Evolution equations for dunes and drumlins

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1 Dunes and drumlins

Dunes and drumlins are examples of *bedforms*, geomorphological patterns in which the surface of the Earth is shaped by the erosive power of wind, water or ice. Aeolian dunes (i.e., desert dunes) are the most familiar such feature. Coastal sand dunes are a feature of common experience, and we are also familiar with the existence of dunes in deserts. Dunes are formed by the erosive power of the wind, which transports the sand particles at the surface, and sculpts them into a variety of shapes. These can be very large (hundreds of metres in elevation), and take a variety of geometric forms depending on the prevailing wind speed and direction(s) (Tsoar 2001). A spectacular example is shown in figure 1.

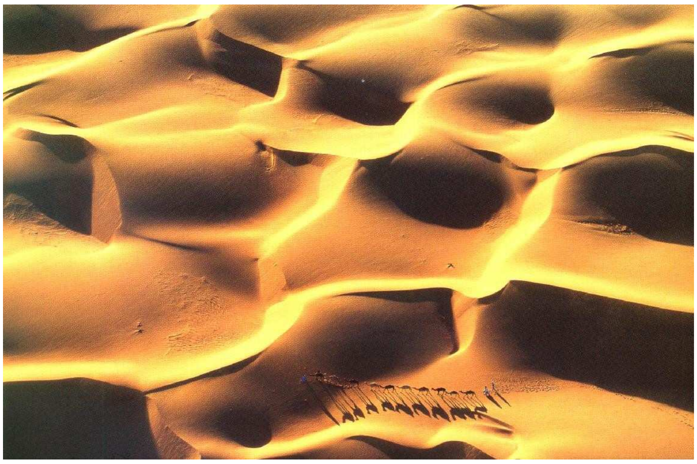

Figure 1: Three-dimensional dunes in Mauritania. Photo ©Yann Arthus-Bertrand, from the book *La Terre Vue du Ciel*, Editions de la Martiniére. Dromedaries for scale.

The simplest kind of dune is the transverse dune, which forms when prevailing winds are uni-directional. Transverse dunes are orthogonal to the wind direction, and form in a regular, more or less periodic, array; they have a gentle upstream face and a steeper downstream face, also known as a slip face, because its slope is limited by the angle of friction of the sand. The prevailing turbulent wind flow over the dune separates at the dune crest, which is therefore sharp, and the flow is illustrated schematically in figure 2.

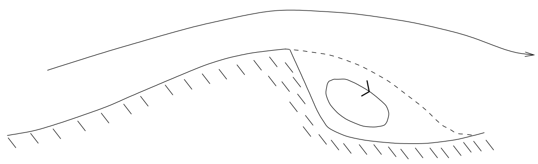

Figure 2: Separation in the lee of a transverse dune.

Dunes also occur in the flow of water over sand, in rivers and estuaries (Allen 1985). Typical wavelengths are of the order of metres, but apart from this, they are essentially similar to transverse aeolian dunes. Since water flow is typically unidirectional, these fluvial dunes do not come in the exotic (barchan, star, seif) shapes which one finds in deserts.

Dunes in both rivers and deserts are a low Froude number phenomenon. The Froude number is given by

$$Fr = \frac{u}{(gh)^{1/2}}, \tag{1}$$

where u is the velocity, g is the acceleration due to gravity, and h is the fluid depth. In the atmosphere, this might be taken as the atmospheric boundary layer thickness, about a kilometre, so that a wind speed of 20 m s^{-1} gives a Froude number of 0.2. In streams, the Froude number can occasionally (e. g., in stormflow) become larger than one, and in this case anti-dunes are formed. These are time-dependent bed waves which interact with the fluid to form surface waves, but are essentially transient features. They can be seen on beach streams (for example, I have seen them on streams of a few centimetres depth on a beach in Normandy).

Drumlins are small, typically oval hills, of length some hundreds of metres, and typical elevation thirty metres. They occur in swarms, and are found widely over formerly glaciated areas of Northern Europe and North America. Figure 3 shows a view of a typical drumlinised topography which is found in Northern Ireland, while figure 4 shows a typical aerial relief of similar terrain near Seattle in the U. S. A. 'Drumlin' is an Irish word, meaning a small hill, and the early literature on these bedforms concerned their occurrence in Ireland (Kinahan and Close 1872); later authors extended the discussion to North America (Davis 1884).

Although the geological literature on drumlins is over a hundred years old, there is still no consensus as to how they are formed, although it is generally accepted that they are the results of the erosive action of an overlying ice flow on a deformable

Figure 3: Drumlins in the Ards peninsula, Northern Ireland.

substrate. This substrate consists of an angular mixture of coarse rock fragments with a finer grained mixture of sediments, and is known as till. Its easily recognised presence is an indicator of the former presence of ice sheets.

2 The Exner equation

In order to construct a mathematical model of either dune or drumlin formation, we consider a two-dimensional flow in the x direction, where x is essentially horizontal, and z is the upwards coordinate orthogonal to x. The bed profile is then denoted by $z = s(x, t)$, and the basic equation to describe its evolution is

$$(1-\phi)\frac{\partial s}{\partial t} + \frac{\partial q}{\partial x} = 0, \tag{2}$$

where ϕ is the bed porosity, and q is the bedload (or till) flux. This equation simply represents conservation of bed material. The restriction to two dimensions is feasible for transverse dunes, although not for other types of dune. For drumlins, it seems less appropriate, but in fact it appears that drumlins represent a three-dimensional modification of a two-dimensional 'ribbed' moraine pattern known as *Rogen moraine* (Sugden and John 1976) after the region in Sweden (Lake Rogen) where it was first identified (Lundqvist 1989). In any case, the extension to a three-dimensional theory is easily made.

Our basic strategy is to identify dunes and drumlins as the result of an instability, and the primary modelling issue is to understand how this instability arises. To get some idea of this, consider the case of river flow. It is generally accepted that bedload

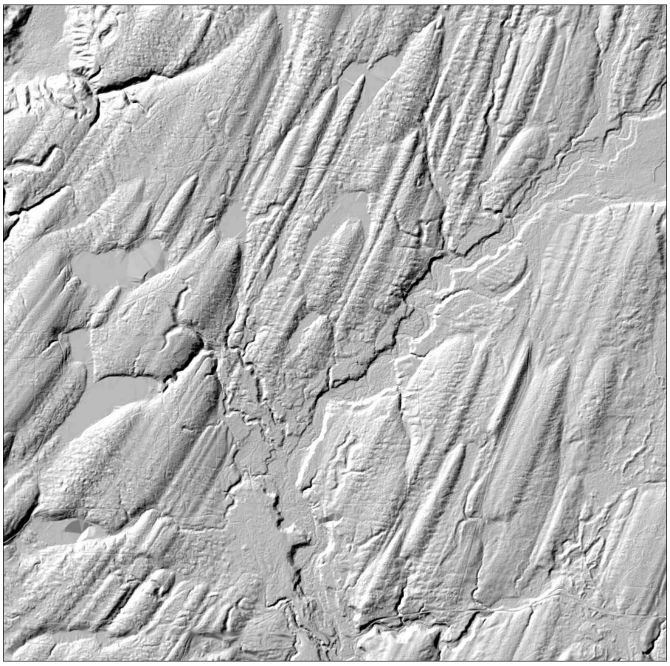

Figure 4: An aerial view of drumlinised topography in the environs of Seattle, Washington. The area shown is 6 km × 6 km; ice flow was from the upper right to the lower left, to the southwest. The image was calculated from the Puget Sound Lidar Consortium 6 foot digital elevation model, and is supplied courtesy of Ralph Haugerud.

transport of sand is prescribed by a functional dependence,

$$q = q(\tau), \tag{3}$$

where τ is the basal shear stress, and a typical form of this is the prescription due to Meyer-Peter and Müller (1948):

$$q(\tau) = C(\tau - \tau_c)_+^{3/2}, \tag{4}$$

where $[x]_+ = \max(x, 0)$, and τ_c is a yield stress, called the *Shields stress* (after Shields 1936); bedload transport only occurs for stresses above this value. The important point is that q increases with τ.

Consider a simple flow in which the surface $z = \eta$ is undisturbed (this is appropriate at low Froude number), so that the flow depth is

$$h = \eta - s. \tag{5}$$

If the mean velocity is u, then

$$uh = Q, \tag{6}$$

where Q is the water flux per unit channel width. Finally, a common empirical estimate for the boundary shear stress in the turbulent flow appropriate to rivers (Reynolds numbers typically $\gtrsim 10^5$) is

$$\tau = f\rho u^2, \tag{7}$$

where ρ is density and f is a dimensionless friction factor. If we put these together, we find that $q = q(s)$ is an increasing function of s, so that (2) is a first order hyperbolic equation for s. Thus disturbances propagate forwards as waves and form shocks (perhaps analogous to dune slip faces), but there is no mechanism for instability.

We can follow essentially the same discussion for ice sheets, except that here the Reynolds number is effectively zero. Flow in an ice sheet is driven by surface slope (through its effect on the cryostatic pressure field); we could then write

$$\tau = \rho g(\eta - s)\sin\alpha \tag{8}$$

(Paterson 1994), where ρ is ice density, and α is the surface slope. If, as we expect, till flux increases with increasing shear stress, then we would have q as a decreasing function of s, and disturbances would propagate backwards.

Neither of these theories can explain instability, but neither of them is very useful in calculating the perturbation to the shear stress at the bed. Equation (8) is particularly suspect because the basic model of slow flow is elliptic, and we would expect τ to depend on a convolution integral of s over the whole x axis. Somewhat surprisingly, this is also the case for the high Reynolds number flow, although for different reasons.

Instability

Kennedy (1963) first advanced the notion that instability would arise because the maximum of the shear stress occurs upstream of the maximum of the bed profile. In the simple theory above for dunes, we have $\tau \propto 1/(\eta - s)^2$, and the maxima in s and τ coincide. Kennedy suggested taking τ as a function of $s(x + \delta, t)$, so that with $\delta > 0$, the maximum of τ occurs on the upstream face of a nascent bump, and this causes instability. However, Kennedy's model is unphysical (and ill-posed) as it stands, and it was not until the work of Smith (1970) and Engelund (1970) that it was shown that the phase shift in bed shear stress could be predicted on the basis of a fully two-dimensional turbulent shear flow calculation. Their analysis consisted of solving the Orr-Sommerfeld equation describing small perturbations (due to the bed profile) of a unidirectional flow in order to calculate the perturbed shear stress.

Despite the apparent similarity of the problem, the linear stability theory of aeolian dunes has not followed the same track. Stam (1996) used a parameterisation of the bed stress due to Jackson and Hunt (1975) (of which more below) in order to demonstrate instability of the bed. For drumlins, the stability theory is even more recent, being initated by Hindmarsh (1998) and Fowler (2000). The important point is that the instability arises through the same upstream phase shift of the maximum shear stress.

3 Bed stress parameterisations

For a mathematician, the next step beyond a linear stability theory is the development of a nonlinear theory. In fluvial dune theory, this appears not to have been done, although such theories are now appearing in the context of aeolian dunes (Kroy *et al.* 2002). The Exner equation is essentially nonlinear, and the difficulty with developing the theory lies in the problem of providing an analytic parameterisation of the bed shear stress in terms of the bed profile.

The reason this is at all possible is because the typical slope of dunes and drumlins is relatively small. We can thus consider the effect of the bed as a small perturbation of the overlying flow, and hence construct an approximation to the resulting shear stress. For dunes, the difficulty we face in doing this is that the flow is turbulent, and the result we get is dependent on the turbulent flow model we choose.

The simplest model to choose is one in which the Reynolds stresses are modelled by an eddy viscosity, which is taken to be constant. If the bed slope is small (and the bed varies slowly compared to the flow's convective time scale), then the perturbed flow is described by a stationary Orr-Sommerfeld equation. This is still intractable, unless one uses the fact that the turbulent Reynolds number is relatively large (essentially, $f \ll 1$ in (7)), and in this case one can develop an approximate expression for the bed stress, using asymptotic methods of Reid (1972) (see also Drazin and Reid 1980). This work is reported by Fowler (2001); the expression for the shear stress takes the dimensionless form

$$\tau \approx \frac{1}{1-\varepsilon s} + \frac{\varepsilon}{(1-\varepsilon s)^2}\int_{-\infty}^{\infty} K(x-\xi)\frac{\partial s}{\partial \xi}(\xi,t)\,d\xi, \tag{9}$$

where we have written horizontal lengths in terms of the mean flow depth h, and the bed stress in terms of that of the undisturbed flow; thus when the bed is flat, $s = 0$ and $\tau = 1$. ε is the (unknown) aspect ratio of the evolving bedform (i.e., its height is $O(\varepsilon h)$. The kernel K is defined by

$$\begin{aligned} K(x) &= \frac{\alpha}{x^{1/3}}, \quad x > 0, \\ &= \quad 0, \quad x < 0, \end{aligned} \tag{10}$$

and the numerical parameter $\alpha \approx 6$, depending on the precise value of the turbulent Reynolds number.

It is fairly clear that this model distributes the effect of the bed slope on the stress upstream of the current position, and thereby allows instability. In fact it is easy to show that waves of (dimensionless) wave number k travel downstream, and grow at a rate proportional to $k^{4/3}$. The resulting infnite growth rate at high wave number is an indicator of ill-posedness, but if one takes into account the fact that the effective stress to drive bed transport is the difference between the stress delivered from the flow and that offered by the slope of the bed (it is harder to move sand uphill), then we find that the effective stress that needs to be incorporated in the bedload formula is

$$\tau = \frac{1}{1-\varepsilon s} + \frac{\varepsilon\alpha}{(1-\varepsilon s)^2}\int_0^{\infty} \xi^{-1/3}\frac{\partial s}{\partial x}(x-\xi,t)\,d\xi - \varepsilon\beta\frac{\partial s}{\partial x}, \tag{11}$$

where

$$\beta = \frac{\Delta\rho g D_s}{\tau_0}, \tag{12}$$

where $\Delta\rho$ is the density difference between solid grains and fluid, D_s is the mean particle diameter, and τ_0 is the dimensional mean bed stress. This parameter is the inverse of the dimensionless Shields stress, and typical values of the parameters suggest $\beta \sim O(1)$. Its inclusion allows the unstable growth rate $\propto k^{4/3}$ to be damped at large k (growth rate $\propto -k^2$): the slope effect is in fact diffusive, as might be expected.

The above discussion shows that an analytic description of the bed stress can indeed be obtained, and that it predicts instability, as we seek. However, the assumption of a constant eddy viscosity is crude, at best. In fact, there is a substantial literature in the context of atmospheric boundary layer meteorology which has been devoted to precisely this question, of ascertaining the correct form of bed stress for turbulent flow over low hills. The basic paper is by Jackson and Hunt (1975), and later editions of the theory are presented by Hunt *et al.* (1988) and Weng *et al.* (1991); a recent review is by Belcher and Hunt (1998). The essential modification to the constant eddy viscosity model is to allow Prandtl's mixing length to depend on distance from the bed, so that the eddy viscosity takes the dimensional form

$$\eta = \rho l^2 \left| \frac{\partial u}{\partial z} \right|, \quad l = \kappa(z - s). \tag{13}$$

(In the constant eddy viscosity theory, we take $\eta = \frac{1}{3} f \rho \bar{u} \bar{h}$, where the overbars denote mean values.) This allows for a logarithmic velocity profile to be obtained, which in the atmosphere is observed to be appropriate for some thirty metres above the surface. There is also a natural dimensionless parameter which arises, which we can define to be

$$\varepsilon = \frac{u_*}{U_\infty} = \frac{\kappa}{\ln(d/z_0)}. \tag{14}$$

d is the relevant macroscopic length scale, while z_0 is the roughness scale (of thickness millimetres), over which the velocity is reduced to zero at the bed; $u_* = (\tau/\rho)^{1/2}$ is known as the friction velocity. The Jackson–Hunt theory uses the smallness of ε to construct an approximate theory which has a basic similarity to the constant eddy viscosity theory discussed earlier. There is an outer inviscid (but rotational) flow and an inner shear layer. Matching this to the bed roughness layer shows that the dimensionless bed stress can be written in the form

$$\sqrt{\tau} = 1 + \varepsilon A_1 + \varepsilon^2 A_2 + \ldots. \tag{15}$$

The different basic velocity structure allows the inviscid pressure field to be found as the solution of Laplace's equation, and we have

$$A_1 = -p_0, \tag{16}$$

where p_0 is the bed pressure, given by

$$p_0 = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{s_\xi \, d\xi}{\xi - x} = H(s_x); \tag{17}$$

H denotes the Hilbert transform.

Inspection of the stability characteristics of the steady state shows that the term A_1 is neutrally stable; it gives rise to dispersive waves with wave speed $\propto |k|$, where k is the wave number. Instability requires determination of A_2. At this point we deviate from the exposition by Jackson and Hunt (1975), and in subsequent papers, because the approximation methods used appear not to be self-consistent. Formal asymptotic approximations do, however, lead to similar qualitative results, of which the essence is that

$$A_2 = \left(\frac{\pi}{\kappa} + 3\kappa\right) s_x + \ldots, \tag{18}$$

where the omitted terms are neutrally stable or nonlinear. The effect of (18) is to give a negative diffusion term in the equation for s, and is thus destabilising as we want, but it is unclear whether the resulting system is well-posed or not.

Drumlins

In studying the same instability in the subglacial context, we have to solve the (Stokes) equations of slow flow, together with suitable boundary conditions on the ice surface and bed. A simplification occurs if we take the surface to be far away, so that the perturbations due to the bedslope die away at large z. The boundary conditions at the bed are complicated by the fact that the stress τ is non-zero, but also the ice will 'slide' and have a non-zero basal velocity. We suppose this is accommodated by deformation of the subglacial till layer on which the ice rests (essentially we think of sliding on lubricated ball bearings). An essential physical feature of this deformation process is that it depends on the *effective pressure* N, which is the difference between ice overburden pressure and the pore water pressure in the till. Physically, then, we expect that basal velocity u will be a function of stress τ, till thickness (of which bed elevation s gives a measure), and N; in addition, we can thus expect till flux q (the analogue of bedload) to depend on the same variables. For simplicity we will suppose that till is sufficiently thick that there is no dependence on s. This is consistent with apparent observations of deforming till thicknesses of order tens of centimetres, and a nearly Coulomb plastic flow law (e. g. Kamb 1991). Thus we suppose

$$u = u(\tau, N), \quad q = q(\tau, N), \tag{19}$$

and we expect the partial derivatives to satisfy $u_\tau > 0$, $u_N < 0$, $q_\tau > 0$, $q_N < 0$: the N derivatives are negative because higher pore water pressure (lower effective pressure) facilitates deformation of the till. Fowler (2001) reports the calculation of the perturbed shear stress at the bed on the basis that the aspect ratio is small: the result can be written in the dimensionless form

$$\begin{aligned} N &= 1 - \varepsilon\alpha H\left[\frac{\partial}{\partial x}\left\{\frac{\partial s}{\partial t} + \frac{\partial s}{\partial x}\right\}\right], \\ \tau &= 1 + \beta(N-1), \end{aligned} \tag{20}$$

where u, τ and N have been scaled with the values u_0, τ_0 and N_0 of the unperturbed state when $s = 0$, x has been scaled with l, s with εl, and t with l/u_0. The equations (20) assume that $2\mu u_0/l\tau_0 \gg 1$, where l is the bedform wavelength: this is a reasonable assumption. The parameters α and β are given by

$$\alpha = \frac{2\mu u_0}{lN_0}, \quad \beta = -\left(\frac{u_N}{u_\tau}\right), \tag{21}$$

and are positive and $O(1)$.

Linear stability of the uniform state can be studied using (20), and the growth rate of perturbations $\propto e^{\sigma t}$ is found to be

$$\mathrm{Re}\,\sigma = \frac{\delta\alpha k^2|k|\gamma}{1+\delta^2\alpha^2k^4\gamma^2}, \tag{22}$$

where

$$\delta = \frac{h_0}{l}, \quad \gamma = q_N + \beta q_\tau, \tag{23}$$

and the till flux q has been scaled with $u_0 h_0$, thus h_0 is essentially the depth of deforming till. Instability occurs if $\gamma > 0$. Satisfaction of this criterion depends on the till flux law, but is plausibly satisfied. Perturbations of wavelength $O(100–1000)$ metres may grow on a time scale of about a year, and the growing ridges move in the ice flow direction at speeds which are less than, but comparable to, the ice velocity.

4 Nonlinear evolution equations

It is only relatively recently that efforts have been made to study the nonlinear evolution of these various bedforms. The principal effort in this direction has been made by Herrmann and his co-workers (Kroy *et al.* 2002, Herrmann *et al.* 2001, and Sauermann *et al.* 2001), who have shown that finite amplitude dune shapes can be predicted on the basis of a combination of the Exner equation with the Jackson–Hunt formula for bed stress. Specifically, for bedforms of aspect ratio ε, the dimensionless Exner equation can be written in the form

$$\varepsilon\frac{\partial s}{\partial t} + \frac{\partial q}{\partial x} = 0, \tag{24}$$

where $q = q(\tau)$ (for dunes) or $q = q(\tau, N)$ (for drumlins), and we use expressions such as (11), (15), or (20) to prescribe τ. The important point about all these recipes for the bed stress is that they require only the bed elevation to be small. Despite this, it is still possible to describe *nonlinear* evolution equations for s.

There is an important scaling issue which could affect the usefulness of this theory for drumlins. In order to obtain the dimensionless Exner equation in the form (24), we have to rescale the dimensionless time $t \sim \varepsilon/\delta = s_0/h_0$, where s_0 is a typical drumlin elevation, and h_0 is the deforming till thickness. For a successful theory, this will be large (e.g. ~ 100) and indicates that it takes of order 100 times the convective time scale to build drumlins. But the drumlins move at the convective speed. So if

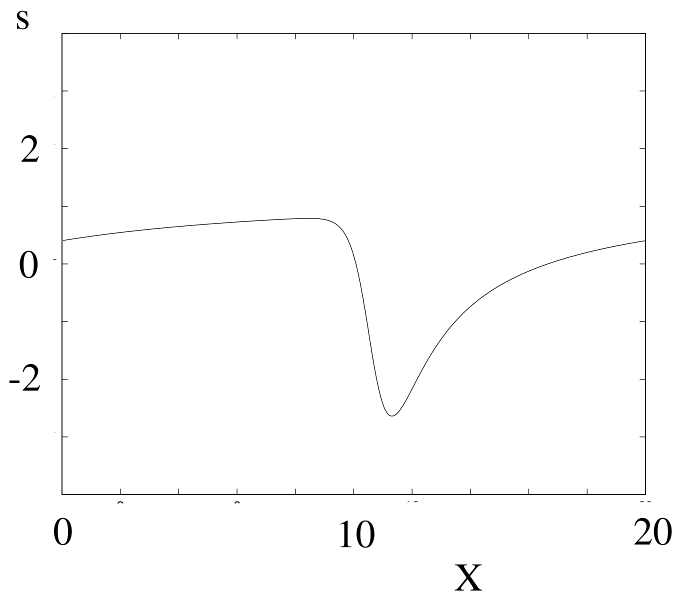

Figure 5: Solution of (26) starting from small amplitude random data, at $T = 80$. The wave shown travels forward at a speed of approximately $dX/dT = 0.73$.

we are to see drumlins at all, they will be at distances of order 100 times their length scale from where they start to form. This is in the range 10–100 km, and suggests that drumlins should be prevalent nearer the margins of ice sheets—as indeed they are. The alternative possibility (and the reason dunes do not share this problem) would be that slip between the ice and the till occurs, so that the ice can rush over the till, much as air or water does over sand.

The expressions (11), (15) and (20) give corrections to the shear stress as power series in the bed amplitude ε; both (11) and (15) contain nonlinear terms, and these may be used to develop nonlinear evolution equations for the bed profile s. We restrict ourselves here to the constant eddy viscosity equation, and write

$$\tau \approx 1 + \varepsilon s + \varepsilon^2 s^2 + \varepsilon\alpha \int_0^\infty \xi^{-1/3} \frac{\partial s}{\partial x}(x - \xi, t)\, d\xi - \varepsilon\beta \frac{\partial s}{\partial x}. \tag{25}$$

We also expand q as $q(\tau) \approx q(1) + q'(1)(\tau - 1) + \ldots$. If $\alpha \sim \varepsilon^{-1/3}$, $\beta \sim O(1)$, then over length scales of $O(1/\varepsilon)$, we find that to leading order, s is given by a non-dispersive travelling wave, but at second order, the wave amplitude satisfies a nonlinear evolution equation which can be written in the scaled form

$$\frac{\partial s}{\partial T} + \frac{\partial}{\partial X}\left[\tfrac{1}{2}s^2 + \int_0^\infty \xi^{-1/3} \frac{\partial s}{\partial X}(X - \xi, T)\, d\xi - \frac{\partial s}{\partial X}\right] = 0. \tag{26}$$

The derivation of the equation is similar to the derivation of the Korteweg-de Vries equation at the order beyond tidal theory (Kevorkian and Cole 1981).

The equation (26) bears resemblance to other nonlinear evolution equations, notably the Benjamin-Ono equation in its singular integral term, and the Kuramoto-Sivashinsky equation in its short wavelength instability and long wavelength stability. It has the form of a Burgers' equation with the extra ingredient of the integral term providing a mechanism for instability. As such, we expect the long time solution to consist of a series of shock waves which develop out of the initial wave instability. Such shock waves in Burgers' equation have a speed dependent on the jump in elevation, and thus larger shocks will capture smaller ones. In a numerical solution of (26) on a large domain, we might expect to see a final state consisting of one travelling wave, and figure 5 shows that this is indeed the case. This is hardly consistent with observations of periodic dunes (as in figure 1).

Fowler (2001) suggested a similar nonlinear evolution equation for drumlins, based on the nonlinearity of $q(\tau, N, s)$, and assuming dependence on s (which we have excluded). No comparison can therefore be made with the nonlinear evolution equation (18.113) of Fowler (2001), but for the record, numerical solution of that drumlin evolution equation (which takes the form in a moving coordinate frame

$$\frac{\partial s}{\partial t} - \frac{\partial s}{\partial Z} + \frac{\partial}{\partial Z}\left[\tfrac{1}{2}s^2 - H\left\{\frac{\partial^2 s}{\partial Z \partial t}\right\}\right] = 0), \tag{27}$$

shows growing instability forming shock like features upstream of bumps (which is what is commonly observed in drumlins, whose steep face is upstream), but no saturation occurs, and the instability leads to unbounded growth.

5 Discussion

The derivation of linear evolution equations for dunes is in a relatively satisfactory state; this may also be the case for drumlins. However, the prescription of their nonlinear evolution is less well developed. We can derive an evolution equation for dunes which takes the form of a modified Burgers' equation, which does form forward travelling waves, but these have no intrinsic length scale, and the solutions evolve to fill the domain.

The situation with drumlins is less good. There is indeed an instability mechanism, but as yet there is no nonlinear evolution equation in which the growing waves saturate.

The remedy for both these ills may lie in the consideration of separation, which is actually essential for a realistic description of both dunes and drumlins. In dunes, the slip face forms behind a sharp ridge, where the turbulent boundary layer separates from the upstream face, thus forming a wake. Previous treatment of this in models has been essentially schematic (Kroy *et al.* 2002). The simplest way to model separation is by assuming that a wake of constant pressure forms in the lee of dunes. If we use the mixing length model in (15) and (17), then s satisfies

$$\frac{\partial s}{\partial t} = \frac{\partial^2 H(s)}{\partial x^2} + O(\varepsilon) \tag{28}$$

outside the separation bubble (note that $\partial H(s)/\partial x = H(\partial s/\partial x)$, but in the bubble, s is determined as a free boundary by the solution of a Hilbert problem. Specifically, $F = p + iw$ is an analytic function of $\zeta = x + iz$ in $\mathrm{Im}\,\zeta > 0$. We define F in $\mathrm{Im}\,\zeta < 0$ by the Schwarz reflection $F(\zeta) = \overline{F(\bar{\zeta})}$; then F satisfies

$$\begin{aligned} F_+ - F_- &= 2is_x \quad \text{in} \quad B', \\ F_+ + F_- &= p_B \quad \text{in} \quad B, \\ F &\to \infty \quad \text{as} \quad \zeta \to \infty, \end{aligned} \tag{29}$$

where B denotes the separated region(s), and p_B is constant. The method of solution for this Hilbert problem subject to suitable continuity assumptions has been given by Fowler (1986), and then s is given on B by $s_x = (F_+ - F_-)/2i$. It remains to be seen whether this introduction of separation bubbles will introduce a natural length scale for the developing dunes.

A similar comment applies to drumlins. Cavities form if the subglacial effective pressure reaches zero, and this always appears to occur as drumlins grow (C. Schoof, private communication). It is then at least plausible that the cavities provide the mechanism which limits the drumlin growth. In addition, many drumlins appear to contain water-borne sediment layers, whose presence can be understood, at least conceptually, through sediment deposition in water-filled cavities.

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