10. WAVE PATTERNS IN ONE-DIMENSIONAL NONLINEAR DEGENERATE DIFFUSION EQUATIONS

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10.1 INTRODUCTION

Several different types of wave patterns occur in physiology, chemistry and biology. In many cases such phenomena are modelled by reactive-diffusive parabolic systems (see, for example, Fisher 1937; Kolmogorov *et al.* 1937; Winfree 1988; Murray 1989; Swinney & Krinsky 1992). In many biological and physical situations, dispersal is modelled by a density-dependent diffusion coefficient, for example, the bacterium *Rhizobium* diffuses through the roots of some *leguminosae* plants according to a nonlinear diffusive law (Lara-Ochoa & Bustos 1990); nonlinear diffusion has been observed in the dispersion of some insects (Okubo 1980) and small rodents (Meyers & Krebs 1974).

Here, we restrict ourselves to analyzing the problem of the existence of travelling wave solutions (TWS) in an one-dimensional domain for the special case of a nonlinear diffusion coefficient which is degenerate at $u = 0$. That is, we look for a solution $u(x,t) = \phi(x - ct)$ for equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[D(u)\frac{\partial u}{\partial x}\right] + g(u); \quad (x,t) \in \mathbf{R} \times \mathbf{R}^+, \tag{10.1}$$

where D and g are defined on $[0, 1]$ and:

1. $g(0) = g(1) = 0, g(u) > 0 \; \forall \; u \in (0,1)$
2. $g \in C^2[0,1]$ with $g'(0) > 0$ and $g'(1) < 0$
3. $D(0) = 0$ with $D(u) > 0 \; \forall \; u \in (0,1]$
4. $D \in C^2[0,1]$.

with $u(x,0) = u^0(x), \;\; 0 \le u^0(x) \le 1 \; \forall \; x \in \mathbf{R}$.

Different types of one-dimensional TWS, for example, fronts, pulses, sharp and oscillatory, have been reported in the literature (Fife 1979; Sánchez-Garduño & Maini 1992).

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The problem of the existence of TWS of (10.1) has been studied for a few particular cases by a number of authors, for example, Aronson (1980); Newman (1980); Murray (1989) and Lara-Ochoa & Bustos (1990). In this communication we generalize previous analysis. The proof of our results uses a qualitative approach in which to look for TWS for (10.1) is equivalent to finding the appropriate parameter space (which includes the speed c) for which there exists a heteroclinic trajectory of a certain system of ordinary differential equations. The boundary conditions for the TWS are given by the coordinates of the equilibrium states connected by the heteroclinic trajectory.

In the following section we only state the results and present some illustrative examples. Proofs, full details and other examples can be found in Sánchez-Garduño & Maini (1992).

10.2 RESULTS AND EXAMPLES

Theorem 1 If the functions D and g in (10.1) satisfy the above conditions and $D'(u) > 0 \; \forall \; u \in [0,1]$ with $D''(0) \neq 0$, then the reaction-diffusion equation (10.1) possesses at most a TWS $u(x,t) = \phi(x - c^* t)$ of sharp type such that for $c^* > 0$: $\phi(-\infty) = 1, \; \phi(\xi) = 0$ for $\xi \geq \xi^*$; $\phi'(-\infty) = 0$, $\phi'(\xi^{*-}) = -\frac{c^*}{D'(0)}$, and $\phi'(\xi^{*+}) = 0$. Moreover

1. For $0 < c < c^*$ there are no TWS
2. For each $c > c^*$, (10.1) has a TWS of front type satisfying the boundary conditions: $\phi(-\infty) = 1$ and $\phi(+\infty) = 0$.

An application of this theorem is illustrated in the following example.

Example 1. Here we consider the equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[(\beta u + u^2)\frac{\partial u}{\partial x}\right] + u(1-u)[1 - u(1-u)], \tag{10.2}$$

where $\beta > 0$. In this case the qualitative behaviour of the solution does not depend on the value of β. We illustrate in Figure 10.1 the phase portrait for the case $\beta = 2.0$, for which $c^* \approx 0.98$.

Theorem 2 If the functions D and g in equation (10.1) satisfy the conditions 1-4 in the Introduction, with the further conditions on D, that, $D'(0) = 0$, $D'(u) > 0 \; \forall \; u \in (0,1]$ and $D''(0) > 0$, then for each c such that

$$c \geq sup\left\{\frac{d}{d\phi}\left[\rho(\phi)D(\phi)\right] + \frac{g(\phi)}{\rho(\phi)}\right\}, \tag{10.3}$$

where ρ is a continuously differentiable non-negative function in the interval $[0,1]$ with $\rho(0) = 0$ and $\rho'(0) > 0$, there exists a TWS of front type for the equation (10.1) satisfying the boundary conditions, $\phi(-\infty) = 1$ and $\phi(+\infty) = 0$. An application of this theorem is illustrated in example 2.

Example 2. Here we consider the equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[u^2\frac{\partial u}{\partial x}\right] + u(1-u). \tag{10.4}$$

Typical phase portraits are illustrated in Figure 10.2.

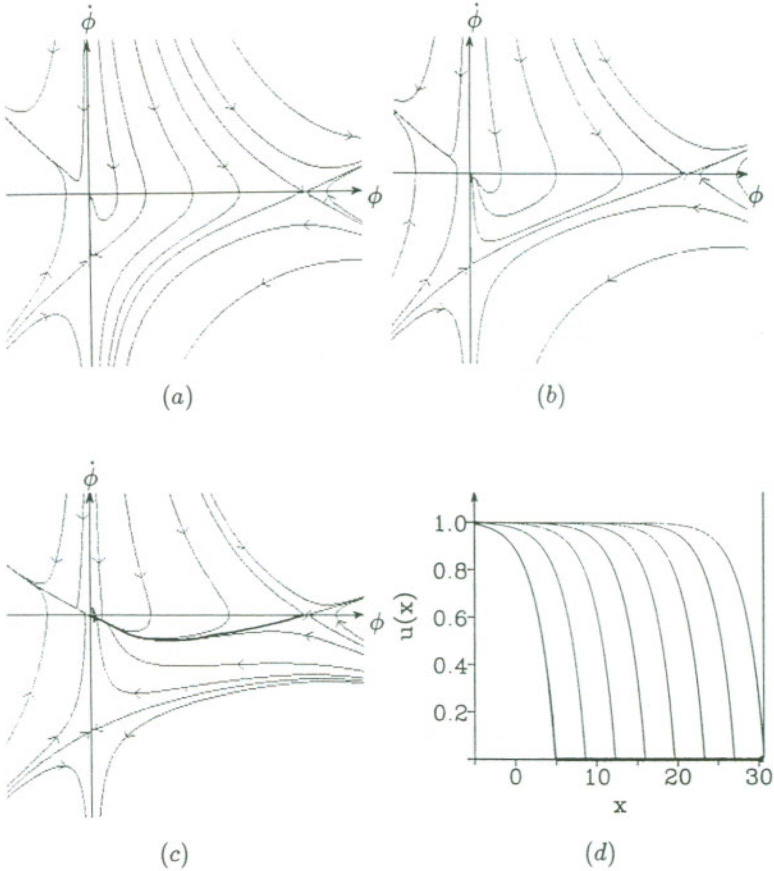

Figure 10.1. Phase portraits for example 1 with $\beta = 2.0$ for different values of the speed c. (In this case $c^* \approx 0.98$.) (a) $c = 0.7$, (b) $c = 0.98$, (c) $c = 1.5$. For $c < c^*$ there is no connection between the equilibrium points and hence no travelling wave. For $c = c^*$, the saddle-saddle connection implies a travelling wave of sharp type, while for each $c > c^*$ we have a travelling wave of front type. The solution for the partial differential equation is shown in (d). The numerically calculated wavespeed is $c^* = 0.9797$.

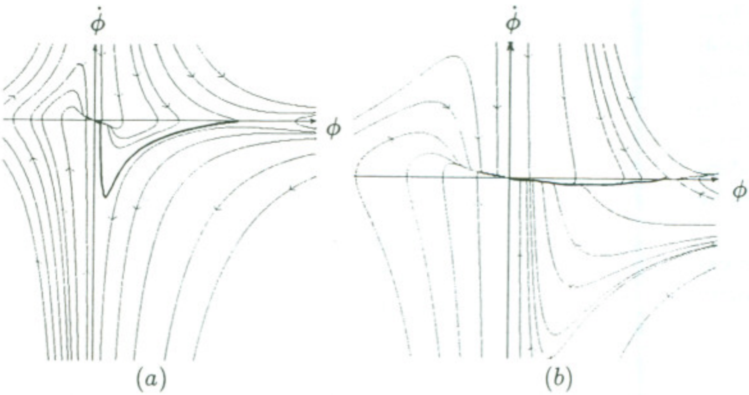

Figure 10.2. Phase portraits for example 2 for different values of the speed c. (a) $c = 0.5$ and (b) $c = 2.0$. In both cases there exists a connection between the two equilibrium points suggesting the existence of a travelling wave of front type.

REFERENCES

Aronson, D. G. 1980. Density-dependent interaction-diffusion systems. In *Dynamics and Modelling of Reactive Systems*, Stewart, Warren E. (ed). Academic Press.

Fife, P. 1979. *Mathematical aspects of reaction diffusing systems*. Vol. 28. Springer-Verlag. Lecture Notes in Biomathematics.

Fisher, R. A. 1937. The wave of advance of advantageous genes. *Ann. Eugenics.*, **7**, 353–369.

Kolmogorov, A., Petrovsy, I., & Piskounov, N. 1937. Study of the diffusion equation with growth of the quantity of matter and its applications to a biological problem. In *Applicable Mathematics of Non-Physical Phenomena*, Oliveira-Pinto, F., & Conolly, B. W. (eds). John Wiley and Sons, 1982 edn.

Lara-Ochoa, F., & Bustos, V. P. 1990. A model for aggregation-dispersion dynamics of a population. *BioSystems*, **24**, 215–222.

Meyers, M. R., & Krebs, C. 1974. Population cycles in rodents. *Sci. Am.*, **230**, 38–46.

Murray, J. D. 1989. *Mathematical biology*. New York: Springer-Verlag, Berlin.

Newman, W. I. 1980. Some exact solutions to a non-linear diffusion problem in population genetics and combustion. *J. Theor. Biol.*, **85**, 325–334.

Okubo, A. 1980. Diffusion and ecological problems: mathematical models. In *Biomathematics*, volume 10. Springer-Verlag.

Sánchez-Garduño, F., & Maini, P. K. 1992. Travelling wave phenomena in some degenerate reaction-diffusion equations. (Submitted for publication).

Swinney, H. L., & Krinsky, V. I. 1992. Waves and Patterns in Chemical and Biological Media. Special Issue of Physica D.

Winfree, A. 1988. *When time breaks down*. Princeton University Press.