larger baskets by the intractability of the mathematics in higher dimensions, we turn to numerics in Chapter 6, enabling us to incorporate a much richer dependence structure.

Chapter 4

Calculating Joint Survival Probabilities

In this chapter we consider the joint survival probability for two correlated Brownian motions with constant default barriers. By a simple change of variables, this case can be used to consider the more general formulation of correlated geometric Brownian motions with exponential default barriers; the results obtained in this chapter are applied to the pricing of corporate bonds and credit default swaps in Chapter 5.

This is the first work to date using a two-dimensional structural model with default contagion to price corporate bonds. Elements of the underlying framework, however, are considered in two previous papers. In the first, Hua et al. (1998) derive the general formula for the joint survival probability and explore its use in the valuation of double lookback options. As outlined in Chapter 3, Zhou (2001a) considers the default correlation that arises between two companies modelled as correlated Brownian motions. Results are primarily given for the simpler, constant-leverage case, with formulæ derived solely for this case.

This chapter is in four sections. Section 4.1 formulates the problem, describing the general case of interest and its transformation into the simpler case of two correlated Brownian motions with constant default barriers that is considered in the rest of this chapter. The joint survival probability is derived, and simple cases are discussed. Section 4.2 considers the numerical evaluation of the joint survival probability, whilst Section 4.3 illustrates the results and analyses the impact of different parameter values; conclusions are in Section 4.4.

This chapter should be considered as the basic foundation for Chapter 5. Here we explain the general framework, derive the main formulæ and consider their implementation. These are then used as the basis for pricing credit instruments in Chapter 5. Much of the detail is in the appendices for ease of exposition.

4.1 Notation & Mathematical Results

4.1.1 Setup

We consider the first passage structural credit model for two companies, firm values V_i, $i = 1, 2$. Each company issues equity and a single homogeneous class of debt, assumed to be a zero coupon bond, $C_i(t, T)$, par value K_i, maturity T.

For each company, firm value is assumed to follow a geometric Brownian motion, with default as the first time that firm value hits a lower default barrier, $b_i(t)$. We assume that a firm's value can be constructed from tradable securities and so the risk-neutral drift can be set to $r_f - q_i$. For $i = 1, 2$,

$$dV_i(t) = (r_f - q_i)V_i dt + \sigma_i V_i dW_i(t)$$

where the risk-free rate, r_f, dividend yield, q_i, and volatility, σ_i, are constants. $W_i(t)$ is a standard Brownian motion, $\mathrm{cov}(W_1(t), W_2(t)) = \rho t$ for some constant correlation ρ and, as in the single-firm case,

$$V_i(t) = V_i(0)e^{(\mu_i t + \sigma_i W_i(t))} \quad \text{where} \quad \mu_i = r_f - q_i - \frac{1}{2}\sigma_i^2.$$

We assume exponential default barriers and denote the default barrier for company i by

$$b_i(t) = b_i(0)e^{\gamma_i t}$$

Taking the logarithm of firm value and absorbing the exponential barrier, we set

$$X_i(t) = \ln\left(\frac{V_i(t)}{V_i(0)}e^{-\gamma_i t}\right) = \alpha_i t + \sigma_i W_i(t),$$

where $\alpha_i = \mu_i - \gamma_i = r_f - q_i - \gamma_i - \frac{1}{2}\sigma_i^2$. $X_i(0) = 0$ and we have a general Brownian motion with drift, $X_i(t)$, with constant default barrier, B_i, where

$$B_i = \ln\left(\frac{b_i(0)}{V_i(0)}\right) \leq 0.$$

For the remainder of this chapter, we therefore consider the two processes

$$X_i(t) = \alpha_i t + \sigma_i W_i(t), \quad t \geq 0 \tag{4.1}$$

such that $\mathrm{cov}(W_1(t), W_2(t)) = \rho t$ where α_i, σ_i and ρ are constants.

We define the running minimum

$$\underline{X}_i(t) = \min_{0 \leq s \leq t} X_i(s),$$

and default time, τ_i, as the first hitting time of the default barrier,

$$\tau_i = \inf\{t : \underline{X}_i(t) = B_i\}.$$

The survival probability at time t is then

$$\mathbb{P}(\tau_i > t) = \mathbb{P}(\underline{X}_i(t) > B_i).$$

4.1.2 Main Result

The main result we use in this chapter is the joint survival probability density function for the two Brownian motions, $X_1(t)$, $X_2(t)$ given in (4.1), which we denote $p(x_1, x_2, t)$, where

$$p(x_1, x_2, t) = \frac{\partial^2}{\partial x_1 \partial x_2} \mathbb{P}(X_1(t) \leq x_1, X_2(t) \leq x_2, \underline{X}_1(t) \geq B_1, \underline{X}_2(t) \geq B_2). \tag{4.2}$$

This function satisfies the Fokker-Planck equation, and as shown in Hua et al. (1998), suitable transformation of the coordinate system leads to separable solutions. These take the form of the infinite series

$$p(x_1, x_2, t) = \frac{2e^{a_1 x_1 + a_2 x_2 + bt}}{\beta t \sigma_1 \sigma_2 \sqrt{1-\rho^2}} \sum_{n=1}^{\infty} e^{-(r^2 + r_0^2)/2t} \sin\left(\frac{n\pi\theta_0}{\beta}\right) \sin\left(\frac{n\pi\theta}{\beta}\right) I_{\left(\frac{n\pi}{\beta}\right)}\left(\frac{rr_0}{t}\right) \tag{4.3}$$

where $I_{\left(\frac{n\pi}{\beta}\right)}\left(\frac{rr_0}{t}\right)$ are modified Bessel's functions, and

$$\begin{aligned}
a_1 &= \frac{\alpha_1\sigma_2 - \rho\alpha_2\sigma_1}{(1-\rho^2)\sigma_1^2\sigma_2}, \quad a_2 = \frac{\alpha_2\sigma_1 - \rho\alpha_1\sigma_2}{(1-\rho^2)\sigma_1\sigma_2^2} \\
b &= -\alpha_1 a_1 - \alpha_2 a_2 + \frac{1}{2}\sigma_1^2 a_1^2 + \rho\sigma_1\sigma_2 a_1 a_2 + \frac{1}{2}\sigma_2^2 a_2^2 \\
\tan\beta &= -\frac{\sqrt{1-\rho^2}}{\rho}, \quad \beta \in [0, \pi] \\
z_1 &= \frac{1}{\sqrt{1-\rho^2}}\left[\left(\frac{x_1 - B_1}{\sigma_1}\right) - \rho\left(\frac{x_2 - B_2}{\sigma_2}\right)\right], \quad z_2 = \left(\frac{x_2 - B_2}{\sigma_2}\right) \\
z_{10} &= \frac{1}{\sqrt{1-\rho^2}}\left[-\frac{B_1}{\sigma_1} + \frac{\rho B_2}{\sigma_2}\right], \quad z_{20} = -\frac{B_2}{\sigma_2} \\
r &= \sqrt{z_1^2 + z_2^2}, \quad \tan\theta = \frac{z_2}{z_1}, \quad \theta \in [0, \beta] \\
r_0 &= \sqrt{z_{10}^2 + z_{20}^2}, \quad \tan\theta_0 = \frac{z_{20}}{z_{10}}, \quad \theta_0 \in [0, \beta].
\end{aligned} \tag{4.4}$$

Full details of the proof are in Appendix B.

By integrating over all possible values of x_1 and x_2, and changing to polar coordinates, (4.3) can be immediately used to give the joint survival probability, $P(t)$, up until some time t,

$$\begin{aligned} P(t) &= \mathbb{P}(\underline{X}_1(t) \geq B_1, \underline{X}_2(t) \geq B_2) \\ &= \frac{2}{\beta t} e^{a_1 B_1 + a_2 B_2 + bt} \sum_{n=1}^{\infty} e^{-r_0^2/2t} \sin\left(\frac{n\pi\theta_0}{\beta}\right) \int_0^{\beta} \sin\left(\frac{n\pi\theta}{\beta}\right) g_n(\theta)\,\mathrm{d}\theta \end{aligned} \quad (4.5)$$

where

$$\begin{aligned} g_n(\theta) &= \int_0^{\infty} r e^{-r^2/2t} e^{A(\theta) r} I_{\left(\frac{n\pi}{\beta}\right)}\left(\frac{r r_0}{t}\right) \mathrm{d}r \\ r_0 &= \frac{1}{\sqrt{1-\rho^2}} \left(\frac{B_1^2}{\sigma_1^2} - \frac{2\rho B_1 B_2}{\sigma_1 \sigma_2} + \frac{B_2^2}{\sigma_2^2}\right)^{1/2} \\ \tan\theta_0 &= \frac{\sigma_1 B_2 \sqrt{1-\rho^2}}{\sigma_2 B_1 - \rho \sigma_1 B_2}, \quad \theta_0 \in [0, \beta] \\ A(\theta) &= a_1 \sigma_1 \sin(\beta - \theta) + a_2 \sigma_2 \sin\theta. \end{aligned}$$

Details of the calculation are in Appendix C. This is the same result as obtained by Zhou (2001a); the different expression he gives for $g_n(\theta)$ is reconciled in Appendix C. Hua et al. (1998) provide the same result, although they appear to have a slight error in their formula.

Numerical evaluation of the survival probability is not straightforward, and it is instructive to consider the roles and ranges of a number of the parameters. This is done in Section 4.2.1. First, however, we consider two special cases for which the survival probability simplifies.

4.1.3 Special Cases

4.1.3.1 Case I

In the special case when the barrier growth rate is set equal to the drift in firm value, the model simplifies. As neither parameter is observable in practice, this is an attractive simplification and in our framework means that we set $\mu_i = \gamma_i$, for $i = 1, 2$. This implies that the expected growth rates in debt value and asset value for a firm are equal, and so company leverage is constant. Over short time periods, this is a reasonable assumption, and has the added benefit of significantly simplifying the joint survival probability, (4.5). Zhou (2001a) shows that this approximation has little impact on default correlations, particularly over short periods.

Under this assumption, for $i = 1, 2$, we have

$$dX_i = \sigma_i dW_i, \qquad X_i(0) = 0,$$

with constant default barrier B_i defined as before. Since $\alpha_i = 0$, a_1, a_2, and b are also all zero, and so

$$p(x_1, x_2, t) = \frac{2}{\beta t \sigma_1 \sigma_2 \sqrt{1-\rho^2}} \sum_{n=1}^{\infty} e^{-(r^2+r_0^2)/2t} \sin\left(\frac{n\pi\theta_0}{\beta}\right) \sin\left(\frac{n\pi\theta}{\beta}\right) I_{\left(\frac{n\pi}{\beta}\right)}\left(\frac{rr_0}{t}\right) \quad (4.6)$$

where $\tan\beta = -\frac{\sqrt{1-\rho^2}}{\rho}$, $\beta \in [0, \pi]$, etc. are defined as before. Similarly,

$$\begin{aligned} &\mathbb{P}(\underline{X}_1(t) \geq B_1, \underline{X}_2(t) \geq B_2) \qquad (4.7) \\ &= \frac{2}{\beta t} \sum_{n=1}^{\infty} e^{-r_0^2/2t} \sin\left(\frac{n\pi\theta_0}{\beta}\right) \int_0^\beta \sin\left(\frac{n\pi\theta}{\beta}\right) \mathrm{d}\theta \int_0^\infty r e^{-r^2/2t} I_{\left(\frac{n\pi}{\beta}\right)}\left(\frac{rr_0}{t}\right) \mathrm{d}r \end{aligned}$$

for r_0 and θ_0 defined as before.

As shown in Appendix D, various identities concerning modified Bessel's functions enable the integrals to be evaluated, giving

$$\begin{aligned} &\mathbb{P}(\underline{X}_1(t) \geq B_1, \underline{X}_2(t) \geq B_2) \qquad (4.8) \\ &= \frac{2r_0}{\sqrt{2\pi t}} e^{-r_0^2/4t} \sum_{\text{n odd}} \frac{1}{n} \sin\left(\frac{n\pi\theta_0}{\beta}\right) \left[I_{\frac{1}{2}\left(\frac{n\pi}{\beta}+1\right)}\left(\frac{r_0^2}{4t}\right) + I_{\frac{1}{2}\left(\frac{n\pi}{\beta}-1\right)}\left(\frac{r_0^2}{4t}\right) \right] \end{aligned}$$

This is the same result as given by Zhou (2001a), and the main result considered in his paper, although the second identity given in his outline proof on page 575 is incorrect.

Equation (4.8) is much quicker and easier to evaluate numerically. Given the difficulty in calibrating the drifts in firm value and default barrier, it therefore makes sense to use this framework in subsequent analysis. Unfortunately, however, as will become evident in Chapter 5, whilst the bond default payment and CDS spread calculation simplify in a similar manner, the same method cannot be applied to simplify the bond maturity payment. We therefore continue to work with the most general framework.

Nevertheless, (4.8) is extremely useful for checking the accuracy of our numerical results for the general case, (4.5). It can also be used to investigate the asymptotic behaviour of the survival probability and some details are given in Appendix E for a couple of cases of interest.

4.1.3.2 Case II

By considering the integral form of the modified Bessel's function,

$$I_{(\frac{n\pi}{\beta})}\left(\frac{rr_0}{t}\right) = \frac{1}{\pi}\int_0^\pi e^{\frac{rr_0}{t}\cos\phi}\cos\left(\frac{n\pi\phi}{\beta}\right)\mathrm{d}\phi - \frac{1}{\pi}\sin\left(\frac{n\pi^2}{\beta}\right)\int_0^\infty e^{-\frac{rr_0}{t}\cosh s - \frac{n\pi s}{\beta}}\,\mathrm{d}s \quad (4.9)$$

we can consider specific values of the correlation parameter, ρ, for which this term simplifies considerably. Setting $\beta = \frac{\pi}{k}$ for integer k, $\sin(n\pi^2/\beta) = 0\ \forall\, n$, and hence

$$I_{(\frac{n\pi}{\beta})}\left(\frac{rr_0}{t}\right) = \frac{1}{\pi}\int_0^\pi e^{\frac{rr_0}{t}\cos\phi}\cos(nk\phi)\,\mathrm{d}\phi.$$

The correlation $\rho = -\cos\beta = -\cos(\pi/k)$ can then only take a fixed set of values which are almost entirely negative. The case $k = 2$ corresponds to $\rho = 0$ and hence independence; $k = 3, 4, \ldots$ correspond to negative correlation and as k tends to infinity, ρ tends to minus one and hence perfectly negative correlation.

Substituting $\beta = \frac{\pi}{k}$ in (4.9), the joint survival probability, (4.5), becomes

$$\begin{aligned}\mathbb{P}(\underline{X}_1(t) \geq B_1, \underline{X}_2(t) \geq B_2) = {} & \frac{2k}{\pi^2 t}e^{a_1B_1+a_2B_2+bt}e^{-r_0^2/2t}\int_0^{\frac{\pi}{k}}\int_0^\infty\int_0^\pi re^{-r^2/2t} \\ & \times e^{(A(\theta)+\frac{r_0}{t}\cos\phi)r}\sum_{n=1}^\infty \sin(nk\theta_0)\sin(nk\theta)\cos(nk\phi)\,\mathrm{d}\phi\,\mathrm{d}r\,\mathrm{d}\theta. \quad (4.10)\end{aligned}$$

As detailed in Appendix F this simplifies to

$$\begin{aligned}\mathbb{P}(\underline{X}_1(t) \geq B_1, \underline{X}_2(t) \geq B_2) = {} & \sum_{p=-\infty}^{\infty}\frac{k}{\pi}e^{a_1B_1+a_2B_2+bt}e^{-r_0^2/2t}\times \quad (4.11) \\ & \int_0^{\frac{\pi}{k}}\left\{\left(1+\sqrt{2\pi t}f_-e^{f_-^2t/2}\Phi(f_-\sqrt{t})\right)\mathbb{I}_1 - \left(1+\sqrt{2\pi t}f_+e^{f_+^2t/2}\Phi(f_+\sqrt{t})\right)\mathbb{I}_2\right\}\mathrm{d}\theta\end{aligned}$$

where

$$\begin{aligned} k &= 1, 2, 3, \ldots \\ f_- &= A(\theta) + \frac{r_0}{t}\cos(\theta - \theta_0 - 2\pi p/k) \\ f_+ &= A(\theta) + \frac{r_0}{t}\cos(\theta + \theta_0 - 2\pi p/k) \\ \mathbb{I}_1 &= \mathbb{I}_{\{\theta-\theta_0-2\pi p/k\in[0,\pi]\}} \\ \mathbb{I}_2 &= \mathbb{I}_{\{\theta+\theta_0-2\pi p/k\in[0,\pi]\}}. \end{aligned}$$

The advantage of this approach is that it can be extended to the valuation of bond and CDS spreads analogously. A major disadvantage, however, is that it only applies to negatively correlated companies. This is of interest when considering the dependence between reference entities in a CDS basket, but less so when considering the impact of dependent defaults on bond spreads.

The simplification $\rho = -\cos(\pi/k)$ is also considered in Hua et al. (1998). They derive a formula for the probability density function (4.2) as a sum of bivariate normal densities. The derivation is similar to that for the general case in Appendix B, but rather that solving for $H(z_1, z_2, t)$ by separation of variables, a method of images solution is possible, akin to that used in Chapter 2 for the single-firm case. Since the solution is defined in a wedge formed by the two lines L_1 and L_2, the angle between which is π/k, it can be derived by summing k positive and k negative images such that zero is obtained on the boundaries.

Solutions are only possible in this way when the angle of the wedge, β, neatly divides π. For general values of correlation, $\rho = -\cos\beta$, this is not the case and an infinite number of images is needed. In theory, solutions may be possible following methodology in Skipper (2004) and Skipper and Buchen (2003), but it is not obvious that the series of image solutions would always converge and it is not easy to see how to deal with satisfying the condition at maturity, or what constraints there might be on the form it could take.

4.2 Numerical Implementation

The integrals are evaluated in C++ by numerical quadrature using a sparse grid with up to 20 refinements. For further information regarding sparse grid methods, see Gerstner and Griebel (1998).

This section is organised as follows. We begin by discussing the properties of the various parameters before outlining the methods used to evaluate the survival probability, both in the general case and in the simpler case of equal drifts. Since the numerical evaluation proved to be rather involved, we provide some detail of the various approaches we tried. The goal was a method producing stable, accurate results, and it is highly probable that faster techniques exist.

4.2.1 Parameters

As discussed in Chapters 2 and 5, we assume that the default barriers are of the form $K_i e^{-\gamma_i(T-t)}$. The independent constants are then

Correlation	$\rho \in [-1, 1]$
Volatility	σ_i
Maturity	T
Time	$t \in [0, T]$
Par value	$K_i = 100$
Write-down factor	$\omega_i \in [0, 1]$
Risk-free rate	r_f
Dividend yield	q_i
Barrier growth rate	γ_i
Base credit quality	QB_i.

ω_i is as defined in Chapter 2; QB_i is discussed below. The following dependent parameters are then calculated directly from these independent constants,

Credit quality $V_i(0)/b_i(0)$	$QB_i e^{(\gamma_i - r_f + q_i + \sigma_i^2/2)T}$
Initial firm value $V_i(0)$	$QB_i K_i e^{(-r_f + q_i + \sigma_i^2/2)T}$
Barrier B_i	$\ln(\frac{1}{\text{Credit quality}})$
$\tan\beta$	$-\frac{\sqrt{1-\rho^2}}{\rho}, \quad \beta \in [0, \pi]$
α_i	$r_f - q_i - \sigma_i^2/2 - \gamma_i$
a_1	$\frac{\alpha_1\sigma_2 - \rho\alpha_2\sigma_1}{(1-\rho^2)\sigma_1^2\sigma_2}$
a_2	$\frac{\alpha_2\sigma_1 - \rho\alpha_1\sigma_2}{(1-\rho^2)\sigma_1\sigma_2^2}$
b	$-\alpha_1 a_1 - \alpha_2 a_2 + \frac{1}{2}\sigma_1^2 a_1^2 + \rho\sigma_1\sigma_2 a_1 a_2 + \frac{1}{2}\sigma_2^2 a_2^2$
r_0	$\frac{1}{\sqrt{1-\rho^2}}\left(\frac{B_1^2}{\sigma_1^2} - \frac{2\rho B_1 B_2}{\sigma_1\sigma_2} + \frac{B_2^2}{\sigma_2^2}\right)^{1/2}$
$\tan\theta_0$	$\frac{\sigma_1 B_2\sqrt{1-\rho^2}}{\sigma_2 B_1 - \rho\sigma_1 B_2}, \quad \theta_0 \in [0, \beta]$

Credit quality represents the initial creditworthiness of a company – it is initial firm value divided by the initial level of the barrier. We parameterise it in this way, with a base quality QB_i, so that the slope of the barrier, γ_i can be varied. If initial credit quality is a fixed input to the calculation, varying γ_i causes $V_i(0)$ to change giving meaningless results. The base case scenario is specified here to be the case when

$\mu_i = \gamma_i$. With this formulation, results can be consistently compared to the simple case of equal drifts in a meaningful way.

Correlation lies between plus and minus one, with perfect correlation corresponding to $\beta = \pi$ and perfect negative correlation corresponding to $\beta = 0$; $\beta = \pi/2$ represents independence. In the case when the drifts in firm value and the default barrier are equal, $\alpha_i = 0$, leading to $a_i = 0 = b$. Lastly, r_0 is positive and $r_0 \to +\infty$ as $\rho \to \pm 1$.

4.2.2 Evaluation details

In order to evaluate the joint survival probability, we use the integral form of the modified Bessel's function,

$$I_{(\frac{n\pi}{\beta})}\left(\frac{rr_0}{t}\right) = \frac{1}{\pi}\int_0^\pi e^{\frac{rr_0}{t}\cos\phi}\cos\left(\frac{n\pi\phi}{\beta}\right)\mathrm{d}\phi - \frac{1}{\pi}\sin\left(\frac{n\pi^2}{\beta}\right)\int_0^\infty e^{-\frac{rr_0}{t}\cosh s - \frac{n\pi s}{\beta}}\,\mathrm{d}s \quad (4.12)$$

and transform each integral so that the range of integration is $[0, 1]$. The equations are summarised below.

4.2.2.1 Simplified Survival Probability: Equal Drift Case

For the simple case that the drifts in the barrier and firm value are equal, namely $\mu_i = \gamma_i$, the joint survival probability is given by

$$\begin{aligned}&\mathbb{P}(\underline{X}_1(t) \geq B_1, \underline{X}_2(t) \geq B_2) \\ &= \frac{2r_0}{\sqrt{2\pi t}} e^{-r_0^2/4t} \sum_{\text{n odd}} \frac{1}{n} \sin\left(\frac{n\pi\theta_0}{\beta}\right)\left[I_{\frac{1}{2}(\frac{n\pi}{\beta}+1)}\left(\frac{r_0^2}{4t}\right) + I_{\frac{1}{2}(\frac{n\pi}{\beta}-1)}\left(\frac{r_0^2}{4t}\right)\right].\end{aligned}$$

Using the integral form of the modified Bessel's equation and various simple trigonometric identities, this becomes

$$\frac{4r_0}{\pi\sqrt{2\pi t}} e^{-r_0^2/4t} \sum_{\text{n odd}} \frac{1}{n} \sin\left(\frac{n\pi\theta_0}{\beta}\right)\left[\int_0^\pi G_n(\phi)\,\mathrm{d}\phi + \int_0^\infty H_n(s)\,\mathrm{d}s\right] \quad (4.13)$$

where

$$G_n(\phi) = \cos\left(\frac{n\pi\phi}{2\beta}\right)\cos\left(\frac{\phi}{2}\right) e^{r_0^2\cos\phi/(4t)} \quad (4.14)$$

$$H_n(s) = \cos\left(\frac{n\pi^2}{2\beta}\right)\sinh\left(\frac{s}{2}\right) e^{-r_0^2\cosh s/(4t) - n\pi s/(2\beta)} \quad (4.15)$$

To change the regions of integration to $[0,1]$, we make the substitutions

$$\begin{aligned} x &= \frac{\phi}{\pi}, \\ x &= \frac{s}{s+1}, \end{aligned}$$

in (4.14) and (4.15), respectively. The function to be integrated over $[0,1]$ is then

$$\frac{4r_0}{\sqrt{2\pi t}} e^{-r_0^2/4t} \sum_{\text{n odd}} \frac{1}{n} \sin\left(\frac{n\pi\theta_0}{\beta}\right) \left[\cos\left(\frac{n\pi^2 x}{2\beta}\right) \cos\left(\frac{\pi x}{2}\right) e^{(r_0^2 \cos(\pi x))/(4t)} + \frac{1}{\pi(1-x)^2} \cos\left(\frac{n\pi^2}{2\beta}\right) \sinh\left(\frac{x}{2(1-x)}\right) e^{-r_0^2 \cosh(\frac{x}{1-x})/(4t)} e^{-n\pi x/(2\beta(1-x))} \right].$$

As x tends to one, the second part of this function becomes difficult to evaluate numerically, so we exclude it from the numerical integration for $|1-x| < \epsilon$; $\epsilon = 0.007$ proves sufficient.

The alternative substitution

$$x = \frac{s}{D}$$

in (4.15) for some (large) constant D gives the function

$$\frac{4r_0}{\sqrt{2\pi t}} e^{-r_0^2/4t} \sum_{\text{n odd}} \frac{1}{n} \sin\left(\frac{n\pi\theta_0}{\beta}\right) \left[\cos\left(\frac{n\pi^2 x}{2\beta}\right) \cos\left(\frac{\pi x}{2}\right) e^{(r_0^2 \cos(\pi x))/(4t)} + \frac{D}{\pi} \cos\left(\frac{n\pi^2}{2\beta}\right) \sinh\left(\frac{Dx}{2}\right) e^{-(r_0^2 \cosh(Dx))/(4t)} e^{-n\pi Dx/(2\beta)} \right]. \tag{4.16}$$

Calculating simplified survival probabilities for 3, 5, 7 and 10 years using each method (with D=30) gives identical results to 9 decimal places and takes the same amount of time. Using 16 integration partitions, the calculation takes about 15 seconds on an Intel Xeon CPU 3.4 Ghz x2 machine. By truncating the integral in this way, it is straightforward to obtain an upper bound on the error by approximating the integral from D to ∞, and hence to ensure that D is chosen such that the error is sufficiently small.

4.2.2.2 Simplified Survival Probability: $\beta = \pi/k$

(4.11) is quick and easy to evaluate with just the substitution $x = \theta k/\pi$ required to change the range of integration to $[0,1]$. Few terms in the infinite sum are required and calculations are very fast – for a given value of k, the joint survival probability takes just a few seconds to calculate.

4.2.2.3 General Survival Probability

Writing

$$\begin{aligned} A(\theta) &= a_1\sigma_1 \sin(\beta-\theta) + a_2\sigma_2 \sin\theta \\ f_1 &= A(\theta) + \frac{r_0}{t}\cos\phi \\ f_2 &= A(\theta) - \frac{r_0}{t}\cosh s, \end{aligned}$$

and using the integral form of the modified Bessel's function, (4.12), the joint survival probability, (4.5), is

$$\frac{2}{\pi\beta t} e^{a_1B_1+a_2B_2+bt} \sum_{n=1}^{\infty} e^{-r_0^2/2t} \sin\left(\frac{n\pi\theta_0}{\beta}\right) \int_0^{\beta}\int_0^{\infty} \sin\left(\frac{n\pi\theta}{\beta}\right) r e^{-r^2/2t} \times \left[\int_0^{\pi} e^{rf_1} \cos\left(\frac{n\pi\phi}{\beta}\right) \mathrm{d}\phi - \sin\left(\frac{n\pi^2}{\beta}\right) \int_0^{\infty} e^{rf_2 - n\pi s/\beta}\, \mathrm{d}s\right] \mathrm{d}r\, \mathrm{d}\theta. \tag{4.17}$$

Since it is clear that the integrals in r and s tend to zero rapidly as r and s tend to infinity, respectively, we evaluate the integrals between zero and some constant D, making the substitutions

$$\begin{aligned} x &= \frac{\theta}{\beta} \\ y &= \frac{\phi}{\pi} \quad \text{and} \quad y = \frac{s}{D} \\ z &= \frac{r}{D} \end{aligned}$$

for θ, ϕ, s and r in (4.17). The function we need to integrate over $[0,1]^3$ is then

$$\frac{2D^2}{\pi t} e^{a_1B_1+a_2B_2+bt} \sum_{n=1}^{\infty} e^{-r_0^2/2t} \sin\left(\frac{n\pi\theta_0}{\beta}\right) \sin(n\pi x) z e^{-D^2z^2/2t} \times \left[\pi e^{Dzf_1} \cos\left(\frac{n\pi^2 y}{\beta}\right) - D\sin\left(\frac{n\pi^2}{\beta}\right) e^{Dzf_2 - n\pi Dy/\beta}\right] \tag{4.18}$$

where

$$\begin{aligned} A(x) &= a_1\sigma_1 \sin(\beta(1-x)) + a_2\sigma_2 \sin(\beta x) \\ f_1 &= A(x) + \frac{r_0}{t}\cos(\pi y) \\ f_2 &= A(x) - \frac{r_0}{t}\cosh(Dy). \end{aligned} \tag{4.19}$$

$D = 30$ or $D = 50$ is generally more than sufficient to capture the value of the integral. Rather than cutting off the infinite integrals, it is possible to make the substitutions

$$\begin{aligned} x &= \frac{\theta}{\beta} \\ y &= \frac{\phi}{\pi} \quad \text{and} \quad y = \frac{s}{s+1} \\ z &= \frac{r}{r+1}. \end{aligned}$$

The function to be integrated over $[0,1]^3$ then contains terms of the form $\frac{y}{(1-y)}$ and $\frac{z}{(1-z)}$, and so must be truncated for y and z close to one. This method, however, proved unstable and led to worse results than direct truncation of the infinite integrals at D. One other method we explored was to reduce the function to a double integral through completing the square in the r-integral. Again, the infinite integral in s could be dealt with by the substitutions $y = \frac{s}{(s+1)}$ or $y = s/D$. As the former led to problems with stability we used the second, simpler method. The function to be integrated over $[0,1]^2$ then became

$$\frac{2\sqrt{2\pi t}}{\pi} e^{a_1 B_1 + a_2 B_2 + bt} \sum_{n=1}^{\infty} e^{-r_0^2/2t} \sin\left(\frac{n\pi\theta_0}{\beta}\right) \sin(n\pi x) \times$$
$$\left[\pi \cos\left(\frac{n\pi^2 y}{\beta}\right) f_1 e^{tf_1^2/2} \Phi(f_1\sqrt{t}) - D \sin\left(\frac{n\pi^2}{\beta}\right) f_2 e^{tf_2^2/2 - n\pi Dy/\beta} \Phi(f_2\sqrt{t})\right]$$

with $A(x)$, f_1 and f_2 as defined in (4.19). Whilst the two-dimensional numerical integration was considerably faster than the three-dimensional one, this advantage was offset by the presence of the cumulative normal distribution term, in particular the combined term $e^{\cosh^2(Dy)}\Phi(f_2\sqrt{t})$ which was highly problematic to evaluate numerically. Attempts to use either the asymptotic expansion of the cumulative normal, or rational approximations to it, failed to achieve the required degree of accuracy. They were also extremely involved algebraically. Since our primary focus is the general theory and form of the results rather than the numerical techniques, we stick with the slower, more accurate method.

4.3 Results & Analysis

The aim of this section is two-fold – firstly to consider the accuracy of our numerical calculations, and secondly to investigate the impact of various parameters on the joint survival probability.

There is little previous work in this area. Hua et al. (1998) provide a surface plot of the joint density of the maximum and minimum of two correlated Brownian motions for $\sigma_1 = \sigma_2 = 0.2\,\text{yr}^{-1/2}$, $t = 1\,\text{yr}$, $\alpha_1 = \alpha_2 = 0$ and $\rho = 0.5$. This has a similar form to the density function, (4.3), of interest to us. Hua et al. (1998) also provide values for some double lookback options for discrete sets of parameter values. They assume that $r_f = 0.05\,\text{yr}^{-1}$ and $q_1 = q_2 = 0$, and consider values for a few different strikes, volatilities and correlation values. They do not go as far as to plot the impact that changing parameters has on their results, however, and parameters are chosen such that their formulæ simplify. No work is done for the general case.

Zhou (2001a) plots the relation between default correlation and time for three different values of firm-value correlation (ρ = -0.1, 0,2, 0.4) and three different values of initial credit quality ($V_i(0)/b_i(0)$ = 2, 2.5, 3). Other assumptions are credit quality = 1.8 for the former, $\rho = 0.4$ for the latter, and $\sigma_1 = \sigma_2 = 0.4\,\text{yr}^{-1/2}$. All results are given for the simpler equal-drift case. In the general case, Zhou considers various time periods from one to ten years and for $\sigma_i = 0.3$, $\rho = 0.4$ and initial credit quality of 5, investigates whether setting $\mu_i = \gamma_i$ is a realistic simplifying assumption. He finds that for $\mu_i = 0.05$, varying γ_i from 0 to 0.03 to 0.05 has little impact on default correlations for shorter time periods (less than five years).

We consider the survival probability for the full range of firm-value correlations for a number of different time periods and for several values of initial credit quality. We do this for both the simple equal-drift case of Section 4.1.3.1 and also the general case. We use the same set of parameter values in each in order to evaluate the accuracy of the calculation for the general case. We then consider the impact of changing the barrier growth rate on the survival probability for different levels of correlation. Finally, we illustrate the use of the simplification in Section 4.1.3.2 to show the dependence of the survival probability on firm volatilities for various correlations.

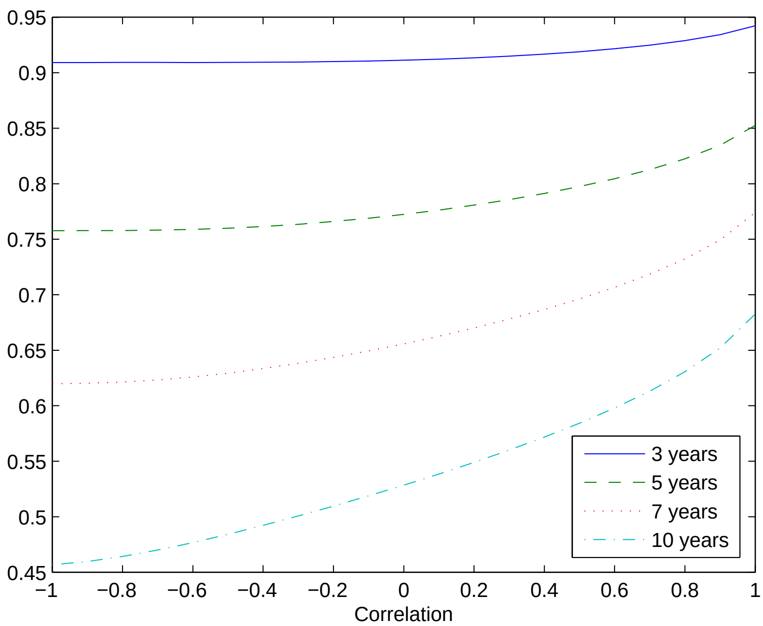
4.3.1 Time Period

Figure 4.1 illustrates the dependence of survival probability on time period for the general case, (4.5). Figure 4.2 shows the same result for the simplified case (4.8). Parameters are the same in each case, so we are modelling the situation when drifts in firm value and the barrier are equal, enabling us to confirm the stability and accuracy of our numerical calculations for the general case. As shown in Figure 4.3, the calculations are the same to within four decimal places at worst – the vertical scale has been changed for this graph to $[-6 \times 10^{-5}, 6 \times 10^{-5}]$ as otherwise the differences are too small to be seen. The differing accuracy of the numerical integration for different values of correlation is evident in the jagged shape of Figure 4.3. However, the errors are very small and by modifying the point at which the integrals in (4.17) are truncated, or increasing the refinement level of the grid, they can be decreased still further.

For an initial credit quality of 2, Figures 4.1 and 4.2 show that, as expected, survival probability declines with time. We also see that survival probability increases with correlation. This is not quite as obvious, but is also the result one would expect. The probability of at least one of the companies defaulting in a given time period is higher when they are negatively correlated than when they are positively correlated.

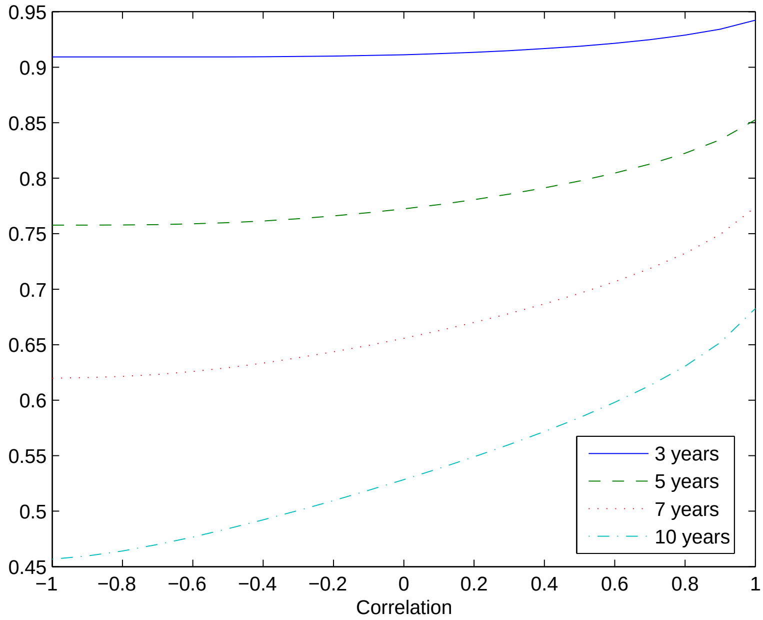
Figure 4.4 shows the same story, illustrating how fast survival probability drops off with time for different values of correlation. This figure also captures one of the main weaknesses of the structural model, namely that default probability over very small time periods is basically zero, and survival probability tends to one as time tends to zero. This then leads to zero short spreads when pricing corporate bonds.

Figure 4.1: Survival probability for different time periods, general case



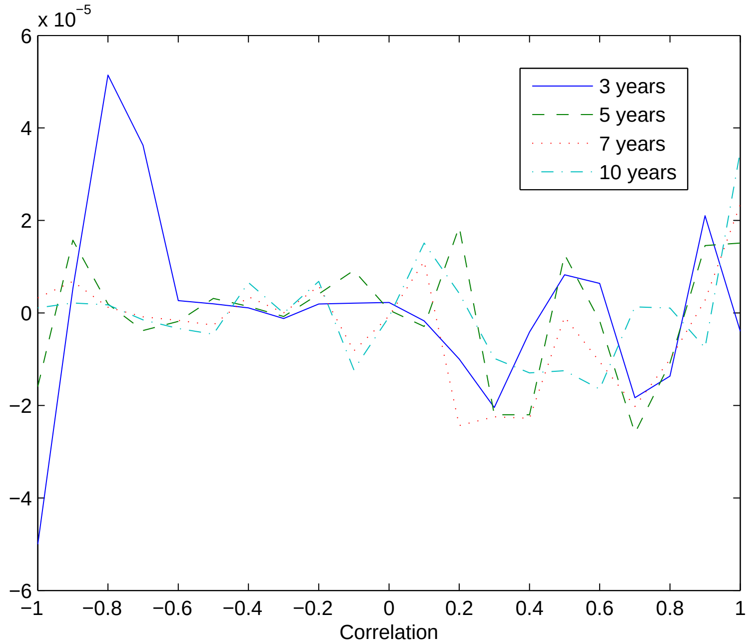
$\sigma_1 = \sigma_2 = 0.2$, $r_f = 0.05$, $q_1 = q_2 = 0$
$\gamma_1 = \gamma_2 = 0.03$, initial credit quality $= 2$

Figure 4.2: Survival probability for different time periods, equal drifts



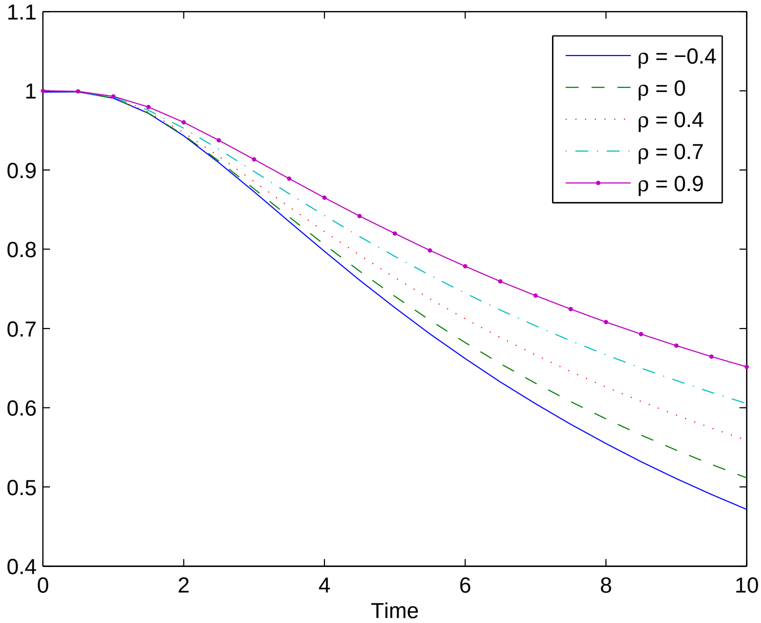
$\sigma_1 = \sigma_2 = 0.2$, $r_f = 0.05$, $q_1 = q_2 = 0$
$\gamma_1 = \gamma_2 = 0.03$, initial credit quality $= 2$

Figure 4.3: Difference between Figures 4.1 and 4.2



$\sigma_1 = \sigma_2 = 0.2$, $r_f = 0.05$, $q_1 = q_2 = 0$
$\gamma_1 = \gamma_2 = 0.03$, initial credit quality $= 2$

Figure 4.4: Survival probability for different correlations



$\sigma_1 = \sigma_2 = 0.2$, $r_f = 0.05$, $q_1 = q_2 = 0$,
$\gamma_1 = \gamma_2 = 0.03$, initial credit quality $= 2$

4.3.2 Initial Credit Quality

Initial credit quality is an input to the survival probability calculation and is equal to initial firm value divided by the initial level of the barrier,

$$\text{Initial credit quality} = \frac{V_i(0)}{b_i(0)}.$$

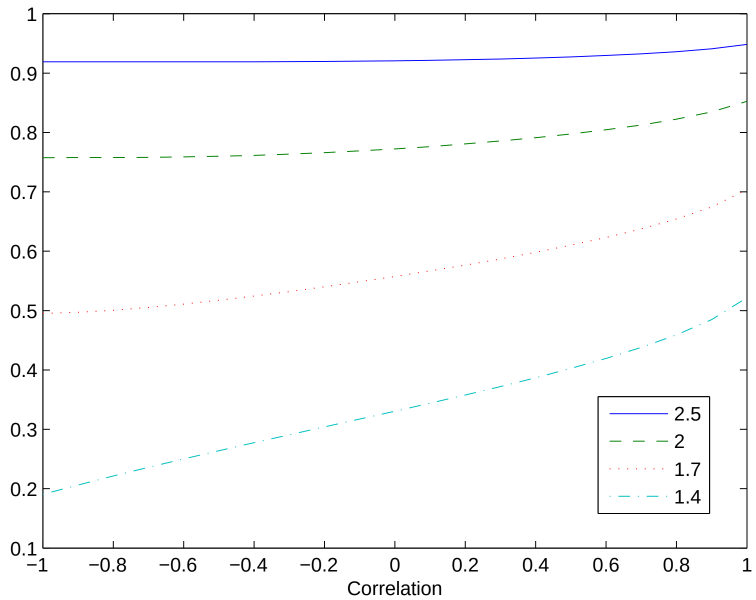
It is a measure of company strength – a stronger company is further from its default barrier, and is therefore represented by a higher number. It can be directly related to a company's distance-to-default, usually calculated as

$$\text{Distance-to-default} = \frac{\ln(V_i(0)) - \ln(b_i(0))}{\sigma_i} = \frac{1}{\sigma_i} \ln\left(\text{Initial credit quality}\right).$$

For a volatility of $\sigma_i = 0.2$, initial credit qualities of 1.4 and 2.5 then correspond to distances-to-default of 1.7 and 4.6 respectively. Figure 4.5 illustrates the dependence of survival probability on initial credit quality for the general case, (4.5), for values of initial credit quality between 1.4 and 2.5. Figure 4.6 shows the same results for the simplified equation (4.8). Again, we use the same parameters in order to compare the accuracy of the calculations, and Figure 4.7 shows the error to be of the order of

10^{-5}. Figures 4.5 and 4.6 show that, as expected, survival probability is greater for stronger companies and increases with correlation.

Figure 4.5: Survival probability for varying initial credit quality, general case



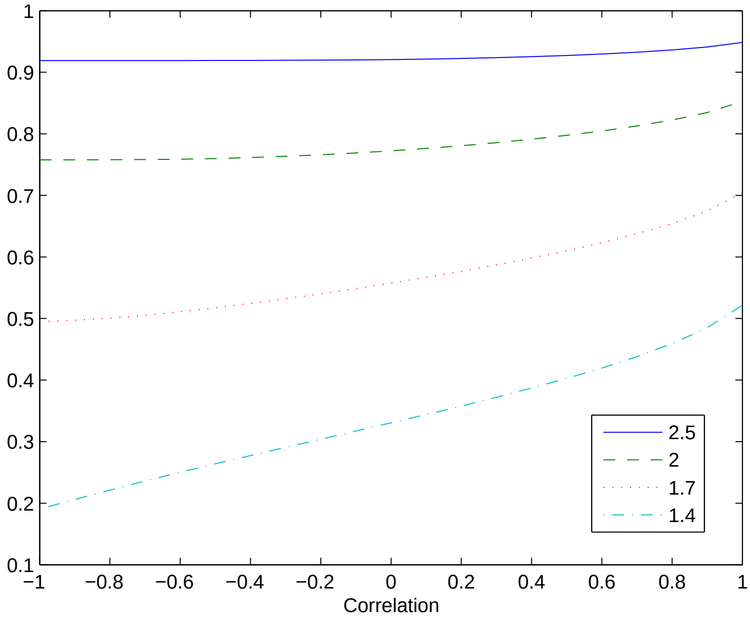
$\sigma_1 = \sigma_2 = 0.2$, $r_f = 0.05$,
$q_1 = q_2 = 0$, $\gamma_1 = \gamma_2 = 0.03$, $T = 5$ years

Figures 4.3 and 4.7 illustrate the accuracy of our numerical integration for the general case which is applicable when drifts are not necessarily equal. It is possible to further reduce the error at the expense of increased calculation time, but for the purposes of our calculations, the accuracy here is more than sufficient.

4.3.3 Barrier Growth Rate

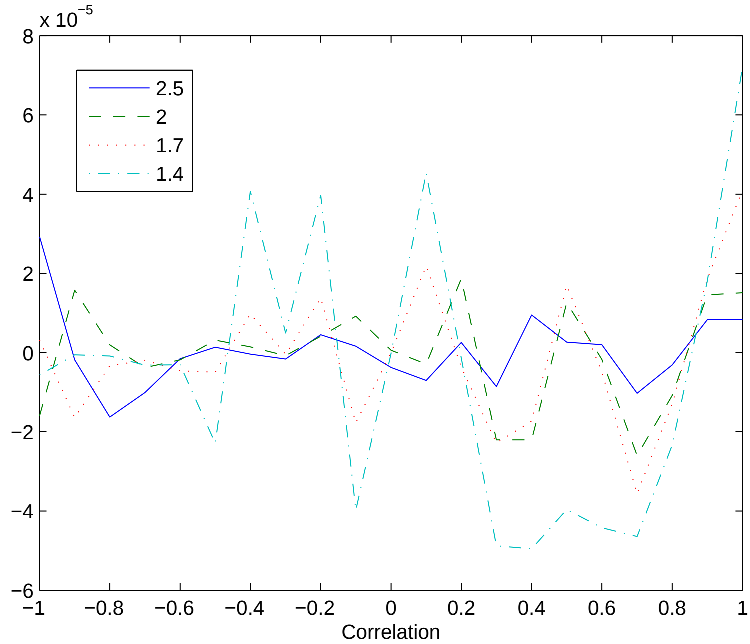
Returning to the most general case, (4.5), we now consider the impact that changing the growth rate of the default barrier has on survival probability. Results are shown in Figures 4.8, 4.9 and 4.10. In each case we keep the firm growth rate, $\mu_i = 0.03$ constant and we look at survival probabilities for values of γ_i between 0.01 and 0.08 for three different time periods. Figure 4.8 shows that for a 3-year time horizon, there is very little difference in survival probabilities as the barrier growth rate, γ_i, varies. This is in line with results obtained by Zhou (2001a). Figures 4.9 and 4.10 show the same results for 5 and 10-year time horizons, respectively, and illustrate that as the length of the time period increases, changing the growth rate of the barrier

Figure 4.6: Survival probability for varying initial credit quality, equal drifts



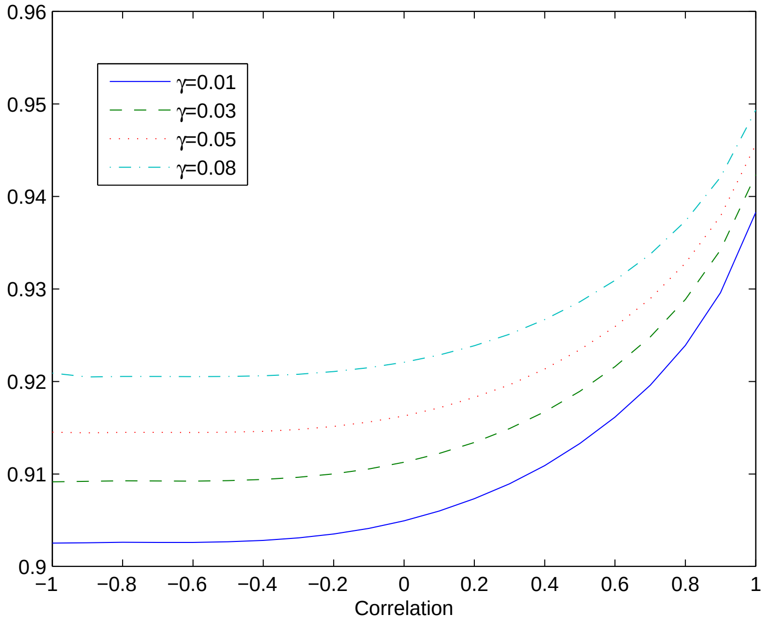
$\sigma_1 = \sigma_2 = 0.2$, $r_f = 0.05$,
$q_1 = q_2 = 0$, $\gamma_1 = \gamma_2 = 0.03$, $T = 5$ years

Figure 4.7: Difference between Figures 4.5 and 4.6



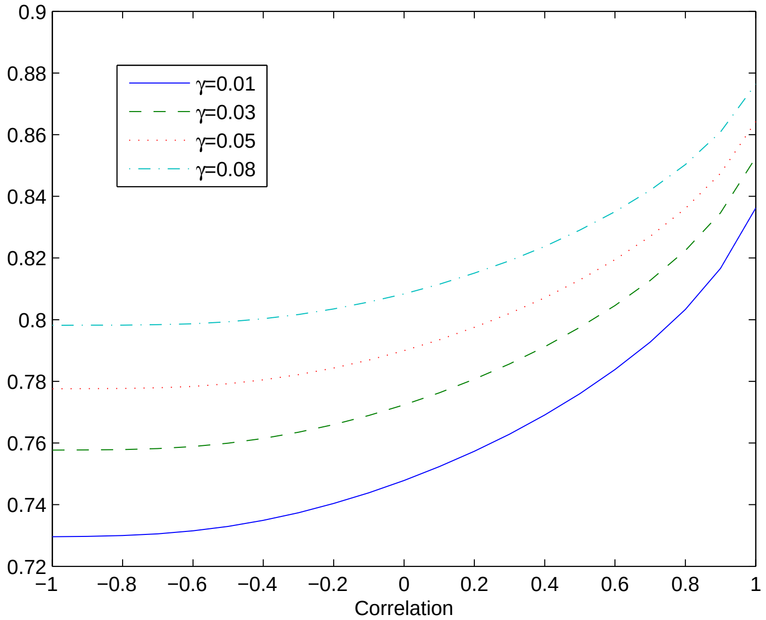
$\sigma_1 = \sigma_2 = 0.2$, $r_f = 0.05$,
$q_1 = q_2 = 0$, $\gamma_1 = \gamma_2 = 0.03$, $T = 5$ years

Figure 4.8: Survival probability for varying γ_i over a 3-year period



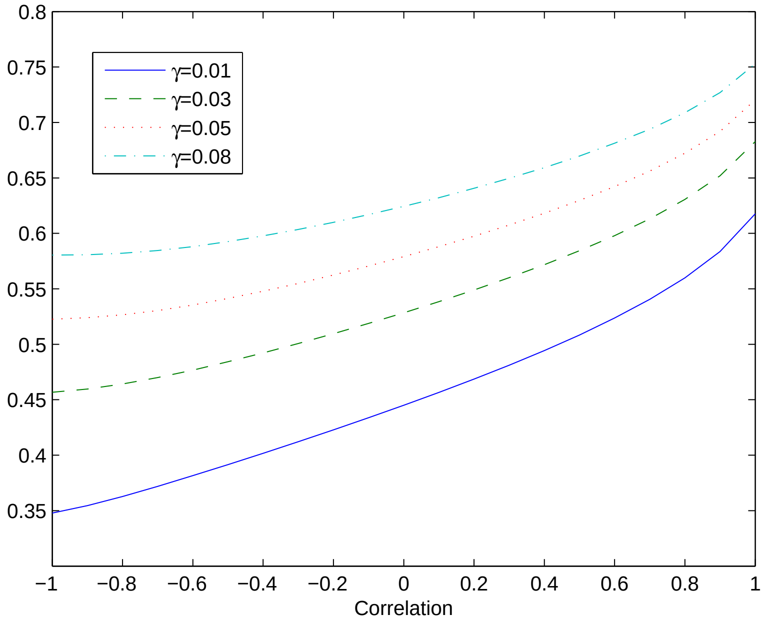
$\sigma_1 = \sigma_2 = 0.2$, $r_f = 0.05$,
$q_1 = q_2 = 0$, initial credit quality $= 2$

Figure 4.9: Survival probability for varying γ_i over a 5-year period



$\sigma_1 = \sigma_2 = 0.2$, $r_f = 0.05$,
$q_1 = q_2 = 0$, initial credit quality $= 2$

Figure 4.10: Survival probability for varying γ_i over a 10-year period



$\sigma_1 = \sigma_2 = 0.2$, $r_f = 0.05$,
$q_1 = q_2 = 0$, initial credit quality $= 2$

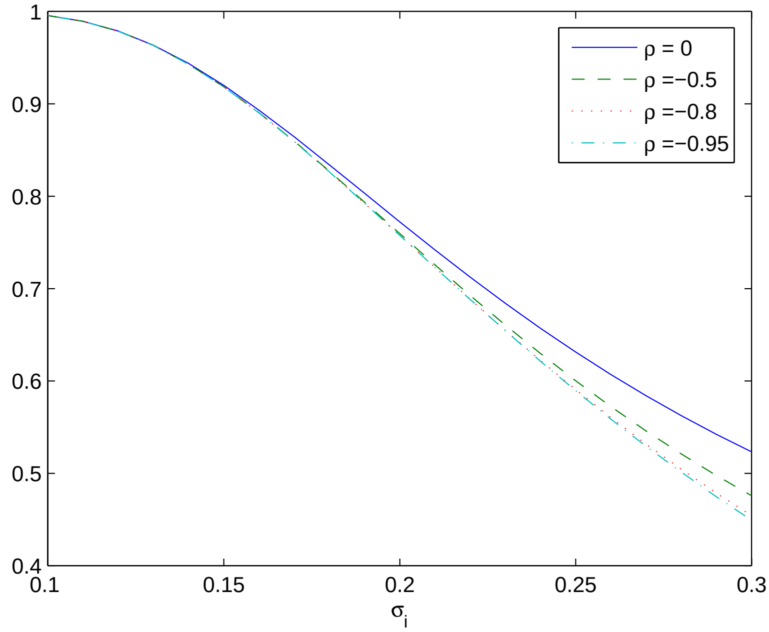
has an increasing impact on survival probabilities. The lower γ_i, the flatter, and therefore the higher the barrier and the more likely the firm is to default. As time to maturity increases, this has an increasing impact on the probability of default. Over longer time periods, survival probabilities are lower and more highly dependent on the growth rate of the barrier, as one would expect.

In each case, initial credit quality varies depending on the amount of discounting of the barrier (the size of γ_i). The model is set so that initial credit quality for $\mu_i = \gamma_i$ is 2, consistent with previous results. This differs from Zhou (2001a) in that, in his work, initial credit quality is fixed by construction and the final level of his barrier varies. However, our conclusions are similar. For short maturities, $\mu_i = \gamma_i$ is a good approximation, but it becomes harder to justify as T increases.

4.3.4 Volatility

Figure 4.11 illustrates the impact of volatility on 5-year survival probabilities for negative values of correlation, calculated using the simplification $\beta = \pi/k$. We consider $k = 2, 3, 5$ and 10, corresponding to $\rho = 0$, -0.5, -0.8 and -0.95, respectively. As

Figure 4.11: Survival probability for varying σ_i over a 5-year period



$r_f = 0.05$, $q_1 = q_2 = 0$, $T = 5$
$\gamma_1 = \gamma_2 = 0.03$, initial credit quality $= 2$

mentioned before, the downside of this simplification is that it only applies to negative values of correlation; the upside is that the calculation is extremely quick and easy to implement. For negative correlation, volatility clearly has a large impact on survival probabilities, much more so than the degree of correlation. In fact, only for larger values of volatility does changing correlation result in noticeably different survival probabilities and as $\rho \to -1$, the impact of changing correlation declines further.

4.4 Conclusion

In this chapter, we have provided the derivation of the survival probability for a two-firm structural model, both in the general case and for two simplified scenarios. We discussed numerical methods for evaluating the function, and illustrated the accuracy of our calculations. We finished with a summary of the dependence of the joint survival probability on various input parameters and showed it to behave as we would expect.

Whilst Zhou (2001a) and Hua et al. (1998) have used the joint survival probability, (4.3), for calculating default correlations and double lookbacks, respectively, neither provided a full numerical evaluation for all values of correlation, or considered sensitivity to the main input parameters.

In Chapter 5, we use the underlying results and methods derived in this chapter to calculate corporate bond yields and CDS spreads. Our ability to evaluate the joint survival probability for the general case, $\mu_i \neq \gamma_i$, for all parameter values is key in its application to modelling credit spreads.

Chapter 5

Calculating Credit Spreads

The aim of this chapter is to apply the results of the previous chapter to price corporate bonds in the presence of default contagion and two-company credit default swap baskets. We begin by putting Chapter 4 into context, describing the problem of interest and our approach. For bonds and credit default swaps in turn, we then derive the relevant mathematical formulæ, outline their numerical evaluation and discuss the results.

5.1 Framework

Throughout this chapter, we consider two companies, the values of which evolve as correlated geometric Brownian motions. We assume that each company has an exponential default barrier, reflecting the existence of safety covenants and assume that the barrier for company i is $K_i e^{-\gamma_i(T-t)}$ for $i = 1, 2$. As noted before, this is of a similar form to the barrier definition in Black and Cox (1976). We keep a general discount factor, γ_i, rather than using the risk-free rate as in Black and Cox for added flexibility. In particular, this enables us to set the drift in firm value equal to the growth rate of the barrier, a scenario that makes a lot of practical sense, as outlined in Section 4.1.3.1. In the notation of Chapter 4,

$$\begin{aligned} b_i(t) &= b_i(0)e^{\gamma_i t} = K_i e^{-\gamma_i(T-t)} \\ \Rightarrow \quad b_i(0) &= K_i e^{-\gamma_i T}. \end{aligned}$$

We use the following standard result regarding the distribution function of the minimum of a Brownian motion, as given in Musiela and Rutkowski (1998),

$$\mathbb{P}(\underline{X}_i(t) \geq B_i) = \Phi\left(\frac{-B_i + \alpha_i t}{\sigma_i \sqrt{t}}\right) - e^{2\alpha_i B_i/\sigma_i^2}\Phi\left(\frac{B_i + \alpha_i t}{\sigma_i \sqrt{t}}\right), \quad B_i \leq 0 \tag{5.1}$$

where $\Phi(\cdot)$ is the standard cumulative normal distribution function.

5.2 Bond Yield Calculation

We now apply the results of Section 4.1.2 to value a corporate bond in the presence of contagion. We consider the $t = 0$ value of a zero coupon bond issued by company one, maturing at time T. We denote this $C_1(0, T)$.

The value of a zero-coupon corporate bond to a bondholder arises from two components – its value on maturity, should it mature, and its value in the event of default. We calculate the maturity payment and the payment on default separately, discounted in each case to time zero. Adding them together then gives the price of the bond at $t = 0$, $C_1(0, T)$. This can be converted to a bond yield $y_1(0, T)$ by

$$y_1(0, T) = -\frac{1}{T} \ln\left(\frac{C_1(0, T)}{K_1}\right). \tag{5.2}$$

We incorporate default contagion by assuming that company one defaults on its outstanding debt the first time that the value of either company reaches its default barrier. In this way, there are two components to the default mechanism. If the value of firm one declines sufficiently, the company is forced into bankruptcy – this is due to the direct performance of the company itself and is exactly the framework used in normal first passage structural models. The second way in which company one can default is if company two goes bankrupt, modelled as the time when the value of company two reaches its default barrier – default contagion. An example of such a scenario would be if company two was the only supplier of a key component in company one's business, with company one unable to operate without it.

It is worth noting that company two need not default automatically if company one does. It can continue to operate regardless of the financial viability of company one with dependence solely through the asset correlation, ρ. *As a result, the model is asymmetric with respect to default risk, in stark contrast with the majority of previous models incorporating a credit dependence structure, and there is a degree of causality* *introduced into the default event.* The natural extension of this framework to larger portfolios of companies would be the type of situation considered in Jarrow and Yu (2001) in which 'primary' companies impact 'secondary' companies but not vice versa. A 'primary' company is likely to be larger with a greater market impact than a 'secondary' company. For example Microsoft or General Motors compared to a small, local IT or auto component manufacturer.

This framework is only really realistic for $\rho \geq 0$. It is highly unlikely that the bankruptcy of one company would lead to the immediate default of another, negatively correlated company. Whilst it is possible to contrive a theoretical example (e.g. a highly diversified company like General Electric might be key to one of its suppliers, but negatively correlated with it overall), economically it is rather improbable in practice and so we restrict ourselves to consideration of $0 \leq \rho \leq 1$.

We proceed by calculating the maturity payment and default payment in turn. For ease of notation, as before we denote the joint survival density function by

$$p(x_1, x_2, t) = \frac{\partial^2}{\partial x_1 \partial x_2} \mathbb{P}(X_1(t) \leq x_1, X_2(t) \leq x_2, \underline{X}_1(t) \geq B_1, \underline{X}_2(t) \geq B_2).$$

5.2.1 Payment at Maturity

As in Section 2.3.2.1 we assume that

$$\text{Payment at maturity} = \min(\omega_1 V_1(T), K_1) \quad \text{provided} \quad \tau_1 > T,\, \tau_2 > T$$

where K_1 is the par value of the bond, τ_i denotes the default time of company i, and ω_1 is a constant write down factor. This factor is the same as used in the specification of the payment on default in Section 5.2.2 and represents the fact that a portion of the defaulting company's value is lost to bondholders in the event of a default or restructuring.

Changing to $X_i(t)$ coordinates, the barriers become $B_i = \ln(K_i/V_i(0)) - \gamma_i T$. The payment at maturity can therefore be represented as

$$\text{payment} = \begin{cases} K_1 & \text{if} \quad \underline{X}_1(T) \geq B_1, \underline{X}_2(T) \geq B_2, X_1(T) \geq d \\ \omega_1 V_1(0) e^{X_1(T)+\gamma_1 T} & \text{if} \quad \underline{X}_1(T) \geq B_1, \underline{X}_2(T) \geq B_2, X_1(T) \leq d \\ 0 & \text{if} \quad \text{either} \quad \underline{X}_1(T) < B_1 \quad \text{or} \quad \underline{X}_2(T) < B_2 \end{cases}$$

where

$$d = \ln \frac{K_1}{\omega_1 V_1(0)} - \gamma_1 T = B_1 - \ln \omega_1 \geq B_1.$$

Since we know the joint density function of the Brownian motions given that they have not yet defaulted from (4.3), we can evaluate the expected value of this payoff, discounted back to $t = 0$, by integrating over the possible final values of $X_1(t)$ and $X_2(t)$. Denoting the expected discounted maturity payoff by DMP,

$$\begin{aligned}\mathrm{DMP} &= e^{-r_f T}\int_{B_2}^{\infty}\int_{d}^{\infty} K_1 p(x_1, x_2, T)\,\mathrm{d}x_1\,\mathrm{d}x_2 \\ &+ \; e^{-r_f T}\int_{B_2}^{\infty}\int_{B_1}^{d} \omega_1 V_1(0) e^{x_1+\gamma_1 T} p(x_1, x_2, T)\,\mathrm{d}x_1\,\mathrm{d}x_2\end{aligned}$$

For $\omega_1 = 1$, $d = B_1$ so the second term is zero and the value of the bond is simply its discounted par value multiplied by the probability that both companies survive until the maturity date of the bond.

Changing variables and integrating exactly as for (4.5),

$$\mathrm{DMP} = \sum_{n=1}^{\infty}\frac{C_1}{\beta}\int_0^{\beta}\sin\left(\frac{n\pi\theta}{\beta}\right) g_n^{+}(\theta)\,\mathrm{d}\theta \tag{5.3}$$

$$+\sum_{n=1}^{\infty}\frac{C_2}{\beta}\int_0^{\beta}\sin\left(\frac{n\pi\theta}{\beta}\right) g_n^{*}(\theta)\,\mathrm{d}\theta \tag{5.4}$$

where

$$\begin{aligned}C_1 &= \frac{2K_1}{T} e^{-r_f T} e^{a_1 B_1 + a_2 B_2 + bT} e^{-r_0^2/2T}\sin\left(\frac{n\pi\theta_0}{\beta}\right)\\ C_2 &= \frac{2\omega_1 V_1(0)}{T} e^{(\gamma_1 - r_f)T} e^{(a_1+1)B_1 + a_2 B_2 + bT} e^{-r_0^2/2T}\sin\left(\frac{n\pi\theta_0}{\beta}\right)\\ g_n^{+}(\theta) &= \int_{d^*(\theta)}^{\infty} r e^{-r^2/2T} e^{A(\theta) r} I_{(\frac{n\pi}{\beta})}\left(\frac{r r_0}{T}\right)\,\mathrm{d}r\\ g_n^{*}(\theta) &= \int_{0}^{d^*(\theta)} r e^{-r^2/2T} e^{[A(\theta)+\sigma_1\sin(\beta-\theta)]r} I_{(\frac{n\pi}{\beta})}\left(\frac{r r_0}{T}\right)\,\mathrm{d}r\\ d^*(\theta) &= \frac{d - B_1}{\sigma_1\left[\sqrt{1-\rho^2}\cos\theta + \rho\sin\theta\right]} = \frac{\ln\omega_1}{\sigma_1\sin(\theta-\beta)} \geq 0.\end{aligned}$$

Details are in Appendix C. In the case that there is zero recovery on default, this represents the value of the bond, and hence can be converted into a yield using (5.2). The method used for evaluating the maturity payment is given in Section 5.2.3 and results are examined in Section 5.2.4. The presence of $d^*(\theta)$ as one of the limits on the integration means that it is not possible to simplify the maturity payment formulæ in the same way as for the joint survival probability in Section 4.1.3.1 in the case of equal drifts.

5.2.2 Payment on Default

In this section we calculate the payment on default by company one. We assume that default occurs the first time that either company hits its default barrier and that in the event of default, the bondholder receives a percentage of discounted par value. In other words, the payment on default is $\omega_1 K_1 e^{-r_f(T-\tau)}$ at $\tau = \min(\tau_1, \tau_2)$ provided $\tau < T$; here, ω_1 is the same constant write-down factor as before.

This is the same payoff as used by Black and Cox (1976) and is highly attractive since the discounted default payment becomes

$$\begin{aligned} & e^{-r_f\tau}\omega_1 K_1 e^{-r_f(T-\tau)} \times \mathbb{P}(\tau \in [0,T]) \\ = \; & \omega_1 K_1 e^{-r_f T}(1 - \mathbb{P}(\underline{X}_1(T) \geq B_1, \underline{X}_2(T) \geq B_2) \end{aligned} \tag{5.5}$$

We can calculate the joint survival probability as in Chapter 4 and so we are able to accurately calculate the default payment for general drifts. No further numerical calculations are required.

By construction, the default payment is worth less than discounted par, and so we just need to ensure that it is worth less than the value of the firm at default. Since company one must be worth at least as much as its default barrier, a sufficient condition is

$$\begin{aligned} \omega_1 K_1 e^{-r_f(T-\tau)} &\leq K_1 e^{-\gamma_1(T-\tau)} \\ \Rightarrow \omega_1 &\leq e^{(r_f-\gamma_1)(T-\tau)}. \end{aligned}$$

5.2.3 Numerical Details

The default payment, (5.5), is evaluated exactly as in Section 4.2.2.3. To evaluate the maturity payment, we proceed in the same way, using the integral form of the modified Bessel's function, (4.12). Making the substitution $x = \theta/\beta$, line (5.3) becomes

$$C_1 \int_0^1 \int_{d^*}^{\infty} \sin(n\pi x) r e^{-r^2/(2T)} e^{A(x)r} \times \tag{5.6}$$
$$\left[\int_0^1 e^{\frac{rr_0}{T}\cos(\pi y)} \cos\left(\frac{n\pi^2 y}{\beta}\right) \mathrm{d}y - \frac{1}{\pi}\sin\left(\frac{n\pi^2}{\beta}\right)\int_0^{\infty} e^{-\frac{rr_0}{T}\cosh s - \frac{n\pi s}{\beta}} \mathrm{d}s\right] \mathrm{d}r\,\mathrm{d}x$$

The second term tends rapidly to zero as s tends to infinity, so we consider $s \in [0, D]$, for some large constant D. Setting $y = s/D$ and writing

$$\begin{aligned}
A(x) &= a_1\sigma_1 \sin(\beta(1-x)) + a_2\sigma_2 \sin(\beta x) \\
F_1 &= A(x) + \frac{r_0}{T}\cos(\pi y) \\
F_2 &= A(x) - \frac{r_0}{T}\cosh(Dy),
\end{aligned}$$

(5.6) becomes

$$C_1 \int_0^1 \int_{d^*}^{\infty} \sin(n\pi x) r e^{-r^2/(2T)} \times \left[\int_0^1 e^{F_1 r} \cos\left(\frac{n\pi^2 y}{\beta}\right) \mathrm{d}y - \frac{D}{\pi}\sin\left(\frac{n\pi^2}{\beta}\right)\int_0^1 e^{F_2 r - n\pi D y/\beta}\,\mathrm{d}y\right] \mathrm{d}r\,\mathrm{d}x. \quad (5.7)$$

Similarly, writing

$$\begin{aligned}
A_1(x) &= A(x) + \sigma_1 \sin(\beta(1-x)) \\
F_3 &= A_1(x) + \frac{r_0}{T}\cos(\pi y) \\
F_4 &= A_1(x) - \frac{r_0}{T}\cosh(Dy),
\end{aligned}$$

line (5.4) is

$$C_2 \int_0^1 \int_0^{d^*} \sin(n\pi x) r e^{-r^2/(2T)} \times \left[\int_0^1 e^{F_3 r} \cos\left(\frac{n\pi^2 y}{\beta}\right) \mathrm{d}y - \frac{D}{\pi}\sin\left(\frac{n\pi^2}{\beta}\right)\int_0^1 e^{F_4 r - n\pi D y/\beta}\,\mathrm{d}y\right] \mathrm{d}r\,\mathrm{d}x.$$

Setting $z = r/d^*$, this becomes

$$C_2 \int_0^1 \int_0^1 \int_0^1 \sin(n\pi x) z {d^*}^2 e^{-{d^*}^2 z^2/(2T)} \times \left[e^{d^* z F_3} \cos\left(\frac{n\pi^2 y}{\beta}\right) - \frac{D}{\pi}\sin\left(\frac{n\pi^2}{\beta}\right) e^{d^* z F_4 - n\pi D y/\beta}\right] \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \quad (5.8)$$

Now $x = 1$ corresponds to $d^* = \infty$, but both (5.7) and (5.8) are zero for $x = 1$, so we set the integrand to zero at $x = 1$.

Finally, since (5.7) tends to zero as r tends to infinity, we consider $r \in [d^*, D^*]$, for some large constant D^*, and define z such that $r = d^* + (D^* - d^*)z$. (5.7) is then

$$C_1 \int_0^1 \int_0^1 \int_0^1 \sin(n\pi x) z^*(D^* - d^*) e^{-{z^*}^2/(2T)} \times \left[e^{F_1 z^*} \cos\left(\frac{n\pi^2 y}{\beta}\right) - \frac{D}{\pi}\sin\left(\frac{n\pi^2}{\beta}\right) e^{F_2 z^* - n\pi D y/\beta}\right] \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

where we are denoting $z^* = d^* + (D^* - d^*)z$.

To evaluate the discounted maturity payment, we therefore integrate

$$\begin{aligned} & C_1 \sin(n\pi x) z^* (D^* - d^*) e^{-{z^*}^2/(2T)} \left[e^{F_1 z^*} \cos\left(\frac{n\pi^2 y}{\beta}\right) - \frac{D}{\pi} \sin\left(\frac{n\pi^2}{\beta}\right) e^{F_2 z^* - n\pi D y/\beta} \right] \\ + \quad & {d^*}^2 C_2 \sin(n\pi x) z e^{-{d^*}^2 z^2/(2T)} \left[e^{d^* z F_3} \cos\left(\frac{n\pi^2 y}{\beta}\right) - \frac{D}{\pi} \sin\left(\frac{n\pi^2}{\beta}\right) e^{d^* z F_4 - n\pi D y/\beta} \right] \end{aligned}$$

over $[0, 1]^3$ for $x \neq 1$.

Finally, we require $D^* > d^*$. Taking $D = 50$, since $d^* \to \infty$ as $x \to 1$, we take

$$d^* = \begin{cases} \min\left\{\frac{\ln \omega_1}{\sigma_1 \sin[\beta(x-1)]}, D\right\} & x \neq 1 \\ D & x = 1 \end{cases}$$

and we let $D^* = \max(D, d^*) = D$.

5.2.4 Results

In this section, we give an overview of the results generated by our model for different parameter values. We do this in two stages. We first consider yields implied by the maturity payment alone, and then taking into account payment on default. In this way, it is easier to see how each is contributing to the overall yield profile and its sensitivity to parameter values. In all cases, yields are given in percent.

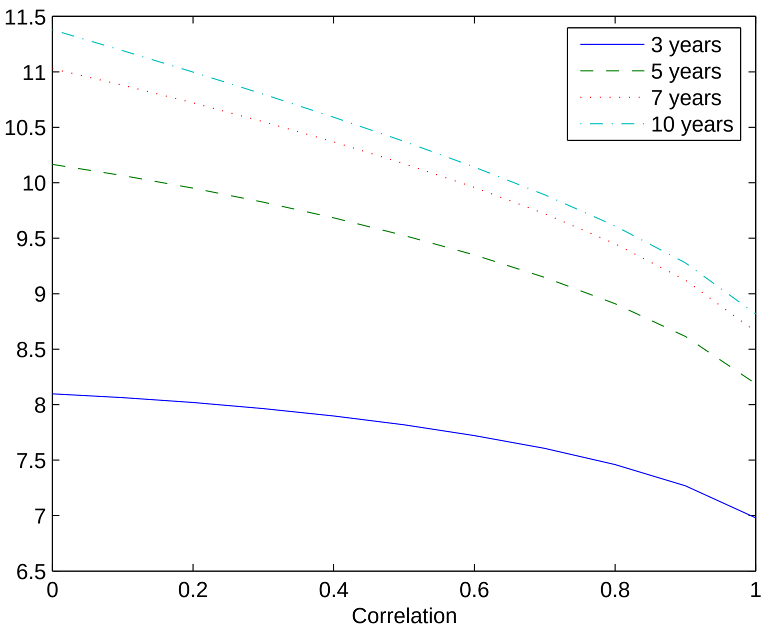
5.2.4.1 Maturity Payment

We begin by considering the form of the maturity payment and the resultant implied yield. This would be the yield on a bond in the event that there was zero recovery on default. We begin with the case when $\omega_1 = 1$. This is the simplest possible scenario since $d = B_1$ and therefore there is no contribution from line (5.4).

Figure 5.1 shows the implied yield for bonds with 3, 5, 7 and 10 years to maturity. As expected, longer-maturity bonds have higher yields. In addition, yields clearly decline as correlation increases. Since default is less likely with increasing correlation, the bond is less risky, bondholders are not rewarded with such high returns, increasing the price and reducing the yield.

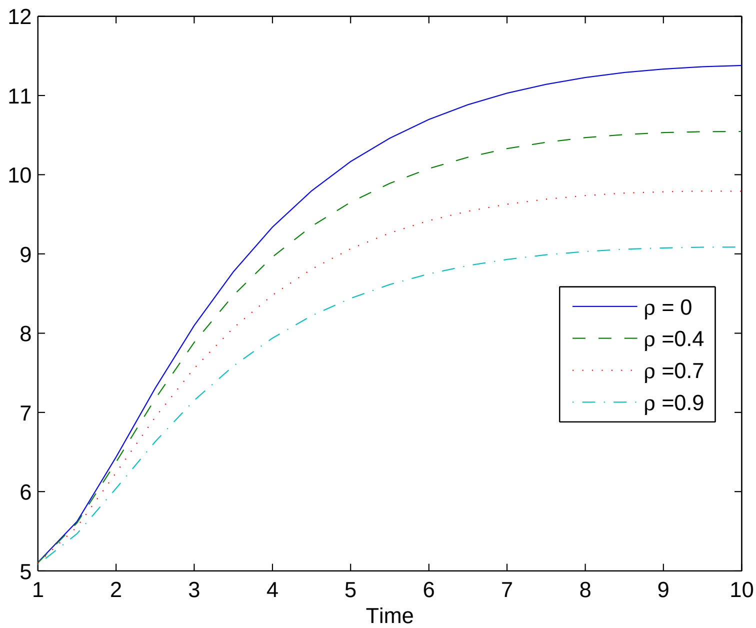
This result is readily apparent in Figure 5.2 – yields are highest when the companies are independent. The yield differential for different values of correlation is greatest

Figure 5.1: Implied yield for different maturities, $\omega_1 = 1$



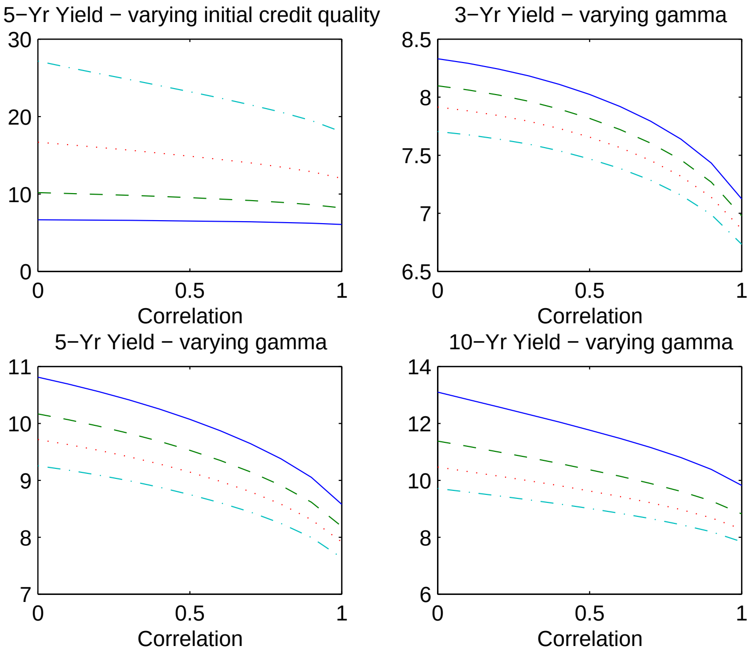
$\sigma_1 = \sigma_2 = 0.2$, $K_1 = 100$, $r_f = 0.05$,
$q_1 = q_2 = 0$, $\gamma_1 = \gamma_2 = 0.03$, initial credit quality $= 2$

Figure 5.2: Implied yield for different values of ρ, $\omega_1 = 1$



$\sigma_1 = \sigma_2 = 0.2$, $K_1 = 100$, $r_f = 0.05$,
$q_1 = q_2 = 0$, $\gamma_1 = \gamma_2 = 0.03$, initial credit quality $= 2$

Figure 5.3: Implied yield for changing credit quality and γ, $\omega_1 = 1$

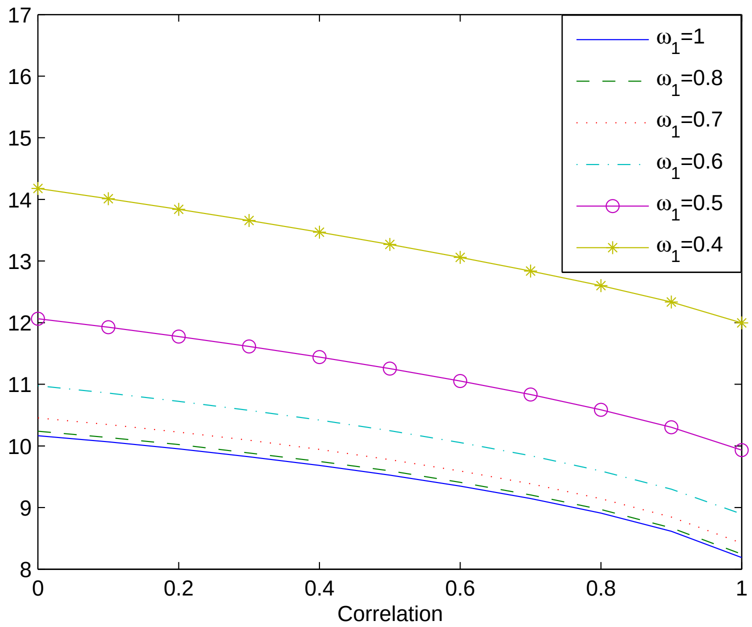


for the longest maturities and so the impact of contagion is greatest for longer-dated bonds. Again this is as expected. As time to maturity tends to zero, yields tend to the risk-free rate which is 5% in this case.

Figure 5.3 shows how initial credit quality (the initial value of the firm divided by the initial level of the barrier as described in Section 4.3.2) and the shape of the default barrier impact yields. The first plot shows implied yields for initial credit qualities of 1.4, 1.7, 2 and 2.5 for a 5-year bond (the solid line represents initial credit quality of 2.5). Clearly the initial proximity of the value of the firm to the default barrier has a large impact on yields – the closer to default initially, the higher the yield. The subsequent three plots illustrate how yields are affected by the rate of growth of the default barrier for 3, 5 and 10-year bonds. We take $\gamma = 0.01$, 0.03, 0.05, and 0.08 in each case, with $\gamma = 0.01$ the solid line. Yields increase with time-to-maturity and decline with the growth rate of the barrier. This is because a steeper barrier means the firm is initially further from default and so less risky. The degree of correlation has a larger impact for longer maturities.

So, in the simplest case of no payment on default and $\omega_1 = 1$, bond yields generated by our model behave as one would expect. We now consider the impact that changing ω_1 has on the yield implied by the maturity payment. We continue to assume that there is no payment on default prior to maturity for the time being.

Figure 5.4: Implied yield, different values of ω_1, no payment on default

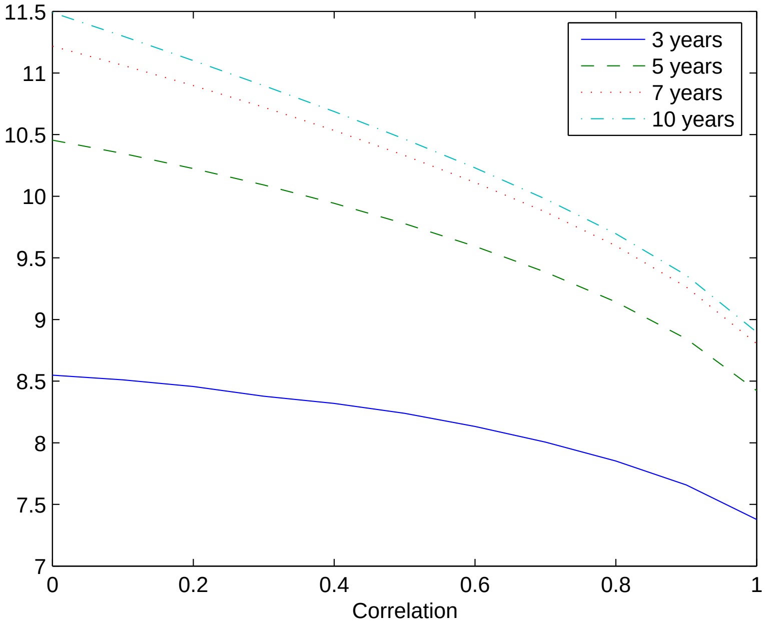


$\sigma_1 = \sigma_2 = 0.2$, $K_1 = 100$, $r_f = 0.05$, $T = 5$
$q_1 = q_2 = 0$, $\gamma_1 = \gamma_2 = 0.03$, initial credit quality $= 2$

Figure 5.4 illustrates yields for a five-year bond with initial credit quality of two. Since we are assuming zero recovery on default, ω_1 here acts to alter the minimum value of the company at bond maturity, T, for which bondholders are repaid in full, K_1/ω_1. The lower ω_1, the greater the anticipated refinancing and/or liquidation costs of the company and the greater the firm must be worth at maturity for bondholders to receive the full face value of their holdings. We therefore expect yields to increase with decreasing ω_1. As ω_1 decreases, not only does the minimum value of $V_1(T)$ required for full repayment of par increase, but for $K_1 < V_1(T) < K_1/\omega_1$, the value of the maturity payment $\omega_1 V_1(T)$ also decreases. As a result we see that decreasing ω_1 has an increasing impact on yields for lower ω_1.

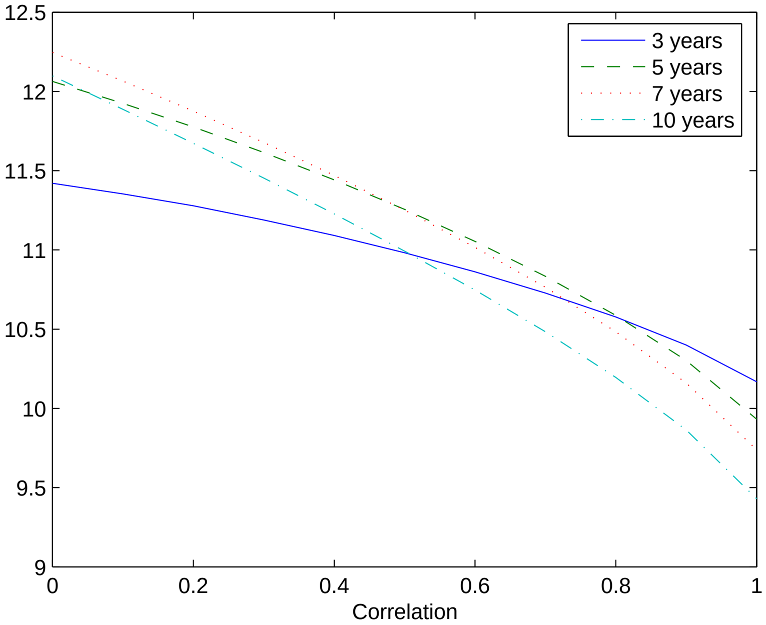
Figures 5.5 – 5.7 illustrate implied yields for different maturity bonds for decreasing values of ω_1. In all cases, yields decline with increasing correlation as we expect. The relative behaviour of different maturity bonds, however, is not necessarily as one would initially anticipate. Figure 5.5 has the form expected – yields on 3-year bonds are the lowest, 10-year bonds the highest. As ω_1 decreases, however, to 0.5 and then 0.4, we see the yields of shorter-maturity bonds crossing over those of longer-maturity bonds, until the situation is inverted with 3-year bonds having the highest yields in

Figure 5.5: Implied yield, $\omega_1 = 0.7$, no payment on default



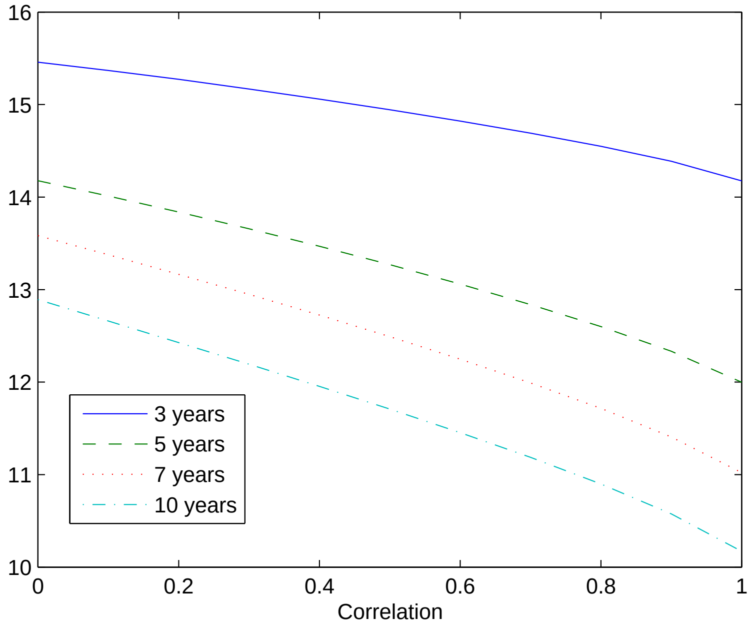
$\sigma_1 = \sigma_2 = 0.2$, $K_1 = 100$, $r_f = 0.05$,
$q_1 = q_2 = 0$, $\gamma_1 = \gamma_2 = 0.03$, initial credit quality $= 2$

Figure 5.6: Implied yield, $\omega_1 = 0.5$, no payment on default



$\sigma_1 = \sigma_2 = 0.2$, $K_1 = 100$, $r_f = 0.05$,
$q_1 = q_2 = 0$, $\gamma_1 = \gamma_2 = 0.03$, initial credit quality $= 2$

Figure 5.7: Implied yield, $\omega_1 = 0.4$, no payment on default

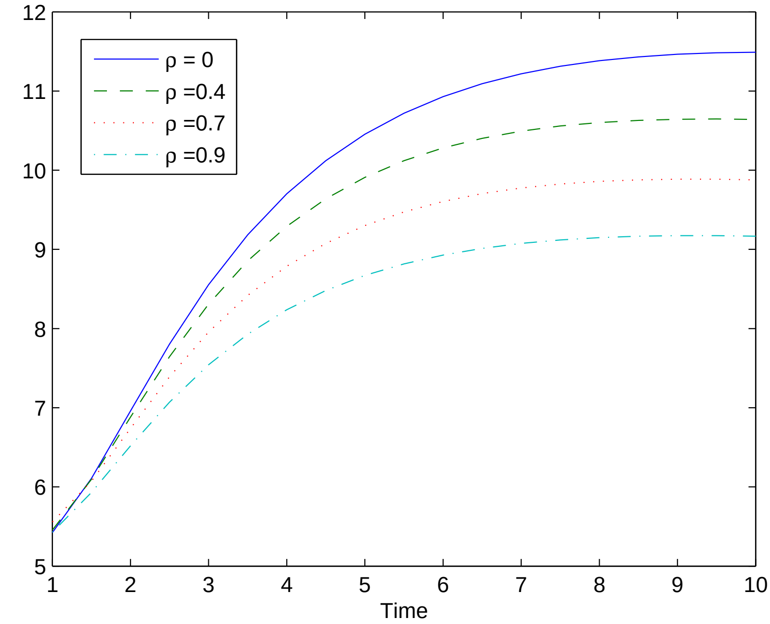


$\sigma_1 = \sigma_2 = 0.2$, $K_1 = 100$, $r_f = 0.05$,
$q_1 = q_2 = 0$, $\gamma_1 = \gamma_2 = 0.03$, initial credit quality = 2

Figure 5.7. What we are seeing is a change in the shape of the yield curve from a regular, upward-sloping yield curve, to a more complex one whose shape depends on the initial riskiness of the company. Figures 5.8 and 5.9 illustrate two yield curves with very different shapes.

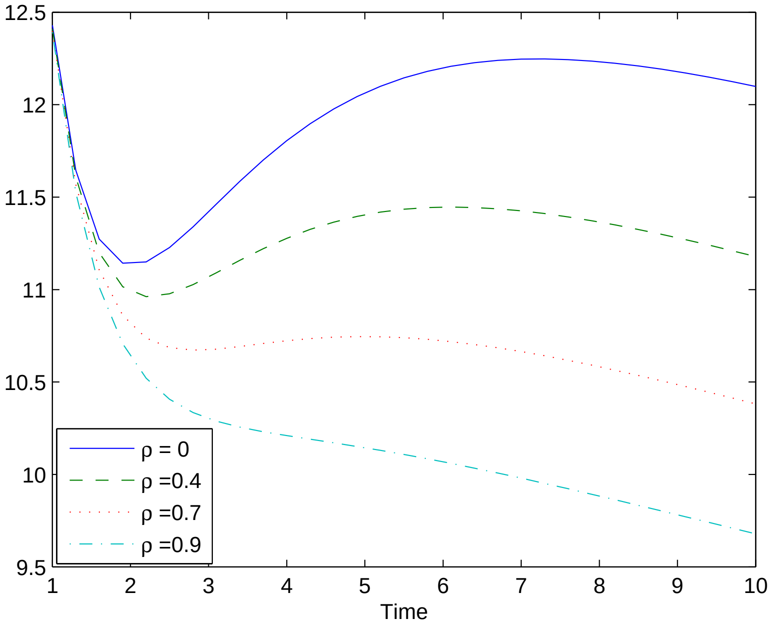
Figure 5.8 illustrates the yield curve for $\omega_1 = 0.7$ and has a fairly standard shape with yields increasing with time to maturity. Figure 5.9 is very different. In all cases, higher correlation leads to a less risky scenario and therefore lower yields. However, the yield curve is now downward sloping for highly correlated companies, and changes from downward sloping at the shorter end to upward sloping for longer-maturity bonds with lower correlations. In practical terms, as ω_1 decreases, the debt becomes much riskier, until the point that very short-term bonds are riskier than medium-term bonds. Since $V_1(T) \geq K_1/\omega_1$ is required for the bondholder to be repaid in full, for low values of ω_1, it is possible for $V_1(0) \leq K_1/\omega_1$. In this situation, the longer the company survives without defaulting, the more likely it is that firm value will increase sufficiently during the life of the bond to enable full repayment at maturity. This leads to a downward sloping yield curve as credit risk and therefore yields decline with time-to-maturity. Offsetting this is the fact that firm-value is more

Figure 5.8: Implied yield curve, $\omega_1 = 0.7$, no payment on default



$\sigma_1 = \sigma_2 = 0.2$, $K_1 = 100$, $r_f = 0.05$,
$q_1 = q_2 = 0$, $\gamma_1 = \gamma_2 = 0.03$, initial credit quality = 2

Figure 5.9: Implied yield curve, $\omega_1 = 0.5$, no payment on default



$\sigma_1 = \sigma_2 = 0.2$, $K_1 = 100$, $r_f = 0.05$,
$q_1 = q_2 = 0$, $\gamma_1 = \gamma_2 = 0.03$, initial credit quality = 2

likely to fall below the default level over longer periods of time. The combination of these opposing forces leads to the yield curve shapes shown in Figure 5.9.

Take the case $\rho = 0$ in Figure 5.9 as an example. Initially $V_1(0) \leq K_1/\omega_1$, and so very short maturity bonds are very risky. The firm is unlikely to increase in value sufficiently during the lifetime of the bond to repay the bondholder in full, and this factor offsets the fact the default is less likely to happen in the short-term, leading to a downward sloping yield curve. In the medium-term, for bonds of maturities from three to seven years, the greater likelihood of default over the longer term is the predominant factor, and the yield curve increases. For seven to ten year maturities, the yield curve tails off as default risk remains fairly similar.

The model is thus flexible enough to be able to capture different yield-curve shapes. This is a highly desirable attribute, and often a problem with more simple models[1]. An upward sloping yield curve is the most common in practice, but in certain scenarios it can change shape drastically. As shown in Figures 5.5 to 5.9, a variety of different shapes can be obtained using our model by changing the input parameters.

5.2.4.2 Bond Yield

We now consider yields implied by our model in its entirety, with both payment at maturity and payment on default contributing to the value of the bond. Results are in Figures 5.10 – 5.20.

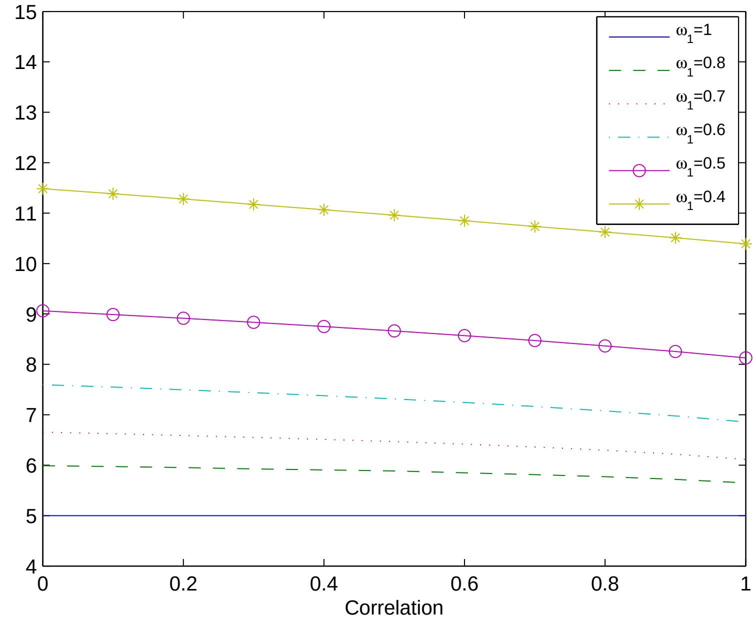
Figure 5.10 shows the impact of varying the write-down factor, ω_1, and has the same form as Figure 5.4. Of note, the case when $\omega_1 = 1$ corresponds to no write-down on default and in effect the bond becomes risk-free, yielding as we would expect, the risk-free rate of 5%, regardless of correlation.

Figures 5.11 and 5.12 show bond yields for a variety of maturities for $\omega_1 = 0.7$ and $\omega_1 = 0.5$. As in the case of no payment on default, we see that the yield curve inverts for low values of ω_1. This is further illustrated in Figures 5.13 and 5.14 and it is clear that the extent to which payments are written down has a large impact on yields, in particular at the short end of the yield curve.

Figures 5.15 – 5.16 then illustrate the impact of varying ω_1 on five-year yields for different values of initial credit quality. This time, results take exactly the form

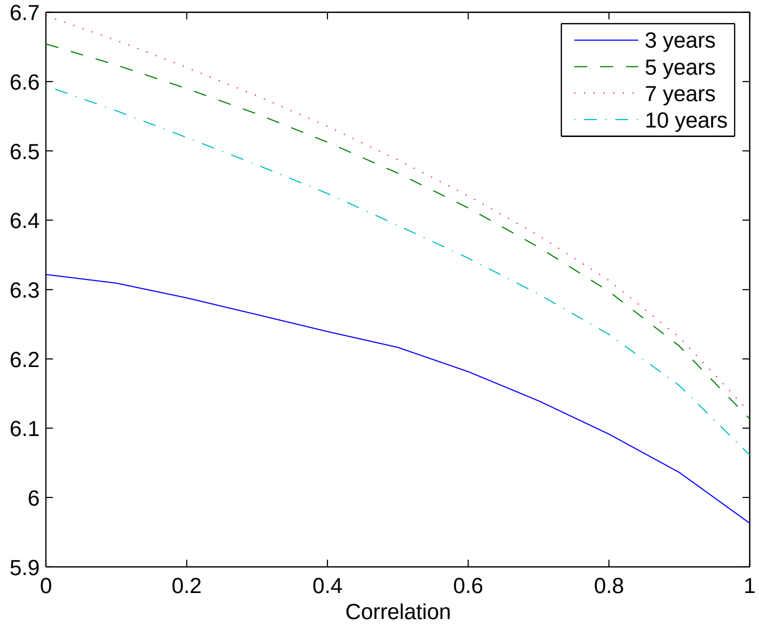
[1]See Schönbucher (2003), Chapter 9.6 for a discussion of the shape of credit curves in structural models.

Figure 5.10: Implied bond yield, different values of ω_1



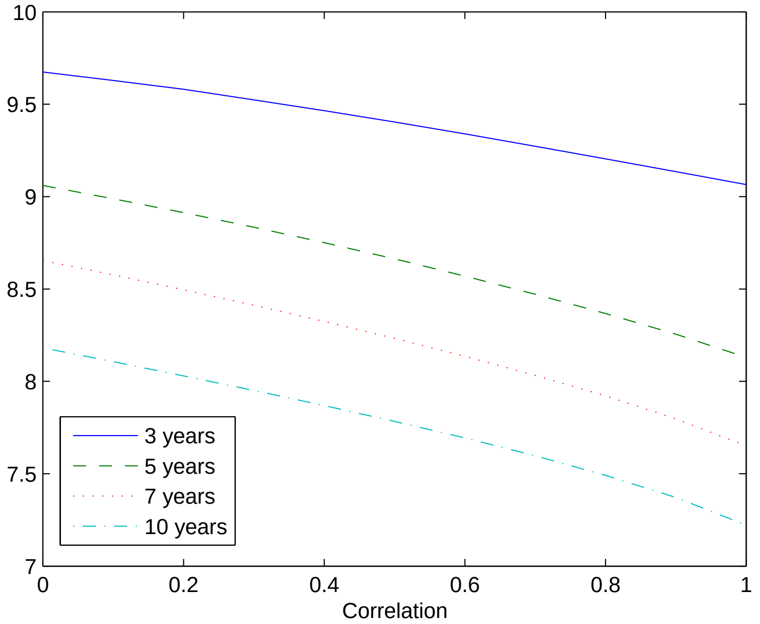
$\sigma_1 = \sigma_2 = 0.2$, $K_1 = 100$, $r_f = 0.05$, $q_1 = q_2 = 0$,
$\gamma_1 = \gamma_2 = 0.03$, initial credit quality $= 2$, T=5 years

Figure 5.11: Implied bond yield for different maturities, $\omega_1 = 0.7$



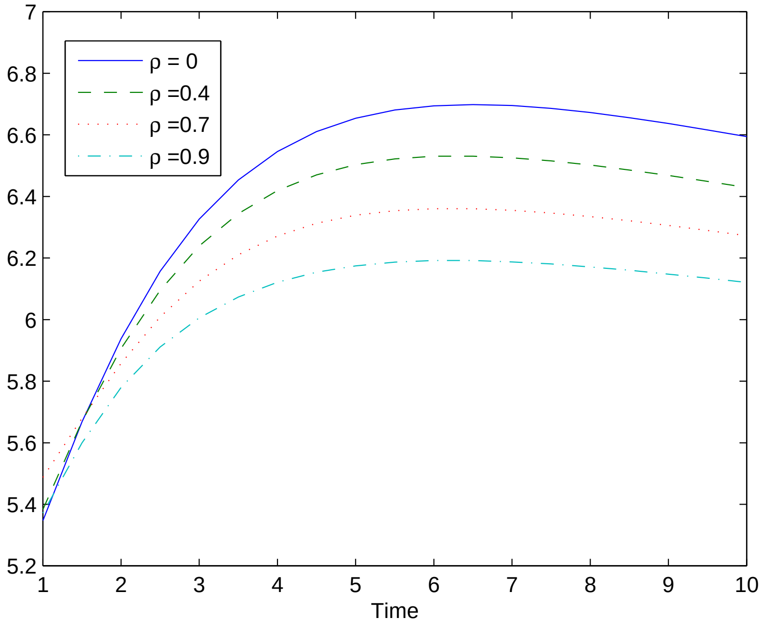
$\sigma_1 = \sigma_2 = 0.2$, $K_1 = 100$, $r_f = 0.05$,
$q_1 = q_2 = 0$, $\gamma_1 = \gamma_2 = 0.03$, initial credit quality $= 2$

Figure 5.12: Implied bond yield for different maturities, $\omega_1 = 0.5$



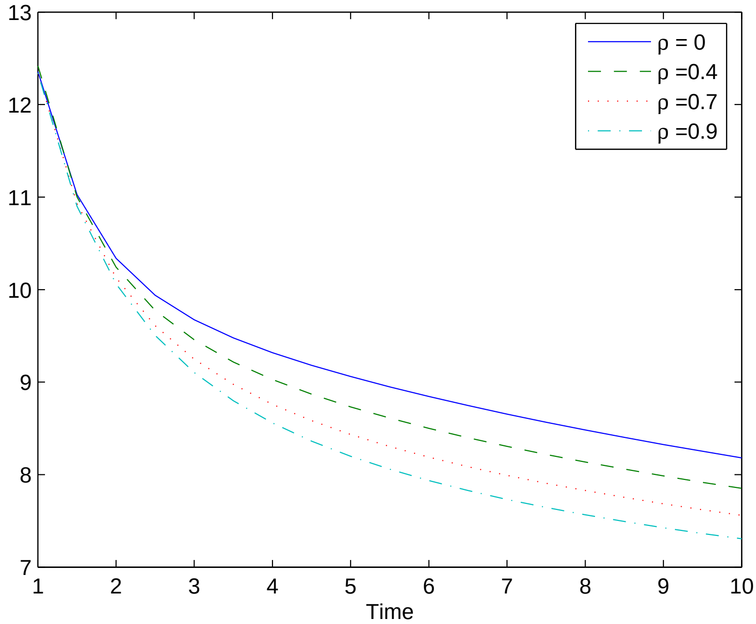
$\sigma_1 = \sigma_2 = 0.2$, $K_1 = 100$, $r_f = 0.05$,
$q_1 = q_2 = 0$, $\gamma_1 = \gamma_2 = 0.03$, initial credit quality $= 2$

Figure 5.13: Implied yield curve, $\omega_1 = 0.7$



$\sigma_1 = \sigma_2 = 0.2$, $K_1 = 100$, $r_f = 0.05$,
$q_1 = q_2 = 0$, $\gamma_1 = \gamma_2 = 0.03$, initial credit quality $= 2$

Figure 5.14: Implied yield curve, $\omega_1 = 0.5$



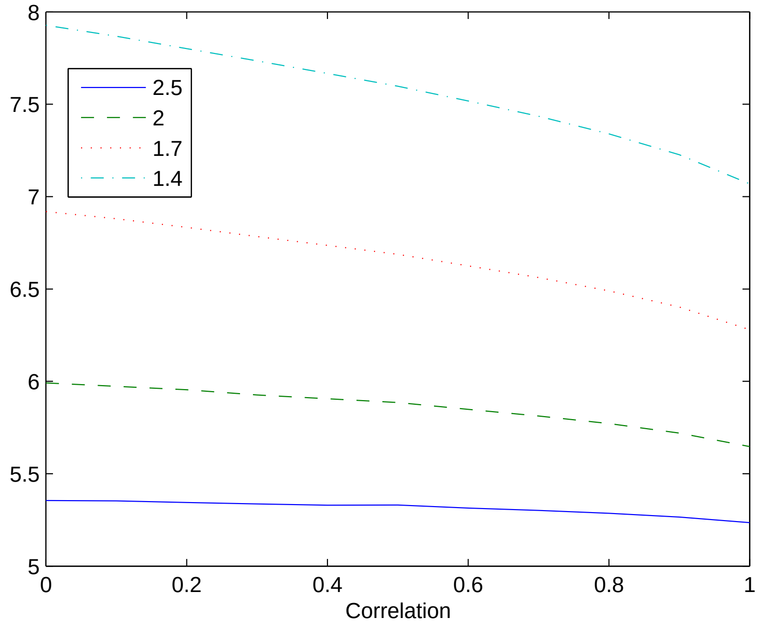
$\sigma_1 = \sigma_2 = 0.2$, $K_1 = 100$, $r_f = 0.05$,
$q_1 = q_2 = 0$, $\gamma_1 = \gamma_2 = 0.03$, initial credit quality $= 2$

we would expect – yields are greatest for lowest initial credit quality, decline with increasing correlation and increase as ω_1 decreases.

Figures 5.17 and 5.18 consider the sensitivity of yields to the shape of the default barrier and the risk-free rate. Both consider 5-year bonds with initial credit quality of two and $\omega_1 = 0.7$. Changing the slope of the default barrier has minimal impact on yields – as the slope increases, default is less likely and yields decrease, but the impact is fairly small, particularly when considering the dependence on other parameters. Similarly, changing the risk-free rate has almost no impact on credit spreads – as r_f increases in value, yields increase by a very similar amount for all values of correlation. In practice spreads tend to increase with interest rates as the economic environment is usually more risky when rates are higher. This relationship is not captured in this framework, but would be an attractive extension.

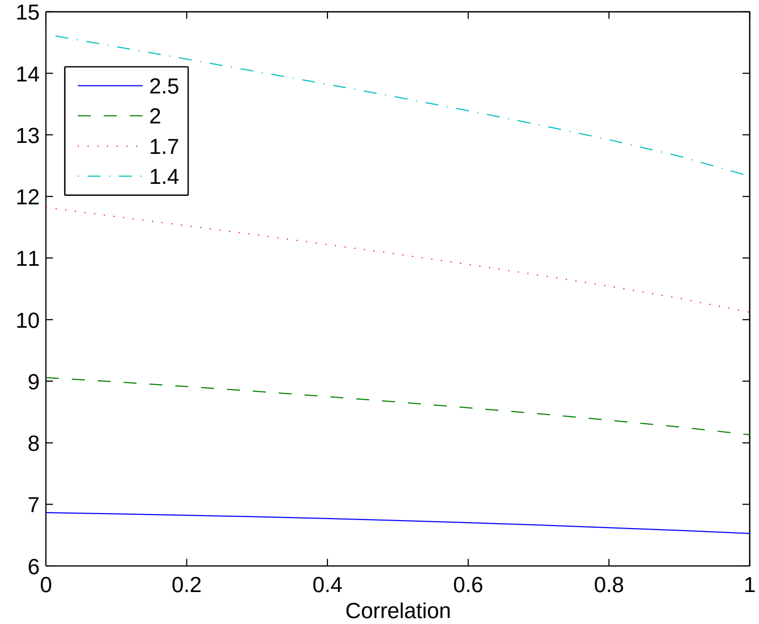
Finally, we consider the impact of varying the volatility of firm value. Figure 5.19 shows how yields behave when the volatilities of both firms are changed simultaneously. As expected, higher volatilities lead to a higher likelihood of default and higher yields. In Figure 5.20 we assume that the volatility of firm one remains fixed at 0.2 and we increase the volatility in firm two. Since we are considering the yield

Figure 5.15: Implied bond yield, varying initial credit quality, $\omega_1 = 0.8$



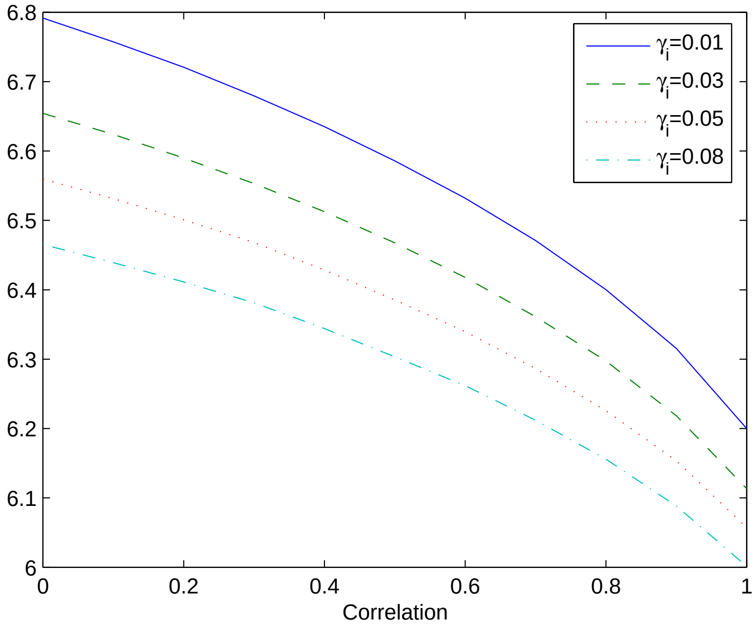
$\sigma_1 = \sigma_2 = 0.2$, $K_1 = 100$, $r_f = 0.05$,
$q_1 = q_2 = 0$, $\gamma_1 = \gamma_2 = 0.03$, T=5 years

Figure 5.16: Implied bond yield, varying initial credit quality, $\omega_1 = 0.5$



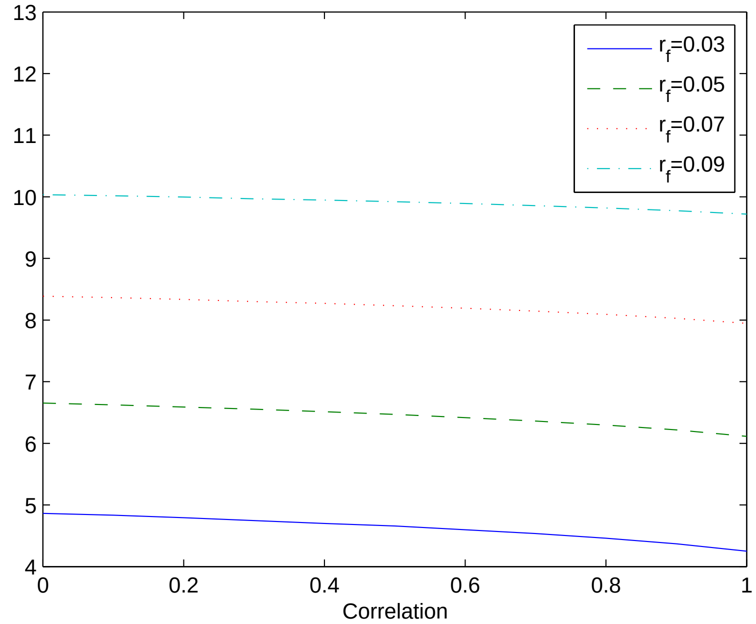
$\sigma_1 = \sigma_2 = 0.2$, $K_1 = 100$, $r_f = 0.05$,
$q_1 = q_2 = 0$, $\gamma_1 = \gamma_2 = 0.03$, $T = 5$ years

Figure 5.17: Implied bond yield, varying γ_i



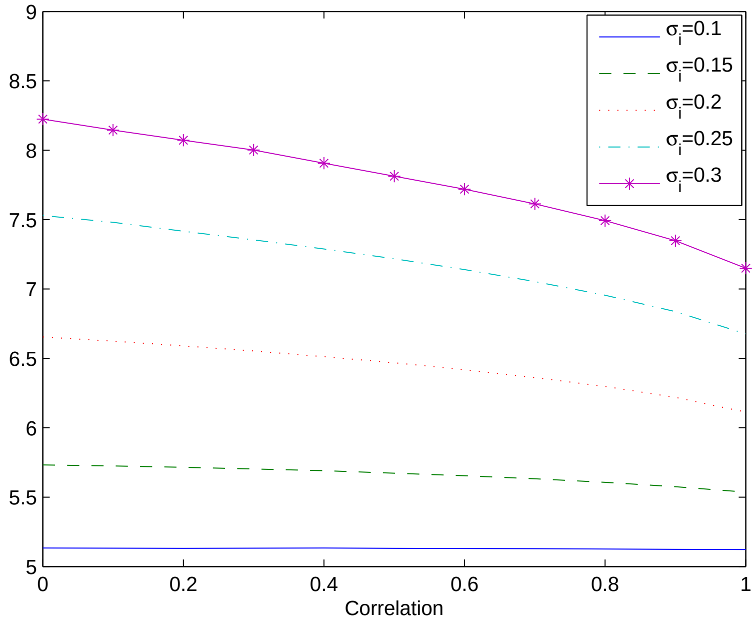
$\sigma_1 = \sigma_2 = 0.2$, $K_1 = 100$, $r_f = 0.05$, $\omega_1 = 0.7$,
$q_1 = q_2 = 0$, $T = 5$ years, $QB_1 = QB_2 = 2$

Figure 5.18: Implied bond yield, varying r_f



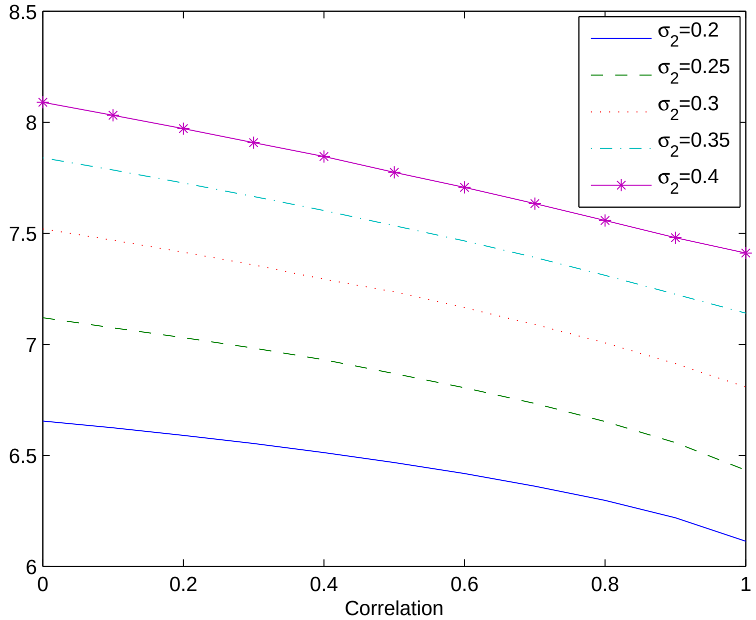
$\sigma_1 = \sigma_2 = 0.2$, $K_1 = 100$, $QB_1 = QB_2 = 2$
$\omega_1 = 0.7$, $q_1 = q_2 = 0$, $\gamma_1 = \gamma_2 = 0.03$, $T = 5$ years

Figure 5.19: Implied bond yield, varying σ_i



$\gamma_1 = \gamma_2 = 0.3$, $K_1 = 100$, $r_f = 0.05$, $\omega_1 = 0.7$,
$q_1 = q_2 = 0$, initial credit quality $= 2$, $T = 5$ years

Figure 5.20: Implied bond yield, varying σ_2



$\sigma_1 = 0.2$, $\gamma_i = 0.03$, $K_1 = 100$, $r_f = 0.05$, $\omega_1 = 0.7$,
$q_1 = q_2 = 0$, initial credit quality $= 2$, $T = 5$ years

on firm one's bonds, the increasing riskiness of firm two impacts yields through the correlation between the two companies and the possibility of default contagion. As expected, the more volatile firm two is, the riskier firm one and the higher yielding its bonds.

From the above analysis, we see that the underlying correlation structure impacts yields of all maturities by a meaningful amount. Of all the model parameters, however, ω_1 and the value of the initial credit quality have the most significant influence. The latter can be obtained from accounting data, however the value of ω_1 is hard to measure and is necessarily a fairly arbitrary assumption.

5.3 Credit Default Swap Calculations

Using a similar approach to that in Section 5.2, we evaluate first and second-to-default credit default swap (CDS) spreads for a two-company basket. For each, we describe the product, derive formulæ for CDS spreads, outline the numerical methods used and present results. We conclude the section by considering the application of the framework to the evaluation of a single-name CDS with counterparty risk.

5.3.1 Spread Calculations

5.3.1.1 First-to-default CDS Basket

We consider a basket of two related companies. The buyer of a first-to-default CDS on this underlying basket pays a premium, the CDS spread, for the life of the CDS – until maturity or the first default, whichever happens first. In the event of default by one of the underlying reference companies, the buyer receives a default payment and the contract terminates. Denoting the default time of company i by τ_i, we write τ_{first} for the time of the first default,

$$\tau_{\text{first}} = \min\{\tau_1, \tau_2\}$$

where, using the same notation as before,

$$\tau_i = \inf\{t : \underline{X}_i(t) = B_i\}.$$

If bond recovery on default is R, and the protection buyer makes continuous spread payments, c, on a par value K, then the discounted spread payment (DSP) and discounted default payment (DDP) on the first-to-default basket are

$$\begin{aligned}
\text{DSP} &= cK\int_0^T e^{-r_f s}\mathbb{P}(\tau_{\text{first}} > s)\,\mathrm{d}s \\
\text{DDP} &= (1-R)K\int_0^T e^{-r_f s}\mathbb{P}(s \le \tau_{\text{first}} < s+ds)\,\mathrm{d}s \\
&= (1-R)K\int_0^T -e^{-r_f s}\frac{\partial}{\partial s}\mathbb{P}(\tau_{\text{first}} > s)\,\mathrm{d}s \\
&= (1-R)K\left\{1 - e^{-r_f T}\mathbb{P}(\tau_{\text{first}} > T) - r_f\int_0^T e^{-r_f s}\mathbb{P}(\tau_{\text{first}} > s)\,\mathrm{d}s\right\}.
\end{aligned} \tag{5.9}$$

With $P(s)$ as defined in (4.5), the market spread, c_{first}, is therefore

$$\begin{aligned}
c_{\text{first}} &= \frac{(1-R)\left\{1 - e^{-r_f T}\mathbb{P}(\tau_{\text{first}} > T) - \int_0^T r_f e^{-r_f s}\mathbb{P}(\tau_{\text{first}} > s)\,\mathrm{d}s\right\}}{\int_0^T e^{-r_f s}\mathbb{P}(\tau_{\text{first}} > s)\,\mathrm{d}s} \\
&= \frac{(1-R)\left\{1 - e^{-r_f T}P(T) - \int_0^T r_f e^{-r_f s}P(s)\,\mathrm{d}s\right\}}{\int_0^T e^{-r_f s}P(s)\,\mathrm{d}s}
\end{aligned} \tag{5.10}$$

since

$$\mathbb{P}(\tau_{\text{first}} > s) = \mathbb{P}(\tau_1 > s, \tau_2 > s) = P(s),$$

the joint survival probability.

5.3.1.2 Second-to-default CDS Basket

A second-to-default CDS spread is evaluated in the same way. The purchaser of the swap receives a payment in the event that both companies default during the life of the swap, at which point the contract terminates. Denoting τ_{second} as the time of the second default, exactly as for (5.10), the market spread, c_{second} is

$$c_{\text{second}} = \frac{(1-R)\left\{1 - e^{-r_f T}\mathbb{P}(\tau_{\text{second}} > T) - \int_0^T r_f e^{-r_f s}\mathbb{P}(\tau_{\text{second}} > s)\,\mathrm{d}s\right\}}{\int_0^T e^{-r_f s}\mathbb{P}(\tau_{\text{second}} > s)\,\mathrm{d}s}, \tag{5.11}$$

where

$$\mathbb{P}(\tau_{\text{second}} > s) = \mathbb{P}(\tau_1 > s) + \mathbb{P}(\tau_2 > s) - \mathbb{P}(\tau_1 > s, \tau_2 > s). \tag{5.12}$$

5.3.2 Numerical Implementation

Since we are considering the impact of the relationship between reference entities on first and second-to-default CDS spreads, we are interested in the full range of correlation values, $\rho \in [-1, 1]$. The standard maturity of a credit default swap is five years, and so we restrict our analysis to fives years or less and consider just the case in which firm value and the default barrier have the same drifts. Numerically this is much simpler and faster, and as shown by Zhou (2001a) and from our results in Section 4.3.3, results are very similar for time periods of up to five years.

To calculate first-to-default CDS spreads the only new term that we need to be able to evaluate is

$$\int_0^T e^{-r_f s}\mathbb{P}(\underline{X}_1(s) \geq B_1, \underline{X}_2(s) \geq B_2)\,\mathrm{d}s$$

Substituting (4.16) for the joint survival probability and setting

$$\begin{aligned} E_1 &= \frac{4r_0}{n\sqrt{2\pi}}\sin\left(\frac{n\pi\theta_0}{\beta}\right)\cos\left(\frac{n\pi^2 x}{2\beta}\right)\cos(\frac{\pi x}{2}) \\ E_2 &= \frac{4r_0 D}{n\pi\sqrt{2\pi}}\sin\left(\frac{n\pi\theta_0}{\beta}\right)\cos\left(\frac{n\pi^2}{2\beta}\right)\sinh\left(\frac{Dx}{2}\right)e^{-n\pi Dx/(2\beta)}, \end{aligned}$$

$$\begin{aligned} &\int_0^T e^{-r_f s}\mathbb{P}(\underline{X}_1(s) \geq B_1, \underline{X}_2(s) \geq B_2)\,\mathrm{d}s \\ &= \sum_{\text{n odd}}\int_0^1\int_0^T \frac{1}{\sqrt{s}}e^{-r_f s}\left[E_1 e^{-\frac{r_0^2}{4s}(1-\cos(\pi x))} + E_2 e^{-\frac{r_0^2}{4s}(1+\cosh(Dx))}\right]\mathrm{d}s\,\mathrm{d}x \\ &= \sum_{\text{n odd}}\int_0^1\int_{\frac{r_0}{2\sqrt{T}}}^{\infty} \frac{r_0}{u^2}e^{\frac{-r_f r_0^2}{4u^2}}\left[E_1 e^{-u^2(1-\cos(\pi x))} + E_2 e^{-u^2(1+\cosh(Dx))}\right]\mathrm{d}u\,\mathrm{d}x \end{aligned}$$

where $u = \frac{r_0}{2\sqrt{s}}$. Making the additional substitution $u = (y + r_0/(2\sqrt{T}))/(1-y)$ and letting

$$\begin{aligned} z &= \frac{1-y}{y + \frac{r_0}{2\sqrt{T}}} \\ E &= \frac{4r_0^2}{n\sqrt{2\pi}}\sin\left(\frac{n\pi\theta_0}{\beta}\right)\frac{(1+\frac{r_0}{2\sqrt{T}})}{(y+\frac{r_0}{2\sqrt{T}})^2}e^{-\frac{r_f r_0^2 z^2}{4}}, \end{aligned}$$

the function to be integrated over $[0, 1]^2$ is

$$\begin{aligned} &\sum_{\text{n odd}} E\Bigg[\cos\left(\frac{n\pi^2 x}{2\beta}\right)\cos(\frac{\pi x}{2})e^{-(1-\cos(\pi x))/z^2} \\ &\quad + \frac{D}{\pi}\cos\left(\frac{n\pi^2}{2\beta}\right)\sinh\left(\frac{Dx}{2}\right)e^{-n\pi Dx/(2\beta)}e^{-(1+\cosh(Dx))/z^2}\Bigg]. \end{aligned}$$

The integral is truncated for $1 - y \leq \epsilon$, where ϵ is taken to be 7×10^{-7}.

To calculate second-to-default CDS spreads, we need to calculate $\mathbb{P}(\tau_i > s)$ (by (5.12)) and

$$\int_0^T e^{-r_f s} \mathbb{P}(\tau_i > s)\, \mathrm{d}s$$

Now, from (5.1), we know that

$$\mathbb{P}(\tau_i > s) = 1 - 2\Phi\left(\frac{B_i}{\sigma_i \sqrt{s}}\right) \tag{5.13}$$

since $\alpha_i = 0$ by construction. Hence

$$\int_0^T e^{-r_f s} \mathbb{P}(\tau_i > s)\, \mathrm{d}s = \frac{1}{r_f}\left(1 - e^{-r_f T}\right) - 2T \int_0^1 e^{-r_f T u} \Phi\left(\frac{B_i}{\sigma_i \sqrt{Tu}}\right) \mathrm{d}u \tag{5.14}$$

where $s = Tu$. The code used for calculating the cumulative normal distribution is given in Appendix G.

5.3.3 Results & Analysis

In Figures 5.21 – 5.32 we consider the impact of correlation on first and second-to-default CDS spreads (expressed in percent) for different parameter values. Numerical evaluation is done by numerical quadrature on a sparse grid as before.

In all cases, with increasing correlation between the two reference entities, first-to-default CDS spreads decrease, whilst second-to-default CDS spreads increase. This is because the probability of at least one company defaulting in a given period is higher for negative correlations, whilst the probability of both defaulting is greater for positive correlations.

Figures 5.21 and 5.22 show spreads for first and second-to-default CDS baskets with maturities of up to 5 years. Initial credit quality is 2 (i.e., as before, firm value is initially twice the level of the barrier). Figures 5.23 and 5.24 show the same results for five-year CDSs for different values of initial credit quality. Spreads are greater for longer-maturity swaps and significantly higher for weaker companies – those companies with lower initial credit quality who are initially closer to default. As we would expect, first-to-default spreads are everywhere greater than second-to-default spreads.

Figures 5.25 and 5.26 illustrate the extent to which CDS spreads depend on our recovery rate assumption. As would be expected, moving from a 30% recovery rate

Figure 5.21: First-to-default CDS

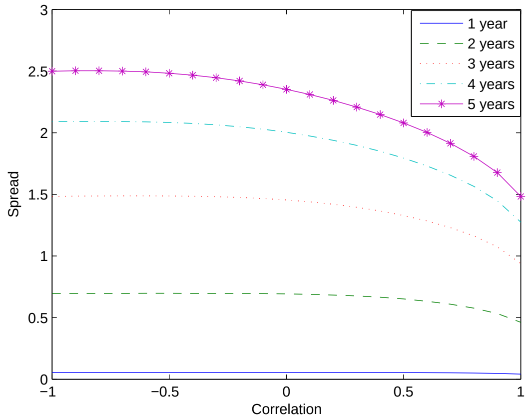
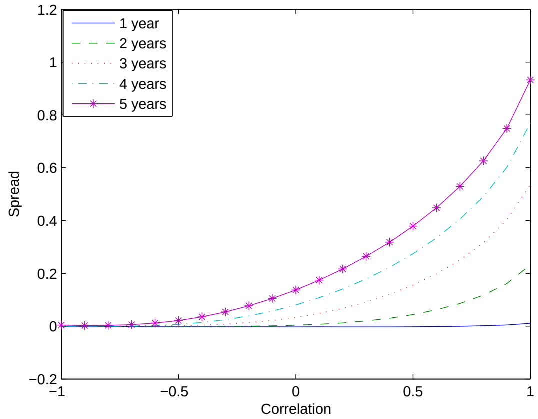


Figure 5.22: Second-to-default CDS



Sensitivity of spreads to maturity, T

$\sigma_i = 0.2$, $r_f = 0.05$, $q_i = 0$, $\gamma_i = 0.03$, initial credit quality $= 2$, $R = 0.5$

Figure 5.23: First-to-default CDS

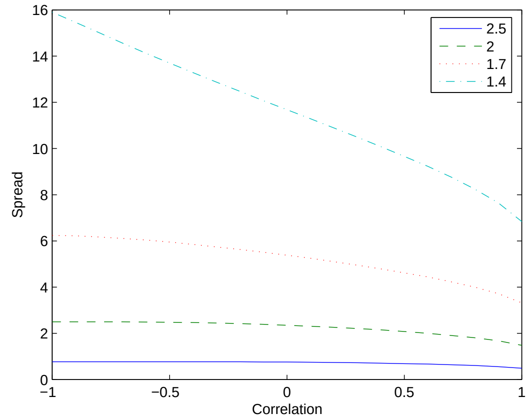
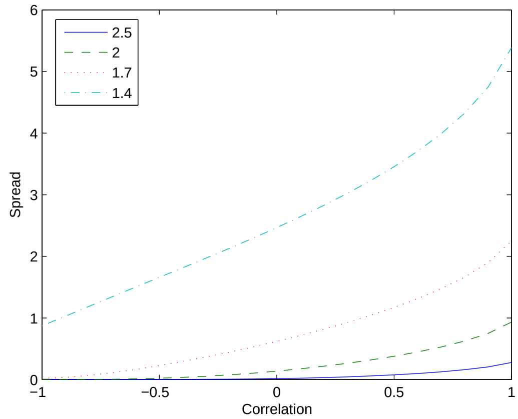


Figure 5.24: Second-to-default CDS



Sensitivity of spreads to initial credit quality

$\sigma_i = 0.2$, $r_f = 0.05$, $R = 0.5$, $q_i = 0$, $\gamma_i = 0.03$, $T = 5$ years

Figure 5.25: First-to-default CDS

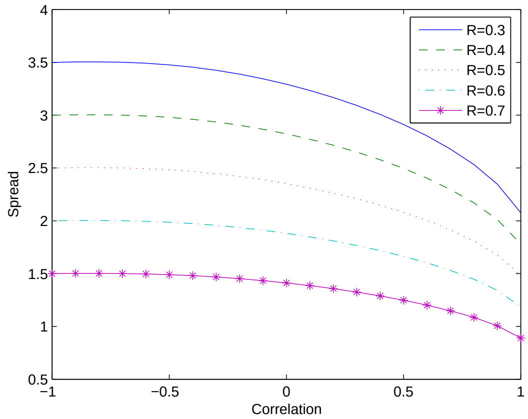
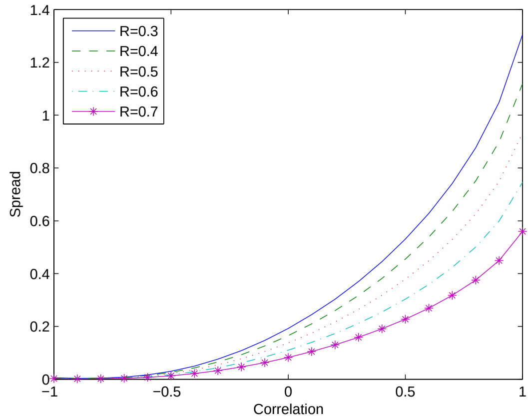


Figure 5.26: Second-to-default CDS



Sensitivity of spreads to recovery rate, R

$\sigma_i = 0.2$, $r_f = 0.05$, $q_i = 0$, $\gamma_i = 0.03$, initial credit quality $= 2$, $T = 5$

to a 70% recovery rate has a large impact. However, the overall form of spreads and their variation with changing correlation is the same. In general, taking R=50% is representative of the levels seen in practice (see, for example, Bakshi et al. (2006)) and is in line with the value used in the popular CreditGrades^{TM} approach to modelling credit as described in Finger et al. (2002).

It is straightforward to extend the analysis to consider the case when the recovery rate of the reference entities is a percentage of discounted par value, in line with our assumption in Section 5.2.2 when modelling credit-risky bonds. If recovery on default is $\omega e^{-r_f(T-\tau_{\text{first}})}$ rather than R, the first-to-default CDS spread becomes

$$c_{\text{first}} = \frac{\left(1 - \omega e^{-r_f T}\right) + P(T)e^{-r_f T}\left(\omega - 1\right) - \int_0^T r_f e^{-r_f s} P(s)\,\mathrm{d}s}{\int_0^T e^{-r_f s} P(s)\,\mathrm{d}s} \tag{5.15}$$

with a similar result for the second-to-default spread. It should be noted that we are assuming that the recovery rate is homogeneous across reference entities. If either R or ω depended on the identity of the defaulting entity, the calculation would be considerably more involved.

Figure 5.27 shows the impact that varying ω has on first-to-default CDS spreads. The results for constant recovery on default, R, are reproduced in Figure 5.28 for reference. The same results are illustrated in Figures 5.29 and 5.30 for second-to-default spreads. In each case it is clear that spreads are very similar in both formulations, both in level and dependence on correlation. When $\omega = 1$ there is no write-down on default and recovery is purely discounted par value.

Figure 5.27: Varying ω

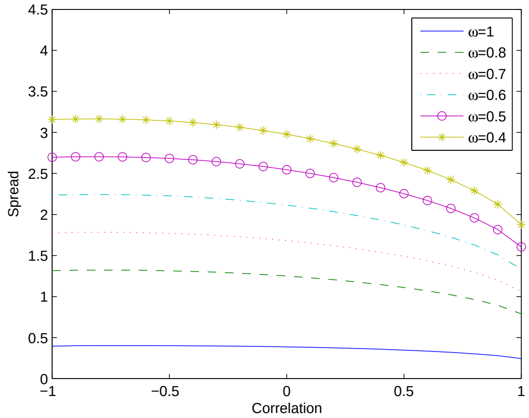
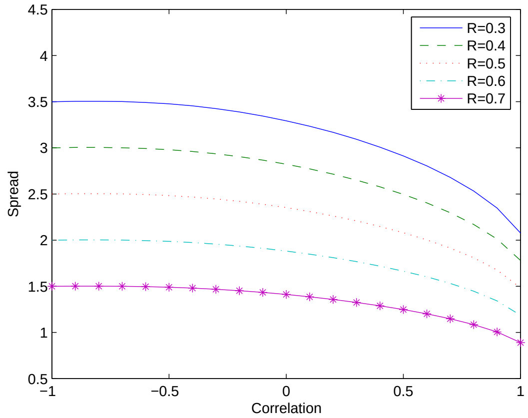


Figure 5.28: Varying R



First-to-default CDS spreads

$\sigma_i = 0.2$, $r_f = 0.05$, $q_i = 0$, $\gamma_i = 0.03$, initial credit quality $= 2$, $T = 5$

Figure 5.29: Varying ω

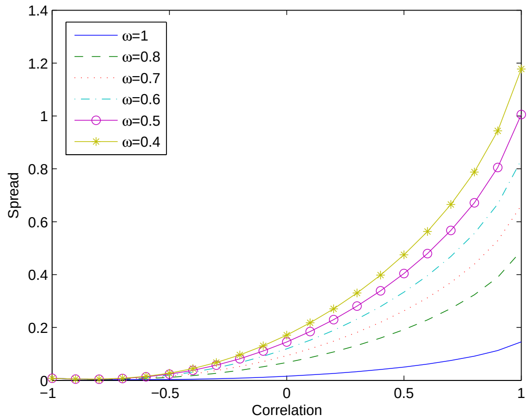
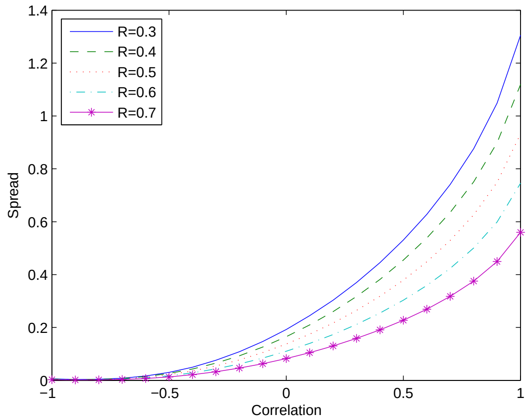


Figure 5.30: Varying R



Second-to-default CDS spreads

$\sigma_i = 0.2$, $r_f = 0.05$, $q_i = 0$, $\gamma_i = 0.03$, initial credit quality $= 2$, $T = 5$

From Figures 5.21 – 5.26, it is evident that positive correlation has a greater overall effect than negative correlation on spreads. Having reference entities that are either negatively correlated or independent has only a limited spread impact, with the exception of very weak companies (as shown by the case when initial credit quality is 1.4 in Figures 5.23 and 5.24). As the reference entities become increasingly positively correlated, first-to-default CDS spreads decline as we would expect, but at an increasing rate, whilst increases in second-to-default spreads are greater than for negative correlation. An increase in ρ from 0.5 to 1 has a much larger impact on spreads than an increase in ρ from 0 to 0.5.

Figure 5.31: First-to-default CDS

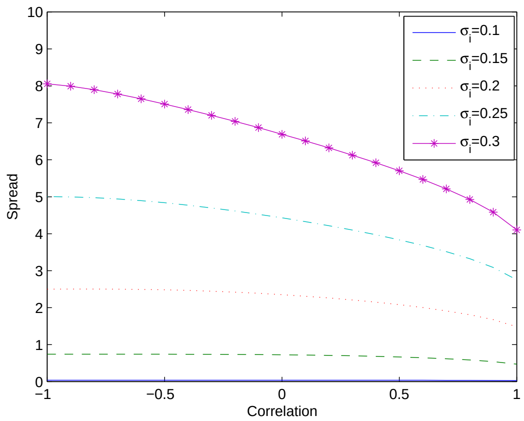
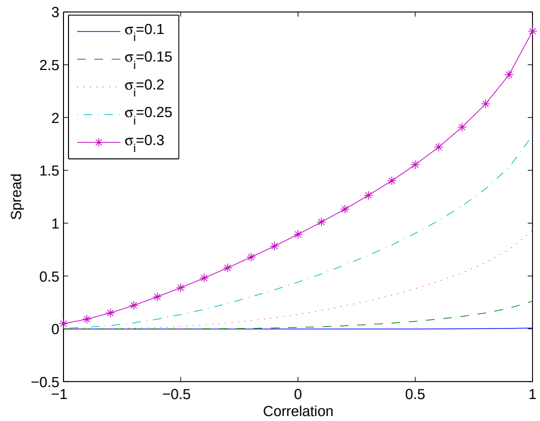


Figure 5.32: Second-to-default CDS



Sensitivity of spreads to volatility, σ_i

$r_f = 0.05$, $q_i = 0$, $R = 0.5$, $\gamma_i = 0.03$, initial credit quality $= 2$, $T = 5$

Figures 5.31 and 5.32 show that firm volatility has a considerable impact on spreads. As the reference entities become more volatile, both credit default swaps become much more risky and spreads increase significantly.

In this section, we have illustrated the impact of correlation between reference assets in a CDS basket on first and second-to-default CDS spreads. We have investigated parameter sensitivity and our model demonstrates behaviour expected of CDS spreads – they increase with time to maturity, increasing volatility and weakening credit quality. Of all the parameters, it is clear that assumptions about the value of initial credit quality and the volatility of firm value are the most influential.

5.3.3.1 CDS with Counterparty Risk

Consider now a single-name CDS, face-value K, maturity T, on reference company one bought from a counterparty company two. The purchaser of the CDS makes spread payments for the life of the CDS – until either the reference company or the counterparty defaults. If the reference entity defaults during the life of the CDS and before the counterparty, the purchaser receives a default payment. If, however, the counterparty defaults first, they receive nothing, irrespective of whether or not the reference company later defaults. Denoting the default time of company i by τ_i, if bond recovery on default is R and the purchaser of protection makes continuous spread payments, c, for the life of the CDS, then using the same notation as before, the protection buyer

- makes spread payments for $t < \min\{\tau_1, \tau_2, T\}$,
- receives a default payment if $\tau_1 < \min\{\tau_2, T\}$,
- receives nothing if $\tau_2 < \min\{\tau_1, T\}$.

In other words, expected discounted spread and default payments are

$$
\begin{aligned}
\text{Spread} &= cK \int_0^T e^{-r_f s} \mathbb{P}(\tau_1 > s, \tau_2 > s)\, \mathrm{d}s \\
&= cK \int_0^T e^{-r_f s} \mathbb{P}(\underline{X}_1(s) \geq B_1, \underline{X}_2(s) \geq B_2)\, \mathrm{d}s \\
\text{Default} &= (1-R)K \Big\{ \int_0^T e^{-r_f s} \mathbb{P}(s \leq \tau_1 < s + ds, \tau_2 > s)\, \mathrm{d}s
\end{aligned}
$$

Considering the default payment for a first-to-default CDS, (5.9),

$$
\begin{aligned}
\text{DDP} &= (1-R)K \int_0^T e^{-r_f s} \mathbb{P}(s \leq \tau_{\text{first}} < s + ds)\, \mathrm{d}s \\
&= (1-R)K \int_0^T e^{-r_f s} \mathbb{P}(s \leq \tau_1 < s + ds, \tau_2 > s)\, \mathrm{d}s \\
&\quad + (1-R)K \int_0^T e^{-r_f s} \mathbb{P}(\tau_1 > s, s \leq \tau_2 < s + ds)\, \mathrm{d}s, \qquad (5.16)
\end{aligned}
$$

we see that the default payment for a CDS with counterparty, added to its image when the identity of the reference entity and the counterparty are swapped, gives the value of the first-to-default swap payment. In the homogeneous case, when both entities can be assumed to have the same parameters, (5.16) can therefore be used to calculate the value of a CDS spread with counterparty risk for all values of correlation, ρ. In some applications this would not be too onerous an assumption and so this provides a neat little trick for evaluating CDS spreads in the presence of a counterparty. A further set of relations between first and second-to-default CDS payments and payments with counterparty risk can be obtained by breaking down the second-to-default CDS payment in a similar way to (5.16) and using the fact that

$$\mathbb{P}(\tau_1 < s, s \leq \tau_2 < s + ds) = \frac{\partial}{\partial s}\mathbb{P}(\tau_1 < s, \tau_2 < s) - \int_0^s \frac{\partial^2}{\partial s \partial u}\mathbb{P}(\tau_1 < s, \tau_2 < u)\,\mathrm{d}u.$$

In the more general inhomogeneous situation, the evaluation becomes more complicated and cannot be tackled using this framework.

5.4 Conclusions & Future Work

In this chapter, we have modelled bonds and basket credit default swaps using a two-dimensional structural model incorporating a credit dependence structure. Working with a Black and Cox (1976) type first passage framework, we have built on the work by Zhou (2001a) to derive analytical formulæ for both bond yields and CDS spreads. We have modified the default barrier to better reflect reality and have incorporated default contagion within the structural framework for the first time. The result is a credit model that is asymmetric with respect to default risk and which has a dependence structure based on both long-term asset correlation and default contagion.

Results illustrate that the sensitivity of yields to input parameters is as expected, and clearly demonstrate the importance of credit correlation. In our model of bond yields, we see that our rather simplistic dependence structure can have a large impact on yields. Our specification of default contagion is clearly not very realistic – default by one company very rarely leads to direct default by another, although it is possible. More likely, the impact of a corporate bankruptcy causes a ripple of credit weakness through the market as related companies are impacted. Nevertheless, the importance of taking into account credit interactions is, once again, clearly highlighted.

Dependence modelling is most critical in the analysis and pricing of large basket credit derivatives, such as k^{th}-to-default credit default swap baskets and CDO tranches.

These require the framework to be extended to considerably more than two dimensions. To use the approach of the last two chapters would therefore require solving for the n-dimensional survival probability density function for correlated Brownian motions with lower default thresholds. To our knowledge, the analytical solution to this problem for more than two dimensions remains unsolved. By extending the method given in Appendix B for two dimensions, it is possible to reduce the problem to solving the heat equation,

$$\frac{\partial H}{\partial t} = \frac{1}{2}\nabla^2 H,$$

subject to an initial condition and boundary conditions

$$H(P_1, t) = H(P_2, t) = H(P_3, t) = 0,$$

for three planes P_1, P_2 and P_3. Full details are given in Appendix H. However, at this stage, there are no obvious simple cases for which a solution is possible, and the problem becomes seemingly intractable. Independent analysis by Escobar and Seco (2004) claiming to have solved the problem in n dimensions turned out to be flawed.

From a mathematical standpoint, this is an interesting problem and is likely to have many applications elsewhere in finance. However, from the point of view of credit modelling, it is not clear that an analytical solution would be particularly useful. Even should one be possible, the evaluation of spreads in three or more dimensions may prove too numerically intensive to be feasible. The two-dimensional case was highly involved, both analytically and numerically, and we were restricted in our ability to specify a realistic formulation of contagion. Rather, we would like a framework in which we can incorporate a network of asymmetric dependences between companies, enabling the impact of a credit event at one company to propagate as a ripple of credit contagion across other entities.

In Chapter 6 we develop a model that addresses many of these limitations, and provides a much easier way to approach the problem. With the same underlying dynamics for firm value, we develop a framework for quickly and easily calculating default probabilities and valuing basket CDS spreads. The model allows for a much richer dependence structure, asymmetry in the propagation of credit contagion is possible and the model is not restricted to two dimensions.

Chapter 6

Numerical Solutions with Contagion

In this chapter we extend the first passage structural framework of Chapters 2 – 5 to incorporate a more realistic contagion structure. We begin by considering the two-firm case, describing the characterisation of default contagion and the implementation of the model before discussing results. We illustrate results for a basket of three firms and then consider an extension of the framework that allows the effects of contagion to dissipate over time. We follow with a discussion of some of the numerical issues related to applying the model to larger portfolios, outlining the form of a general n-dimensional model and its possible applications.

In contrast to previous work in which we obtained analytical formulæ for survival probabilities and credit spreads, that we then had to evaluate numerically, this work is wholly numerical. We proceed by specifying the PDE and boundary conditions that describe the default behaviour we are interested in and solve them using finite-difference methods. This approach is a lot more flexible and enables us to incorporate a far richer credit dependence structure than before. The framework is also straightforward to extend to higher dimensions, although we are limited in our ability to implement it by the computational complexity.

6.1 Two-Firm Model

We begin by illustrating the use and flexibility of the model for the case of two companies. This enables us to highlight the main attributes quickly and easily and allows results to be related back to those arrived at using the analytical approach of Chapters 4 and 5. Exactly as done analytically, we assume that companies are modelled as correlated geometric Brownian motions with default as the first hitting time of an exponential default barrier, $b_i(t)$. Using previous notation, firm value dynamics are as before,

$$
\begin{aligned}
dV_i(t) &= (r_f - q_i)V_i(t)dt + \sigma_i V_i(t) dW_i(t) \quad (6.1)\\
b_i(t) &= K_i e^{-\gamma_i (T-t)}. \quad (6.2)
\end{aligned}
$$

for $i = 1, 2$ and $\mathrm{cov}(W_i(t), W_j(t)) = \rho_{ij} t$.

For a function $U(\mathbf{V}, t)$, where $\mathbf{V}$ is the vector of firm values, the infinitesimal generator, $\mathcal{L}$, of (6.1) is

$$
\mathcal{L}U = \frac{\partial U(\mathbf{V},t)}{\partial t} + \sum_{i=1}^{2} \beta_i V_i \frac{\partial U(\mathbf{V},t)}{\partial V_i} + \frac{1}{2} \sum_{i,j=1}^{2} a_{ij} V_i V_j \frac{\partial^2 U(\mathbf{V},t)}{\partial V_i \partial V_j} \quad (6.3)
$$

where $\beta_i = r_f - q_i$ and $a_{ij} = \rho_{ij} \sigma_i \sigma_j$. If $U(\mathbf{V}, t)$ is such that $\mathcal{L}U = 0$ with terminal condition $U(\mathbf{V}, T) = \Psi(\mathbf{V})$, then applying the Feynman-Kac formula,[1]

$$
U(\mathbf{V}, t) = \mathbb{E}\{\Psi(\mathbf{V}(T)) | \mathbf{V}(t) = \mathbf{V}\}. \quad (6.4)
$$

The event that companies $i_1, \ldots, i_k$ default by time T (but no others) is given by the set[2]

$$
\Omega_{\{i_1,\ldots,i_k\}} = \left\{ V_{i_j}(T) \leq b_{i_j}(T) \quad \text{for} \quad j = 1, \ldots, k \quad \text{and} \quad V_{i_j}(T) > b_{i_j}(T) \quad \text{otherwise} \right\}.
$$

The set Ω_k corresponding to exactly k companies defaulting in $[0, T]$ is then[3]

$$
\Omega_k = \bigcup_{I \subset \{1,\ldots,n\},\, |I| = k} \Omega_I.
$$

If $\mathbb{I}_E$ denotes the indicator function of an event E, then since

$$
\mathbb{P}(\mathbf{V}(T) \in \Omega_k) = \mathbb{E}(\mathbb{I}_{\Omega_k}(\mathbf{V}(T))), \quad (6.5)
$$

we can calculate the probability of k defaults in $[0, T]$ by solving

$$
\begin{aligned}
\frac{\partial U(\mathbf{V},t)}{\partial t} &+ \sum_{i=1}^{2} \beta_i V_i \frac{\partial U(\mathbf{V},t)}{\partial V_i} + \frac{1}{2} \sum_{i,j=1}^{2} a_{ij} V_i V_j \frac{\partial^2 U(\mathbf{V},t)}{\partial V_i \partial V_j} = 0 \\
U(\mathbf{V}, T) &= \mathbb{I}_{\Omega_k}(\mathbf{V}) = \Psi(\mathbf{V}), \quad (6.6)
\end{aligned}
$$

[1]See, e.g., Bjork (2004) for a definition of the Feynman-Kac formula.

[2]As described here, the model allows for recovery − if a firm hits its default barrier prior to maturity, but recovers so that $V_i(T) > b_i(T)$, it does not default. We remove this possibility by setting the parameters β_i and a_{ij} to zero on the barrier as explained in Section 6.1.2.

[3]We are describing the general n-dimensional situation. For the two-firm case, $n = 2$ and $k \leq 2$.

with boundary conditions defined such that once a company hits its default barrier, it remains there. Numerically we can do this by setting a firm's volatility and drift to zero on the barrier. In two dimensions, as explained in Section 6.3, we can also solve the remaining one-firm problem on the barrier and use the solution as the boundary condition. By carefully specifying parameter values on the barrier, this framework is extremely flexible and can be used to calculate a wide range of default and survival probabilities of interest.

6.1.1 Introducing Credit Contagion

As specified so far, the dependence structure is driven purely by the correlation in firm values. Default by a company has no direct impact on the behaviour of the remaining companies; there is no credit or default contagion.

In Chapter 5, default by one company led directly to default by a related company. This simplified formulation of contagion was necessary to generate analytical solutions, but does not reflect reality particularly well. It is usual to see the impact of a default ripple through the market causing jumps in the credit spreads of related companies. Generally these would be increases in spreads at companies deemed to be exposed to similar risks, but could also be manifest as tightening spreads at companies likely to benefit from the default.

We now incorporate this type of credit contagion by allowing the factor weightings, a_{ij}, to jump on default. If company i defaults, then specifying a jump in the value of σ_j, $i \neq j$, causes a jump in the volatility of the remaining firm, thereby introducing credit contagion. By relating the size and direction of the jump in σ_j to the correlation between companies i and j, both positive and negative effects and differing degrees of contagion can be incorporated. The result is a dependence structure incorporating both mechanisms described in Section 3.1. Exposure to common factors arises through the specification of correlated firm value processes in (6.1) whilst the contagion mechanism reflects the existence of a network of inter-company links. Asymmetric ties are easy to incorporate, allowing one company to have greater influence than the other, as in the model of Jarrow and Yu (2001).

6.1.2 Implementation

We solve (6.6) backwards in time on $[0, T] \times \mathbb{R}_+^2$ using a finite-difference method with Crank-Nicolson time-stepping and a multigrid solver as in Reisinger and Wittum (2004). Further discussion of some of the numerical issues is contained in Section 6.4 and some details of the discretisation are in Appendix I. Our approach is based on the application of PDE techniques and so it was natural for us to implement the framework using finite-difference methodology, with results carefully verified against the exact solutions in Chapters 4 and 5 wherever possible. There is, however, no reason why Monte Carlo simulation could not be used, and in higher dimensions this may be the better approach.

To force a company i to default once it hits the default barrier, we define the coefficients β_i and a_{ij} in equation (6.6) such that

$$\widetilde{\beta}_i(V_i, t) := \beta_i \mathbb{I}_{\{V_i(t) > b_i(t)\}} \tag{6.7}$$

$$\widetilde{a}_{ij}(V_i, V_j, t) := a_{ij} \mathbb{I}_{\{V_i(t) > b_i(t),\, V_j(t) > b_j(t)\}}. \tag{6.8}$$

In this way, once a firm reaches its barrier, its value remains constant. We then count the number of companies whose values are at or below their default barriers and define the initial condition accordingly. For example, if we want the probability that all companies survive, we set the initial condition to be one when the value of this counting function is zero, and zero elsewhere. If we want the probability of k defaults, we set it to be one when the counting function equals k, zero otherwise.

Simultaneously, we are able to evaluate the integral over time of a function of the default or survival probability of interest, enabling easy evaluation of k^{th}-to-default credit default swaps. If τ_k represents the time of the k^{th} default, then as in Section 5.3.1, calculating $\mathbb{P}(\tau_k > T)$ and $\int_0^T e^{-r_f s}\mathbb{P}(\tau_k > s)\,\mathrm{d}s$ enables us to value a k^{th}-to-default CDS spread, c_k,

$$c_k = \frac{(1-R)\left\{1 - e^{-r_f T}\mathbb{P}(\tau_k > T) - \int_0^T r_f e^{-r_f s}\mathbb{P}(\tau_k > s)\,\mathrm{d}s\right\}}{\int_0^T e^{-r_f s}\mathbb{P}(\tau_k > s)\,\mathrm{d}s}. \tag{6.9}$$

We can also calculate the expected number of defaults in a certain time period, $\sum_{k=0}^{n} k\,\mathbb{P}(k \text{ firms default})$ by setting the initial condition to be the number of firms in default.

We introduce credit contagion through a jump in the value of the volatility parameter, σ_i, of the firms still alive. We relate the direction and size of the jump to the degree of correlation between the firms, so for correlation parameter ρ_{ij}, if company i defaults, the volatility of company j jumps by

$$\sigma_j \to \sigma_j F^{\rho_{ij}} \tag{6.10}$$

for some constant $F \geq 1$. In this way, a positively correlated company experiences an increase in volatility, and hence an increase in credit spreads, whilst a negatively correlated company becomes less volatile and consequently less risky. The degree of correlation has an impact on the size of the jump in volatility, and by changing the value of F, default can be assumed to have a bigger or smaller influence on the strength of the remaining company. This formulation makes intuitive sense and serves to illustrate the impact company relationships can have on credit spreads. Of course (6.10) is just one of many possible specifications of the contagion mechanism that could be used and others could be handled similarly.

6.1.3 Results

By modifying the initial condition, we can calculate a number of different survival probabilities as discussed in Section 6.1.2. These can then be used to calculate first and second-to-default CDS spreads using (6.9). Unless otherwise stated, all spreads are in percent.

Our results are generated using a regular grid with $2^{10}+1$ grid-points in each direction and 200 time-steps. On an Intel Xeon CPU 3.5 Ghz x2 machine, it takes about eight minutes to generate default probabilities for a given value of correlation at 200 points in a ten-year period. The integral over time of the discounted default probability is generated simultaneously enabling immediate calculation of CDS spreads. Comparing default probabilities with the analytical approach in Chapter 4, using 10 uniform refinements and 200 time-steps gives results accurate to five decimal places. We have gone for accuracy over speed; to generate the same results accurate to three decimal places takes just thirty seconds.

Figure 6.1 shows the joint survival probability for five different levels of correlation parameter[4], ρ, and Figure 6.2 shows the resultant first-to-default CDS spreads. These can be compared with Figure 5.21 in Chapter 5. For the sake of comparison, the same parameter values have been used, and results are identical. It is also worth noting that the presence of contagion has no impact on first-to-default spreads or joint survival

[4]Since we have only one correlation parameter, in this section we denote $\rho_{ij} = \rho$ for $i \neq j$.

probabilities. These values are driven solely by the probability of one of the companies defaulting; what happens later is irrelevant and by definition, default contagion comes into play following the default of the first company. Similarly, the probability of at least one default is independent of the level of contagion.

Figure 6.1: Joint survival probability

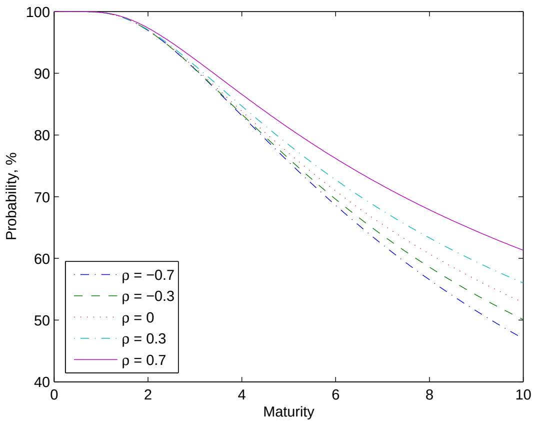
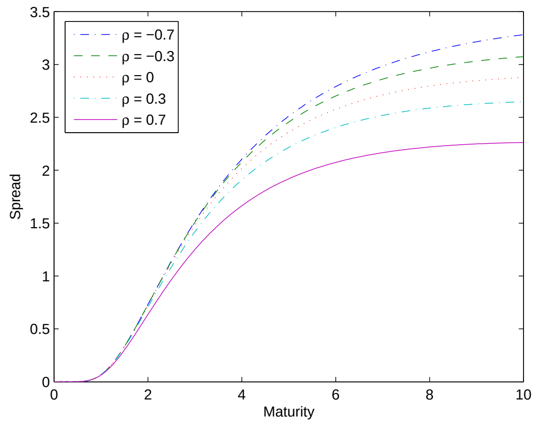


Figure 6.2: First-to-default CDS spread



$\sigma_i = 0.2$, $r_f = 0.05$, $q_i = 0$, $\gamma_i = 0.03$, initial credit quality $= 2$, $R = 0.5$

In Figures 6.3 – 6.10 we consider the impact contagion has on the probability of either one or two defaults occurring. Figures 6.3 and 6.4 show the probabilities of one and two defaults, respectively, against time for correlation between the companies of -0.5. Figures 6.5 and 6.6 show the same results for correlation of $+0.5$. In each case, $F = 1$ corresponds to no default contagion, whilst increasing F corresponds to an increasing degree of contagion, with volatility jumping on default according to (6.10).

Figure 6.3: Probability of 1 default

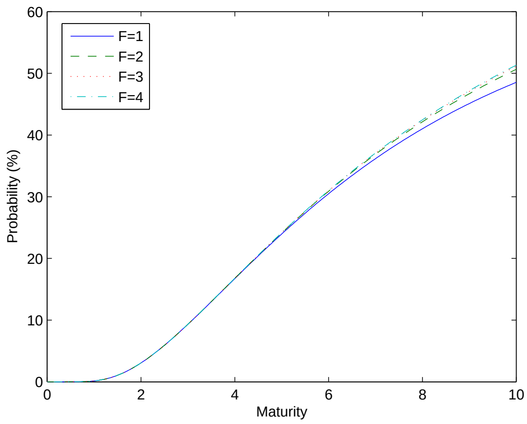
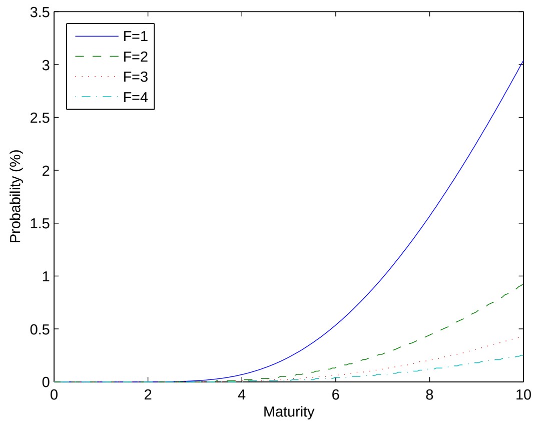


Figure 6.4: Probability of 2 defaults



Default probabilities for $\rho = -0.5$

$\sigma_i = 0.2$, $r_f = 0.05$, $q_i = 0$, $\gamma_i = 0.03$, initial credit quality $= 2$

Figure 6.5: Probability of 1 default

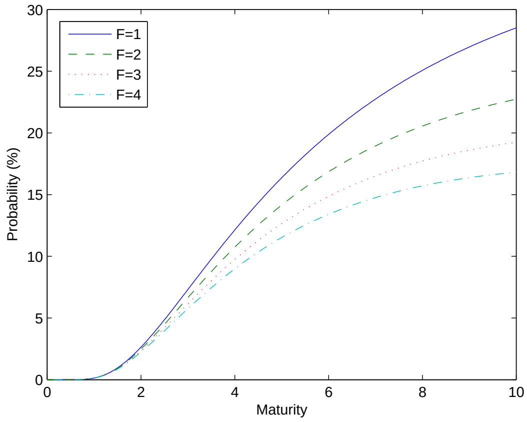
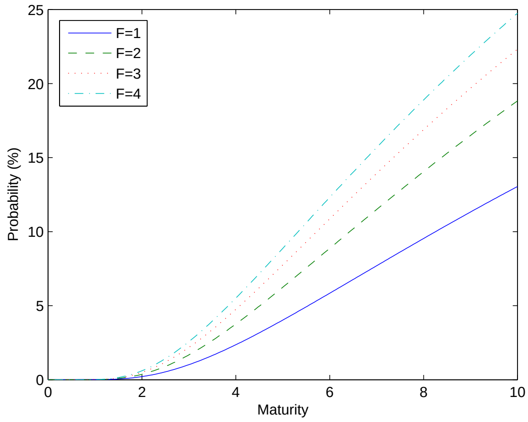


Figure 6.6: Probability of 2 defaults



Default probabilities for $\rho = +0.5$

$\sigma_i = 0.2$, $r_f = 0.05$, $q_i = 0$, $\gamma_i = 0.03$, initial credit quality $= 2$

Contagion when the companies are negatively correlated has little impact on the probability of one default since the likelihood of a second default is slim. A second default is, however, slightly more likely when there is less contagion, and so the probability of exactly one default is slightly lower for lower values of F. When the companies are positively correlated, contagion has a greater impact, and since a second default is more likely for higher F, the probability of one default is lower for higher F, while the probability of two defaults is higher.

Figures 6.7 – 6.10 illustrate this more clearly, showing the probabilities of either one or two defaults against correlation for five-year and ten-year time periods. The forms are the same for five and ten years, and in each case, since zero correlation corresponds to no contagion, all curves cross at this point. The probability of a second default being higher for positive correlation and contagion, and lower for negative correlation and contagion, explains the position of the curves relative to one another. In Figure 6.9, for example, for negative correlation, a greater degree of contagion means a second default is less likely, so the probability of exactly one default is greater for higher values of F. Contrastingly, a second default is more likely with higher F for positive values of correlation and so the probability of one default decreases with increasing F for positive correlation. The same argument explains the results for the probability of two defaults.

In Figures 6.8 and 6.10 we see that the probability of two defaults peaks for a value of $\rho < 1$ for higher values of F. The same effect is evident and better highlighted in

Figure 6.7: Probability of 1 default

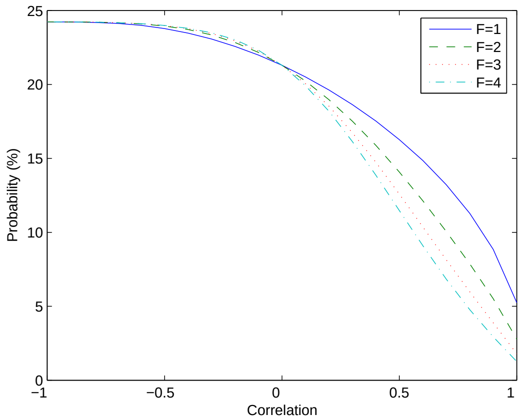
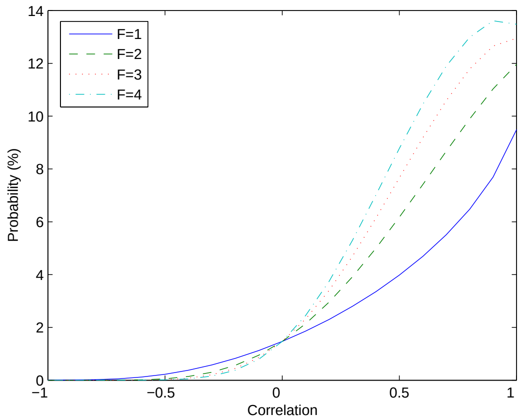


Figure 6.8: Probability of 2 defaults



Five-year default probabilities

$\sigma_i = 0.2$, $r_f = 0.05$, $q_i = 0$, $\gamma_i = 0.03$, initial credit quality $= 2$

Figure 6.9: Probability of 1 default

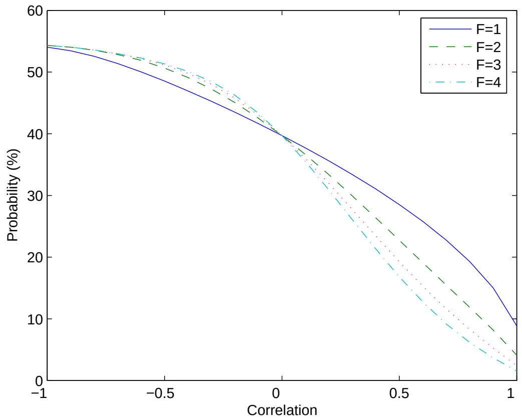
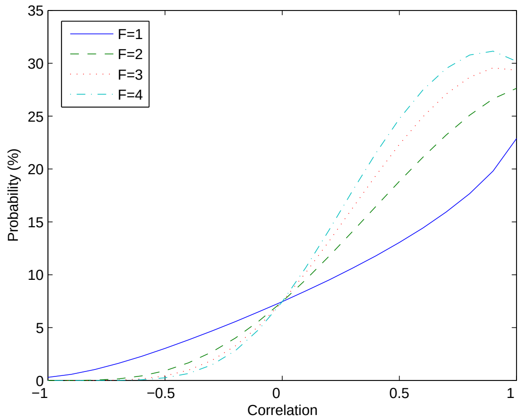


Figure 6.10: Probability of 2 defaults



Ten-year default probabilities

$\sigma_i = 0.2$, $r_f = 0.05$, $q_i = 0$, $\gamma_i = 0.03$, initial credit quality $= 2$

Figures 6.11 and 6.12 which illustrate the impact of correlation and contagion on the expected number of defaults over five and ten years.

Figure 6.11: 5 years

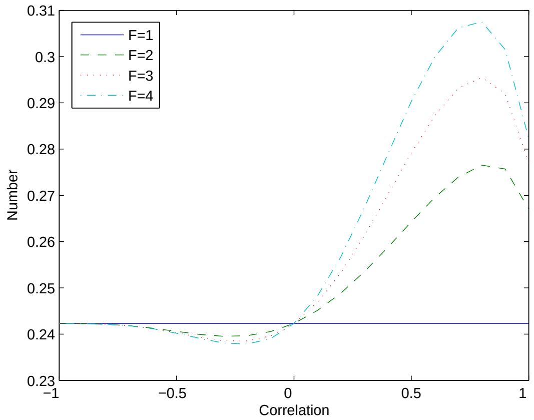
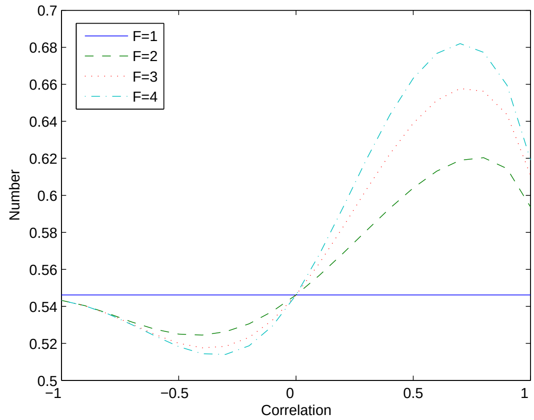


Figure 6.12: 10 years



Expected number of defaults

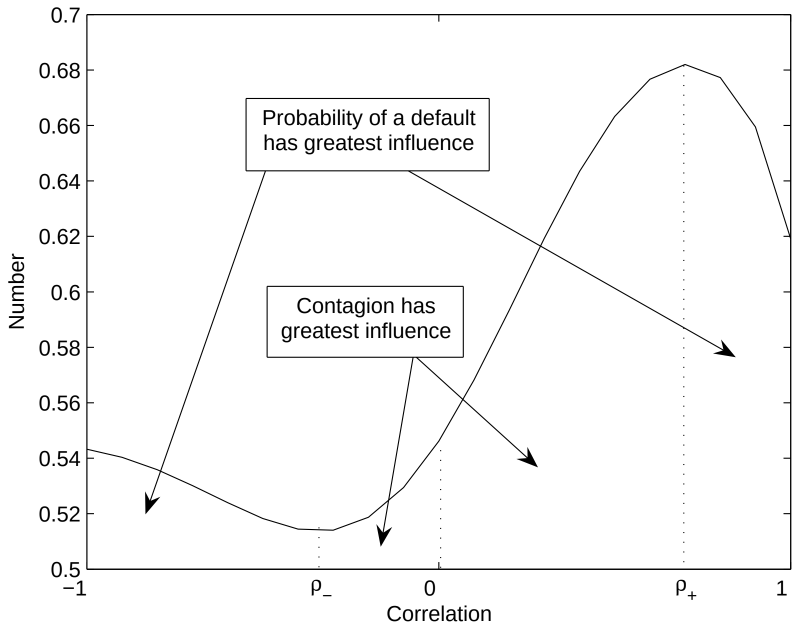
$\sigma_i = 0.2$, $r_f = 0.05$, $q_i = 0$, $\gamma_i = 0.03$, initial credit quality $= 2$

For a given value of ρ, the higher F is, the greater the impact the contagion has on the expected number of defaults, as expected. However, the greatest impact occurs for values of $\rho = \rho_\pm$ where $-1 < \rho_- < 0$ and $0 < \rho_+ < 1$, and not for $\rho = \pm 1$. This is illustrated in Figure 6.13. As ρ increases, there are conflicting influences on default probabilities. With increasing correlation, the probability of a default becomes less likely, however, if one occurs, the presence of contagion then means that a second default is much more likely for high values of ρ and very unlikely for low values of ρ. For $\rho_- < \rho < \rho_+$, the presence of contagion is the driving factor with spreads increasing with correlation. For correlation nearer ± 1, the probability of there being a default in the first place for contagion to have an effect is more important. In other words, contagion has the biggest impact around $\rho = 0$, for $\rho \in (\rho_-, \rho_+)$, the most likely levels to occur in reality.

Whilst we expect contagion to increase the likelihood of a second default for positive correlation and decrease it for negative correlation, it is not necessarily so obvious on first glance that the expected number of defaults should be independent of the level of correlation when there is no contagion. We can see this, however, either as a direct consequence of the linearity of the expectation operator,

$$\mathbb{E}(\text{Number of defaults}) = \mathbb{E}\left(\sum_{i=1}^{2} N_i\right) = \sum_{i=1}^{2} \mathbb{P}(N_i = 1),$$

Figure 6.13: Expected number of defaults



where N_i is the default indicator process for firm i, or by considering the default events. For two companies and a time period $[0, T]$, denoting the event that firm i has defaulted by T by D_i, and writing D_i^c for its complement,

$$\begin{aligned}\mathbb{P}(1 \text{ default}) &= \mathbb{P}(D_1 \cap D_2^c) + \mathbb{P}(D_1^c \cap D_2) \\ \mathbb{P}(2 \text{ defaults}) &= \mathbb{P}(D_1 \cap D_2),\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}(\text{Number of defaults}) &= \mathbb{P}(1 \text{ default}) + 2\,\mathbb{P}(2 \text{ defaults}) \qquad (6.11)\\ &= \{\mathbb{P}(D_1 \cap D_2^c) + \mathbb{P}(D_1 \cap D_2)\} + \{\mathbb{P}(D_1^c \cap D_2) + \mathbb{P}(D_1 \cap D_2)\} \\ &= \mathbb{P}(D_1) + \mathbb{P}(D_2).\end{aligned}$$

Since individual default probabilities do not depend on the degree of correlation between companies, neither does the expected number of defaults. As correlation increases, the probability of a default happening decreases but the probability of two defaults occurring when one does, increases. These two changes offset one another to have no overall impact on the expected number of defaults.

Indeed, it is straightforward to calculate the expected number of defaults in the case of no contagion from individual default probabilities. We take the parameter values used in Figures 6.11 and 6.12. Making the transformation

$$X_i(t) = \ln\left(\frac{V_i(t)}{V_i(0)}e^{-\gamma_i t}\right) = \alpha_i t + \sigma_i W_i(t), \tag{6.12}$$

to (6.1) exactly as in Chapter 4, $\alpha_i = r_f - q_i - \gamma_i - \frac{1}{2}\sigma_i^2$ and the default barrier becomes

$$B_i = \ln\left(\frac{b_i(0)}{V_i(0)}\right).$$

Since we have chosen $\alpha_i = 0$ by construction, by (5.1), the survival probability for company i is

$$\mathbb{P}(\underline{X}_i(T) \geq B_i) = 1 - 2\Phi\left(\frac{B_i}{\sigma_i\sqrt{T}}\right), \tag{6.13}$$

and thus by (6.11),

$$\mathbb{E}(\text{Number of defaults}) = 2\Phi\left(\frac{B_1}{\sigma_1\sqrt{T}}\right) + 2\Phi\left(\frac{B_2}{\sigma_2\sqrt{T}}\right) \tag{6.14}$$

Since $B_i = \ln(1/(\text{Initial credit quality}))$, the expected number of defaults is therefore 0.24232 for five years, and 0.54619 for ten years, corresponding to the values obtained numerically (to 5 decimal places).

The results in Figures 6.11 and 6.12 were generated directly by setting the initial condition to be the number of firms in default, however they could have been deduced directly from the default probabilities given in Figures 6.7 – 6.10 using (6.11).

Figures 6.14 and 6.15 show the impact of contagion on five and ten-year second-to-default CDS spreads for different values of correlation. Again, we see the same form, with spreads higher in the presence of contagion for positive correlation and lower for negative correlation, with a peak for some $0 < \rho < 1$ for positive values of contagion. The more likely a second default is to occur, the riskier the default swap and the greater the spread. Figures 6.16 and 6.17 show the same results against maturity for $\rho = \pm 1/2$. It is clear from both sets of graphs that both correlation between firms and default contagion can have a large impact on CDS spreads.

Our framework for incorporating contagion is flexible enough to enable us to assume that only default by certain companies leads to contagion. This we illustrate in Figures 6.18 and 6.19 for the case of two correlated companies, only one of which directly influences the other. For $F = 2$ and $F = 4$, we compare the case in which bankruptcy of either company causes a jump in volatility at the remaining company (labelled 'Double' in the graphs) to the case in which only one company impacts the

Figure 6.14: 5-year 2nd-to-default CDS

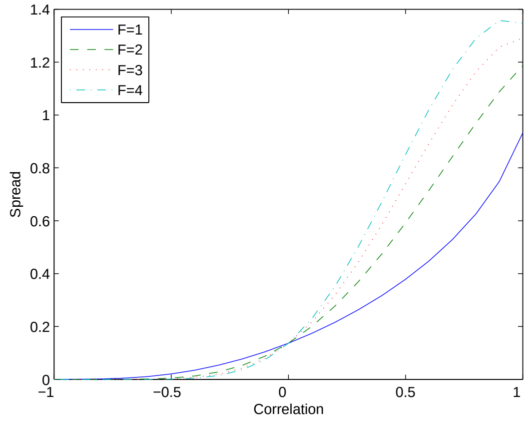
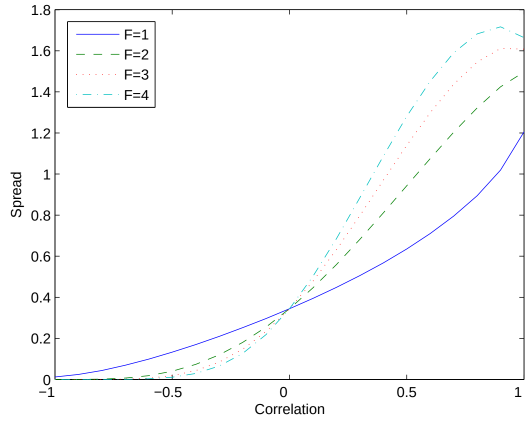


Figure 6.15: 10-year 2nd-to-default CDS



$\sigma_i = 0.2$, $r_f = 0.05$, $q_i = 0$, $\gamma_i = 0.03$, initial credit quality $= 2$, $R = 0.5$

Figure 6.16: $\rho = -0.5$

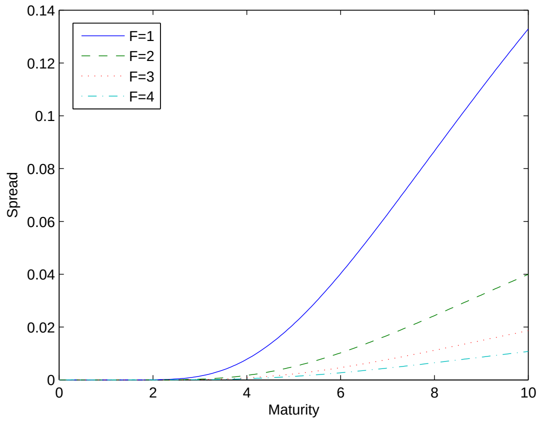
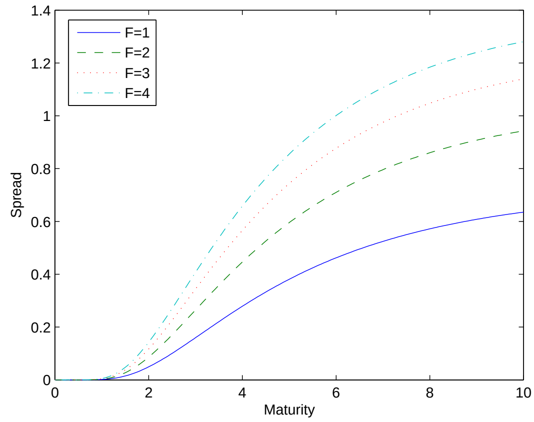


Figure 6.17: $\rho = +0.5$



Second-to-default spreads with credit contagion

$\sigma_i = 0.2$, $r_f = 0.05$, $q_i = 0$, $\gamma_i = 0.03$, initial credit quality $= 2$, $R = 0.5$

Figure 6.18: $\rho = -0.5$

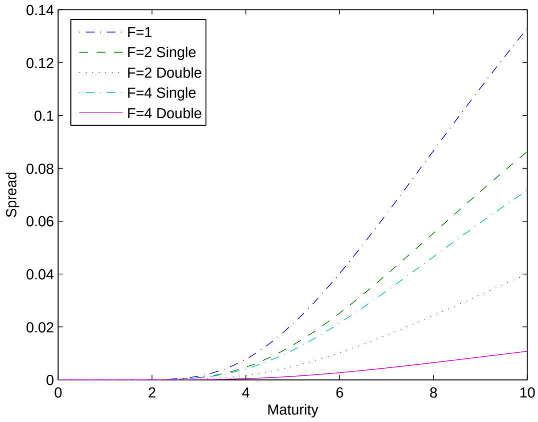
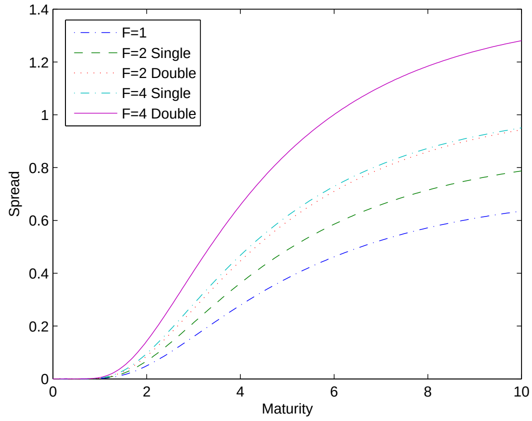


Figure 6.19: $\rho = +0.5$



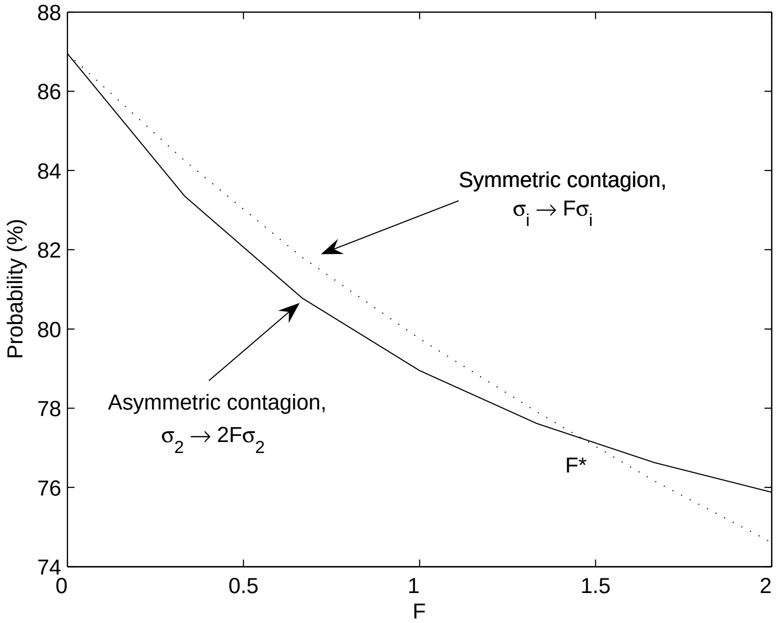
Second-to-default spreads with asymmetric credit contagion
$\sigma_i = 0.2$, $r_f = 0.05$, $q_i = 0$, $\gamma_i = 0.03$, initial credit quality $= 2$, $R = 0.5$

other (labelled 'Single' in the graphs). In the latter case, if company one defaults, the volatility of company two jumps, but if company two defaults, company one carries on as usual, its volatility unaffected by the default.

Results are as we would expect with asymmetric contagion having less impact on spreads. Considering the relative impact of $F = 2$ Double (symmetric contagion) and $F = 4$ Single (asymmetric contagion) also highlights the non-linear relationship between changes in volatility and changes in spreads. For $\rho = -0.5$, symmetric contagion with $F = 2$ has more spread impact than asymmetric contagion with $F = 4$. The reverse is true for $\rho = +0.5$. For a given value of ρ, there is a value of F for which contagion in the symmetric case is equal to twice the level of contagion in the asymmetric case, but this just represents the point at which their relative impact crosses over. This is easier to illustrate if we ignore ρ in the contagion term, and assume that volatility jumps by $\sigma_i \to F\sigma_i$. Figure 6.20 illustrates the probability of less than two defaults for $\rho = +0.5$ and shows the point, F^*, where the default probability crosses over. In the asymmetric case, the first default event leads to contagion half of the time, but since the impact of volatility on default probabilities and spreads is non-linear, doubling the increase in volatility does not lead to the same results as having contagion occur every time there is a default.

Calculating the probability that a given company defaults is also straightforward in this framework and so we can consider the impact of correlated firm values and contagion on the level of default contagion using equations (3.3) – (3.6) in Chapter 3. Zhou (2001a) examines the impact of firm value correlation and firm credit quality on

Figure 6.20: Probability of less than 2 defaults



default correlation in the non-contagion setting. Figures 6.21 and 6.22 illustrate the impact of contagion for five and ten-year horizons. As expected, the presence of credit contagion increases the range of possible default correlations. When asset correlation is positive, contagion increases default correlation, and when it is negative, contagion causes default correlation to become more negative.

Figure 6.21: 5-year default correlation

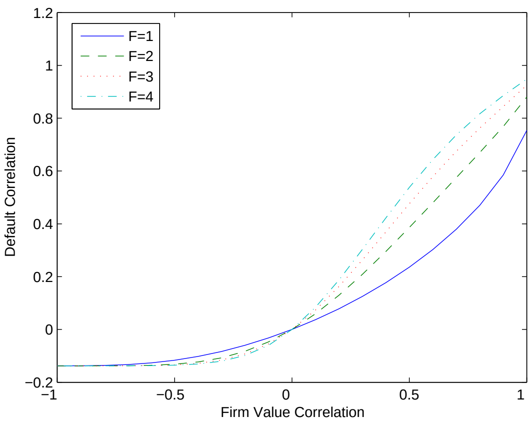
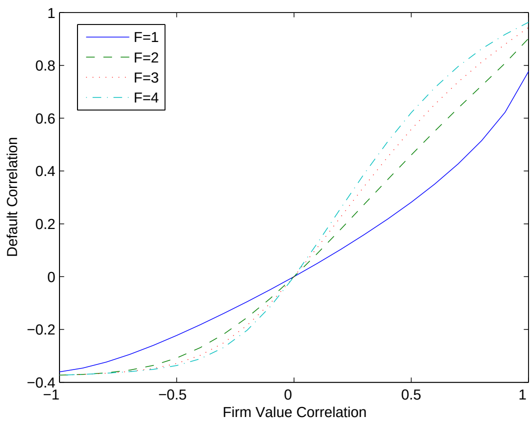


Figure 6.22: 10-year default correlation



$\sigma_i = 0.2$, $r_f = 0.05$, $q_i = 0$, $\gamma_i = 0.03$, initial credit quality $= 2$

In Figures 6.23 and 6.24 we illustrate the term structure of default correlation, with and without contagion. Results can be compared with Figure 1 in Zhou (2001a).

Using identical parameters to Zhou, we are able to exactly replicate his results using our method. Of note, for our parameter values, over the ten-year time horizon shown, we see no peak in the magnitude of default correlation. This is as a result of the lower value we assign to firm volatility – 0.2 compared to 0.4 in Zhou's work. As he explains, the magnitude of default correlation declines over the longer-term since beyond a certain point, the event of non-default becomes idiosyncratic. The effect of reducing volatility or increasing initial credit quality is to move the peak in default correlation further out – beyond ten years in our case.

Figure 6.23: No contagion, $F = 1$

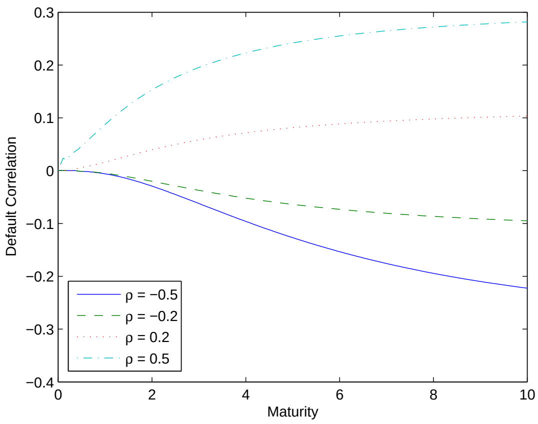
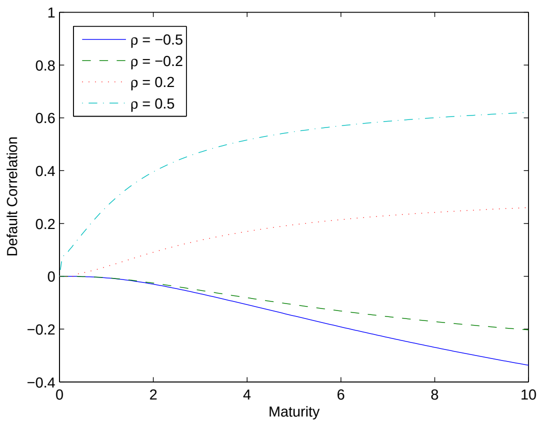


Figure 6.24: Contagion, $F = 4$



Default correlation with and without contagion

$\sigma_i = 0.2$, $r_f = 0.05$, $q_i = 0$, $\gamma_i = 0.03$, initial credit quality $= 2$

6.2 Results for Three Firms

The framework of Section 6.1 extends straightforwardly to higher dimensions and using identical numerical techniques we are able to generate k^{th}-to-default spreads for baskets of three companies. We begin, as in the two-firm case, by showing results for the joint survival probability and first-to-default CDS spread in Figures 6.25 and 6.26. Comparing with Figures 6.1 and 6.2 for the two-company basket, we see the same shaped curves. As would be expected, survival probabilities are lower and spreads are higher when there are three firms as the extra company increases the risk of the product.

In Figures 6.25 – 6.32, we are assuming that all three firms have identical characteristics, given by the parameters shown, and that there is just one correlation variable,

Figure 6.25: Joint survival probability

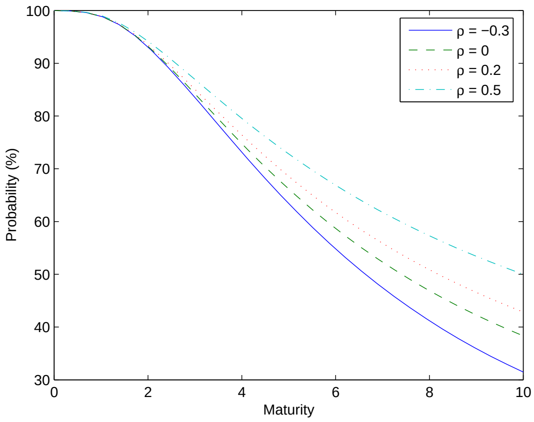
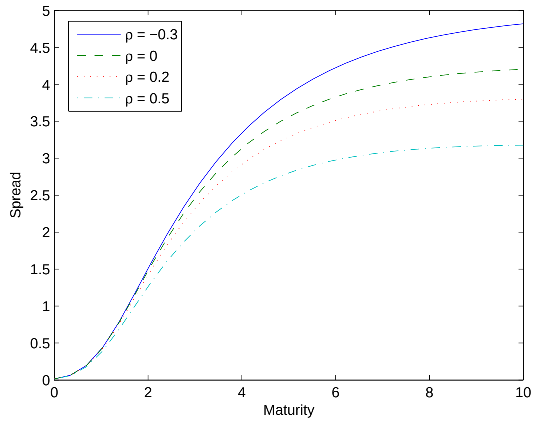


Figure 6.26: First-to-default CDS spread



$\sigma_i = 0.2$, $r_f = 0.05$, $q_i = 0$, $\gamma_i = 0.03$, initial credit quality $= 2$, $R = 0.5$

denoted by ρ. By considering homogeneous firms we are able to see the direct impact of changes in correlation and default contagion on spreads.

Figure 6.27: No contagion, $F = 1$

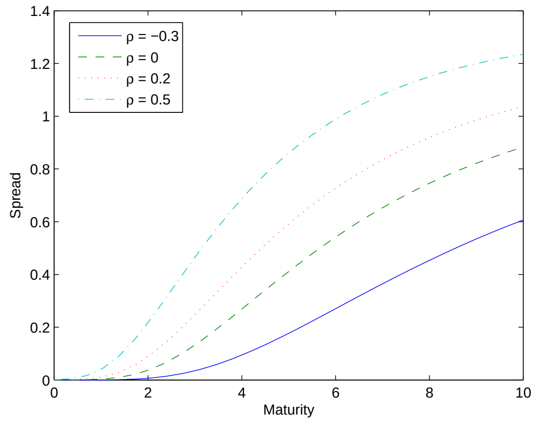
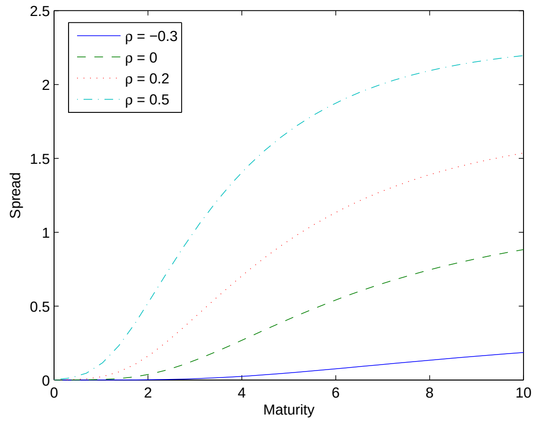


Figure 6.28: Contagion, $F = 4$



2nd-to-default CDS spreads

$\sigma_i = 0.2$, $r_f = 0.05$, $q_i = 0$, $\gamma_i = 0.03$, initial credit quality $= 2$, $R = 0.5$

Figures 6.27 and 6.28 show second-to-default spreads for various values of correlation with and without contagion. As before, spreads are the same in each case for zero correlation, lower with contagion for negatively correlated firms and higher with contagion when firms are positively correlated. The same results are illustrated in Figures 6.29 – 6.30 for third-to-default spreads. Whilst the shapes of the spread curves are the same in both cases, correlation and contagion have a much bigger impact on third-to-default spreads. Finally, spreads are greatest for first-to-default products and lowest for third-to-default products, as they should be. The impact of

Figure 6.29: No contagion, $F = 1$

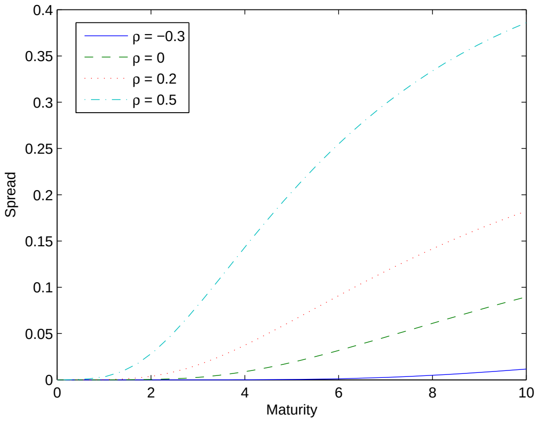
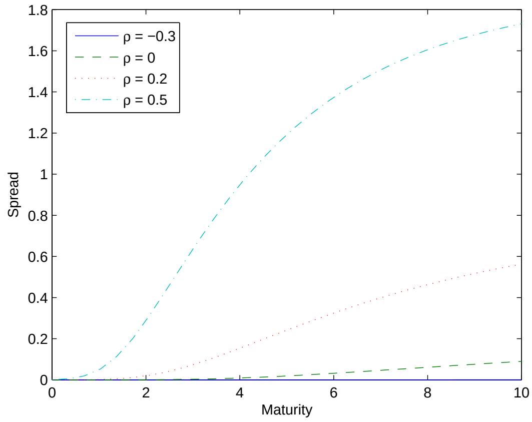


Figure 6.30: Contagion, $F = 4$



3rd-to-default CDS spreads

$\sigma_i = 0.2$, $r_f = 0.05$, $q_i = 0$, $\gamma_i = 0.03$, initial credit quality $= 2$, $R = 0.5$

contagion on second and third-to-default spreads is highlighted directly in Figures 6.31 and 6.32 for firm value correlation of $\rho = +0.5$.

Figure 6.31: 2nd-to-default CDS

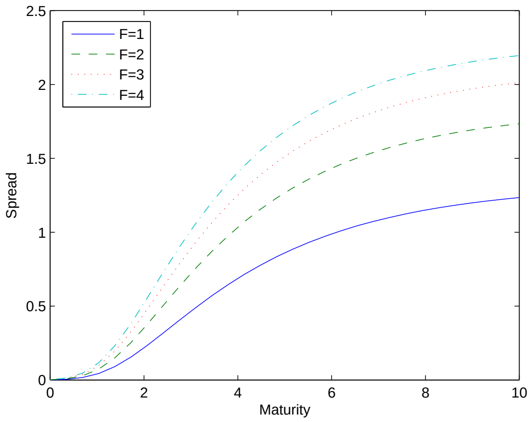
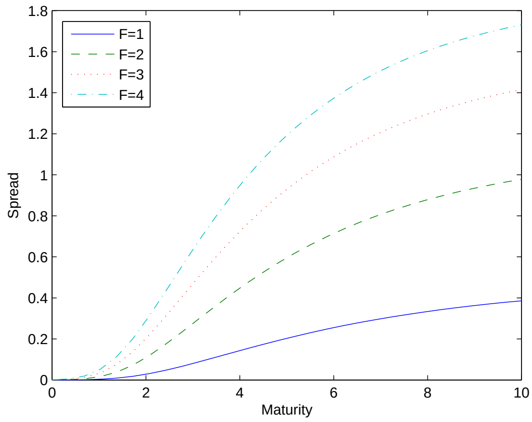


Figure 6.32: 3rd-to-default CDS



CDS spreads for $\rho = 0.5$

$\sigma_i = 0.2$, $r_f = 0.05$, $q_i = 0$, $\gamma_i = 0.03$, initial credit quality $= 2$, $R = 0.5$

Finally, in Figures 6.33 and 6.34, we provide a couple of results for different correlation structures. In order of increasing spreads, Figure 6.33 compares first-to-default swap spreads for a two-firm basket with correlation of 0.5 ($2D : \rho = 0.5$), a three-firm basket with correlation of 0.5 between all firms ($3D : \rho = 0.5$), a three-firm basket with two correlated and one completely uncorrelated firm ($3D : \rho_{12} = 0.5$, $\rho_{23} = \rho_{13} = 0$) and a three-firm basket with positively and negatively correlated firms ($3D : \rho_{12} = 0.5$, $\rho_{13} = -0.5$, $\rho_{23} = -0.25$). The order is intuitive, but it is interesting to note the

Figure 6.33: 1st-to-default CDS spread

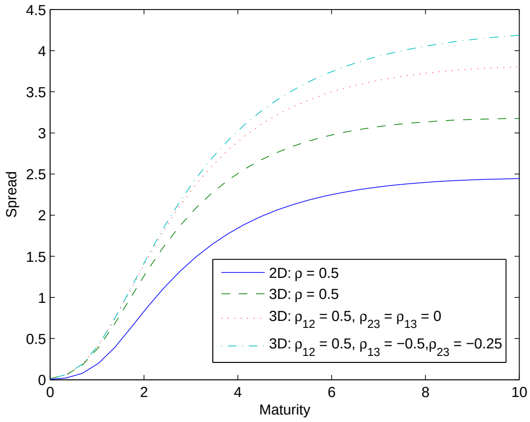
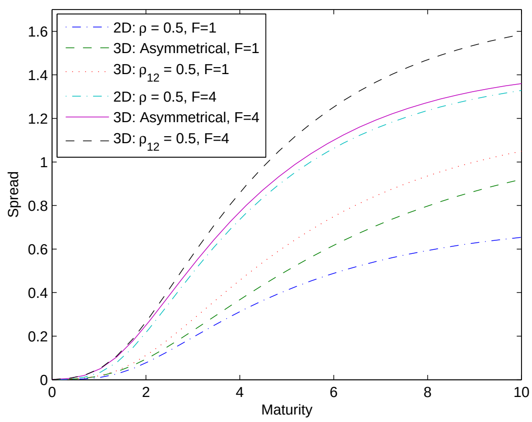


Figure 6.34: 2nd-to-default CDS spread



$\sigma_i = 0.2$, $r_f = 0.05$, $q_i = 0$, $\gamma_i = 0.03$, initial credit quality $= 2$, $R = 0.5$

relative differences in spreads and the impact that varying the correlation structure can have. Figure 6.34 gives similar results for second-to-default swap spreads, considering, in addition, the impact of contagion ($F = 4$ versus $F = 1$). 3D: Asymmetrical corresponds to $\rho_{12} = 0.5$, $\rho_{13} = -0.5$, $\rho_{23} = -0.25$, whilst $3D : \rho_{12} = 0.5$ represents $\rho_{12} = 0.5$, $\rho_{23} = \rho_{13} = 0$. Again, it is interesting to see the relative impact of the different correlation structures and the presence of contagion on spreads, and once again, these graphs highlight the flexibility of the model for investigating the relationship between the dependence structure and spreads. Whilst, as we discuss below, the model is not quite accurate enough for pricing three-firm baskets, it nonetheless represents a powerful tool, whether for assessing the impact of the dependence assumption incorporated in another model, or for an investor choosing between a number of investment options.

6.3 Contagion with Decay

Currently, with contagion as specified by (6.10), the knock-on effect of a default has a lasting impact on remaining companies. Once a firm's volatility jumps to a new level, it stays there forever. This is not necessarily particularly realistic unless the default event serves to permanently change the corporate environment. Rather, it is more usual for contagion to take the form of a decaying spike in spreads or volatility, much as seen in the case of Telus in Figure 1.1 of the introduction. In the absence of a second default or more news, spreads tend to trend back towards their original level over time.

In the event of default by company i, we can incorporate this behaviour into our model by having company j's volatility jump according to

$$\sigma_j \rightarrow \sigma_j(1 + \Delta e^{-\zeta\tau}). \tag{6.15}$$

In this way, volatility jumps by an amount Δ, and then trends exponentially back to its original level, at a rate determined by the parameter ζ; τ here denotes time since the default of firm i and $j \neq i$.

Davis and Lo (2001b) allow for something similar in their model of infectious defaults. They consider large homogeneous portfolios with two risk states – normal and enhanced. Default by one of the companies causes the hazard rate of all remaining companies to jump to an elevated level where it remains for an exponentially distributed length of time before returning to its original level. In their framework there are therefore two jumps in the hazard rate, marking the beginning and the end of the enhanced-risk state. Here, we assume an initial jump in volatility on default, the effect of which then dissipates over time.

Writing τ_i for the time of default of company i, and τ as the time elapsed since company i defaulted, for $j \neq i$, the volatility of firm j as given by (6.15) is now time-dependent, $\sigma_j(\tau)$. As outlined in Wilmott (1998), we can remove the time-dependence by replacing $\sigma_j^2(\tau)$ with its average over the remaining time-to-maturity, $\bar{\sigma}_j^2$, where

$$\begin{aligned}
\bar{\sigma}_j^2 &= \frac{1}{T-\tau_i}\int_0^{T-\tau_i} \sigma_j^2(\tau)\,\mathrm{d}\tau \\
&= \frac{\sigma_j^2}{T-\tau_i}\int_0^{T-\tau_i} \left(1 + 2\Delta e^{-\zeta\tau} + \Delta^2 e^{-2\zeta\tau}\right)\,\mathrm{d}\tau \\
&= \sigma_j^2 + \frac{2\Delta\sigma_j^2}{\zeta(T-\tau_i)}\left(1 - e^{-\zeta(T-\tau_i)}\right) + \frac{\sigma_j^2\Delta^2}{2\zeta(T-\tau_i)}\left(1 - e^{-2\zeta(T-\tau_i)}\right).
\end{aligned}$$

For a basket of two firms, we can then solve the PDE, (6.6), by breaking the time period into two. We consider the usual two-firm problem on $[0, \tau_i]$ and a one-company problem on $[\tau_i, T]$. By standard first passage theory, as given in Musiela and Rutkowski (1998), the probability that company j survives and does not fall below its default threshold before maturity T given that company i defaults at time $\tau_i < T$ is

$$\Phi\left(\frac{X_j(\tau_i) + \bar{\alpha}_j(T-\tau_i)}{\bar{\sigma}_j\sqrt{T-\tau_i}}\right) - e^{\{-2\bar{\alpha}_j X_j(\tau_i)/\bar{\sigma}_j^2\}}\Phi\left(\frac{-X_j(\tau_i) + \bar{\alpha}_j(T-\tau_i)}{\bar{\sigma}_j\sqrt{T-\tau_i}}\right), \tag{6.16}$$

where $\bar{\alpha}_j = r_f - q_j - \gamma_j - \bar{\sigma}_j^2/2$ and $X_j(t) = \ln\left(V_j(t)e^{-\gamma_j t}/b_j(0)\right)$.

We then solve (6.6) on $[0, \tau_i]$ exactly as before, subject to the additional boundary conditions at $X_i(t) = 0$ given by (6.16). By modifying the initial condition and specifying the boundary conditions according to whether we are interested in company j surviving or defaulting, we are able to calculate a range of default probabilities as before, and asymmetric default contagion can be easily incorporated.

For consistency, we set $\Delta = F^{\rho_{ij}} - 1$, so that the initial jump in volatility is driven by both the degree of contagion, F, and the correlation between firms as before. By specifying the process in this manner we can directly compare results with and without decay as shown in Figures 6.35 – 6.42.

In Figures 6.35 – 6.38, we consider results for no contagion ($F = 1,\ \zeta = 0$), contagion but no decay ($F = 4,\ \zeta = 0$) and contagion with varying rates of decay ($F = 4$, $\zeta = 0.5, 2, 4, \& 8$). Values of ζ of 8, 4 and 2 correspond to volatility reverting to its pre-default level over roughly six months, one year and two years, respectively. For the purpose of comparison, we also include the fairly extreme case of $\zeta = 0.5$ for which the effects of the default take nearly ten years to dissipate. In practice for a company's default to have such a lasting impact, it would have to have been a key company in the sector or market of interest, or a particularly momentous default. The possible bankruptcy of General Motors and the fall-out following the Enron debacle spring to mind as incidences that might lead to such a long-drawn out settling down of spreads.

Figures 6.35 and 6.36 show the probability of one and two defaults, respectively, for a ten-year period. We see that results with a decay in default contagion, $\zeta > 0$, have the same form and lie closer to results with no contagion (shown by the solid black line) than those with a high degree of permanent contagion, $\zeta = 0$. As ζ increases, the rate at which the contagion effects dissipate increases and probabilities tend more quickly towards their non-contagion level.

Since the rate of decay is exponential, it makes sense that the impact of contagion is fairly limited when compared to the situation in which the jump in volatility is permanent. That said, as we see from the 2nd-to-default spreads in Figures 6.37 and 6.38, the effect is still large enough to warrant attention. For highly correlated companies, even if the knock-on impact of the default dissipates in six months ($\zeta = 8$), spreads are five to ten basis points higher with contagion. The spread impact of contagion for negative values of ρ is considerably lower than for positive ρ. This is because for $\rho < 0$ contagion serves to reduce volatility and as we saw in Figure 5.32,

Figure 6.35: Probability of 1 default

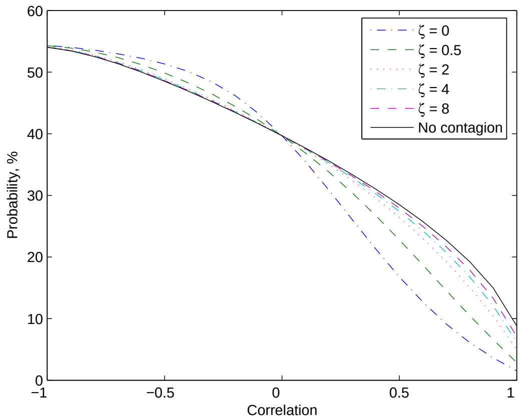
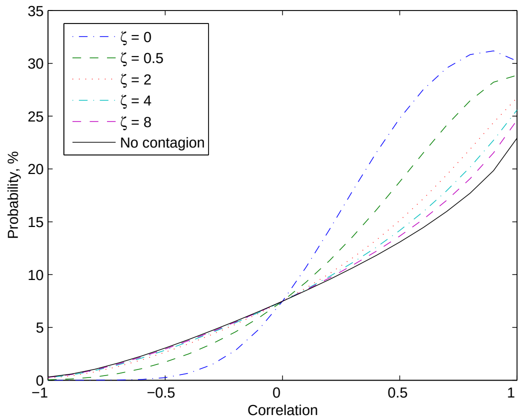


Figure 6.36: Probability of 2 defaults



Ten-year default probabilities with decaying contagion

$\sigma_i = 0.2$, $r_f = 0.05$, $q_i = 0$, $\gamma_i = 0.03$, initial credit quality = 2

Figure 6.37: 10-year 2nd-to-default CDS

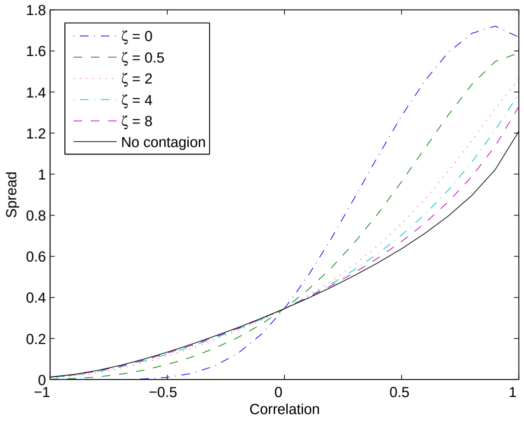
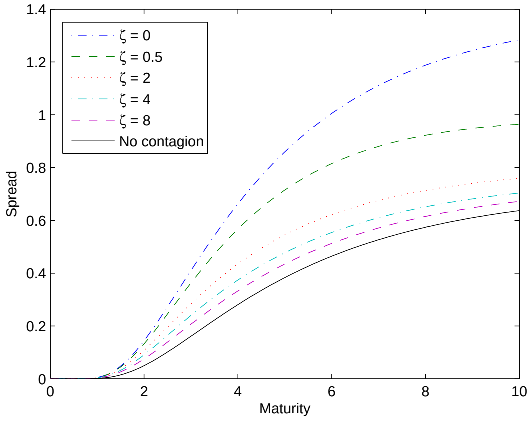


Figure 6.38: Spread curve for $\rho = 0.5$



2nd-to-default CDS spreads with decaying contagion

$\sigma_i = 0.2$, $r_f = 0.05$, $q_i = 0$, $\gamma_i = 0.03$, initial credit quality = 2, $R = 0.5$

increasing volatility by a given amount (e.g. from 0.2 to 0.3) has a much bigger impact on spreads than reducing volatility by the same amount (0.2 to 0.1).

Figures 6.39 – 6.42 show the direct impact of contagion on spreads for positive values of correlation. Figures 6.39 – 6.40 give results for $\rho = 0.5$, Figures 6.41 – 6.42 for $\rho = 0.75$. In each case we illustrate the extra spread in basis points (1/100ths of a percent) for 2nd-to-default CDSs with contagion compared to the base case of correlated firm values but no default contagion. For reference, in the absence of contagion, the five-year spread with $\rho = 0.5$ is 38 basis points, whilst for $\rho = 0.75$ it is 63 basis points. The extra spreads we are seeing due to the existence of contagion are therefore of a meaningful size.

Figure 6.39:

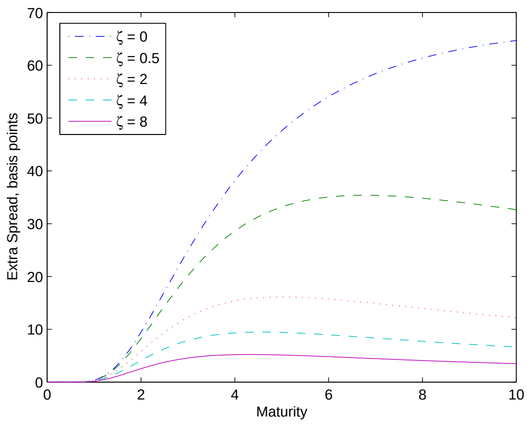
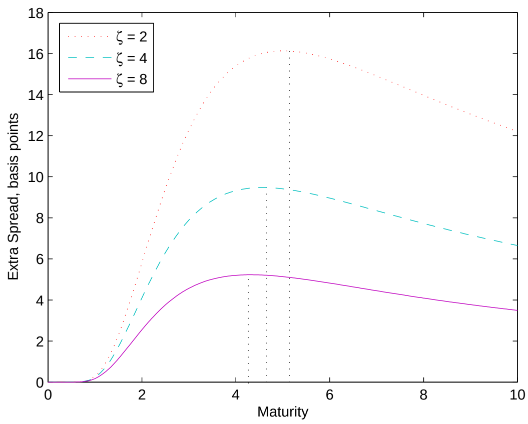


Figure 6.40:



Additional spread on 2nd-to-default CDS due to contagion, $\rho = 0.5$
$\sigma_i = 0.2$, $r_f = 0.05$, $q_i = 0$, $\gamma_i = 0.03$, initial credit quality = 2, R=0.5

From the left-hand graphs, we see that when contagion results in a permanent increase in volatilities ($\zeta = 0$), spreads increase with maturity – the longer dated the CDS, the greater the impact of contagion. This is not the case when the jump in volatility declines over time. As would be expected, the faster the rate of decay, the shorter the maturity of the product that sees the greatest spread impact, and the earlier we see a peak in the spread difference. This is easier to see in the right-hand graphs, Figures 6.40 and 6.42. These reproduce the lower three curves of the left-hand graphs and show clearly the peaks for $\zeta = 2, 4$ and 8. For $\rho = 0.75$, contagion with $\zeta = 2$ has the greatest impact on spreads for a 4.9-year maturity CDS whilst when $\zeta = 8$, the difference is greatest for a maturity of 3.9 years.

Extending this model with decaying contagion to three or higher dimensions is not straightforward. Having enjoyed manipulating infinite sums of modified Bessel func-

Figure 6.41:

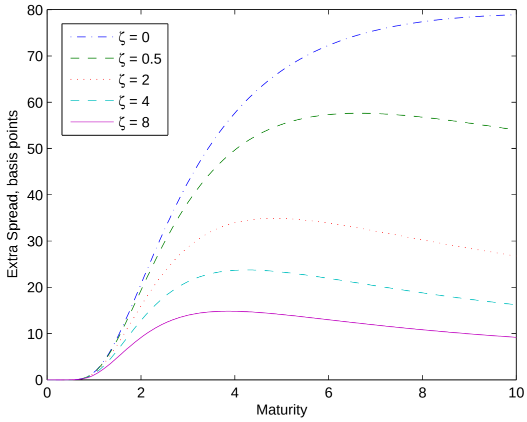
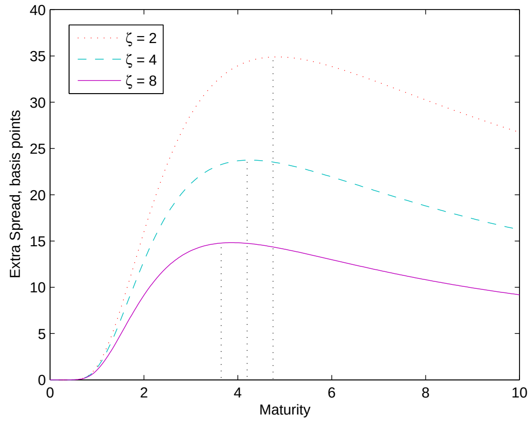


Figure 6.42:



Additional spread on 2nd-to-default CDS due to contagion, $\rho = 0.75$
$\sigma_i = 0.2$, $r_f = 0.05$, $q_i = 0$, $\gamma_i = 0.03$, initial credit quality $= 2$, $R = 0.5$

tions in Chapters 4 and 5, we leave any attempt to incorporate them into a (6.16)-type boundary condition to a braver soul. Alternatively, in theory it would also be possible to tackle the problem recursively, starting with the case when all bar one of the firms have defaulted to define the boundary conditions for the boundary planes (as done here for the two-dimensional case), and then solving the two-firm case on these planes to give the boundary conditions in three-dimensions. Having seen the behaviour of contagion with decay in two dimensions, however, by comparing two and three-dimensional results without decay, it is possible to infer roughly how much a temporary jump in firm volatility might impact spreads.

6.4 Numerical Issues

For a basket of n firms, our code requires coordinates to be specified on $[0,1]^n$, with default barriers at zero. In order to implement the general case with correlated driving Brownian motions, we transform (6.1) to

$$dX_i(t) = \alpha_i dt + \sigma_i dW_i(t). \tag{6.17}$$

using

$$X_i(t) = \ln\left(\frac{V_i(t)}{b_i(0)} e^{-\gamma_i t}\right) = \alpha_i t + \sigma_i W_i(t) \tag{6.18}$$

for $\alpha_i = r_f - q_i - \gamma_i - \frac{1}{2}\sigma_i^2$ defined as before. The default barriers are then $X_i(t) = 0$ and, reversing time, $\mathcal{L}U = 0$ becomes

$$\frac{\partial U}{\partial t} = \sum_{i=1}^{n} \alpha_i \frac{\partial U}{\partial X_i} + \frac{1}{2} \sum_{i,j=1}^{n} \rho_{ij}\sigma_i\sigma_j \frac{\partial^2 U}{\partial X_i \partial X_j}. \tag{6.19}$$

As in the analytical approach, we define the starting point for firm value in terms of an initial credit quality as per Section 4.2.1.

Finally, to move coordinates to $[0,1]^n$, we divide $X_i(t)$ by a constant value, D, such that $\mathbb{P}(X_i(t) > 1) \approx 0$, chosen by considering the accuracy of results. For the two and three-firm cases, we use $D = 5$, although results are stable for D in the range 3 – 15. For two firms, results obtained in this way converge to a stable, accurate solution in a reasonable time frame. However, dealing with higher numbers of firms becomes rapidly more problematic and our numerical approach is unable to cope with four dimensions or higher.

6.4.1 The Situation in 3D

Increasing the problem to consider three firms means that we are limited to using a maximum of six refinements when using a full grid. This is because for n refinements, an additional company increases the number of nodes by a factor of 2^n and so we run into memory constraints. On the positive side, the calculation is very fast, producing results in around thirty seconds with thirty time-steps. The number of refinements is the limiting factor in the accuracy of results rather than the number of time-steps and results are the same to four decimal places whether using 30 or 200 time-steps. On the downside, results are not quite as accurate as we would like.

Since in the absence of default contagion, $\mathbb{P}(\tau_1 < T)$ is the same regardless of the number of companies, we can use results generated in two dimensions to evaluate the accuracy of results using six partitions in three dimensions. A couple of comparisons are given in Table 6.1 for five and ten years.

These levels of accuracy seem to be indicative, with default probabilities accurate to (at least) two decimal places and integrals less than 0.01 different. The important consideration is the impact this level of error has on CDS spreads. For the spreads illustrated in Figures 6.26 – 6.32, we stressed both the survival probabilities and integrals used in (6.9) by ± 0.01. Across the eight different possible cases (each were either kept fixed, increased or decreased by 0.01), the maximum absolute error was

	$\mathbb{P}(\tau_1 < T)$		$\int_0^T e^{-r_f s}\mathbb{P}(\tau_1 < s)$	
	T = 5 years	T = 10 years	T = 5 years	T = 10 years
2D 10 partitions 200 time steps	0.121162	0.273097	0.165742	0.853578
3D 6 partitions 30 time steps	0.122469	0.273592	0.171685	0.862082

Table 6.1: *Examples of accuracy of 3D calculation vs 2D*
$\sigma_i = 0.2, r_f = 0.05, q_i = 0, \gamma_i = 0.03$, initial credit quality = 2

around 10 basis points, usually less for 10-year spreads, sometimes a little higher for 5-year spreads. Whilst this should represent a strong upper bound on the error, it is, unfortunately, a little too high for the model to be used for pricing. It is, however, by far accurate enough to be used as a tool to consider the sensitivity of CDS spreads to input parameter assumptions. The approach is quick and easy to implement, providing a very flexible framework in which to assess realistic pricing bounds.

The other limitation to the model is that the numerical discretisation is unstable in three dimensions for larger values of ρ and in the homogeneous case, we are restricted to $|\rho| \leq 0.5$. We are using the seven-point stencil introduced in Hackbusch (1992) to discretise the cross-derivatives, further details of which are contained in Appendix I. The limits on ρ represent the values for which the off-diagonals of the discretisation of the second-order terms become negative.

6.4.2 Can the Numerics be Improved?

One method that would speed up the process and allow it to be implemented for larger baskets would be to use a sparse grid as we did for the calculations in Chapters 4 – 5. However, even for two firms, whilst using the simple coordinate transformation $\frac{1}{D}X_i(t)$ from (6.18) produces stable results using a full grid, there is no convergence with a sparse grid. The problem arises from the discontinuity in the initial condition near $\mathbf{X} = 0$.

6.4.2.1 Coordinate Transformation

In an effort to implement the sparse grid approach, we tried a number of coordinate transformations to better deal with this discontinuity. The goal was a transformation that would lead to the numerical scheme sampling progressively more points as the space variables approached zero. The obvious first transformation was to take the natural logarithm, $Y_i(t) = \ln X_i(t)$, thereby sending the singularity to $-\infty$. Unfortunately, despite trying a number of mappings of the resulting domain $(-\infty, +\infty)^n \to [0,1]^n$, we were unable to generate any results that were as accurate as (6.18), even using a full grid. Results with a sparse grid did not even converge.

To avoid having to map the domain back to $[0,1]^n$, we then tried transformations of the form $Y_i(t) = X_i(t)^{1/q}$. Only small values of q gave good results and those using a sparse grid were slightly more stable than before. However, they still did not converge, tending to oscillate, and results with the full grid were not only less accurate than those using our simple first transformation, but they took twice as long to calculate.

6.4.2.2 Singularity Removal

The other tack we tried was to remove the singularity at zero. By writing $U(t) = V(t) + W(t)$, we tried to solve (6.19) in the case $n = 2$ by separating the solution into a function $V(t)$ and the solution to the uncorrelated problem, $W(t)$. Solving

$$\frac{\partial W}{\partial t} = \frac{1}{2}\sigma_1^2 \frac{\partial^2 W}{\partial X_1^2} + \frac{1}{2}\sigma_2^2 \frac{\partial^2 W}{\partial X_2^2}. \tag{6.20}$$

subject to

$$\begin{aligned} W(X_1, X_2, 0) &= 0 \\ W(0, 0, t) &= 1 \end{aligned}$$

to give the joint default probability, $\mathbb{P}(\tau_1 < T, \tau_2 < T)$, was straightforward analytically by splitting the problem into two one-dimensional problems, and using the method in Rozier (1984). Since

$$W(X_1, X_2, t) = \frac{4}{\pi} \int_{\frac{X_1}{\sigma_1\sqrt{2t}}}^{\infty} e^{-s^2}\,\mathrm{d}s \int_{\frac{X_2}{\sigma_2\sqrt{2t}}}^{\infty} e^{-s^2}\,\mathrm{d}s, \tag{6.21}$$

$V(X_1, X_2, t)$ then satisfied

$$\frac{\partial V}{\partial t} = \sum_{i=1}^{2} \left(\alpha_i \frac{\partial V}{\partial X_i} + \frac{1}{2}\sigma_i^2 \frac{\partial^2 V}{\partial X_i^2} \right) \tag{6.22}$$

$$- \quad \frac{2\sqrt{2}\alpha_1}{\pi\sigma_1\sqrt{t}} e^{-X_1^2/(2t\sigma_1^2)} \int_{\frac{X_2}{\sigma_2\sqrt{2t}}}^{\infty} e^{-s^2}\, \mathrm{d}s - \frac{2\sqrt{2}\alpha_2}{\pi\sigma_2\sqrt{t}} e^{-X_2^2/(2t\sigma_2^2)} \int_{\frac{X_1}{\sigma_1\sqrt{2t}}}^{\infty} e^{-s^2}\, \mathrm{d}s \tag{6.23}$$

$$+ \quad \rho_{12}\sigma_1\sigma_2 \left(\frac{\partial^2 V}{\partial X_1 \partial X_2} + \frac{2}{\pi\sigma_1\sigma_2 t} e^{-X_2^2/(2t\sigma_2^2)} e^{-X_1^2/(2t\sigma_1^2)} \right). \tag{6.24}$$

$W(X_1, X_2, t)$ is quick and easy to evaluate, and so solving numerically for $V(X_1, X_2, t)$ subject to zero initial and boundary conditions would enable us to evaluate the default probability $U(X_1, X_2, t)$. However, the $O(\frac{1}{t})$ term in line (6.24) continued to cause problems for $\mathbf{X} = \mathbf{0}$ as $t \to 0$, being $O(\ln t)$ in the expression for V. The algebra simplifies considerably in the case that $\alpha_i = 0$ for $i = 1, 2$, however since the problematic term in (6.24) remains, we were not able to generate anything useful using this approach.

The question of whether the numerics can be improved and hence whether our methodology can be applied to larger baskets of credits remains open.

6.5 General n-Dimensional Framework

Numerical problems aside, the framework has the flexibility to enable a very general specification of firm value dynamics and underlying correlation structure. Supposing that firm values are driven by both a number of macro factors and an idiosyncratic factor, with correlation between firms arising from their individual exposures to the macro driving factors, then for n companies, values $V_i(t)$, with idiosyncratic factors $W_i(t)$ and m independent macro driving processes $Y_j(t)$,

$$\begin{aligned} dV_i(t) &= (r_f - q_i)V_i(t)dt + \delta_i V_i(t) dW_i(t) + \sum_{j=1}^{m} \gamma_{ij} V_i(t) dY_j(t) \\ &= \beta_i V_i(t)dt + \sum_{j=1}^{d} \eta_{ij} V_i(t) dZ_j(t) \end{aligned} \tag{6.25}$$

where $d = m + n$, $\beta_i = r_f - q_i$ and

$$\eta_{ij} = \begin{cases} \delta_i & \& \quad dZ_j(t) = dW_j(t) \quad \text{for} \quad 1 \le j \le n, \quad i = j \\ 0 & \& \quad dZ_j(t) = dW_j(t) \quad \text{for} \quad 1 \le j \le n, \quad i \ne j \\ \gamma_{i(j-n)} & \& \quad dZ_j(t) = dY_{(j-n)}(t) \quad \text{for} \quad n+1 \le j \le d. \end{cases}$$

For each i and all j, we assume that the Brownian motions $W_i(t)$ and $Y_j(t)$ are uncorrelated with one another. β_i represents the expected growth rate of the value of firm i and the weightings γ_{ij} and δ_i represent the exposure of company i to the various macro and idiosyncratic factors. The larger a given weighting is, the greater that factor's influence and the potentially more volatile the company. As before, we assume that company i defaults the first time that its value drops below the level of its default barrier $b_i(t)$. Defined in this way, company values can be considered to be driven by both company-specific and global factors. The latter could, for example, be at the country, sector or industry level, introducing a rich correlation structure between companies.

If $\mathbf{V}$ is the vector of firm values, the infinitesimal generator of (6.25) is

$$\mathcal{L}U = \frac{\partial U(\mathbf{V},t)}{\partial t} + \sum_{i=1}^{n} \beta_i V_i \frac{\partial U(\mathbf{V},t)}{\partial V_i} + \frac{1}{2}\sum_{i,j=1}^{n} a_{ij} V_i V_j \frac{\partial^2 U(\mathbf{V},t)}{\partial V_i \partial V_j} \tag{6.26}$$

where

$$a_{ij}\,\mathrm{dt} = \sum_{k,l=1}^{d} \eta_{ik}\eta_{jl} \langle dZ_k, dZ_l \rangle , \tag{6.27}$$

and $\langle dZ_k, dZ_l \rangle$ represents the quadratic covariation between dZ_k and dZ_l.

Applying the Feynman-Kac formula as before, we can then calculate the probability of k defaults in $[0, T]$ by solving

$$\begin{aligned} \frac{\partial U(\mathbf{V},t)}{\partial t} &+ \sum_{i=1}^{n} \beta_i V_i \frac{\partial U(\mathbf{V},t)}{\partial V_i} + \frac{1}{2}\sum_{i,j=1}^{n} a_{ij} V_i V_j \frac{\partial^2 U(\mathbf{V},t)}{\partial V_i \partial V_j} = 0 \\ U(\mathbf{V},T) &= \mathbb{I}_{\Omega_k}(\mathbf{V}) = \Psi(\mathbf{V}), \end{aligned} \tag{6.28}$$

where Ω_k is the set corresponding to exactly k companies defaulting in $[0, T]$.

The basic structure of (6.25) is related to that considered by Hull et al. (2005), described in Section 3.3.1. However, whilst Hull et al. model firm value in a factor framework with the resultant correlation structure, there is no mechanism for introducing any type of default contagion or inter-company ties in their model. To our knowledge, there has been no other work to date incorporating a dependence structure in a multi-asset structural framework representing anything other than a correlation between firm value processes.

6.6 Conclusion

By taking a purely numerical approach to modelling firm value dynamics and the default event, we have made significant progress in valuing basket credit default swap spreads in a first passage framework with both asset correlation and default contagion. The approach is easy to implement and enables specification of a rich dependence structure incorporating asymmetries and default causality. By modifying the initial condition, there is great flexibility to calculate many different default and survival probabilities of interest, allowing evaluation of k^{th}-to-default CDS spreads, the expected number of defaults and default correlation.

Results reiterate the need for credit models to take into account a full dependence structure, with default contagion having a very clear impact on spreads. Whilst accuracy in the current approach is not quite good enough for pricing baskets of three companies, the model could be used as a powerful tool for analysing the spread impact of different dependence assumptions and parameter values. We have focused on providing results to highlight the importance of the specification of the dependence structure on spreads, leaving the discussion of parameter sensitivities to our analysis of the analytical model in Chapters 4 and 5. Incorporating extensions such as allowing recovery rate to be a fraction of discounted par value as we did earlier would, of course, be straightforward.

The numerical approach is a great improvement on the analytical method. Not only is it easier to specify and implement, but it deals with many of the limitations of the latter outlined in Section 5.4. For the first time, we have been able to value CDS spreads in the structural framework with default contagion for baskets of up to three firms.

The goal remains to extend the approach to cope with bigger baskets in order to price large k^{th}-to-default CDS products and CDO tranches. We have the framework for specifying the dynamics and a realistic dependence structure, but extending the numerics to deal with higher dimensions has proven elusive. In order to model large portfolios of up to 125 companies, principal component analysis could be used to reduce the dimensions to something more manageable, but this would still require numerical methods capable of coping with more than three dimensions. More sophisticated techniques seem to be necessary to make progress on this front.

One numerical approach that we have not tried but which may be more easily applicable to larger baskets is Monte Carlo simulation. Our focus has been on the application of PDE techniques, and hence finite-difference methods, but there is no reason why results should not be obtainable using Monte Carlo simulation and dealing with the conditions on the boundaries should be relatively straightforward. In the case of baskets of two or three companies as considered here, finite-difference methods are generally faster, but Monte Carlo becomes more efficient when extending beyond four dimensions. Simulating results should therefore overcome some of the problems we have faced in extending our framework to higher dimensions, although obtaining accurate results is likely to be slow.

Chapter 7

Concluding Remarks

We concluded Chapter 3 with an outline of the desirable attributes of a good multi-firm credit model, namely that it should

1. be intuitive,
2. capture real-world default dynamics,
3. incorporate a realistic dependence structure,
4. be implementable,
5. enable calibration.

As we recognised then, developing a model with all five characteristics in their entirety is a highly ambitious proposition and never the goal of this thesis. Rather, we sought to investigate ways of implementing a more realistic dependence structure within a multi-firm first passage framework – satisfying points one, two and three. In this regard, we have made significant advances on existing models, particularly in relation to points two and three, and have raised some interesting questions in the process.

Starting with the analytical approach of Chapters 4 and 5, we incorporated default contagion within the structural framework for the first time, developing a model with a dependence structure driven by both macro and firm-level influences. The model was by necessity simplistic due to the complicated nature of the mathematics, but introduced both causality and asymmetry into the default specification, a significant advance on prior firm value models.

In addition to requiring an overly restrictive contagion mechanism, however, the numerical solution of these 'analytical' solutions was highly involved and time consuming, albeit considerably easier in certain simplifying cases. Whilst we therefore made some progress on points two and three, it was at the expense of point four. In all, it was a good starting point and provided the direction for the development of our numerical model in Chapter 6.

Despite being fully numerical, this approach was arguably less numerically involved than the analytical approach and certainly much easier to implement. Whilst there were no closed-form solutions for credit spreads, the model was significantly more flexible, enabling a far more realistic specification of the dependence structure. In addition to having a macro driving influence incorporated through the correlation in firm values, credit contagion was specified so that a default event had a knock-on impact on the credit quality of remaining companies. We provided a number of examples incorporating both significant asymmetry in the dependence structure and allowing for the dissipating nature of credit contagion over time. A wide range of default probabilities and credit spreads was straightforward and quick to calculate with relative accuracy.

The process of developing this model, both analytically and numerically, raised some interesting points regarding the form of a good model. Whilst the analytical approach enabled the derivation of exact solutions, they were pretty horrid, the framework was inflexible, and the numerical evaluation of the solutions was often more complicated than solving the entire problem numerically in the first place.

As implemented in Chapter 6, our model can be used as a powerful tool for evaluating the sensitivity of credit spreads to assumptions regarding the nature of the dependence structure and parameter values. To date, there seems to have been limited academic interest in the application of firm value models for pricing credit derivatives, and yet our framework is extremely flexible and easy to implement. It also begins to address the problem inherent in most multi-firm models of having just a single correlation parameter. Going back to our desirable attributes, we have therefore made significant progress in developing a model satisfying points one to four.

The next stage would be to consider its applicability as a pricing tool, numerical limitations aside for the moment. We have illustrated in detail the characteristics of the spread curves generated by our model; these need to be compared with those seen in practice, in terms of both level and shape, in order to ensure that the dependence structure can be formulated to give realistic spreads. In terms of calibration, standard techniques can be used to obtain individual company parameters from market data and through calibration to single-name CDS spreads. There are then two or three free parameters (ρ_{ij} and either F, or Δ and ζ) depending on whether or not default contagion decays, that could be used to calibrate the multi-asset model.

The more pressing issue, however, is to resolve the numerical limitations so that the framework can be applied to larger baskets. It should be possible to improve our finite-difference approach to enable more accurate, faster pricing of small baskets. Alternatively, Monte Carlo techniques are frequently used in credit modelling, and should enable results to be generated in higher dimensions. Whether or not they can do so sufficiently fast and accurately, however, is not certain.

In terms of future extensions to the model, it would be relatively straightforward to extend the framework to a non-geometric Brownian motion setting. It would be interesting to see the impact of incorporating stochastic correlation or stochastic volatility, although this would raise the added issue of market incompleteness. By relating firm correlations or volatilities to a global state variable, it would be possible to have default correlations depend on the state of the economy as considered by Hull et al. (2005), reflecting the fact that defaults tend to be more highly correlated when default probabilities are higher. Another desirable extension would be to incorporate jumps into the specification of the underlying firm value dynamics to remove the predictable nature of default and the resultant zero short spreads. As ever, there is a difficult trade-off between having realistic dynamics for individual firms, both in terms of their own specific characteristics and their relationships with other companies, and having a model that is scalable to higher dimensions.

Appendix A

Method of Images

Proposition A.1 *Consider the option $C(V,t,T)$ such that*

$$\begin{aligned} \mathcal{L}C(V,t,T) &= 0 \quad V(t) \geq b(t), \quad t \leq T \\ C(V,T,T) &= C_T(V) \quad \textit{terminal condition} \\ C(b(t),t,T) &= 0 \quad \textit{barrier condition} \end{aligned} \tag{A.1}$$

for a lower barrier $b(t) = b(0)e^{\gamma t}$ and linear operator $\mathcal{L}$,

$$\mathcal{L}C(V,t,T) = \frac{1}{2}\sigma^2 V^2 C_{VV} + (r_f - q)VC_V - r_f C + C_t. \tag{A.2}$$

If $C_B(V,t)$ satisfies

$$\begin{aligned} \mathcal{L}C_B(V,t,T) &= 0 \quad V \geq 0, \quad t \leq T \\ C_B(V,T,T) &= C_T(V)\mathbb{I}_{\{V(T) \geq b(T)\}}, \end{aligned}$$

where $\mathbb{I}_E$ is the indicator function of an event, E, then the solution to A.1 is

$$C(V,t,T) = C_B(V,t,T) - C_B^i(V,t,T)$$

where the image solution $C_B^i(V,t,T)$ is

$$C_B^i(V,t,T) = \left(\frac{b(t)}{V(t)}\right)^{\frac{2(\mu-\gamma)}{\sigma^2}} C_B\left(\frac{b(t)^2}{V(t)}, t, T\right) \tag{A.3}$$

for $\mu = r_f - q - \frac{1}{2}\sigma^2$.

Proof

Making the change of variables,

$$X(t) = \ln\left(\frac{V(t)}{V(0)}e^{-\gamma t}\right) \quad \text{(A.4)}$$

$$\tau = T - t, \quad \text{(A.5)}$$

the barrier becomes $B = \ln\left(\frac{b(0)}{V(0)}\right) \leq 0$ and (A.2) becomes

$$\mathcal{L}C(X,\tau) = \frac{1}{2}\sigma^2 C_{XX} + (\mu - \gamma)C_X - r_f C - C_\tau = 0. \quad \text{(A.6)}$$

Writing $C(X,\tau) = e^{-\delta X - \epsilon\tau}F(X,\tau)$, it is possible to define δ and ϵ such that

$$F_\tau = \frac{1}{2}\sigma^2 F_{XX} \quad \text{(A.7)}$$

with barrier $X = B$. Solving for δ gives

$$\delta = \frac{1}{\sigma^2}(\mu - \gamma). \quad \text{(A.8)}$$

Since $C_B(V,t)$ satisfies (A.1) in the absence of a barrier,

$$F_B(X,\tau) = e^{\delta X + \epsilon\tau}C_B(X,\tau)$$

is a solution of (A.7). By the invariance of (A.7) under translation and changes of sign,

$$F_B^i(X,\tau) = F_B(2B - X, \tau) \quad \text{(A.9)}$$

also satisfies (A.7) and is the image of $F_B(X,\tau)$ with respect to the barrier $X = B$. By standard method of images arguments,

$$C(X,\tau) = e^{-\delta X - \epsilon\tau}\left(F_B(X,\tau) + F_B^i(X,\tau)\right) \quad \text{(A.10)}$$

$$= C_B(X,\tau) + e^{2\delta(B-X)}C_B(2B - X, \tau) \quad \text{(A.11)}$$

is then the solution to (A.1) with a barrier. Reversing the change of variables gives (A.3).

□

N.B. In the absence of dividends and with a constant barrier, $q = 0$ and $\gamma = 0$, and the image solution (A.3) collapses down to the standard solution

$$C_B^i(V,t) = \left(\frac{b}{V(t)}\right)^{\frac{2}{\sigma^2}(r_f - \frac{1}{2}\sigma^2)} C_B\left(\frac{b^2}{V(t)}, t\right),$$

as given, for example, in Wilmott et al. (1995).

Proposition A.2 *For Black-Scholes operator,*

$$\mathcal{L}_{BS}C(V,t,T) = \frac{1}{2}\sigma^2 V^2 C_{VV} + r_f V C_V - r_f C + C_t, \tag{A.12}$$

and exponential barrier $b(t) = b(0)e^{\gamma t}$*,*

$$\begin{aligned} \mathcal{L}_{BS}C(V,t,T) &= 0 \qquad V > 0, \quad t < T \\ C(V,T,T) &= \min\{\omega V(T), K\}\mathbb{I}_{\{V(T)>b(T)\}} \end{aligned} \tag{A.13}$$

has solution

$$C(V,t,T) = \omega V(t)\big(\Phi(d_b) - \Phi(d_1)\big) + Ke^{-r_f(T-t)}\Phi(d_2) \tag{A.14}$$

where

$$d_1 = \frac{\ln(\omega V(t)/K) + (r_f + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \tag{A.15}$$

$$d_2 = d_1 - \sigma\sqrt{T-t} \tag{A.16}$$

$$d_b = \frac{\ln(V(t)/b(T)) + (r_f + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}. \tag{A.17}$$

Proof

Since

$$V(T) = V(t)e^{(r_f - \frac{1}{2}\sigma^2)(T-t) + \sigma W_{T-t}},$$

by Feynman-Kac,

$$\begin{aligned} C(V,t,T) &= \mathbb{E}\left[e^{-r_f(T-t)}\min\{\omega V(T), K\}\mathbb{I}_{\{V(T)>b(T)\}}|V(t)\right] \\ &= \frac{1}{\sqrt{2\pi}}e^{-r_f(T-t)}\Bigg[\int_{b(T)<V(T)<K/\omega} \omega V(t)e^{(r_f-\frac{1}{2}\sigma^2)(T-t)+\sigma x\sqrt{T-t}}e^{-x^2/2}\,\mathrm{d}x \\ &\quad + \int_{K/\omega<V(T)} Ke^{-x^2/2}\,\mathrm{d}x\Bigg]. \end{aligned} \tag{A.18}$$

The limits of the integration give d_1, d_2 and d_b on rearranging and the result follows.

□

Proposition A.3 *With* $\mathcal{L}_{BS}$ *defined as in Proposition A.2 and exponential default barrier* $b(t) = b(0)e^{\gamma t}$*,*

$$\begin{aligned} \mathcal{L}_{BS}C(V,t,T) &= 0 \qquad V > 0, \quad t < T \\ C(V,T,T) &= \omega K^{1-c}V(T)^c\mathbb{I}_{\{V(T)>b(T)\}} \end{aligned} \tag{A.19}$$

has solution

$$C(V,t,T) = \omega K\left(\frac{V(t)}{K}\right)^{c} e^{(r_f+\frac{1}{2}\sigma^2 c)(c-1)(T-t)}\Phi\left(d_c\right) \tag{A.20}$$

where

$$d_c = \frac{\ln\left(V(t)/b(t)\right) + (r_f - \sigma^2/2 - \gamma + c\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

Proof

Standard application of Feynman-Kac as in Proposition A.2.

□

Solution to Problem 1, (2.20):

$$\begin{aligned} \mathcal{L}C_1(V,t,T) &= 0 \qquad V(t) > b(t), \quad t < T \\ C_1(V,T,T) &= \min\{\omega V(T), K\} \\ C_1(b(t),t,T) &= 0 \\ C_1(V,t,T) &\to Ke^{-r_f(T-t)} \quad \text{as} \quad V \to \infty. \end{aligned} \tag{A.21}$$

Suppose that $C_{1B}(V,t,T)$ satisfies

$$\begin{aligned} \mathcal{L}C_{1B}(V,t,T) &= 0 \qquad V(t) > 0, \quad t < T & \text{(A.22)} \\ C_{1B}(V,T,T) &= \min\{\omega V(T), K\}\mathbb{I}_{\{V(T)>b(T)\}}. & \text{(A.23)} \end{aligned}$$

Then $\widetilde{C}_{1B}(V,t,T) = e^{q(T-t)}C_{1B}(V,t,T)$ satisfies

$$\mathcal{L}_{BS}^{q}\widetilde{C}_{1B}(V,t,T) = 0$$

where $\mathcal{L}_{BS}^{q}$ is the Black-Scholes operator $\mathcal{L}_{BS}$ with r_f replaced by $r_f - q$. Hence, by Proposition A.2,

$$\begin{aligned} C_{1B}(V,t,T) &= e^{-q(T-t)}\widetilde{C}_{1B}(V,t,T) \\ &= e^{-q(T-t)}\omega V(t)\big(\Phi\left(d_{bq}\right) - \Phi\left(d_{1q}\right)\big) + Ke^{-r_f(T-t)}\Phi\left(d_{2q}\right). \end{aligned}$$

Thus, by the method of images, Proposition A.1,

$$\begin{aligned} C_1(V,t,T) &= C_{1B}(V,t,T) - \left(\frac{b(t)}{V(t)}\right)^{\frac{2(\mu-\gamma)}{\sigma^2}} C_{1B}\left(\frac{b(t)^2}{V(t)},t,T\right) \\ &= \omega V(t)e^{-q(T-t)}\left[\Phi(d_{bq}) - \Phi(d_{1q}) - \left(\frac{b(t)}{V(t)}\right)^{\kappa+2}\big(\Phi(d_{bq}^{i}) - \Phi(d_{1q}^{i})\big)\right] \\ &+ Ke^{-r_f(T-t)}\left[\Phi(d_{2q}) - \left(\frac{b(t)}{V(t)}\right)^{\kappa}\Phi(d_{2q}^{i})\right] \end{aligned} \tag{A.24}$$

where notation is defined in (2.23), Section 2.3.3. As $V \to \infty$, $C_1(V,t,T) \to Ke^{-r_f(T-t)}$ as required.

$\square$

Solution to Problem 2, (2.21):

$$\begin{aligned}
\mathcal{L}C_2(V,t,T) &= 0 \qquad V(t) > b(t), \quad t < T \\
C_2(V,T,T) &= 0 \\
C_2(b(t),t,T) &= \omega K e^{-\zeta(T-t)} \\
C_2(V,t,T) &\to 0 \quad \text{as} \quad V \to \infty.
\end{aligned} \tag{A.25}$$

Defining $\widetilde{C}_2(V,t,T) = e^{q(T-t)}C_2(V,t,T)$,

$$\begin{aligned}
\mathcal{L}^q_{BS}\widetilde{C}_2(V,t,T) &= 0 \qquad V(t) > b(t), \quad t < T \\
\widetilde{C}_2(V,T,T) &= 0 \\
\widetilde{C}_2(b(t),t,T) &= \omega K e^{-(\zeta-q)(T-t)} \\
\widetilde{C}_2(V,t,T) &\to 0 \quad \text{as} \quad V \to \infty,
\end{aligned}$$

with $\mathcal{L}^q_{BS}$ defined as above. Solving the associated rebate problem,

$$\begin{aligned}
\mathcal{L}^q_{BS}R(V,t,T) &= 0 \\
R(b(t),t,T) &= \omega K e^{-(\zeta-q)(T-t)},
\end{aligned}$$

gives

$$R(V,t,T) = \omega K \left(\frac{V(t)}{K}\right)^c e^{-(\zeta-q-c\gamma)(T-t)},$$

where

$$c = \frac{1}{\sigma^2}\left(\gamma - \mu - \sqrt{(\mu-\gamma)^2 + 2\sigma^2(r_f - \zeta)}\right).$$

Setting $\widehat{C}_2(V,t,T) = R(V,t,T) - \widetilde{C}_2(V,t,T)$,

$$\begin{aligned}
\mathcal{L}^q_{BS}\widehat{C}_2(V,t,T) &= 0 \qquad V(t) > b(t), \quad t < T \\
\widehat{C}_2(V,T,T) &= \omega K \left(\frac{V(T)}{K}\right)^c \\
\widehat{C}_2(b(t),t,T) &= 0,
\end{aligned}$$

which can be solved by the method of images. Suppose that $\widehat{C}_{2B}(V,t,T)$ satisfies

$$\begin{aligned} \mathcal{L}^q_{BS}\widehat{C}_{2B}(V,t,T) &= 0 \qquad V(t)>0, \quad t<T & \text{(A.26)} \\ \widehat{C}_{2B}(V,T,T) &= \omega K\left(\frac{V(T)}{K}\right)^c \mathbb{I}_{\{V(T)>b(T)\}}, & \text{(A.27)} \end{aligned}$$

then by Proposition A.3,

$$\widehat{C}_{2B}(V,t,T) = \omega K\left(\frac{V(t)}{K}\right)^c e^{(r_f-q+\frac{1}{2}\sigma^2 c)(c-1)(T-t)}\Phi\left(d_{cq}\right).$$

Thus by the method of images, Proposition A.1,

$$\widehat{C}_2(V,t,T) = \widehat{C}_{2B}(V,t,T) - \left(\frac{b(t)}{V(t)}\right)^\kappa \widehat{C}_{2B}\left(\frac{b(t)^2}{V(t)},t,T\right)$$

and hence,

$$\begin{aligned} C_2(V,t,T) &= e^{-q(T-t)}\left[R(V,t,T)-\widehat{C}_2(V,t,T)\right] \\ &= \omega K e^{-\zeta(T-t)}\left(\frac{V(t)}{b(t)}\right)^c\left[\Phi(-d_{cq})+\left(\frac{b(t)}{V(t)}\right)^{\kappa+2c}\Phi(d^i_{cq})\right] \quad \text{(A.28)} \end{aligned}$$

□

The value of $C(V,t,T)$ in (2.22) is then just the sum of (A.24) and (A.28).

Appendix B

Survival Probability Derivation

Consider two correlated Brownian motions with drifts,

$$X_i(t) = \alpha_i t + \sigma_i W_i(t), \quad t \geq 0, \quad i = 1, 2$$

where α_i, σ_i are constants, $W_i(t)$ are Brownian motions with $\text{cov}(W_1(t), W_2(t)) = \rho t$ and $X_i(0) = 0$. For constant lower default thresholds $B_i \leq 0$, the joint survival probability density function, $p(x_1, x_2, t)$, is given by

$$\begin{aligned} p(x_1, x_2, t) &= \frac{\partial^2}{\partial x_1 \partial x_2} \mathbb{P}(X_1(t) \leq x_1, X_2(t) \leq x_2, \underline{X}_1(t) \geq B_1, \underline{X}_2(t) \geq B_2) \quad \text{(B.1)} \\ &= \frac{2e^{a_1 x_1 + a_2 x_2 + bt}}{\beta t \sigma_1 \sigma_2 \sqrt{1-\rho^2}} \sum_{n=1}^{\infty} e^{-(r^2+r_0^2)/2t} \sin\left(\frac{n\pi\theta_0}{\beta}\right) \sin\left(\frac{n\pi\theta}{\beta}\right) I_{\left(\frac{n\pi}{\beta}\right)}\left(\frac{rr_0}{t}\right) \end{aligned}$$

where $I_{\left(\frac{n\pi}{\beta}\right)}\left(\frac{rr_0}{t}\right)$ is a modified Bessel's function, and

$$\begin{aligned} a_1 &= \frac{\alpha_1\sigma_2 - \rho\alpha_2\sigma_1}{(1-\rho^2)\sigma_1^2\sigma_2}, \quad a_2 = \frac{\alpha_2\sigma_1 - \rho\alpha_1\sigma_2}{(1-\rho^2)\sigma_1\sigma_2^2} \\ b &= -\alpha_1 a_1 - \alpha_2 a_2 + \frac{1}{2}\sigma_1^2 a_1^2 + \rho\sigma_1\sigma_2 a_1 a_2 + \frac{1}{2}\sigma_2^2 a_2^2 \\ \tan\beta &= -\frac{\sqrt{1-\rho^2}}{\rho}, \quad \beta \in [0, \pi] \\ z_1 &= \frac{1}{\sqrt{1-\rho^2}}\left[\left(\frac{x_1 - B_1}{\sigma_1}\right) - \rho\left(\frac{x_2 - B_2}{\sigma_2}\right)\right], \quad z_2 = \left(\frac{x_2 - B_2}{\sigma_2}\right) \\ z_{10} &= \frac{1}{\sqrt{1-\rho^2}}\left[-\frac{B_1}{\sigma_1} + \frac{\rho B_2}{\sigma_2}\right], \quad z_{20} = -\frac{B_2}{\sigma_2} \\ r &= \sqrt{z_1^2 + z_2^2}, \quad \tan\theta = \frac{z_2}{z_1}, \quad \theta \in [0, \beta] \\ r_0 &= \sqrt{z_{10}^2 + z_{20}^2}, \quad \tan\theta_0 = \frac{z_{20}}{z_{10}}, \quad \theta_0 \in [0, \beta]. \quad \text{(B.2)} \end{aligned}$$

Proof

The full proof, on which this is based, is given in Hua et al. (1998).

For each X_i, denote the running minimum,

$$\underline{X}_i(t) = \min_{0 \le s \le t} X_i(s).$$

The covariance matrix, C, of the system is

$$\begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

and we denote the ij^{th} element $C_{ij} = \rho_{ij}\sigma_i\sigma_j$ for ease of notation later.

For $i = 1, 2$, $B_i \le 0$ and $x_i \ge B_i$, the transition density, $p(x_1, x_2, t)$, defined in (B.1) satisfies the Kolmogorov forward equation (also known as the Fokker-Planck equation),

$$\frac{\partial p}{\partial t} = -\sum_{i=1}^{2} \alpha_i \frac{\partial p}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{2} \rho_{ij}\sigma_i\sigma_j \frac{\partial^2 p}{\partial x_i \partial x_j} \tag{B.3}$$

with initial condition

$$p(x_1, x_2, t = 0) = \delta(x_1)\delta(x_2)$$

where $\delta(x_i)$ is a Dirac's delta function. The absorbing boundary conditions are

$$\begin{aligned} p(x_1 = B_1, x_2, t) &= 0 \\ p(x_1, x_2 = B_2, t) &= 0. \end{aligned}$$

For a derivation of the boundary conditions see Zhou (1997a), Lemma 1.

To derive (B.1), we start by making the substitution,

$$p(x_1, x_2, t) = e^{a_1 x_1 + a_2 x_2 + bt} q(x_1, x_2, t) = e^{\mathbf{a}.\mathbf{x} + bt} q(\mathbf{x}, t)$$

where $\mathbf{a} = (a_1, a_2)'$, $\mathbf{x} = (x_1, x_2)'$. Choosing $\mathbf{a}$ such that

$$-\boldsymbol{\alpha} + C\mathbf{a} = 0,$$

and setting

$$b = -\sum_{i=1}^{2} \alpha_i a_i + \frac{1}{2}\sum_{i,j=1}^{2} \rho_{ij}\sigma_i\sigma_j a_i a_j,$$

a_1, a_2 and b are as defined in (B.2) and equation (B.3) becomes

$$\frac{\partial q}{\partial t} = \frac{1}{2}\sum_{i,j=1}^{2} \rho_{ij}\sigma_i\sigma_j \frac{\partial^2 q}{\partial x_i \partial x_j} \tag{B.4}$$

with the same initial condition and boundary conditions as before,

$$\begin{aligned} q(x_1, x_2, t=0) &= \prod_{i=1}^{2} \delta(x_i) \\ q(x_1 = B_1, x_2, t) &= 0 \\ q(x_1, x_2 = B_2, t) &= 0. \end{aligned}$$

Defining new coordinates $z_i(\mathbf{x})$, $i = 1, 2$ and setting

$$q(x_1, x_2, t) = \frac{1}{J} H\left(z_1(\mathbf{x}), z_2(\mathbf{x}), t\right)$$

where J is the Jacobian for the change of variables, we can let

$$\begin{aligned} z_1 &= \frac{1}{\sqrt{1-\rho^2}}\left[\left(\frac{x_1 - B_1}{\sigma_1}\right) - \rho\left(\frac{x_2 - B_2}{\sigma_2}\right)\right] \\ z_2 &= \left(\frac{x_2 - B_2}{\sigma_2}\right) \end{aligned}$$

so that

$$J = \sigma_1 \sigma_2 \sqrt{1-\rho^2}$$

and H satisfies

$$\frac{\partial H}{\partial t} = \frac{1}{2}\nabla^2 H, \tag{B.5}$$

with initial and boundary conditions,

$$\begin{aligned} H(z_1, z_2, t=0) &= \delta(z_1 - z_{10})\delta(z_2 - z_{20}) \\ H(L_1, t) &= 0 = H(L_2, t). \end{aligned}$$

z_{10} and z_{20} are defined in (B.2) and $H = 0$ on the lines

$$\begin{aligned} L_1 &= \{(z_1, z_2) : z_2 = 0\} \\ L_2 &= \left\{(z_1, z_2) : z_2 = \frac{-\sqrt{1-\rho^2}}{\rho} z_1\right\} \end{aligned}$$

By choosing suitable polar coordinates, the Laplacian can then be solved by separation of variables.

Writing, $r^2 = z_1^2 + z_2^2$ and $\tan\theta = z_2/z_1$ for $\theta \in [0, \beta]$ and $\tan\beta = \frac{-\sqrt{1-\rho^2}}{\rho}$ where $\beta \in [0, \pi]$, (B.5) becomes

$$\frac{\partial H}{\partial t} = \frac{1}{2}\left(\frac{\partial^2 H}{\partial r^2} + \frac{1}{r}\frac{\partial H}{\partial r} + \frac{1}{r^2}\frac{\partial^2 H}{\partial \theta^2}\right) \tag{B.6}$$

with initial and boundary conditions,

$$H(r,\theta,t=0) = \frac{1}{r_0}\delta(r-r_0)\delta(\theta-\theta_0)$$
$$H(r,\theta=0,t) = 0 = H(r,\theta=\beta,t).$$

Making the substitution $H(r,\theta,t) = R(r)\Theta(\theta)T(t)$ in (B.6), since the transition density must decay as $t\to\infty$, we get

$$\begin{aligned}\frac{T'}{T} &= -\frac{\lambda^2}{2}\\ \frac{\Theta''}{\Theta} &= -k_n^2\\ r^2R''+rR'+\lambda^2r^2R &= k_n^2R\end{aligned} \tag{B.7}$$

where $\lambda\in\mathbb{R}^+$, and k_n are chosen so that the boundary conditions $\Theta(0)=\Theta(\beta)=0$. Setting $y=\lambda r$ in (B.7),

$$T(t) \sim e^{-\frac{1}{2}\lambda^2 t} \tag{B.8}$$

$$\Theta(\theta) \sim \sin\left(\frac{n\pi\theta}{\beta}\right) \tag{B.9}$$

$$y^2\frac{d^2R}{dy^2}+y\frac{dR}{dy}+(y^2-(\frac{n^2\pi^2}{\beta^2}))R=0 \tag{B.10}$$

for $n=1,2,3,\ldots$ Equation (B.10) is Bessel's equation, and since we require R to be well-behaved at zero, $R(r)\sim J_{(\frac{n\pi}{\beta})}(\lambda r)$, a Bessel's function of the first kind. From (B.8) – (B.10), the general solution satisfying $H(r,0,t)=H(r,\beta,t)=0$ is then

$$H(r,\theta,t) = \int_0^\infty \sum_{n=1}^\infty c_n(\lambda)e^{-\frac{1}{2}\lambda^2 t}\sin\left(\frac{n\pi\theta}{\beta}\right)J_{(\frac{n\pi}{\beta})}(\lambda r)\,\mathrm{d}\lambda \tag{B.11}$$

where $c_n(\lambda)$ are chosen such that the initial condition, $H(r,\theta,0)=\frac{1}{r_0}\delta(r-r_0)\delta(\theta-\theta_0)$ is satisfied. Substituting the initial condition directly gives

$$\int_0^\beta \sin\left(\frac{m\pi\theta}{\beta}\right)H(r,\theta,0)\,\mathrm{d}\theta = \frac{1}{r_0}\delta(r-r_0)\sin\left(\frac{m\pi\theta_0}{\beta}\right), \tag{B.12}$$

whilst by the orthogonality of $\sin(n\theta)$, substituting $H(r,\theta,0)$ from (B.11) gives

$$\int_0^\beta \sin\left(\frac{m\pi\theta}{\beta}\right)H(r,\theta,0)\,\mathrm{d}\theta = \frac{\beta}{2}\int_0^\infty c_m(\lambda)J_{(\frac{m\pi}{\beta})}(\lambda r)\;\mathrm{d}\lambda \tag{B.13}$$

Multiplying (B.12) and (B.13) by $rJ_{(\frac{m\pi}{\beta})}(\lambda' r)$ and using the fact that

$$\int_0^\infty xJ_\nu(ax)J_\nu(bx)\,\mathrm{d}x = \frac{1}{a}\delta(a-b),$$

integrating over r gives

$$\int_0^\infty \frac{1}{r_0}\delta(r-r_0)\sin\left(\frac{m\pi\theta_0}{\beta}\right) r J_{\left(\frac{m\pi}{\beta}\right)}(\lambda' r)\ \mathrm{d}r = \sin\left(\frac{m\pi\theta_0}{\beta}\right) J_{\left(\frac{m\pi}{\beta}\right)}(\lambda' r_0)$$

$$= \frac{\beta}{2}\int_0^\infty\int_0^\infty c_m(\lambda) r J_{\left(\frac{m\pi}{\beta}\right)}(\lambda r)\, J_{\left(\frac{m\pi}{\beta}\right)}(\lambda' r)\ \mathrm{d}r\,\mathrm{d}\lambda = \frac{\beta}{2\lambda'}c_m(\lambda').$$

Hence

$$c_m(\lambda) = \frac{2\lambda}{\beta}\sin\left(\frac{m\pi\theta_0}{\beta}\right) J_{\left(\frac{m\pi}{\beta}\right)}(\lambda r_0)$$

and

$$H(r,\theta,t) = \int_0^\infty \frac{2\lambda}{\beta}\sum_{n=1}^{\infty} e^{-\lambda^2 t/2}\sin\left(\frac{n\pi\theta_0}{\beta}\right)\sin\left(\frac{n\pi\theta}{\beta}\right) J_{\left(\frac{n\pi}{\beta}\right)}(\lambda r_0)\, J_{\left(\frac{n\pi}{\beta}\right)}(\lambda r)\ \mathrm{d}\lambda.$$

This can be simplified using

$$\int_0^\infty \lambda e^{-\lambda^2 t/2} J_{\left(\frac{n\pi}{\beta}\right)}(\lambda r_0)\, J_{\left(\frac{n\pi}{\beta}\right)}(\lambda r)\ \mathrm{d}\lambda = \frac{1}{t}e^{-(r_0^2+r^2)/(2t)} I_{\left(\frac{n\pi}{\beta}\right)}\left(\frac{r r_0}{t}\right),$$

from Gradshteyn and Ryzhik (1980), p.718, to give

$$H(r,\theta,t) = \frac{2}{\beta t}\sum_{n=1}^{\infty} e^{-(r^2+r_0^2)/2t}\sin\left(\frac{n\pi\theta_0}{\beta}\right)\sin\left(\frac{n\pi\theta}{\beta}\right) I_{\left(\frac{n\pi}{\beta}\right)}\left(\frac{r r_0}{t}\right).$$

Since

$$p(x_1,x_2,t) = e^{a_1x_1+a_2x_2+bt}\frac{H(r,\theta,t)}{\sigma_1\sigma_2\sqrt{1-\rho^2}},$$

the result follows.

Appendix C

Integrating the Transition Probability

As before, we denote the joint survival probability transition density by $p(x_1, x_2, t)$ where

$$p(x_1, x_2, t) = \frac{\partial^2}{\partial x_1 \partial x_2} \mathbb{P}(X_1(t) \leq x_1, X_2(t) \leq x_2, \underline{X}_1(t) \geq B_1, \underline{X}_2(t) \geq B_2).$$

We prove a general result regarding the integration of $p(x_1, x_2, t)$, specific cases of which then give us the formula for the joint survival probability distribution, (4.5) in Section 4.1.2, and the discounted value of the maturity payment for a corporate bond, (5.3) + (5.4) in Section 5.2.1.

Proposition C.1 *For general constants A, B and ϵ,*

$$\begin{aligned} &\int_{B_2}^{\infty} \int_{A}^{B} e^{\epsilon x_1} p(x_1, x_2, t) \, \mathrm{d}x_1 \, \mathrm{d}x_2 \\ &\qquad = \frac{2}{\beta t} e^{(a_1+\epsilon)B_1 + a_2 B_2 + bt} \sum_{n=1}^{\infty} e^{-r_0^2/2t} \sin\left(\frac{n\pi\theta_0}{\beta}\right) \int_0^{\beta} \sin\left(\frac{n\pi\theta}{\beta}\right) g_n(\theta) \, \mathrm{d}\theta \end{aligned}$$

where

$$
\begin{aligned}
g_n(\theta) &= \int_{d_A}^{d_B} re^{-r^2/2t} e^{[A(\theta)+\epsilon\sigma_1\sin(\beta-\theta)]r} I_{\left(\frac{n\pi}{\beta}\right)}\left(\frac{rr_0}{t}\right) \mathrm{d}r \\
r_0 &= \frac{1}{\sqrt{1-\rho^2}}\left(\frac{B_1^2}{\sigma_1^2} - \frac{2\rho B_1 B_2}{\sigma_1\sigma_2} + \frac{B_2^2}{\sigma_2^2}\right)^{1/2} \\
\tan\theta_0 &= \frac{\sigma_1 B_2\sqrt{1-\rho^2}}{\sigma_2 B_1 - \rho\sigma_1 B_2}, \quad \theta_0 \in [0,\beta] \\
d_A &= \frac{A - B_1}{\sigma_1\sin(\beta-\theta)} \\
d_B &= \frac{B - B_1}{\sigma_1\sin(\beta-\theta)} \\
A(\theta) &= a_1\sigma_1\sin(\beta-\theta) + a_2\sigma_2\sin\theta.
\end{aligned}
$$

Proof

From Section 4.1.2,

$$
\begin{aligned}
&p(x_1, x_2, t) \\
&= \frac{2e^{a_1x_1+a_2x_2+bt}}{\beta t\sigma_1\sigma_2\sqrt{1-\rho^2}} \sum_{n=1}^{\infty} e^{-(r^2+r_0^2)/2t} \sin\left(\frac{n\pi\theta_0}{\beta}\right) \sin\left(\frac{n\pi\theta}{\beta}\right) I_{\left(\frac{n\pi}{\beta}\right)}\left(\frac{rr_0}{t}\right).
\end{aligned}
$$

Changing variables to r and θ as defined in (4.4), the Jacobian for the transformation is $\sqrt{(1-\rho^2)}r\sigma_1\sigma_2$ and we can write

$$
\begin{aligned}
x_1 &= B_1 + \sqrt{(1-\rho^2)}\sigma_1 r\cos\theta + \rho\sigma_1 r\sin\theta = B_1 + \sigma_1 r\sin(\beta-\theta) \qquad \text{(C.1)} \\
x_2 &= B_2 + \sigma_2 r\sin\theta,
\end{aligned}
$$

since

$$
\tan\beta = -\frac{\sqrt{1-\rho^2}}{\rho} \quad \Rightarrow \quad \cos\beta = -\rho \quad \& \quad \sin\beta = \sqrt{1-\rho^2},
$$

by consideration of the region of definition of β for positive and negative values of ρ.

Hence, integrating over x_1 and x_2 gives

$$
\begin{aligned}
&\int_{B_2}^{\infty}\int_A^B e^{\epsilon x_1} p(x_1,x_2,t)\,\mathrm{d}x_1\,\mathrm{d}x_2 \\
&= \int_{B_2}^{\infty}\int_A^B \frac{2e^{(a_1+\epsilon)x_1+a_2x_2+bt}}{\beta t\sigma_1\sigma_2\sqrt{1-\rho^2}} \sum_{n=1}^{\infty} e^{-(r^2+r_0^2)/2t} \times \\
&\quad \sin\left(\frac{n\pi\theta_0}{\beta}\right)\sin\left(\frac{n\pi\theta}{\beta}\right) I_{\left(\frac{n\pi}{\beta}\right)}\left(\frac{rr_0}{t}\right)\,\mathrm{d}x_1\,\mathrm{d}x_2 \\
&= \frac{2e^{bt}}{\beta t\sigma_1\sigma_2\sqrt{1-\rho^2}} \sum_{n=1}^{\infty} e^{-r_0^2/2t}\sin\left(\frac{n\pi\theta_0}{\beta}\right)\times \\
&\quad \int_{B_2}^{\infty}\int_A^B e^{(a_1+\epsilon)x_1+a_2x_2} e^{-r^2/2t}\sin\left(\frac{n\pi\theta}{\beta}\right) I_{\left(\frac{n\pi}{\beta}\right)}\left(\frac{rr_0}{t}\right)\,\mathrm{d}x_1\,\mathrm{d}x_2 \\
&= \frac{2e^{bt}}{\beta t}\sum_{n=1}^{\infty} e^{-r_0^2/2t}\sin\left(\frac{n\pi\theta_0}{\beta}\right)\times \\
&\quad \int_r\int_\theta e^{(a_1+\epsilon)(B_1+\sigma_1 r\sin(\beta-\theta))+a_2(B_2+\sigma_2 r\sin\theta)} e^{-r^2/2t}\sin\left(\frac{n\pi\theta}{\beta}\right) I_{\left(\frac{n\pi}{\beta}\right)}\left(\frac{rr_0}{t}\right) r\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \frac{2}{\beta t} e^{(a_1+\epsilon)B_1+a_2B_2+bt}\sum_{n=1}^{\infty} e^{-r_0^2/2t}\sin\left(\frac{n\pi\theta_0}{\beta}\right)\int_\theta \sin\left(\frac{n\pi\theta}{\beta}\right) g_n(\theta)\,\mathrm{d}\theta
\end{aligned}
$$

where

$$g_n(\theta) = \int_r r e^{-r^2/2t} e^{[A(\theta)+\epsilon\sigma_1\sin(\beta-\theta)]r} I_{\left(\frac{n\pi}{\beta}\right)}\left(\frac{rr_0}{t}\right)\,\mathrm{d}r$$

and

$$A(\theta) = a_1\sigma_1\sin(\beta-\theta) + a_2\sigma_2\sin\theta.$$

Writing $X = \frac{x_1-B_1}{\sigma_1}$ and $Y = \frac{x_2-B_2}{\sigma_2}$, from (C.1)

$$
\begin{aligned}
X &= r\sin(\beta-\theta) \\
Y &= r\sin\theta,
\end{aligned}
\tag{C.2}
$$

and

$$
\begin{aligned}
0 \le \frac{A-B_1}{\sigma_1} &\le X \le \frac{B-B_1}{\sigma_1} \\
0 &\le Y \le \infty.
\end{aligned}
$$

Since

$$\tan\theta = \frac{\sqrt{1-\rho^2}}{X/Y-\rho},$$

we have $\theta \in [0,\beta]$ where $\beta \in [0,\pi]$ and $\tan\beta = -\sqrt{1-\rho^2}/\rho$. Letting

$$d_A = \frac{A-B_1}{\sigma_1\sin(\beta-\theta)}$$

$$d_B = \frac{B-B_1}{\sigma_1\sin(\beta-\theta)},$$

then $d_A \leq r \leq d_B$ from (C.2), giving us the limits on the integration, and the result follows.

□

Survival Probability Derivation

Using Proposition C.1 with $\epsilon = 0$, $A = B_1$ and $B = \infty$,

$$\begin{aligned} &\mathbb{P}(\underline{X}_1(t) \geq B_1, \underline{X}_2(t) \geq B_2) \\ &= \frac{2}{\beta t} e^{a_1 B_1 + a_2 B_2 + bt} \sum_{n=1}^{\infty} e^{-r_0^2/2t} \sin\left(\frac{n\pi\theta_0}{\beta}\right) \int_0^{\beta} \sin\left(\frac{n\pi\theta}{\beta}\right) g_n(\theta)\, \mathrm{d}\theta \end{aligned} \tag{C.3}$$

where

$$g_n(\theta) = \int_0^{\infty} r e^{-r^2/2t} e^{A(\theta) r} I_{\left(\frac{n\pi}{\beta}\right)}\left(\frac{r r_0}{t}\right) \mathrm{d}r.$$

Zhou (2001a) has exactly the same result with the exception that in his paper, $g_n(\theta)$ is written

$$g_n(\theta) = \int_0^{\infty} r e^{-r^2/2t} e^{(-a_1\sigma_1 - \rho a_2 \sigma_2) r \sin(\theta - \beta) + a_2 \sigma_2 \sqrt{1-\rho^2} r \cos(\theta - \beta)} I_{\left(\frac{n\pi}{\beta}\right)}\left(\frac{r r_0}{t}\right) \mathrm{d}r. \tag{C.4}$$

Using double angle formulæ for $\sin(\theta - \beta)$ and $\cos(\theta - \beta)$, and the fact that $\cos\beta = -\rho$ and $\sin\beta = \sqrt{1-\rho^2}$, (C.4) can be reduced to our formulation. N.B. the definitions of a_1 and a_2 in Zhou (2001a) are the negative of the definitions we use.

Hua et al. (1998) also have the same result and proof, with the exception that their paper has an error in the definition of $g_n(\theta)$ on page 207. In their notation, it should be

$$g_n(\theta) = \int_0^{\infty} r e^{-r^2/2t} e^{-b_1 r \sin(\theta - \beta) + b_2 r \cos(\theta - \beta)} I_{\left(\frac{n\pi}{\beta}\right)}\left(\frac{r r_0}{t}\right) \mathrm{d}r.$$

Maturity Payment Derivation

From Section 5.2.1, the discounted maturity payment, DMP, is

$$\text{DMP} = e^{-r_f T} \int_{B_2}^{\infty} \int_d^{\infty} K_1 p(x_1, x_2, T)\, \mathrm{d}x_1\, \mathrm{d}x_2 \tag{C.5}$$

$$+ \quad e^{-r_f T} \int_{B_2}^{\infty} \int_{B_1}^{d} \omega_1 V_1(0) e^{x_1 + \gamma_1 T} p(x_1, x_2, T)\, \mathrm{d}x_1\, \mathrm{d}x_2. \tag{C.6}$$

Using Proposition C.1 with $\epsilon = 0$, $A = d$ and $B = \infty$ for line (C.5) and $\epsilon = 1$, $A = B_1$ and $B = d$ for line (C.6), it becomes

$$\text{DMP} = \sum_{n=1}^{\infty} \frac{C_1}{\beta} \int_0^{\beta} \sin\left(\frac{n\pi\theta}{\beta}\right) g_n^{+}(\theta)\,\mathrm{d}\theta + \sum_{n=1}^{\infty} \frac{C_2}{\beta} \int_0^{\beta} \sin\left(\frac{n\pi\theta}{\beta}\right) g_n^{*}(\theta)\,\mathrm{d}\theta$$

where

$$\begin{aligned}
C_1 &= \frac{2K_1}{T} e^{-r_f T} e^{a_1 B_1 + a_2 B_2 + bT} e^{-r_0^2/2T} \sin\left(\frac{n\pi\theta_0}{\beta}\right), \\
C_2 &= \frac{2\omega_1 V_1(0)}{T} e^{(\gamma_1 - r_f)T} e^{(a_1+1)B_1 + a_2 B_2 + bT} e^{-r_0^2/2T} \sin\left(\frac{n\pi\theta_0}{\beta}\right) \\
g_n^{+}(\theta) &= \int_{d^*(\theta)}^{\infty} r e^{-r^2/2T} e^{A(\theta) r} I_{\left(\frac{n\pi}{\beta}\right)}\left(\frac{r r_0}{T}\right) \mathrm{d}r \\
g_n^{*}(\theta) &= \int_0^{d^*(\theta)} r e^{-r^2/2T} e^{[A(\theta) + \sigma_1 \sin(\beta - \theta)] r} I_{\left(\frac{n\pi}{\beta}\right)}\left(\frac{r r_0}{T}\right) \mathrm{d}r \\
d^*(\theta) &= \frac{d - B_1}{\sigma_1 \left[\sqrt{1-\rho^2}\cos\theta + \rho \sin\theta\right]} \\
&= \frac{\ln \omega_1}{\sigma_1 \sin(\theta - \beta)} \geq 0,
\end{aligned}$$

since $d - B_1 = -\ln \omega_1$.

Appendix D

Simplified Survival Probability I

We show here how to derive the simplified formula for the joint survival probability, (4.8), in the case that the barrier growth rate equals the drift in firm value.

Proposition D.1

$$\int_0^\infty re^{-r^2/2t} I_{(\frac{n\pi}{\beta})}\left(\frac{rr_0}{t}\right)\,\mathrm{d}r = \frac{r_0\sqrt{2\pi t}}{4} e^{r_0^2/4t}\left[I_{\frac{1}{2}(\frac{n\pi}{\beta}+1)}\left(\frac{r_0^2}{4t}\right) + I_{\frac{1}{2}(\frac{n\pi}{\beta}-1)}\left(\frac{r_0^2}{4t}\right)\right]$$

Proof

For modified Bessel function $I_\nu(br)$,

$$\frac{d}{dr}I_\nu(br) = \frac{b}{2}\Big(I_{\nu+1}(br) + I_{\nu-1}(br)\Big),$$

so setting $\nu = n\pi/\beta$ and $b = r_0/t$, writing $a = 1/(2t)$ and integrating by parts,

$$\int_0^\infty re^{-r^2/2t} I_{(\frac{n\pi}{\beta})}\left(\frac{rr_0}{t}\right)\,\mathrm{d}r = \frac{b}{4a}\int_0^\infty e^{-ar^2}\big(I_{\nu+1}(br) + I_{\nu-1}(br)\big)\,\mathrm{d}r. \tag{D.1}$$

Now

$$\int_0^\infty e^{-ar^2} I_\nu(br)dr = \frac{\sqrt{\pi}}{2\sqrt{a}} e^{b^2/8a} I_{\frac{\nu}{2}}\left(\frac{b^2}{8a}\right)$$

from Gradshteyn and Ryzhik (1980), page 711, and hence

$$\int_0^\infty re^{-r^2/2t} I_{(\frac{n\pi}{\beta})}\left(\frac{rr_0}{t}\right)\,\mathrm{d}r = \frac{b\sqrt{\pi}}{8a\sqrt{a}} e^{b^2/8a}\left[I_{\frac{1}{2}(\nu+1)}\left(\frac{b^2}{8a}\right) + I_{\frac{1}{2}(\nu-1)}\left(\frac{b^2}{8a}\right)\right]$$

Substituting the values of ν, a and b then gives the result.

$\square$

Using Proposition D.1, combined with the fact that

$$\int_0^\beta \sin\left(\frac{n\pi\theta}{\beta}\right) d\theta = \begin{cases} 2\beta/n\pi & \text{for} \quad \text{n odd} \\ 0 & \text{for} \quad \text{n even,} \end{cases}$$

it is then straightforward to show that (4.7) simplifies to

$$\begin{aligned} &\mathbb{P}(\underline{X}_1(t) \geq B_1, \underline{X}_2(t) \geq B_2) \\ &= \frac{2r_0}{\sqrt{2\pi t}} e^{-r_0^2/4t} \sum_{\text{n odd}} \frac{1}{n} \sin\left(\frac{n\pi\theta_0}{\beta}\right) \left[I_{\frac{1}{2}(\frac{n\pi}{\beta}+1)}\left(\frac{r_0^2}{4t}\right) + I_{\frac{1}{2}(\frac{n\pi}{\beta}-1)}\left(\frac{r_0^2}{4t}\right) \right] \end{aligned}$$

Appendix E

Some Survival Probability Asymptotics

We consider the asymptotic behaviour of the simplified survival probability (4.8) from Section 4.1.3.1,

$$P(z) = \frac{4\sqrt{z}e^{-z}}{\sqrt{2\pi}} \sum_{\text{n odd}} \frac{1}{n} \sin\left(\frac{n\pi\theta_0}{\beta}\right) \left[I_{\frac{1}{2}(\frac{n\pi}{\beta}+1)}(z) + I_{\frac{1}{2}(\frac{n\pi}{\beta}-1)}(z)\right] \tag{E.1}$$

where

$$\begin{aligned} z &= \frac{r_0^2}{4t} \\ r_0 &= \frac{1}{\sqrt{1-\rho^2}} \left(\frac{B_1^2}{\sigma_1^2} - \frac{2\rho B_1 B_2}{\sigma_1\sigma_2} + \frac{B_2^2}{\sigma_2^2}\right)^{1/2} \\ \tan\theta_0 &= \frac{\sigma_1 B_2\sqrt{1-\rho^2}}{\sigma_2 B_1 - \rho\sigma_1 B_2}, \quad \theta_0 \in [0, \beta] \end{aligned} \tag{E.2}$$

and other notation is given in Section 4.1.

Case I : $\mathbf{z \to 0}$

We consider first the behaviour of $P(z)$ as $z \to 0$. By (E.2), for $\rho \neq \pm 1$, this covers two cases of interest,

1. $t \to \infty$
2. $\sigma_i \to \infty,\ i = 1, 2.$

Since $P(z)$ is the joint survival probability, as the time horizon or firm volatilities tend to infinity, $P(z)$ should tend to zero. By Abramowitz and Stegun (1964), p.375,

$$I_\nu(z) = (z/2)^\nu \sum_{s=0}^{\infty} \frac{(z/2)^{2s}}{s!\Gamma(\nu+s+1)}.$$

Using $\Gamma(x+1) = x\Gamma(x)$, it therefore follows that

$$\begin{aligned} I_\nu(z) &= \frac{(z/2)^\nu}{\Gamma(\nu+1)}\left[1 + \frac{(z/2)^2}{\nu+1} + \frac{(z/2)^4}{2!(\nu+2)(\nu+1)} + \frac{(z/2)^6}{3!(\nu+3)(\nu+2)(\nu+1)} + \cdots\right] \\ &\leq \frac{(z/2)^\nu}{\Gamma(\nu+1)} \sum_{s=0}^{\infty} \left(\frac{z^2}{4}\right)^s = \frac{(z/2)^\nu}{\Gamma(\nu+1)} \left(\frac{1}{1-z^2/4}\right) \end{aligned} \tag{E.3}$$

for $\nu = \frac{1}{2}(\frac{n\pi}{\beta} \pm 1) \geq 0$ and small z.

By Gradshteyn and Ryzhik (1980), p.38,

$$\sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1} = \frac{\pi}{4} \quad \text{for} \quad 0 < x < \pi. \tag{E.4}$$

Since $\theta_0 \in [0, \beta]$, we therefore know that $\sum_{\text{n odd}} \frac{1}{n} \sin\left(\frac{n\pi\theta_0}{\beta}\right)$ converges so since the modified Bessel's functions are all bounded by (E.3), (E.1) converges by Dirichlet's test[1] and letting $z \to 0$,

$$\lim_{z\to 0} P(z) = 0,$$

as expected.

Case II : $\mathbf{z} \to \infty$

We now consider the behaviour as $z \to \infty$, corresponding to $t \to 0$. Since we are in a diffusion setting, survival probabilities tend to one as the time horizon tends to zero, and so we expect $P(z) \to 1$ as $z \to \infty$. For fixed order, ν, by Abramowitz and Stegun (1964), p.377,

$$I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}} \left\{1 + O\left(\frac{\nu^2}{z}\right)\right\} \quad \text{as} \quad z \to \infty. \tag{E.5}$$

Since this expansion only holds for fixed values of the order, and not as $\nu \to \infty$, we need to split the infinite sum in (E.1) into two. Taking N to be large and writing $P(z) = P_1(z) + P_2(z)$, where $P_1(z)$ is the sum for $n \leq N$, n odd, then from (E.5),

$$P_1(z) \sim \sum_{\substack{n<N \\ \text{n odd}}} \frac{4}{n\pi} \sin\left(\frac{n\pi\theta_0}{\beta}\right) \left\{1 + O\left(\frac{N^2}{z}\right)\right\}. \tag{E.6}$$

[1] Apostol (1974) Theorem 8.28 p.194

Since we know that $\sum_{\text{n odd}} \frac{1}{n} \sin\left(\frac{n\pi\theta_0}{\beta}\right)$ converges to $\frac{\pi}{4}$ by (E.4), given $\epsilon > 0$, we can pick $N = N_\epsilon$ such that

$$\left| \sum_{\substack{n<N_\epsilon \\ \text{n odd}}} \frac{1}{n} \sin\left(\frac{n\pi\theta_0}{\beta}\right) - \frac{\pi}{4} \right| < \epsilon, \tag{E.7}$$

and provided $z \gg N_\epsilon^2$,

$$P_1(z) \sim \frac{4}{\pi} \sum_{\substack{n<N_\epsilon \\ \text{n odd}}} \frac{1}{n} \sin\left(\frac{n\pi\theta_0}{\beta}\right). \tag{E.8}$$

Since by (E.7),

$$\left| \sum_{\substack{n>N_\epsilon \\ \text{n odd}}} \frac{1}{n} \sin\left(\frac{n\pi\theta_0}{\beta}\right) \right| < \epsilon, \tag{E.9}$$

if

$$\frac{4\sqrt{z}e^{-z}}{\sqrt{2\pi}} \left[I_{\frac{1}{2}(\frac{n\pi}{\beta}+1)}(z) + I_{\frac{1}{2}(\frac{n\pi}{\beta}-1)}(z) \right] \tag{E.10}$$

is uniformly bounded $\forall\, n$ and z, then given $\epsilon > 0$, by picking N_ϵ such that (E.7) holds, we can let $\epsilon \to 0$, to give $P_2(z) \to 0$ and $P_1(z) \to 1$, choosing $z \gg N_\epsilon^2$ for the latter. By considering the maximum value of the integrand in the integral representation

$$I_\nu(z) = \frac{(z/2)^\nu}{\sqrt{\pi}\,\Gamma(\nu + 1/2)} \int_0^\pi e^{z\cos\theta} \sin^{2\nu}\theta \, \mathrm{d}\theta \tag{E.11}$$

(Abramowitz and Stegun (1964), p.376), and using Stirling's formula it can be shown that (E.10) is bounded and hence,

$$\lim_{z\to\infty} P(z) = 1, \tag{E.12}$$

as required.

Appendix F

Simplified Survival Probability II

We derive the joint survival probability for certain values of correlation for which the modified Bessel's function simplifies, $\rho = -(\cos \pi/k)$, $k \in \mathbb{R}$.

Proposition F.1

$$\sum_{n=1}^{\infty} \cos(nk\alpha)\cos(nk\beta) = \pi \sum_{p=-\infty}^{\infty} \delta(k(\alpha-\beta) - 2\pi p) \quad \forall\, \alpha,\, \beta$$

for constants n and k, where δ is the delta function.

Proof

Consider a function

$$f(x) = \sum_{m=1}^{\infty} \gamma_m \cos(mx).$$

Then

$$\int_{-\pi}^{\pi} \cos(nx) f(x)\,\mathrm{d}x = \sum_{m=1}^{\infty} \gamma_m \int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,\mathrm{d}x = \pi\gamma_n.$$

Hence

$$\begin{aligned} f(x) &= \frac{1}{\pi} \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} \cos(ny)\cos(nx) f(y)\,\mathrm{d}y \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(y) \sum_{n=1}^{\infty} \cos(ny)\cos(nx)\,\mathrm{d}y. \end{aligned}$$

So

$$\sum_{n=1}^{\infty} \cos(nx)\cos(ny) \sim \pi\delta(x-y) \quad \text{for} \quad x, y \in [-\pi, \pi] \text{ periodic } 2\pi$$

Therefore

$$\sum_{n=1}^{\infty} \cos(nx)\cos(ny) = \pi \sum_{p=-\infty}^{\infty} \delta(x-y-2\pi p)$$

Writing $x = k\alpha$ and $y = k\beta$ then gives the result.

□

Survival Probability Calculation

For $\theta \in [0, \pi]$,

$$\begin{aligned}
& \sum_{n=1}^{\infty} 2\sin(nk\theta_0)\sin(nk\theta)\cos(nk\phi) \\
= & \sum_{n=1}^{\infty} \cos(nk\phi)\big(\cos(nk(\theta-\theta_0)) - \cos(nk(\theta+\theta_0))\big) \\
= & \ \pi \sum_{p=-\infty}^{\infty} \delta\big(k(\theta-\theta_0-\phi)-2\pi p\big) - \pi \sum_{p=-\infty}^{\infty} \delta\big(k(\theta+\theta_0-\phi)-2\pi p\big)
\end{aligned}$$

Hence, (4.10) in Section 4.1.3.2 becomes

$$\begin{aligned}
& \mathbb{P}(\underline{X}_1(t) \geq B_1, \underline{X}_2(t) \geq B_2) \qquad \text{(F.1)} \\
= & \ \frac{2k}{\pi^2 t} e^{a_1 B_1 + a_2 B_2 + bt} e^{-r_0^2/2t} \int_0^{\frac{\pi}{k}} \int_0^{\infty} \int_0^{\pi} r e^{-r^2/2t} \times \\
& e^{(A(\theta)+\frac{r_0}{t}\cos\phi)r} \sum_{n=1}^{\infty} \sin(nk\theta_0)\sin(nk\theta)\cos(nk\phi)\,\mathrm{d}\phi\,\mathrm{d}r\,\mathrm{d}\theta \\
= & \ \frac{k}{\pi t} e^{a_1 B_1 + a_2 B_2 + bt} e^{-r_0^2/2t} \int_0^{\frac{\pi}{k}} \int_0^{\infty} \int_0^{\pi} r e^{-r^2/2t} \times \\
& e^{(A(\theta)+\frac{r_0}{t}\cos\phi)r} \sum_{p=-\infty}^{\infty} \Big[\delta\big(k(\theta-\theta_0-\phi)-2\pi p\big) - \delta\big(k(\theta+\theta_0-\phi)-2\pi p\big)\Big]\mathrm{d}\phi\,\mathrm{d}r\,\mathrm{d}\theta \\
= & \ \frac{k}{\pi t} e^{a_1 B_1 + a_2 B_2 + bt} e^{-r_0^2/2t} \int_0^{\frac{\pi}{k}} \int_0^{\infty} r e^{-r^2/2t} \times \\
& \sum_{p=-\infty}^{\infty} \left[e^{(A(\theta)+\frac{r_0}{t}\cos(\theta-\theta_0-2\pi p/k))r}\mathbb{I}_1 - e^{(A(\theta)+\frac{r_0}{t}\cos(\theta+\theta_0-2\pi p/k))r}\mathbb{I}_2\right]\,\mathrm{d}r\,\mathrm{d}\theta \\
= & \ \frac{k}{\pi t} e^{a_1 B_1 + a_2 B_2 + bt} e^{-r_0^2/2t} \int_0^{\frac{\pi}{k}} \int_0^{\infty} r e^{-r^2/2t} \sum_{p=-\infty}^{\infty} \left[e^{f-r}\mathbb{I}_1 - e^{f+r}\mathbb{I}_2\right]\,\mathrm{d}r\,\mathrm{d}\theta
\end{aligned}$$

where

$$
\begin{aligned}
f_- &= A(\theta) + \frac{1}{t} r_0 \cos(\theta - \theta_0 - 2\pi p/k) \\
f_+ &= A(\theta) + \frac{1}{t} r_0 \cos(\theta + \theta_0 - 2\pi p/k) \\
\mathbb{I}_1 &= \mathbb{I}_{\{\theta - \theta_0 - 2\pi p/k \in [0,\pi]\}} \\
\mathbb{I}_2 &= \mathbb{I}_{\{\theta + \theta_0 - 2\pi p/k \in [0,\pi]\}}.
\end{aligned}
$$

Finally, completing the square in the r integral,

$$
\mathbb{P}(\underline{X}_1(t) \geq B_1, \underline{X}_2(t) \geq B_2) = \sum_{p=-\infty}^{\infty} \frac{k}{\pi} e^{a_1 B_1 + a_2 B_2 + bt} e^{-r_0^2/2t} \times \tag{F.2}
$$

$$
\int_0^{\frac{\pi}{k}} \left\{ \left(1 + \sqrt{2\pi t} f_- e^{f_-^2 t/2} \Phi(f_- \sqrt{t})\right) \mathbb{I}_1 - \left(1 + \sqrt{2\pi t} f_+ e^{f_+^2 t/2} \Phi(f_+ \sqrt{t})\right) \mathbb{I}_2 \right\} \mathrm{d}\theta
$$

for $k = 1, 2, 3, \ldots$

Appendix G

Cumulative Normal Code

To evaluate the cumulative normal distribution we use the following code, as outlined in Marsaglia (2004),

```
#include <math.h>
#include <iostream>
class CNormDist{
public:
  CNormDist(){}
  double Phi(double x)
    {
  double b=x;
  double q=x*x;
  double i=1;
  double s=x;
  double t=0;
  const double Pi = 2*asin(1.);
  if(x>10)
    return 1;
  else{
    if(x<-10)
      return 0;
    else {
      while (s!=t)
    {
      i=i+2;
      b=b*q/i;
      t=s;
      s=s+b;
    }
      return 0.5 + s*exp(-q/2-log(2*Pi)/2);}}
    }
};
```

Appendix H

Survival Probabilities in Three Dimensions

The Problem

Working with the three Brownian motions,

$$X_i(t) \equiv \alpha_i t + \sigma_i W_i(t), \quad t \geq 0, \quad i = 1, 2, 3$$

for constant drifts and volatilities α_i and σ_i such that $X_i(0) = 0$, where the Brownian motions $W_i(t)$ have covariance $\text{cov}(W_i(t), W_j(t)) = \rho_{ij} t$, we are interested in the probability density function, $p(x_1, x_2, x_3, t)$,

$$\frac{\partial^3}{\partial x_1 \partial x_2 \partial x_3} \mathbb{P}(X_1(t) \leq x_1, X_2(t) \leq x_2, X_3(t) \leq x_3, \underline{X}_1(t) \geq B_1, \underline{X}_2(t) \geq B_2, \underline{X}_3(t) \geq B_3)$$

where $i = 1, 2, 3$, $B_i \leq 0$ are constants such that $x_i \geq B_i$ and for each $X_i(t)$ we denote the running minimum by

$$\underline{X}_i(t) = \min_{0 \leq s \leq t} X_i(s).$$

The PDE

As in the two-dimensional case in Appendix B, the transition probability density $p(x_1, x_2, x_3, t)$ satisfies the Fokker-Planck equation,

$$\frac{\partial p}{\partial t} = -\sum_{i=1}^{3} \alpha_i \frac{\partial p}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{3} \rho_{ij} \sigma_i \sigma_j \frac{\partial^2 p}{\partial x_i \partial x_j} \tag{H.1}$$

with initial condition

$$p(x_1, x_2, x_3, t = 0) = \delta(x_1)\delta(x_2)\delta(x_3) = \prod_{i=1}^{3} \delta(x_i),$$

and absorbing boundary conditions

$$\begin{aligned} p(x_1 = B_1, x_2, x_3, t) &= 0 \\ p(x_1, x_2 = B_2, x_3, t) &= 0 \\ p(x_1, x_2, x_3 = B_3, t) &= 0. \end{aligned}$$

Paralleling the solution of Hua et al. (1998) in two dimensions, we can eliminate the drift terms by the change of variables,

$$p(x_1, x_2, x_3, t) = e^{a_1 x_1 + a_2 x_2 + a_3 x_3 + bt} q(x_1, x_2, x_3, t).$$

Equation (H.1) becomes

$$\frac{\partial q}{\partial t} = \frac{1}{2} \sum_{i,j=1}^{3} \rho_{ij}\sigma_i\sigma_j \frac{\partial^2 q}{\partial x_i \partial x_j} \tag{H.2}$$

If $C = (C_{ij})$, where $C_{ij} = \rho_{ij}\sigma_i\sigma_j$, is the covariance matrix, then $\mathbf{a} = (a_1, a_2, a_3)'$ is the solution to

$$-\boldsymbol{\alpha} + C\mathbf{a} = 0,$$

and

$$b = -\sum_{i=1}^{3} \alpha_i a_i + \frac{1}{2} \sum_{i,j=1}^{3} \rho_{ij}\sigma_i\sigma_j a_i a_j.$$

The initial condition and boundary conditions remain unchanged,

$$\begin{aligned} q(x_1, x_2, x_3, t = 0) &= \prod_{i=1}^{3} \delta(x_i) \\ q(x_1 = B_1, x_2, x_3, t) &= 0 \\ q(x_1, x_2 = B_2, x_3, t) &= 0 \\ q(x_1, x_2, x_3 = B_3, t) &= 0. \end{aligned}$$

Transforming the coordinates to normalize the Brownian motions and eliminate the cross-partial derivatives, we define new coordinates $z_i(x_1, x_2, x_3)$, $i = 1, 2, 3$, and let

$$H = H(z_1(\mathbf{x}), z_2(\mathbf{x}), z_3(\mathbf{x}), t),$$

where

$$q(x_1, x_2, x_3, t) = \frac{1}{J} H(z_1(\mathbf{x}), z_2(\mathbf{x}), z_3(\mathbf{x}), t)$$

and J is the Jacobian for the change of variables. Taking the coordinates

$$\begin{aligned}
z_1 &= \frac{1}{a_{11}}\left[\left(\frac{x_1 - B_1}{\sigma_1}\right) + \left(\frac{\rho_{13}\rho_{23} - \rho_{12}}{1-\rho_{23}^2}\right)\left(\frac{x_2 - B_2}{\sigma_2}\right) + \left(\frac{\rho_{12}\rho_{23} - \rho_{13}}{1-\rho_{23}^2}\right)\left(\frac{x_3 - B_3}{\sigma_3}\right)\right] \\
z_2 &= \frac{1}{\sqrt{1-\rho_{23}^2}}\left[\left(\frac{x_2 - B_2}{\sigma_2}\right) - \rho_{23}\left(\frac{x_3 - B_3}{\sigma_3}\right)\right] \\
z_3 &= \left(\frac{x_3 - B_3}{\sigma_3}\right)
\end{aligned}$$

where

$$a_{11}^2 = \frac{1 - \rho_{12}^2 - \rho_{13}^2 - \rho_{23}^2 + 2\rho_{12}\rho_{13}\rho_{23}}{1-\rho_{23}^2},$$

the Jacobian is

$$J = \sigma_1\sigma_2\sigma_3[1 - \rho_{12}^2 - \rho_{13}^2 - \rho_{23}^2 + 2\rho_{12}\rho_{13}\rho_{23}]^{1/2}$$

and we need to solve

$$\frac{\partial H}{\partial t} = \frac{1}{2}\nabla^2 H \tag{H.3}$$

subject to initial condition

$$H(z_1, z_2, z_3, 0) = \prod_{i=1}^{3}\delta(z_i - z_{i0})$$

and boundary conditions

$$H(P_1, t) = H(P_2, t) = H(P_3, t) = 0$$

where

$$\begin{aligned}
z_{10} &= \frac{1}{a_{11}}\left[\left(\frac{-B_1}{\sigma_1}\right) + \left(\frac{\rho_{13}\rho_{23} - \rho_{12}}{1-\rho_{23}^2}\right)\left(\frac{-B_2}{\sigma_2}\right) + \left(\frac{\rho_{12}\rho_{23} - \rho_{13}}{1-\rho_{23}^2}\right)\left(\frac{-B_3}{\sigma_3}\right)\right] \\
z_{20} &= \frac{1}{\sqrt{1-\rho_{23}^2}}\left[\frac{-B_2}{\sigma_2} + \frac{\rho_{23}B_3}{\sigma_3}\right] \\
z_{30} &= \frac{-B_3}{\sigma_3}
\end{aligned}$$

and P_1, P_2 and P_3 are the planes

$$\begin{aligned}
P_1 &= \left\{(z_1, z_2, z_3) : z_3 = \left(\frac{\rho_{13}\rho_{23} - \rho_{12}}{\rho_{13}\sqrt{1-\rho_{23}^2}}\right) z_2 - \frac{a_{11}}{\rho_{13}} z_1\right\} \\
P_2 &= \left\{(z_1, z_2, z_3) : z_3 = \frac{-\sqrt{1-\rho_{23}^2}}{\rho_{23}} z_2\right\} \\
P_3 &= \{(z_1, z_2, z_3) : z_3 = 0\}.
\end{aligned}$$

The 2D case proceeds by changing to polar coordinates and then solving (H.3) by separation of variables. It is not at all obvious how to solve (H.3) in three (or more) dimensions and we have been unable to find a set of coordinates for which we can make any further progress.

A similar analysis to the above was done independently by Escobar and Seco (2004). They basically got to this stage (albeit with a plethora of minor errors) and then picked a coordinate system that is attractive from the point of view of the boundary conditions and wrote the Laplacian as if the system were orthogonal, which unfortunately is not the case.

Appendix I

Details of the 7-point Finite Difference Method

In the discretisation of

$$\frac{\partial U}{\partial t} = \sum_{i=1}^{n} \alpha_i \frac{\partial U}{\partial X_i} + \frac{1}{2} \sum_{i,j=1}^{n} \rho_{ij}\sigma_i\sigma_j \frac{\partial^2 U}{\partial X_i \partial X_j} \tag{I.1}$$

in Chapter 6, we use the seven-point stencil outlined in Hackbusch (1992) for the cross derivative term. This ensures that the solution is always stable in two dimensions which is not the case when using more naive discretisations. However, it has limitations once we consider the three-dimensional situation. For simplicity we illustrate this here for the case when $\alpha_i = 0$ and the firms are assumed to be homogeneous with volatility σ and correlation ρ. The argument in the general case proceeds analogously, although it leads to different bounds on ρ.

In two dimensions, the right-hand side of (I.1) becomes

$$\frac{1}{2}\sigma^2 \frac{\partial^2 U}{\partial X_1^2} + \frac{1}{2}\sigma^2 \frac{\partial^2 U}{\partial X_2^2} + \rho\sigma^2 \frac{\partial^2 U}{\partial X_1 \partial X_2} \tag{I.2}$$

which we discretise on a mesh of size h using

$$\begin{gathered}
\frac{\partial^2}{\partial X_1^2} : \frac{1}{h^2}\begin{bmatrix} 0 & 0 & 0 \\ 1 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \frac{\partial^2}{\partial X_2^2} : \frac{1}{h^2}\begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \\
\frac{\partial^2}{\partial X_1 \partial X_2} : \begin{cases} \frac{1}{2h^2}\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix}, & \text{if} \quad \rho > 0, \\ \frac{1}{2h^2}\begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}, & \text{if} \quad \rho < 0, \end{cases}
\end{gathered} \tag{I.3}$$

to give

$$\begin{cases} \frac{\sigma^2}{2h^2}\begin{bmatrix} 0 & 1-\rho & \rho \\ 1-\rho & -4+2\rho & 1-\rho \\ \rho & 1-\rho & 0 \end{bmatrix} & \text{if} \quad \rho > 0, \\ \frac{\sigma^2}{2h^2}\begin{bmatrix} -\rho & 1+\rho & 0 \\ 1+\rho & -4-2\rho & 1+\rho \\ 0 & 1+\rho & -\rho \end{bmatrix} & \text{if} \quad \rho < 0. \end{cases}$$

Since $|\rho| \leq 1$, the diagonal term is always negative and the off-diagonal entries are all positive. Under these conditions, the discretisation is represented by an M-matrix so the maximum principle holds and error propagation is stable – discretisation errors do not increase over time. Whilst this is without restriction in two dimensions[1], the same is not true in higher dimensions since it is then possible for some of the off-diagonal entries to turn positive. In this situation, the maximum principle does not hold and the system is potentially unstable.

In three dimensions, by obvious extension from the 2D case above, ignoring the σ^2 term, the diagonal term is

$$\begin{aligned} &\frac{-6+6\rho}{2h^2} \quad \text{if} \quad \rho > 0 \\ &\frac{-6-6\rho}{2h^2} \quad \text{if} \quad \rho < 0 \end{aligned}$$

and is always negative. However, by (I.3), the off-diagonal terms are

$$\begin{aligned} 0, \quad \rho \quad \text{or} \quad \frac{1-2\rho}{2h^2} \quad \text{if} \quad \rho > 0, \\ 0, \quad -\rho \quad \text{or} \quad \frac{1+2\rho}{2h^2} \quad \text{if} \quad \rho < 0. \end{aligned}$$

The latter correspond to the six terms for which there is a contribution from the three derivatives $\partial^2/\partial X_i^2$, $\partial^2/\partial X_i \partial X_j, \partial^2/\partial X_i \partial X_k$, for $i \neq j \neq k$. Since all the off-diagonal terms must be positive for the solution to exist and be stable, these are the terms that lead to a restriction on ρ. In the case here, we need $-1/2 \leq \rho \leq 1/2$, whereas in the case of non-homogeneous companies, the limits on ρ_{ij} obviously depend on the relative sizes of the volatilities, σ_i. In higher dimensions the situation is even worse, requiring $\frac{-1}{(n-1)} \leq \rho \leq \frac{1}{(n-1)}$ in the homogeneous case for dimension n.

[1]The approach can be generalised for $\sigma_1 \neq \sigma_2$ by choosing adapted mesh sizes $h_1 \neq h_2$.

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