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Circulation in Inviscid Gas Flows with Shocks

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Abstract—In this note, we show that the circulation $\Gamma = \int_C \mathbf{u}.\mathbf{dx}$ around a closed material curve $C(t)$ in an inviscid gas flow develops according to the equation $\frac{d\Gamma}{dt} = \int_C T\,dS$, even when the curve may cross shocks, with the entropy jumps at the shocks *excluded* from the right-hand side.

Keywords—Circulation, Shocks.

1. MOTIVATION, STATEMENT, AND DERIVATION OF RESULT

If a uniform transonic or supersonic gas flow is incident on a thin wing or a slender rigid body, and the resulting gas flow contains only *weak* shocks, with strengths of order ϵ, then the entropy changes at those shocks are of order ϵ^3. In the flow behind the shocks, Crocco's theorem $(\nabla\wedge\mathbf{u})\wedge\mathbf{u} = T\nabla S$ then shows that the components of vorticity perpendicular to $\mathbf{u}$ are also of order ϵ^3. However, Crocco's theorem does not directly give us the component of vorticity *parallel* to $\mathbf{u}$. One way to see that this is also of order ϵ^3 is to use a circulation theorem that is valid in the presence of shocks, which is the subject of this note.

We consider the flow of a simple inviscid gas, governed by the compressible Euler equation, and with shocks at which the Rankine-Hugoniot relations hold. In such a flow, we consider the circulation $\Gamma = \int_{C(t)} \mathbf{u}.\mathbf{dx}$ around a closed curve $C(t)$ moving with the flow, and we shall show that

$$\frac{d\Gamma}{dt} = \int'_{C(t)} T\,dS, \tag{1}$$

where T is the absolute temperature, S is the entropy per unit mass, and the prime on the integral on the right indicates that it is only taken along the smooth sections of the flow *between* the shocks, excluding the entropy jumps *at* the shocks. The result is, of course, well known if the flow has *no* shocks [1, Chapter I, Exc. 7]: it is the extension to include shocks that we are interested in here. As with Kelvin's circulation theorem, the flow may be unsteady, and the curve $C(t)$ may encircle obstructions in the flow.

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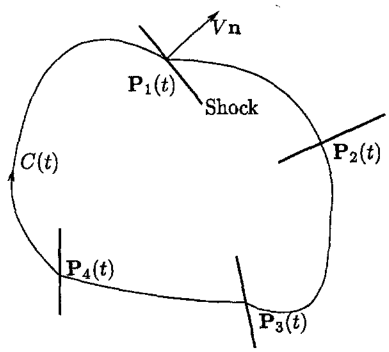

Figure 1. Closed material curve $C(t)$, cut by shocks at point $\mathbf{P}_i(t)$.

Let $C(t)$ be cut by shocks at points $\mathbf{P}_1(t)$,$\mathbf{P}_2(t),\ldots,\mathbf{P}_k(t)$ taken in the same order around C as the sense of integration defining Γ. There will generally be a discontinuity of the tangent to C at each $\mathbf{P}_i$, as in Figure 1.

We shall use Euler's equation to show that

$$\frac{d}{dt}\left(\int_{\mathbf{P}_i}^{\mathbf{P}_{i+1}} \mathbf{u}.d\mathbf{x}\right) = X_{i+1}^- - X_i^+ + \int_{\mathbf{P}_i}^{\mathbf{P}_{i+1}} T\,dS, \tag{2}$$

where the integrations are along the arc of the curve $C(t)$, and the $X_i^\pm$ are certain quantities defined below in (8). Then, we shall show that $X_i^+ = X_i^-$ using the Rankine-Hugoniot equations, and consequently summing (2) over i will give the result (1).

To establish (2), we shall use a transport theorem for an integral along a curve $C(t)$ from $\mathbf{P}(t)$ to $\mathbf{Q}(t)$, where $C(t)$ is convected with the local flow velocity $\mathbf{u}$, but the ends $\mathbf{P}$ and $\mathbf{Q}$ do *not* necessarily move with the local gas velocity. We need this because in our application each $\mathbf{P}_i$ is determined by the intersection of a shock with $C(t)$, and so although the material curve is convected with the flow, $\mathbf{P}$ and $\mathbf{Q}$ are also sliding along it as it moves.

The result we require is

$$\frac{d}{dt}\int_{\mathbf{P}}^{\mathbf{Q}} \mathbf{a}.d\mathbf{x} = \mathbf{a}(\mathbf{Q}).\dot{\mathbf{Q}} - \mathbf{a}(\mathbf{P}).\dot{\mathbf{P}} + \int_{\mathbf{P}}^{\mathbf{Q}}\left(\frac{\partial \mathbf{a}}{\partial t} + (\nabla\wedge\mathbf{a})\wedge\mathbf{u}\right).d\mathbf{x}, \tag{3}$$

which is a slight modification of more familiar transport results in the literature (e.g., [2, p. 269]). To show (3), suppose at some time $t' = t + \delta t$ the point $\mathbf{P}$ has moved to $\mathbf{P}'$, and that the gas element that coincided with $\mathbf{P}$ at t has moved to $\mathbf{P}''$, as indicated in Figure 2. With similar notation for $\mathbf{Q}$, we then have

$$\int_{\mathbf{P}'}^{\mathbf{Q}'} \mathbf{a}(\mathbf{x},t').d\mathbf{x} - \int_{\mathbf{P}}^{\mathbf{Q}} \mathbf{a}(\mathbf{x},t).d\mathbf{x} = \int_{\mathbf{QQ''Q'}} \mathbf{a}(\mathbf{x},t).d\mathbf{x} + \int_{\mathbf{P'P''P}} \mathbf{a}(\mathbf{x},t).d\mathbf{x} \tag{4}$$

$$+\int_{\mathbf{P}'}^{\mathbf{Q}'} (\mathbf{a}(\mathbf{x},t') - \mathbf{a}(\mathbf{x},t)).d\mathbf{x} + \oint_{\mathbf{QPP''Q''Q}} \mathbf{a}(\mathbf{x},t).d\mathbf{x}. \tag{5}$$

The integral around the closed curve in the last term here can be written, by Stokes' theorem, as the flux of $\nabla\wedge\mathbf{a}$ across the surface swept out as the fluid curve $\mathbf{P}$,$\mathbf{Q}$ is convected through the fluid to $\mathbf{P}''$,$\mathbf{Q}''$. When δt is small, we can think of that surface as composed of individual elements with vector area $\mathbf{u}\delta t\wedge d\mathbf{x}$ as illustrated in Figure 2. Thus, when we divide by δt and let $\delta t \to 0$, we obtain

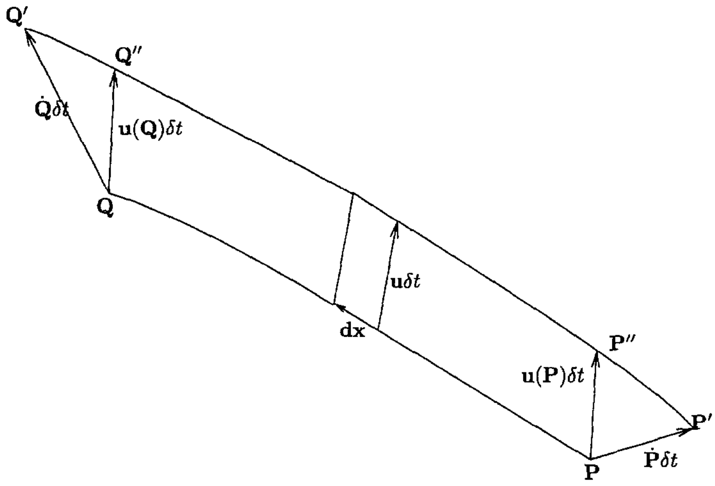

Figure 2. Transport of a material curve with sliding endpoints.

$$\frac{d}{dt}\int_{\mathbf{P}}^{\mathbf{Q}} \mathbf{a}.\mathbf{dx} = \mathbf{a}(\mathbf{Q}).\dot{\mathbf{Q}} - \mathbf{a}(\mathbf{P}).\dot{\mathbf{P}} + \int_{\mathbf{P}}^{\mathbf{Q}} \left(\frac{\partial \mathbf{a}}{\partial t}.\mathbf{dx} + (\nabla \wedge \mathbf{a}).\mathbf{u} \wedge \mathbf{dx}\right), \tag{6}$$

which is equivalent to (3) on rewriting the scalar triple product.

We now apply (3) to the left side of (2) so $\mathbf{a} = \mathbf{u}$, and in the integral on the right we put

$$\frac{\partial \mathbf{u}}{\partial t} + (\nabla \wedge \mathbf{u}) \wedge \mathbf{u} = -\frac{1}{\rho}\nabla p - \nabla\left(\frac{1}{2}|\mathbf{u}|^2\right) = -\nabla\left(H + \frac{1}{2}|\mathbf{u}|^2\right) + T\nabla S. \tag{7}$$

Here, we have used Euler's equation, and then introduced the enthalpy per unit mass H. We thus obtain exactly the required form (2) with

$$X_i^{\pm} = \mathbf{u}\left(\mathbf{P}_i^{\mp}\right).\dot{\mathbf{P}}_i - H\left(\mathbf{P}_i^{\mp}\right) - \frac{1}{2}\left|\mathbf{u}\left(\mathbf{P}_i^{\mp}\right)\right|^2, \tag{8}$$

where $\mathbf{u}(\mathbf{P}_i^{\pm})$, etc. denote values as $\mathbf{P}_i$ is approached from the $\mathbf{P}_{i\pm1}$-side. To show that $X_i^+ = X_i^-$, we focus on a single shock and drop the subscript i. Let the unit normal to the shock at $\mathbf{P}$ be $\mathbf{n}$, and the shock speed V, and then the condition that $\mathbf{P}$ remains on the shock is $\dot{\mathbf{P}}.\mathbf{n} = V$. So we can write

$$X_i^{\mp} = \left(\mathbf{u}\left(\mathbf{P}^{\pm}\right) \wedge \mathbf{n}\right).\left(\dot{\mathbf{P}} \wedge \mathbf{n}\right) + \left(\mathbf{u}\left(\mathbf{P}^{\pm}\right).\mathbf{n}\right)\left(\dot{\mathbf{P}}.\mathbf{n}\right) - H\left(\mathbf{P}^{\pm}\right) - \frac{1}{2}\left(\mathbf{u}\left(\mathbf{P}^{\pm}\right).\mathbf{n}\right)^2 - \frac{1}{2}\left|\mathbf{u}\left(\mathbf{P}^{\pm}\right) \wedge \mathbf{n}\right|^2 \tag{9}$$

$$= \left(\mathbf{u}\left(\mathbf{P}^{\pm}\right) \wedge \mathbf{n}\right).\left(\dot{\mathbf{P}} \wedge \mathbf{n}\right) - \left(H\left(\mathbf{P}^{\pm}\right) + \frac{1}{2}\left(\mathbf{u}\left(\mathbf{P}^{\pm}\right).\mathbf{n} - V\right)^2\right) + \frac{1}{2}V^2 - \frac{1}{2}\left|\mathbf{u}\left(\mathbf{P}^{\pm}\right) \wedge \mathbf{n}\right|^2. \tag{10}$$

Here, the first and last terms are unchanged across the shock, by continuity of tangential velocity, $\mathbf{u}(\mathbf{P}^+) \wedge \mathbf{n} = \mathbf{u}(\mathbf{P}^-) \wedge \mathbf{n}$. The second term is also unaltered across the shock, by the Rankine-Hugoniot energy condition, so $X_i^+ = X_i^-$ as required.

2. DISCUSSION, EXTENSIONS, AND LIMITATIONS

This result is obviously closely related to Kelvin's circulation theorem for barotropic flows, to Bjerknes' theorem, and to Crocco's theorem, but has the added feature of allowing shocks in the flow. We mention some potential applications.

(1) In the circumstances mentioned in the opening paragraph, when we apply (1) to any closed material curve in the incident gas, it shows that the circulation will only become of order ϵ^3 after crossing the weak shocks, and so the vorticity in the resulting flow is of order ϵ^3 as expected.

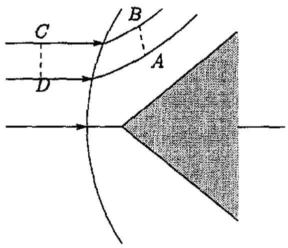

Figure 3. Steady flow with a curved shock.

(2) When a uniform supersonic gas flow is incident on a body and the resulting steady flow contains curved shocks, and therefore, entropy gradients behind the shocks, (1) gives an explicit result for the development of circulation.

For instance, in the situation shown in Figure 3, where DA and CB are streamlines, the circulation Γ around the material curve $ABCD$ develops according to

$$\frac{d\Gamma}{dt} = \int_A^B T\,dS. \tag{11}$$

Of course, Crocco's theorem already tells us that there must be nonzero vorticity behind the shock in this situation.

(3) Another potential use of the result would be as an additional accuracy check on numerical calculations.

The result remains valid if the gas is subject to a *conservative* force field such as gravity, since if U is the gravitational potential, we simply have an additional $-\nabla U$ on the right of Euler's equation, and this integrates around the closed curve $C(t)$ to 0.

We have implicitly assumed that $C(t)$ cuts the shocks transversely. If there is a time t^* for which $C(t^*)$ is somewhere tangential to a shock surface, then the number of shocks cutting $C(t)$ will generally change as t passes through t^*. However, t^* will typically be isolated, and (1) will hold on either side of t^*. But the circulation itself is continuous at t^*, so (1) still holds through t^*, provided that we allow the left- and right-derivatives to differ.

However, if the flow contains vortex sheets (surfaces of tangential discontinuity), then a more fundamental problem arises. Consider, for instance, a shock wave undergoing Mach reflection at a rigid wall, so there is a line L where the three shocks and the vortex sheet meet. If this line L crosses a closed material curve in the incident gas flow, then the curve will no longer be closed: after the first crossing it will form an open arc, whose two ends separate on either side of the vortex sheet, and after the second crossing it will become two open arcs. In this situation, (1) will be valid for a material curve that remains on one side or other of the vortex sheet, but otherwise will only be valid until L meets the curve.

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