

The density of rational points on non-singular hypersurfaces, II

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Abstract

For any integers $d, n \geq 2$, let $X \subset \mathbb{P}^n$ be a non-singular hypersurface of degree d that is defined over \mathbb{Q} . The main result in this paper is a proof that the number $N_X(B)$ of \mathbb{Q} -rational points on X which have height at most B satisfies

$$N_X(B) = O_{d,\varepsilon,n}(B^{n-1+\varepsilon}),$$

for any $\varepsilon > 0$. The implied constant in this estimate depends at most upon d, ε and n .

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1 Introduction

For any $n \geq 2$, let $F \in \mathbb{Z}[X_0, \dots, X_n]$ be a form of degree $d \geq 2$ that defines a non-singular hypersurface $X \subset \mathbb{P}^n$. In this paper we return to the theme of our recent investigation [8] into the distribution of rational points on such hypersurfaces. For any rational point $x = [\mathbf{x}] \in \mathbb{P}^n(\mathbb{Q})$ such that $\mathbf{x} = (x_0, \dots, x_n) \in \mathbb{Z}^{n+1}$ and $\text{h.c.f.}(x_0, \dots, x_n) = 1$, we shall write

$$H(x) = |\mathbf{x}| \tag{1.1}$$

for its height, where $|\mathbf{x}|$ denotes the norm $\max_{0 \leq i \leq n} |x_i|$. With this notation in mind, our primary objective is to understand the asymptotic behaviour of the quantity

$$N_X(B) = \#\{x \in X \cap \mathbb{P}^n(\mathbb{Q}) : H(x) \leq B\},$$

as $B \rightarrow \infty$. We have the following basic conjecture.

Conjecture 1. *Let $\varepsilon > 0$. Then we have*

$$N_X(B) = O_{d,\varepsilon,n}(B^{n-1+\varepsilon}). \tag{1.2}$$

Throughout our work the implied constant in any estimate is absolute unless explicitly indicated otherwise. In the case of Conjecture 1, for example, the constant is permitted to depend only upon d , ε and n . In view of the fact that \mathbf{x} and $-\mathbf{x}$ represent the same point in projective space, it is clear that

$$N_X(B) = \frac{1}{2} \#\{\mathbf{x} \in \mathbb{Z}^{n+1} : F(\mathbf{x}) = 0, \text{ h.c.f.}(x_0, \dots, x_n) = 1, |\mathbf{x}| \leq B\},$$

and so one may equally view Conjecture 1 as a statement about the frequency of integer solutions to certain homogeneous Diophantine equations. In particular, when F is a non-singular quadratic form in 4 variables, with discriminant equal to a square, we have $N_X(B) = c_X B^2 \log B(1 + o(1))$. In all other cases we would suppose that the exponent ε is superfluous in Conjecture 1. In fact there is a conjecture of Batyrev and Manin [2] that predicts one should have $N_X(B) = O_X(B^{n-1-\delta})$, for some $\delta > 0$, provided that $d \geq 3$ and $n \geq 4$.

Conjecture 1 is a special case of a conjecture due to the second author [13, Conjecture 2], which predicts that the same estimate should hold under the weaker assumption that the defining form F is absolutely irreducible. Both of these conjectures have received a significant amount of attention in recent years, to the extent that Conjecture 1 is now known for all values of $d \geq 2$ and $n \geq 2$, except possibly for the eight cases in which $d = 3$ or 4 and $n = 5, 6, 7$ or 8. This comprises the combined work of both the first and second authors [6, 7, 8, 12, 13], as summarised in [8, Corollary 1]. For the exceptional cases the best result available is the estimate $N_X(B) \ll_{d,\varepsilon,n} B^{n-1+\theta_d+\varepsilon}$, for any $\varepsilon > 0$, with

$$\theta_d = \begin{cases} 5/(3\sqrt{3}) - 3/4, & d = 3, \\ 1/12, & d = 4. \end{cases}$$

This follows from joint work of the authors with Salberger [9].

The aim of the present paper is to complete the proof of Conjecture 1, by offering a satisfactory treatment of the eight remaining cases. To this end we define the set

$$\mathcal{E} = \{(d, n) : d = 3 \text{ or } 4, n = 5, 6, 7 \text{ or } 8\}. \quad (1.3)$$

The following is our primary result.

Theorem 1. *Let $\varepsilon > 0$ and $(d, n) \in \mathcal{E}$, and suppose that $F \in \mathbb{Z}[X_0, \dots, X_n]$ is a non-singular form of degree d . Then we have*

$$N_X(B) = O_{d,\varepsilon,n}(B^{n-1+\varepsilon}).$$

Corollary. *Conjecture 1 holds in every case.*

The authors have recently learnt of work due to Salberger [18], which establishes Conjecture 1 in the case $d = 4$. In fact Salberger obtains the estimate (1.2) for any geometrically integral hypersurface $X \subset \mathbb{P}^n$ of degree $d \geq 4$, such that X contains at most finitely many linear subspaces of dimension $n - 2$.

Returning to the setting of non-singular hypersurfaces, the corollary to Theorem 1 is in some sense most significant when $d \leq n$, since it is precisely in this

setting that one expects X to contain a Zariski dense open subset of rational points, possibly defined over some finite algebraic extension of \mathbb{Q} . When $d \leq n$, the conjecture of Manin [16] predicts that one should have an asymptotic formula of the shape $N_U(B) = c_X B^{n+1-d}(1+o(1))$, as $B \rightarrow \infty$. Here $U \subseteq X$ is the open subset formed by deleting certain accumulating subvarieties from X , and c_X is a non-negative constant that has been given a conjectural interpretation by Peyre [17]. Viewed in this light, our main result is most impressive in the case $d = 3$ and $n \geq 4$, in which setting one ought to be able to take $U = X$ in Manin's conjecture.

The proof of Theorem 1 is based upon an application of [7, Theorem 4]. Broadly speaking this shows that every point counted by $N_X(B)$ must lie on one of a small number of linear subspaces contained in X , each of which is defined over \mathbb{Q} . Thus one is naturally led to study the Fano variety

$$F_m(X) = \{\Lambda \in \mathbb{G}(m, n) : \Lambda \subset X\}, \quad (1.4)$$

for $m \leq n - 1$, where $\mathbb{G}(m, n)$ denotes the Grassmannian parametrising m -dimensional linear subspaces $\Lambda \subset \mathbb{P}^n$. Perhaps the most basic example is provided by the case $m = 1$ and $d = n = 3$, for which it is well-known that $F_1(X)$ has dimension 0 and degree 27. The specific facts that we shall need are collected together in §3. It turns out that we have good control over the possible dimension of $F_m(X)$ when $m = 1$ or $m \geq (n - 1)/2$, the latter fact being made available to us by Professor Starr, in the appendix.

We end this introduction by summarising the contents of this paper. The following section is concerned with detailing a number of basic estimates that will be crucial to the proof of Theorem 1. In particular we shall need information about the growth rate of rational points on arbitrary projective varieties. In §3 we shall collect together some facts about the geometry of non-singular hypersurfaces, and the possible linear spaces that they contain. An overview of the proof of Theorem 1 will be given in §4, before being carried out in full within §§5–7.

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2 Preliminary estimates

In this section we collect together some of the basic estimates that we shall need during the course of our work. We begin with a rather easy result from linear programming, whose proof we include for the sake of completeness.

Lemma 1. *Let $H \geq 1$, and let $a, b, c \geq 0$. Then we have*

$$\max_{A, B, C} A^a B^b C^c \leq \max\{H^{(a+b+c)/3}, H^{(b+c)/2}, H^c\},$$

where the maximum on the left hand side is over all real numbers A, B, C for which $1 \leq A \leq B \leq C$ and $ABC \leq H$.

Proof. Let $M = \max A^a B^b C^c$ in the statement of Lemma 1. Then on writing $A = R$, $B = RS$ and $C = RST$, we see that

$$M = \max_{\substack{R, S, T \geq 1 \\ R^3 S^2 T \leq H}} R^{a+b+c} S^{b+c} T^c.$$

Suppose first that $a + b \leq 2c$ and $b \leq c$. Then it follows that

$$M \leq R^{3c} S^{2c} T^c \leq H^c.$$

Next if $a + b > 2c$ or $b > c$, then we substitute $T \leq HR^{-3}S^{-2}$ and deduce that

$$M \leq H^c \max_{\substack{R, S \geq 1 \\ R^3 S^2 \leq H}} R^{a+b-2c} S^{b-c} = M',$$

say. Now it is easy to see that the maximum in the definition of M' is achieved at $S = 1$ (resp. at $R = 1$) if $a + b > 2c$ and $b \leq c$ (resp. if $a + b \leq 2c$ and $b > c$). Thus

$$M' \leq \max\{H^{(a+b+c)/3}, H^{(b+c)/2}\}$$

in either of these two case. Finally if $a + b > 2c$ and $b > c$ then we substitute $S \leq H^{1/2}R^{-3/2}$ and deduce that

$$M' \leq H^{(b+c)/2} \max_{1 \leq R^3 \leq H} R^{a-b/2-c/2} \leq \begin{cases} H^{(b+c)/2}, & 2a \leq b + c, \\ H^{(a+b+c)/3}, & \text{otherwise.} \end{cases}$$

This completes the proof of Lemma 1. \square

We shall also need some facts about the density of rational points on arbitrary locally closed subsets $V \subset \mathbb{P}^N$. We henceforth write $V(\mathbb{Q}) = V \cap \mathbb{P}^N(\mathbb{Q})$ for the set of rational points on V , and recall the definition (1.1) of the projective height function $H : \mathbb{P}^N(\mathbb{Q}) \rightarrow \mathbb{R}_{>0}$, given $x = [\mathbf{x}] \in \mathbb{P}^N(\mathbb{Q})$ such that $\mathbf{x} = (x_0, \dots, x_N) \in \mathbb{Z}^{N+1}$ and $\text{h.c.f.}(x_0, \dots, x_N) = 1$. For any locally closed subset $V \subset \mathbb{P}^N$ and any $B \geq 1$, we define the counting function

$$N_V(B) = \#\{x \in V(\mathbb{Q}) : H(x) \leq B\}. \quad (2.1)$$

This coincides with our definition of $N_X(B)$ for a hypersurface $X \subset \mathbb{P}^n$. When V is a subvariety of \mathbb{P}^N we shall always assume that it is defined over $\overline{\mathbb{Q}}$. Furthermore we shall henceforth refer to such a variety as being integral if it is geometrically integral. We then have the following ‘‘trivial’’ estimate, which is established in [7, Theorem 1].

Lemma 2. *Let $V \subset \mathbb{P}^N$ be a variety of degree d and dimension m . Then we have*

$$N_V(B) = O_{d,N}(B^{m+1}).$$

It is easy to see that Lemma 2 is best possible when V contains a linear subspace of dimension m that is defined over \mathbb{Q} . On the assumption that V is integral and has degree $d \geq 2$, we can do somewhat better than Lemma 2. The following result is extracted from the the introduction to [9].

Lemma 3. *Let $\varepsilon > 0$ and suppose that $V \subset \mathbb{P}^N$ is an integral variety of degree $d \geq 2$ and dimension m . Then we have*

$$N_V(B) \ll_{d,\varepsilon,N} \begin{cases} B^{m+1/4+\varepsilon}, & \text{if } m \geq 4 \text{ and } 3 \leq d \leq 5, \\ B^{m+\varepsilon}, & \text{otherwise.} \end{cases}$$

It will be clear to the reader that when $m \geq 4$ and $3 \leq d \leq 5$, the main result in [9] actually allows one to take a sharper exponent in the statement of Lemma 3. In fact the exponent $m + \delta_d$ is acceptable for any

$$\delta_d > \begin{cases} 5/(3\sqrt{3}) - 3/4, & d = 3, \\ 1/12, & d \geq 4. \end{cases}$$

However it turns out that the estimate provided above is sufficient for the purposes of Theorem 1.

For non-negative integers $m \leq N$, let $\mathbb{G}(m, N)$ denote the Grassmannian which parametrises m -planes contained in \mathbb{P}^N . It is well-known that $\mathbb{G}(m, N)$ can be embedded in \mathbb{P}^ν via the Plücker embedding, where

$$\nu = \binom{N+1}{m+1} - 1,$$

and that $\mathbb{G}(m, N)$ has dimension $(m+1)(N-m)$. If $M \in \mathbb{G}(m, N)(\mathbb{Q}) = \mathbb{G}(m, N) \cap \mathbb{P}^\nu(\mathbb{Q})$, we define the height $H(M)$ of M to be the standard multiplicative height of its coordinates in $\mathbb{G}(m, N)$, under the Plücker embedding. The following result is well-known (see [5, §2.4], for example), and refines Lemma 2 in the case of linear varieties.

Lemma 4. *Let $M \in \mathbb{G}(m, N)(\mathbb{Q})$. Then we have*

$$N_M(B) \ll_N B^m + \frac{B^{m+1}}{H(M)}.$$

Moreover, if M contains $m+1$ linearly independent rational points of height at most B , then M is defined over \mathbb{Q} and

$$\frac{B^{m+1}}{H(M)} \ll_N N_M(B) \ll_N \frac{B^{m+1}}{H(M)}.$$

The following key result allows us to tackle Theorem 1 by restricting attention to the linear spaces that are contained in the hypersurface $F = 0$.

Lemma 5. *Let $\varepsilon > 0$ and suppose that $X \subset \mathbb{P}^n$ is an integral hypersurface of degree d . Then there exist linear spaces $M_1, \dots, M_J \subseteq X$ defined over \mathbb{Q} , with $J = O_{d,\varepsilon,n}(B^{n-1+\varepsilon})$, such that $0 \leq \dim M_j \leq n-1$ for $1 \leq j \leq J$ and*

$$N_X(B) \leq \sum_{j=1}^J N_{M_j}(B).$$

Moreover whenever $\dim M_j \geq 1$, for any $1 \leq j \leq J$, we have

$$H(M_j) = O_{d,\varepsilon,n}(B^{1+\varepsilon}).$$

Proof. Since X is integral there exists an absolutely irreducible form $F \in \overline{\mathbb{Q}}[X_0, \dots, X_n]$ of degree d , such that X is given by the equation $F = 0$. We first assume that F is not proportional to a form defined over \mathbb{Q} and let F^σ be the conjugate of F for any non-trivial $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Then clearly $N_X(B) = N_{X \cap X^\sigma}(B)$, where X^σ is the hypersurface $F^\sigma = 0$. Since $X \cap X^\sigma \subset \mathbb{P}^n$ is a variety of dimension at most $n-2$, we have

$$N_X(B) \ll_{d,n} B^{n-1},$$

by Lemma 2. Thus we may take M_1, \dots, M_J to be the collection of zero dimensional linear subspaces that correspond to precisely these points, in the statement of Lemma 5.

Suppose now that $F \in \mathbb{Z}[X_0, \dots, X_n]$, and let $\|F\|$ denote the maximum modulus of the coefficients of F . We claim that there exists a constant $c_{d,n}$ depending only on d and n , such that if $\log \|F\| > c_{d,n} \log B$, then there is a hypersurface $Y \subset \mathbb{P}^n$ of degree d , different from X , such that $N_X(B) = N_{X \cap Y}(B)$. But this is a direct consequence of [7, Lemma 3]. Thus the case in which $F \in \mathbb{Z}[X_0, \dots, X_n]$ has $\log \|F\| > c_{d,n} \log B$ is also satisfactory for Lemma 5, by Lemma 2.

Finally we suppose that $F \in \mathbb{Z}[X_0, \dots, X_n]$ and that $\log \|F\| \ll_{d,n} \log B$. But then a direct application of [7, Theorems 4 and 5] yields the result. The reader should note that [7] works with forms in $\mathbb{Z}[X_1, \dots, X_n]$, rather than in $\mathbb{Z}[X_0, \dots, X_n]$, so that the two values of the parameter n do not correspond. \square

Let $m \in \mathbb{N}$ and let $V \subset \mathbb{P}^N$ be an integral variety of degree d . Our final result in this section provides a crude upper bound for the dimension of the Fano variety

$$F_m(V) = \{\Lambda \in \mathbb{G}(m, N) : \Lambda \subset V\},$$

which parametrises the m -planes contained in V . In the next section we shall see how much more can be said when V is a non-singular hypersurface. The following estimate may certainly be extracted from the work of Segre [19], although we have provided our own proof for the sake of completeness.

Lemma 6. *We have*

$$\dim F_1(V) \leq \begin{cases} 2 \dim V - 2, & d = 1, \\ 2 \dim V - 3, & d \geq 2. \end{cases}$$

Proof. Let $\delta = \dim V$. In order to establish Lemma 6 we note that the case in which V is isomorphic to \mathbb{P}^δ is easy, since then $F_1(V) \cong \mathbb{G}(1, \delta)$. Assuming therefore that $d \geq 2$, we employ a routine incidence correspondence argument. Let Z be an integral component of $F_1(V)$, and let

$$\Sigma = \{(v, L) \in V \times Z : v \in L\}.$$

Then consideration of the projection onto the second factor shows that $\dim \Sigma = \dim Z + 1$. Now let $V_0 \subseteq V$ be the union of the lines in Z , and let v be a generic point of V_0 . The projection $\Sigma \rightarrow V$ then shows that $\dim \Sigma = \dim V_0 + \dim Z_v$, where $Z_v = \{L \in Z : v \in L\}$. Thus

$$\dim Z = \dim \Sigma - 1 = \dim V_0 - 1 + \dim Z_v \leq \dim V - 1 + \dim Z_v.$$

Now any line $L \in Z_v$ must also lie in the tangent space $\mathbb{T}_v(V_0)$, so that $Z_v \subseteq W_v$, where

$$W_v = \{L \in \mathbb{G}(1, N) : v \in L \subseteq \mathbb{T}_v(V_0)\}.$$

Since v is generic on V_0 it is non-singular, so that W_v is a linear space of dimension $\dim V_0 - 1$.

We proceed to consider two cases. If V_0 is a linear space then it must be a proper subvariety of V , since $d \geq 2$. In this case

$$\dim Z_v \leq \dim W_v = \dim V_0 - 1 \leq \dim V - 2,$$

and the required result follows. On the other hand if V_0 is not linear, then Z_v must be a proper subvariety of W_v , and

$$\dim Z_v \leq \dim W_v - 1 = \dim V_0 - 2 \leq \dim V - 2.$$

Again this suffices for the lemma. □

3 Geometry of non-singular hypersurfaces

For any $d \geq 3$ and $n \geq 4$, let $X \subset \mathbb{P}^n$ be a non-singular projective hypersurface of degree d . The aim of this section is to discuss the geometry of such hypersurfaces, and in particular the possible m -planes in \mathbb{P}^n that are contained in them. Such m -planes are parametrised by the Fano variety $F_m(X)$, given by (1.4). We begin by collecting together some preliminary facts about the degree and dimension of $F_m(X)$, for various values of $m \in \mathbb{N}$.

Lemma 7.

- (i) $F_m(X)$ has degree $O_{d,n}(1)$ when it is non-empty.
- (ii) $F_m(X)$ is empty for $m > (n - 1)/2$.
- (iii) $F_m(X)$ is finite if $m = (n - 1)/2$.

(iv) $F_1(X)$ has dimension $2n - 3 - d$ when $d \leq \min\{6, n\}$.

(v) $F_2(X)$ has dimension $3n - 16$ for $d = 3$ and $n \geq 6$.

Proof. The first part of Lemma 7 is well known, and follows from using the defining equation for X to write down the explicit equations for $F_m(X)$. In the appendix Professor Starr has provided proofs of (ii) and (iii). As indicated there it is very easy to establish part (ii), whereas the boundary case in part (iii) requires subtler methods. Part (iv) is the principal result in recent work of Beheshti [3], and part (v) follows from work of Izadi [15, Prop. 3.4]. \square

In the case $m = 1$ of lines, it is interesting to remark that a standard incidence correspondence argument (see Harris [11, §12.5], for example) reveals that $\dim F_1(X) = 2n - 3 - d$ for a generic hypersurface $X \subset \mathbb{P}^n$ of degree d . Debarre and de Jong have conjectured this to be the true dimension of $F_1(X)$ whenever $d \leq n$. When $m = 2$ the situation seems to be less well understood. While the dimension of $F_2(X)$ is $3n - 6 - (d+1)(d+2)/2$ for a generic hypersurface $X \subset \mathbb{P}^n$ of degree d , which tallies with part (v) of Lemma 7, there exist examples showing that this is not always the true dimension when $d \leq n$. Perhaps the simplest example is provided by the Fermat cubic in \mathbb{P}^5 which contains a finite number of planes. Nonetheless, we shall see below in Lemma 11 that it is possible to prove non-trivial upper bounds for the dimension of $F_2(X)$ when $d = 4$ and X is covered by planes.

It is convenient at this point to raise a question concerning the possible dimension of $F_m(X)$ in the sub-boundary case $m = \lfloor (n-1)/2 \rfloor$, where $\lfloor \alpha \rfloor$ denotes the integer part of $\alpha \in \mathbb{R}$. During the course of our work we have been led to formulate the following conjecture, the resolution of which would simplify the proof of Theorem 1 considerably.

Conjecture 2. *For any $n \geq 5$, let $X \subset \mathbb{P}^n$ be a non-singular hypersurface of degree $d \geq 3$, and let $m = \lfloor (n-1)/2 \rfloor$. Then X is not a union of m -planes.¹*

Conjecture 2 is already included in part (iii) of Lemma 7 when n is odd, since X contains at most finitely many linear subspaces of dimension m in this case. Further evidence is provided by part (v) of the same result. Indeed when $d = 3$ and $n = 6$ we see that $F_2(X)$ has dimension 2, so that $\bigcup_{P \in F_2(X)} P$ has dimension at most 4 and must be a proper subvariety of X . It is clear that Conjecture 2 can be false when $n \leq 4$, since for example a non-singular cubic threefold is covered by its lines.

For any $m \in \mathbb{N}$, and any subvariety $Z \subseteq F_m(X)$, we shall henceforth write

$$D(Z) = \bigcup_{\Lambda \in Z} \Lambda, \tag{3.1}$$

to denote the union of m -planes swept out by Z . It follows from part (ii) of Lemma 7 that $D(Z)$ is empty for $m > (n-1)/2$. Clearly $D(Z) \subseteq X$ and it

¹Since this paper was submitted for publication, the conjecture has been proved when $d \geq 4$ by Roya Beheshti [4].

is not hard to see that the degree of $D(Z)$ is bounded in terms of d, n and the degree of Z . We shall use these facts without further comment throughout this paper.

We proceed by discussing various cones of m -planes contained in X . For any subvariety $Z \subseteq F_m(X)$, and any $x \in X$, we set

$$Z_x = \{\Lambda \in Z : x \in \Lambda\}. \quad (3.2)$$

We shall write

$$C_x(Z) = D(Z_x)$$

for the corresponding cone of m -planes through x . In particular, since the degree of Z_x is bounded in terms of d, n and the degree of Z , it follows from the previous paragraph that the degree of $C_x(Z)$ is also bounded in terms of d, n and the degree of Z . We now observe that if a hypersurface of degree at least 2 is a cone, then its vertex is a singular point. Thus, since X is non-singular and $C_x(Z) \subseteq X$, we must have

$$\dim C_x(Z) \leq n - 2 \quad (3.3)$$

for any $x \in X$ and any subvariety $Z \subseteq F_m(X)$.

Suppose now that $m = 2$, and let $L \in F_1(X)$. Then analogously to (3.2), we define $Z_L = \{P \in Z : L \subset P\}$ and $C_L(Z) = D(Z_L)$ for any closed subset $Z \subseteq F_2(X)$. The following result corresponds to (3.3).

Lemma 8. *Let $d \geq 3$ and let $n \geq 5$. Then we have*

$$\dim C_L(Z) \leq n - 3,$$

for any $L \in F_1(X)$ and any subvariety $Z \subseteq F_2(X)$.

Proof. For fixed $L \in F_1(X)$ it will clearly suffice to establish the upper bound

$$\dim Z_L \leq n - 5, \quad (3.4)$$

under the assumption that $X \subset \mathbb{P}^n$ is a non-singular hypersurface of degree $d \geq 3$ and dimension $n-1 \geq 4$. Pick any distinct points $x, y \in L$ and let $P \in Z_L$. Then the system of inclusions $x \in L \subset P \subset X$ implies that $L \subset P \subset \mathbb{T}_x(X)$, where $\mathbb{T}_x(X) \cong \mathbb{P}^{n-1}$ is the tangent hyperplane to X at x . Similarly we have $L \subset P \subset \mathbb{T}_y(X)$. Hence it follows that

$$\begin{aligned} Z_L &= \{P \in Z : L \subset P \subset \mathbb{T}_x(X) \cap \mathbb{T}_y(X)\} \\ &= Z \cap \{P \in F_2(\mathbb{T}_x(X) \cap \mathbb{T}_y(X)) : L \subset P\} \\ &= Z \cap G_L, \end{aligned}$$

say.

We claim that it is possible to choose x and y in L so that the tangent spaces $\mathbb{T}_x(X)$ and $\mathbb{T}_y(X)$ are different. Suppose the hypersurface X is defined by the form $F(X_0, \dots, X_n)$, and let $x = [\mathbf{x}]$ and $y = [\mathbf{y}]$ be distinct points on L . If the

tangent space is the same for all points on L then there is a fixed vector \mathbf{v} , say, such that $\nabla F(\lambda\mathbf{x} + \mu\mathbf{y})$ is always a scalar multiple of \mathbf{v} . In fact we must have $\nabla F(\lambda\mathbf{x} + \mu\mathbf{y}) = h(\lambda, \mu)\mathbf{v}$ for a certain binary form h of degree $d-1$. Thus there is a pair $(\lambda, \mu) \neq (0, 0)$ for which $h(\lambda, \mu) = 0$. We therefore obtain a singular point on the variety X . This contradiction establishes our claim.

Now, with an appropriate choice of x and y , we see that

$$G_L \cong \{P \in \mathbb{G}(2, n-2) : L \subset P\} \cong \mathbb{P}^{n-4}.$$

In order to complete the proof of (3.4) it plainly suffices to show that G_L is not contained in Z . Arguing by contradiction we suppose that $G_L \subseteq Z$ and form the union of planes $D(G_L)$. But then $D(G_L) \subset X$ is a linear algebraic variety of dimension $n-2$, which is impossible by part (ii) of Lemma 7. \square

We now come to the most important lemma in this section — a kind of stratification result that will form the backbone of our proof of Theorem 1. Let $m \in \mathbb{N}$, let $\Phi \subseteq F_m(X)$ be an integral component and let $Y = D(\Phi) \subseteq X$ so that Y is also integral. We proceed by considering the incidence correspondence

$$I = \{(y, \Lambda) \in Y \times \Phi : y \in \Lambda\}. \quad (3.5)$$

Then the projection onto the first factor is surjective, and by projecting onto the second factor we see that I has dimension $\dim \Phi + m$. Thus it follows that

$$\dim \Phi_y = \dim \Phi - \dim Y + m \quad (3.6)$$

for generic $y \in Y$, in the notation of (3.2). The following result shows how the dimension of Φ_y varies for different choices of $y \in Y$. We henceforth employ the convention that the empty set is the only algebraic set with negative dimension.

Lemma 9. *Let $\Phi \subseteq F_m(X)$ be integral and let $Y = D(\Phi) \subseteq X$ have degree e . Then there exists a stratification of subvarieties*

$$Y = Z_0(\Phi) \supseteq Z_1(\Phi) \supseteq Z_2(\Phi) \supseteq \cdots,$$

such that the following holds.

(i) For $i \geq 1$ we have $\deg Z_i(\Phi) = O_{e,n}(1)$ and

$$\dim Z_i(\Phi) \leq \dim Y - 1 - i.$$

(ii) For $i \geq 0$ and any $y \in Z_i(\Phi) \setminus Z_{i+1}(\Phi)$ we have

$$\dim \Phi_y = \dim \Phi - \dim Y + m + i.$$

Proof. Throughout this proof we shall write δ for the dimension of Φ , and ε for the dimension of Y . Recall the definition (3.5) of the incidence correspondence I . We have already seen that I has dimension $\delta + m$, and that (3.6) holds for generic $y \in Y$. In order to make this more precise we define Y_k to be the set

of $y \in Y$ for which $\dim \Phi_y \geq k$, for any non-negative integer k . Then Y_k is a closed subset of Y and has degree $O_{e,n}(1)$. The first statement here follows from the upper semicontinuity of the dimension of the fibres of a morphism (see [11, Corollary 11.13], for example), while the latter fact can be extracted from an analysis of the proof of [11, Theorem 11.12]. On noting that $\dim \Phi_y \leq \delta$ for any $y \in Y$, it plainly follows that

$$\emptyset = Y_{\delta+1} \subseteq Y_\delta \subseteq \cdots \subseteq Y_0 = Y.$$

Moreover $Y_{\delta-\varepsilon+m+i}$ is a proper subvariety of Y for $i \geq 1$.

We shall take

$$Z_0(\Phi) = Y, \quad Z_i(\Phi) = Y_{\delta-\varepsilon+m+i},$$

for $i \geq 1$, in the statement of Lemma 9. Then part (ii) of the lemma is immediate, and it remains to provide an upper bound for the dimension of $Z_i(\Phi)$ when $i \geq 1$. For this we consider the incidence correspondence

$$I_i = \{(y, \Lambda) \in Z_i(\Phi) \times \Phi : y \in \Lambda\}.$$

Since (3.6) holds for generic $y \in Y$ it follows that $Z_i(\Phi)$ is a proper subvariety of Y for $i \geq 1$. Hence the generic m -plane $\Lambda \in \Phi$ cannot lie completely in $Z_i(\Phi)$. Now let π_2 be the projection from I_i to Φ . If π_2 is onto, then the generic fibre $\{y \in Z_i(\Phi) \cap \Lambda\}$ has dimension at most $m-1$, in which case $\dim I_i \leq \delta + m - 1$. On the other hand, if π_2 is not onto, then $\dim \pi_2(I_i) \leq \delta - 1$ and we again deduce that

$$\dim I_i \leq \delta + m - 1.$$

Turning to the projection to the first factor, we see that it is onto, since $Z_i(\Phi) \subseteq Y = D(\Phi)$, and hence

$$\dim I_i \geq \dim Z_i(\Phi) + \delta - \varepsilon + m + i.$$

Thus it follows that

$$\dim Z_i(\Phi) \leq \varepsilon - 1 - i$$

for $i \geq 1$, as required. This completes the proof of Lemma 9. \square

In the special case $m = 1$ and $Z = F_1(X)$, we can actually write down the equations defining the cone $C_x(Z)$ for any $x \in X$. Let $C_x^1 = C_x(F_1(X))$ for a point $x \in X$, which we assume without loss of generality is given by $x = [1, 0, \dots, 0]$. Then, after a further linear change of variables, X takes the shape

$$X_0^{d-1}X_1 + X_0^{d-2}F_2(X_1, \dots, X_n) + \cdots + F_d(X_1, \dots, X_n) = 0, \quad (3.7)$$

for forms F_i of degree i , for $2 \leq i \leq d$. If $x \in L \in F_1(X)$, then L must be contained in the tangent hyperplane $\mathbb{T}_x(X)$, which is given by the equation $X_1 = 0$. It follows that any point in C_x^1 must be of the form $[a, 0, b_2, \dots, b_n]$, and since $C_x^1 \subset X$ the polynomial

$$a^{d-2}F_2(0, b_2, \dots, b_n) + \cdots + F_d(0, b_2, \dots, b_n)$$

must vanish identically in a . We therefore deduce that

$$C_x^1 = \{[a, 0, \mathbf{b}] \in \mathbb{P}^n : F_2(0, \mathbf{b}) = \cdots = F_d(0, \mathbf{b}) = 0\}, \quad (3.8)$$

where $\mathbf{b} = (b_2, \dots, b_n)$. We see from (3.8) that $\dim C_x^1 \geq n - d$. Moreover it is plain that $C_x^m = C_x(F_m(X)) \subseteq C_x^1$ for any $m \in \mathbb{N}$.

When $d = 3$ and $x \in X$ is generic we can be even more precise about the cone C_x^1 as soon as n is large enough. Given a non-singular cubic hypersurface $X \subset \mathbb{P}^n$, Barth and Van de Ven [1, Prop. 5] have shown that $F_1(X)$ is a non-singular simply connected variety of dimension $2n - 6$, for $n \geq 6$. The fact that $F_1(X)$ is a non-singular variety of dimension $2n - 6$, for $n \geq 6$, is already present in the work of Clemens and Griffiths [10], where it is also shown that X is a union of lines. We may conclude that $F_1(X)$ is integral for $n \geq 6$, and we proceed to establish the following result.

Lemma 10. *Let $d = 3$ and let $n \geq 7$. Then for generic $x \in X$ the cone C_x^1 is integral and has degree 6.*

Proof. We begin by showing that C_x^1 has dimension $n - 3$ for generic $x \in X$. This is an easy consequence of (3.6), since we can take $\Phi = F_1(X)$ and $Y = X$. Thus for a generic point $x \in X$ one has

$$\dim \Phi_x = \dim F_1(X) - \dim X + 1 = n - 4,$$

by part (iv) of Lemma 7. It follows that $\dim C_x^1 = 1 + \dim \Phi_x = n - 3$, as claimed. We shall also make use of the fact for a non-singular cubic hypersurface $X \subset \mathbb{P}^n$, the Hessian H of X does not contain X as a subvariety. This is established by Hooley [14, Lemma 1], for example, and implies in particular that $H \cap X$ is a proper subvariety of X .

To find the degree of C_x^1 for generic $x \in X$ it is convenient to choose coordinates as in (3.7) and (3.8). Thus we may assume without loss of generality that $x = [1, 0, \dots, 0]$, and X is defined by the non-singular cubic form

$$F(X_0, \dots, X_n) = X_0^2 X_1 + X_0 Q(X_1, \dots, X_n) + C(X_1, \dots, X_n), \quad (3.9)$$

for some quadratic and cubic forms Q and C , respectively, and furthermore

$$C_x^1 = \{[a, 0, \mathbf{b}] \in \mathbb{P}^n : Q(0, \mathbf{b}) = C(0, \mathbf{b}) = 0\}.$$

Since C_x^1 has dimension $n - 3$, it follows that neither $Q(0, \mathbf{b})$ nor $C(0, \mathbf{b})$ vanish identically, and that C_x^1 is a complete intersection of pure dimension $n - 3$. In fact $Q(0, \mathbf{b})$ must be a non-singular quadratic form, since the Hessian of F at x is equal to $-4 \det(Q(0, \mathbf{b}))$, and $x \in X$ is generic. We shall think of C_x^1 as lying in \mathbb{P}^{n-1} , by identifying $[a, 0, \mathbf{b}]$ with $[a, \mathbf{b}]$.

In order to complete the proof of Lemma 10 it suffices to show that C_x^1 is reduced and irreducible, by Bézout's theorem. Suppose for a contradiction that $C_x^1 = Y_1 \cup Y_2$, for components Y_1, Y_2 of dimension $n - 3$ in \mathbb{P}^{n-1} . Then any point in the intersection $Y_1 \cap Y_2$ produces a singular point on C_x^1 , so that the singular

locus of C_x^1 has dimension at least $n - 5$. Now write $\nabla' = (\partial/\partial X_2, \dots, \partial/\partial X_n)$, and let V denote the set of $b = [\mathbf{b}] \in \mathbb{P}^{n-2}$ such that $Q(0, \mathbf{b}) = C(0, \mathbf{b}) = 0$ and $\nabla'Q(0, \mathbf{b})$ is proportional to $\nabla'C(0, \mathbf{b})$. Then V is an algebraic subvariety of \mathbb{P}^{n-2} , since the latter condition is defined by the vanishing of various 2×2 determinants. Moreover it is clear that $\dim V \geq n - 6$.

For $1 \leq i \leq n$, let C_i (resp. Q_i) denote the partial derivative $\partial Q/\partial X_i$ (resp. $\partial C/\partial X_i$). We proceed to define the map $\pi : V \rightarrow \mathbb{P}^{n-1}$, via

$$\pi : [\mathbf{b}] \longmapsto [-C_i(0, \mathbf{b}), Q_i(0, \mathbf{b})b_2, \dots, Q_i(0, \mathbf{b})b_n],$$

whenever $C_i(0, \mathbf{b}), Q_i(0, \mathbf{b})$ do not both vanish. It is not hard to check that π is well-defined, since $Q(0, \mathbf{b})$ is non-singular. We claim that $\pi(V)$ has dimension at least $n - 6$, for which it is clearly enough to show that π is generically injective. But if $[u, \mathbf{v}]$ is a generic point in the image $\pi(V)$, then we cannot have $\mathbf{v} = \mathbf{0}$ since $Q(0, \mathbf{b})$ is non-singular. Thus $[u, \mathbf{v}]$ determines $[\mathbf{b}]$, and so $\dim \pi(V) \geq n - 6$, as required. Now any point $[u, \mathbf{v}] \in \pi(V)$ must satisfy

$$Q(0, \mathbf{v}) = 0, \quad u\nabla'Q(0, \mathbf{v}) + \nabla'C(0, \mathbf{v}) = 0,$$

and it follows that the set W of such $[u, \mathbf{v}] \in \mathbb{P}^{n-1}$ has dimension at least $n - 6$. Finally we may conclude that the set of $[u, \mathbf{v}] \in W$ for which

$$u^2 + u \frac{\partial Q}{\partial X_1}(0, \mathbf{v}) + \frac{\partial C}{\partial X_1}(0, \mathbf{v}) = 0,$$

has dimension at least $n - 7$. Since $n \geq 7$, we produce at least one point $(u, 0, \mathbf{v})$ at which the form (3.9) is singular. This contradiction completes the proof of Lemma 10. \square

We now turn to the special case of planes contained in non-singular quartic hypersurfaces, with a view to proving the result alluded to in the paragraph after Lemma 7. With this in mind we have the following result.

Lemma 11. *Let $d = 4$ and let $n \geq 6$. Then for any integral component $\Phi \subseteq F_2(X)$ such that $X = D(\Phi)$, we have*

$$\dim \Phi \leq \begin{cases} 3, & n = 6, \\ 3n - 16, & n \geq 7. \end{cases}$$

Proof. As above we let $C_x^1 = C_x(F_1(X))$ denote the union of all lines contained in X that pass through a point $x \in X$. For generic $x \in X$ we claim that

$$\dim C_x^1 \leq n - 4, \tag{3.10}$$

and that if $n \geq 7$ the cone C_x^1 does not contain any linear space of dimension $n - 4$. The latter claim follows from the fact X contains at most finitely many $(n - 4)$ -planes by parts (ii) and (iii) of Lemma 7, so that a generic point of X cannot lie on such a subvariety. To see the first claim, one notes that if $\Psi \subseteq F_1(X)$ is any integral component such that $D(\Psi)$ is a proper subvariety of

X , then $C_x(\Psi)$ is empty for generic $x \in X$. Alternatively, if $\Psi \subseteq F_1(X)$ is an integral component such that $D(\Psi) = X$ then one combines (3.6) with part (iv) of Lemma 7, just as in the proof of Lemma 10, to deduce (3.10).

We proceed by considering the cone of planes $C_x(\Phi) = D(\Phi_x)$, for any $x \in X$. In particular $C_x(\Phi) \subseteq C_x^1$. Moreover if $H \in \mathbb{P}^{n*}$ is a generic hyperplane, then H does not contain x and it follows that $H \cap P$ is a line for every $P \in \Phi_x$. Thus there is a bijection between planes parametrised by Φ_x and lines contained in $H \cap C_x(\Phi)$, whence

$$\dim \Phi_x = \dim F_1(H \cap C_x(\Phi))$$

for any $x \in X$. Now let $x \in X$ be generic. Then on combining our observation that $C_x(\Phi) \subseteq C_x^1$ with (3.10), we conclude that

$$\dim \Phi_x \leq \dim F_1(H \cap C_x^1), \quad (3.11)$$

where $H \cap C_x^1 \subset \mathbb{P}^n$ is a variety of dimension at most $n-5$. Moreover, since H is generic, the only way that $H \cap C_x^1$ can contain an $(n-5)$ -plane is if C_x^1 contains an $(n-4)$ -plane. We have already seen that this is impossible when $x \in X$ is generic and $n \geq 7$. On applying Lemma 6 to (3.11) we therefore deduce that

$$\dim \Phi_x \leq 2(n-5) - 2 = 0$$

if $n = 6$, whereas

$$\dim \Phi_x \leq 2(n-5) - 3 = 2n - 13$$

if $n \geq 7$. To complete the proof of Lemma 11 it now suffices to apply (3.6) with $Y = X$ and $m = 2$. \square

It is likely that the upper bound in Lemma 11 is not best possible, and the problem of proving sharper versions seems to be an interesting question in its own right.

The final result of this section pertains to the special case $m = 3, n = 8$. We shall use the notation $C_x^3 = C_x(F_3(X))$ for $x \in X$. We have already seen in (3.3) that $\dim C_x^3 \leq n-2$. The following result investigates when the dimension of C_x^3 is maximal.

Lemma 12. *Let $d = 3$ or 4 and let $n = 8$. Suppose that $\dim C_x^3 = 6$ for some $x \in X$. Then we must have $d = 4$ and $\dim G_x = 3$, where*

$$G_x = \{T \in F_3(X) : x \in T\}.$$

Proof. Let $x \in X$ be such that $\dim C_x^3 = 6$, and note that $C_x^3 = D(G_x)$ in the notation of (3.1). In particular it follows that $\dim G_x \geq 3$. Assuming that $x = [1, 0, \dots, 0]$, and that X takes the shape (3.7), it follows from (3.3) and (3.8) that

$$C_x^3 \subseteq C_x^1 = \{[a, 0, \mathbf{b}] \in \mathbb{P}^8 : F_2(0, \mathbf{b}) = \dots = F_d(0, \mathbf{b}) = 0\},$$

where $\dim C_x^1 \leq 6$. If we define the variety

$$Y = \{[\mathbf{b}] \in \mathbb{P}^6 : F_2(0, \mathbf{b}) = \dots = F_d(0, \mathbf{b}) = 0\} \subset \mathbb{P}^6,$$

then it follows that Y must have dimension 5, since C_x^3 has dimension 6. Hence there exists a non-constant form $H \in \mathbb{Q}[X_2, \dots, X_8]$ such that for $2 \leq i \leq d$ we have $H \mid F_i(0, X_2, \dots, X_8)$. Thus the equation for X takes the shape

$$F(X_0, \dots, X_8) = X_1 J(X_0, \dots, X_8) + H(X_2, \dots, X_8) K(X_1, X_2, \dots, X_8) = 0,$$

for some further forms J and K such that $\deg J = d - 1$ and $\deg H + \deg K = d$. This is clearly impossible unless K is in fact a constant, since otherwise we may find a common solution to the system of equations $X_1 = J = H = K = 0$, and this will produce a singular point on X . Thus $F_2(0, \mathbf{b}), \dots, F_{d-1}(0, \mathbf{b})$ must all vanish identically. Taking the constant K to be 1, we then see that X is given by

$$F(X_0, \dots, X_8) = X_1 J(X_0, \dots, X_8) + H(X_2, \dots, X_8) = 0,$$

and that Y is a degree d hypersurface in \mathbb{P}^6 , given by $H(X_2, \dots, X_8) = 0$.

We proceed to show that Y is non-singular. In general we have

$$\nabla F = \left(X_1 \frac{\partial J}{\partial X_0}, J + X_1 \frac{\partial J}{\partial X_1}, X_1 \nabla' J + \nabla' H \right),$$

where

$$\nabla' = (\partial/\partial X_2, \dots, \partial/\partial X_8).$$

Now, given any non-zero vector $\mathbf{x} = (x_2, \dots, x_8)$ corresponding to a singular point on Y , we will have $\nabla' H(\mathbf{x}) = \mathbf{0}$. From this it follows from Euler's identity that $H(\mathbf{x}) = 0$. We can then find $y \in \mathbb{Q}$ such that $\nabla F(y, 0, \mathbf{x}) = \mathbf{0}$. This produces a singular point on X , which is impossible, thereby establishing that $Y \subset \mathbb{P}^6$ is a non-singular hypersurface of degree d . We shall write

$$Z = \{[0, 0, \mathbf{b}] \in \mathbb{P}^8 : [\mathbf{b}] \in Y\},$$

so that C_x^3 is a cone over Z .

Let $T \in G_x$ and define $\pi(T) = T \cap Z$. Then it is clear that $T \cap Z \subset Z$ must be a 2-plane for each $T \in G_x$, so that we have a map $\pi : G_x \rightarrow F_2(Z)$. Since $x \notin Z$, for each given $P \in F_2(Z)$ there is a unique 3-plane which contains P and also passes through x . It follows that π is a bijection. We may now deduce from Lemmas 7 and 11 that

$$\dim G_x = \dim F_2(Z) = \dim F_2(Y) \leq \begin{cases} 2, & d = 3, \\ 3, & d = 4, \end{cases}$$

since $Y \subset \mathbb{P}^6$ is a non-singular hypersurface of degree d and $D(F_2(Y)) = Y$. In particular this is impossible when $d = 3$ since we have already seen that $\dim G_x \geq 3$. This completes the proof of Lemma 12. \square

4 Proof of Theorem 1: the plan of campaign

In this section we set out our framework for the proof of Theorem 1. For any $(d, n) \in \mathcal{E}$, where \mathcal{E} is given by (1.3), let $F \in \mathbb{Z}[X_0, \dots, X_n]$ be a non-singular form of degree d . Then the equation $F = 0$ defines a non-singular

hypersurface $X \subset \mathbb{P}^n$ of degree d . Recall that we have been following the convention that the implied constant in any estimate is absolute unless explicitly indicated otherwise. Since the pairs (d, n) that occur in the remainder of our work will always be restricted to lie in the set \mathcal{E} , we may henceforth assume that $d, n = O(1)$.

At the heart of our argument is an application of Lemma 5. According to this result the points in which we are interested lie on $J = O_\varepsilon(B^{n-1+\varepsilon})$ linear subspaces $M_1, \dots, M_J \subset X$, all of which are defined over \mathbb{Q} . Subspaces of dimension zero are clearly satisfactory for Theorem 1, and so it remains to consider the case in which M_j has dimension $m \geq 1$. In particular we know that $H(M_j) = O_\varepsilon(B^{1+\varepsilon})$ in these cases. As an immediate corollary of parts (i)–(iii) of Lemma 7, it follows that there are only $O(1)$ linear subspaces $M_j \subset X$ for which $m \geq (n-1)/2$. Now Lemma 2 implies that any single m -plane contributes $O(B^{m+1})$ to $N_X(B)$, which is satisfactory for Theorem 1 since $m \leq n-2$. Hence it remains to handle the m -planes $M_j \subset X$ enumerated in Lemma 5, which have dimension

$$1 \leq m < (n-1)/2. \quad (4.1)$$

Since $n \leq 8$ for any $(d, n) \in \mathcal{E}$, we plainly have $m = 1, 2$ or 3 .

Let $1 \leq m < (n-1)/2$, and let Φ be an integral component of the Fano variety $F_m(X)$ of m -planes contained in X . When estimating the contribution from the m -planes parametrised by Φ , we shall always be able to assume that X is a union of m -planes parametrised by Φ , in the sense that $X = D(\Phi)$ in the notation of (3.1). To see that this is permissible we suppose that the closed set $D(\Phi)$ is a proper subvariety of X . Then the degree of $D(\Phi)$ is $O(1)$ by part (i) of Lemma 7, and so Lemma 2 shows that there is a contribution of $O(B^{n-1})$ to $N_X(B)$ from the points of height at most B that lie on $D(\Phi)$. This is satisfactory for Theorem 1. With this in mind we therefore write

$$\tilde{F}_m(X) \subseteq F_m(X)$$

for the union of integral components $\Phi \subseteq F_m(X)$ such that $D(\Phi) = X$.

When n is odd, the possibility that X may contain $(n-1)/2$ -planes is rather inconvenient, even though, as we have already seen, they make an acceptable contribution to $N_X(B)$. We shall write E for the union of such $(n-1)/2$ -planes. If Φ is an integral subvariety of $F_m(X)$ we then write Φ^* for the open subset of Φ obtained by removing any m -planes contained in E . When n is even, so that E is empty, we merely take $\Phi^* = \Phi$.

During the course of our work we shall have cause to estimate the number $N_Y(H)$ of rational points of height at most H , on various locally closed subsets $Y \subseteq X$, in the notation of (2.1). It will be convenient to combine here the information that we gleaned in the previous section about large dimensional linear subspaces of X , together with some of the estimates in §2. Given any $k \in \mathbb{N}$, we define

$$\beta_k = \begin{cases} 0, & k \leq 3, \\ 1/4, & k \geq 4. \end{cases}$$

We then introduce functions $\sigma_n, \tau_n : \mathbb{N} \rightarrow \mathbb{Q}$, given by

$$\sigma_n(k) = \begin{cases} k+1, & k \leq (n-1)/2, \\ k+\beta_k, & k > (n-1)/2, \end{cases} \quad (4.2)$$

and

$$\tau_n(k) = \begin{cases} k+1, & k < (n-1)/2, \\ k+\beta_k, & k \geq (n-1)/2. \end{cases} \quad (4.3)$$

When n is even it is plain that $\sigma_n(k) = \tau_n(k)$ for each $k \in \mathbb{N}$. The following result is a consequence of Lemmas 2, 3 and 7.

Lemma 13. *Let $\varepsilon > 0$ and $H \geq 1$, and suppose that $Y \subseteq X$ is a subvariety of dimension at most k , and degree e . Then we have*

$$(i) \quad N_Y(H) \ll_{\varepsilon, e} H^{\sigma_n(k)+\varepsilon}.$$

$$(ii) \quad N_{Y \setminus E}(H) \ll_{\varepsilon, e} H^{\tau_n(k)+\varepsilon}.$$

Now suppose that $n = 2m+1$ is odd, and let $k < m$. Assume that $\Phi \subseteq F_k(X)$ is integral, and that $\Lambda \in F_{k-1}(X)$ is given. Furthermore, let

$$\Phi_\Lambda = \{\Gamma \in \Phi : \Lambda \subset \Gamma\}, \quad \Phi_\Lambda^* = \{\Gamma \in \Phi^* : \Lambda \subset \Gamma\}.$$

Then with this notation in mind, we proceed by establishing the following result.

Lemma 14. *Let $\varepsilon > 0$ and $H \geq 1$, and suppose that $D(\Phi_\Lambda)$ has dimension $\ell \leq m$ and degree e . Then we have*

$$N_{D(\Phi_\Lambda^*)}(H) \ll_{\varepsilon, e} H^{\tau_n(\ell)+\varepsilon}.$$

Proof. Suppose that $\Pi \subseteq D(\Phi_\Lambda)$ for some m -plane Π , so that we must have $\ell = m$. Since $D(\Phi_\Lambda)$ is a cone with vertex Λ it follows that $\langle \Lambda, \Pi \rangle \subseteq D(\Phi_\Lambda)$, where $\langle \Lambda, \Pi \rangle$ is the linear space spanned by Λ and Π . Since $\dim \Pi = \dim D(\Phi_\Lambda) = m$ this leads to a contradiction unless $\Lambda \subseteq \Pi$.

If $D(\Phi_\Lambda)$ does not contain any m -planes, then the statement of Lemma 14 easily follows from part (ii) of Lemma 13. Alternatively suppose that $D(\Phi_\Lambda)$ contains an m -plane Π , say, so that in particular $\ell = m$. Now let $x \in \Pi \subseteq D(\Phi_\Lambda)$ be any point with $x \notin \Lambda$. We claim that $x \notin D(\Phi_\Lambda^*)$. But this follows from the observation that the k -plane $\langle x, \Lambda \rangle$ is contained in Π , so that $\langle x, \Lambda \rangle \notin \Phi_\Lambda^*$, and the claim follows. We have therefore shown that

$$N_{D(\Phi_\Lambda^*)}(H) \leq N_U(H) + N_\Lambda(H),$$

where $U \subseteq D(\Phi_\Lambda)$ denotes the open subset formed by deleting all of the m -planes from $D(\Phi_\Lambda)$. But then a simple application of part (ii) of Lemma 13 yields the required bound. This completes the proof of Lemma 14. \square

Before embarking on the main thrust of the argument for Theorem 1 we take this opportunity to give an overview of the method. Given an integral component $\Phi \subseteq \tilde{F}_m(X)$, the key idea will be to estimate the contribution from

the m -planes $\Lambda \in \Phi$ defined over \mathbb{Q} , according to their smallest generator. Let $\Lambda \in \mathbb{G}(m, n)$ be any m -plane that is defined over \mathbb{Q} . Then it is well known that Λ contains linearly independent points $a_0, \dots, a_m \in \mathbb{P}^n(\mathbb{Q})$ such that

$$H(a_0) \leq \dots \leq H(a_m), \quad H(\Lambda) \ll \prod_{i=0}^m H(a_i) \ll H(\Lambda).$$

We shall call a rational point on Λ a “smallest generator” for Λ if it has minimal height. Any smallest generator for Λ obviously has height $O(H(\Lambda)^{1/(m+1)})$.

We now turn to the pivotal role that Lemma 9 plays in this work. In fact it helps us to estimate the contribution from the m -planes in $\Phi \subseteq \tilde{F}_m(X)$ according to their smallest generator, a_0 , say, in essentially two different ways. The first approach involves counting the total number of rational points of height at most B that lie on the cone $C_{a_0}(\Phi) = D(\Phi_{a_0})$. (Recall that this is the cone swept out by the m -planes contained in Φ and passing through a_0 .) This will be referred to as “counting by cones”, or the “CC-method” for short. In the second approach one counts the total number of m -planes of height $O_\varepsilon(B^{1+\varepsilon})$ that pass through a_0 , before then estimating the number of rational points of height at most B on each such m -plane. This will be referred to as “counting by linear spaces”, or the “CL-method” for short. In both approaches one then obtains a final estimate for the contribution to $N_X(B)$, by summing over all of the points $a_0 \in X(\mathbb{Q})$ of low height. In doing so we shall need to apply Lemma 9 with the choice $Y = X$, in order to be able to control the dimension of Φ_{a_0} as a_0 varies in X .

We take a moment to analyse the CL-method in more detail. It will become apparent that our implementation of this approach has a distinctly subtler nature than that of the CC-method. We shall focus upon the case $m = 2$ here, the same principle applying in simpler form for the case $m = 1$. Let

$$A_2 \geq A_1 \geq A_0 \geq 1, \tag{4.4}$$

and let $\Phi \subseteq \tilde{F}_2(X)$ be an integral component as above. The method begins by fixing a point $a_0 \in X(\mathbb{Q})$, which has height $A_0/2 < H(a_0) \leq A_0$. One then considers the possible rational points a_1 in the cone $C_{a_0}(\Phi)$, which have height $A_1/2 < H(a_1) \leq A_1$. This then fixes the smallest two generators of the planes we wish to estimate the contribution from. Finally, for fixed a_0, a_1 one forms the variety

$$\Phi_{a_0, a_1} = \{P \in \Phi_{a_0} : a_1 \in P\}$$

of planes in Φ which contain the line $\langle a_0, a_1 \rangle$, and then estimates the number of $a_2 \in D(\Phi_{a_0, a_1})$ such that $A_2/2 < H(a_2) \leq A_2$. Each such value of a_2 determines a plane $P = \langle a_0, a_1, a_2 \rangle \in \Phi$, which it may be assumed has height

$$A_0 A_1 A_2 \ll H(P) \ll A_0 A_1 A_2.$$

One then employs Lemma 4 to estimate the number of rational points of height at most B contained in P . We need to control the dimension of Φ_{a_0} as a_0 varies

in X . Similarly, for fixed a_0 , we need to control the dimension of Φ_{a_0, a_1} as a_1 varies in $C_{a_0}(\Phi)$. In the first case we shall apply Lemma 9 with $Y = X$ as indicated above, and in the second we shall apply the same lemma, but with $Y = D(\Psi)$, where Ψ is an integral component of Φ_{a_0} . Ultimately, since one is only interested in planes of height $O_\varepsilon(B^{1+\varepsilon})$, one sums over dyadic intervals for A_0, A_1, A_2 such that (4.4) holds and $A_0 A_1 A_2 \ll_\varepsilon B^{1+\varepsilon}$. When $m = 1$ the CL-method is plainly simpler since we need only apply Lemma 9 once.

The astute reader will notice that there is a degree of waste in the above description of the CL-method. Thus for fixed $a_0 \in X(\mathbb{Q})$, whereas we are only interested in the number of lines of height $A_0 A_1$ that pass through a_0 , we are actually counting the total number of possible a_1 that serve as generators for the line $\langle a_0, a_1 \rangle$ of height $A_0 A_1$. In fact Lemma 4 implies that any such line contains $\gg A_1/A_0$ rational points a_1 such that $H(a_1) \leq A_1$. Hence it follows that the total number of rational lines of height $A_0 A_1$ contained in Φ_{a_0} is

$$\ll \frac{A_0}{A_1} \#\{a_1 \in C_{a_0}(\Phi)(\mathbb{Q}) : H(a_1) \leq A_1\}, \quad (4.5)$$

in which the first factor allows for the ‘‘over-counting’’ inherent in our method. A similar phenomenon occurs when a_0, a_1 are fixed and one is counting the number of planes of height $A_0 A_1 A_2$ that pass through the line $\langle a_0, a_1 \rangle$. Thus an application of Lemma 4 reveals that each such plane contains $\gg A_2^2/A_0 A_1 \gg A_2/A_0$ suitable points a_2 , whence the total number of planes of height $A_0 A_1 A_2$ contained in Φ_{a_0, a_1} is

$$\ll \frac{A_0}{A_2} \#\{a_2 \in D(\Phi_{a_0, a_1})(\mathbb{Q}) : H(a_2) \leq A_2\}. \quad (4.6)$$

In summary the CL-method has two main ingredients: a stratification argument involving Lemma 9, and a means of rectifying the over-counting that our implementation of the CL-method engenders.

We now have all of the tools with which to complete the proof of Theorem 1. At this stage it is convenient to make a certain hypothesis concerning the quantity $N_X(B)$.

Hypothesis (Projective hypersurface hypothesis). *Let $\varepsilon > 0$ and let $d, n \geq 2$ be integers. Then there exists $\theta_{d, n} \geq 0$ such that*

$$N_X(B) = O_{d, \varepsilon, n}(B^{n-1+\theta_{d, n}+\varepsilon}),$$

for any non-singular hypersurface $X \subset \mathbb{P}^n$ of degree d , that is defined over \mathbb{Q} .

We shall henceforth write $\text{PHH}[\theta_{d, n}]$ to denote the projective hypersurface hypothesis holding with exponent $\theta_{d, n}$. Thus Conjecture 1 is the statement that $\text{PHH}[0]$ holds, and it follows from Lemma 3 that $\text{PHH}[1/4]$ holds. For our purposes it will actually suffice to note that $\text{PHH}[1]$ holds, by Lemma 2.

For any integer m in the range (4.1), we let $X_m \subseteq X$ be the finite union of m -planes $M_j \subset X$ that are enumerated in Lemma 5, and that are parametrised

by $\tilde{F}_m(X)$. In the notation of (2.1), we shall write $N_{X_m}(B)$ for the overall contribution to $N_X(B)$ from the points contained in X_m . Our main task is to prove the following result.

Proposition 1. *Let $(d, n) \in \mathcal{E}$ and let $1 \leq m < (n - 1)/2$. Then we have*

$$N_{X_m}(B) \ll_{\varepsilon} B^{n-1+\theta_{d,n}/2+\varepsilon},$$

provided that $\text{PHH}[\theta_{d,n}]$ holds.

Before establishing Proposition 1, we first indicate how it can be used to prove Theorem 1. Suppose that $\text{PHH}[\theta]$ holds for $\theta = \theta_{d,n} \geq 0$. Then it easily follows from Proposition 1 and our work at the start of this section that

$$N_X(B) \ll_{\varepsilon} B^{n-1+\theta/2+\varepsilon}.$$

But then we see that $\text{PHH}[\theta/2]$ holds. This allows us to deduce the sharper bound

$$N_X(B) \ll_{\varepsilon} B^{n-1+\theta/4+\varepsilon},$$

whence in fact $\text{PHH}[\theta/4]$ holds. By continuing to iterate this procedure sufficiently many times we may clearly conclude that $\text{PHH}[\varepsilon]$ holds for any given $\varepsilon > 0$. This completes the proof of Theorem 1, upon revising the choice of ε .

For the remainder of this paper we shall assume that $\text{PHH}[\theta]$ holds for some $\theta \geq 0$, and that this value of θ is identical for each $(d, n) \in \mathcal{E}$. Furthermore, we shall follow common practice and allow the small positive constant ε to take different values at different parts of the argument.

5 Proof of Theorem 1: lines

We begin the proof of Proposition 1 by handling the contribution from the lines $M_j \subset X_1$, for which we shall employ the CL-method that was introduced in §4. Let $\Phi \subseteq \tilde{F}_1(X)$ be any integral component. On applying Lemma 9 with $Y = X$ we obtain a stratification of subvarieties $X = Z_0(\Phi) \supset Z_1(\Phi)$ such that $\deg Z_1(\Phi) = O(1)$ and

$$\dim Z_1(\Phi) \leq n - 3, \tag{5.1}$$

and

$$\dim \Phi_y = \dim \Phi - n + 2, \tag{5.2}$$

for any $y \in X \setminus Z_1(\Phi)$.

Considering the component $\Phi \subseteq \tilde{F}_1(X)$ as being fixed, it will be convenient to write $Z_1 = Z_1(\Phi)$. Our plan will be to sort the lines in Φ according to whether their smallest generator lies in $X \setminus Z_1$, or in Z_1 . Thus we shall write $M_0(B)$ for the overall contribution to $N_{X_1}(B)$ from the lines contained in X_1 that are parametrised by Φ and have smallest generator $a_0 \in X \setminus Z_1$, and we shall write $M_1(B)$ for the corresponding contribution from the lines with smallest generator

$a_0 \in Z_1$. Now there are $O(1)$ possible integral components of $\tilde{F}_1(X)$. In order to establish Proposition 1 in the case $m = 1$ it will therefore suffice to show that

$$M_i(B) = O_\varepsilon(B^{n-1+\theta/2+\varepsilon}),$$

for $i = 0, 1$. Recall that any line contained in X_1 has height $O_\varepsilon(B^{1+\varepsilon})$. For any real numbers A_0, A_1 such that

$$A_1 \geq A_0 \geq 1, \quad A_0 A_1 \ll_\varepsilon B^{1+\varepsilon}, \quad (5.3)$$

let $M_i(B; A_0, A_1)$ denote the contribution to $M_i(B)$ from the lines of height $A_0 A_1$ that pass through rational points a_0 and a_1 , such that a_0 is a smallest generator for the line and

$$A_0/2 < H(a_0) \leq A_0, \quad A_1/2 < H(a_1) \leq A_1.$$

On summing over $O_\varepsilon(B^\varepsilon)$ dyadic intervals for A_0, A_1 , it will therefore suffice to show that

$$M_i(B; A_0, A_1) = O_\varepsilon(B^{n-1+\theta/2+\varepsilon}), \quad (5.4)$$

for $i = 0, 1$, and each choice of A_0, A_1 such that (5.3) holds.

We begin by establishing (5.4) in the case $i = 0$. For any $a_0 \in X \setminus Z_1$, recall the definitions of the cones Φ_{a_0} and $C_{a_0}(\Phi) = D(\Phi_{a_0})$, as given by (3.1) and (3.2). Since $(d, n) \in \mathcal{E}$, it follows from part (iv) of Lemma 7 that Φ has dimension at most $2n - 3 - d$. Hence, we deduce from (5.2) that

$$\dim C_{a_0}(\Phi) \leq n - d.$$

Moreover we have already seen that $\deg C_{a_0}(\Phi) = O(1)$. We claim that

$$\#\{a_1 \in C_{a_0}(\Phi^*)(\mathbb{Q}) : H(a_1) \leq A_1\} = O_\varepsilon(A_1^{n-3+\varepsilon}),$$

where $C_{a_0}(\Phi^*)$ is the union of those lines in Φ^* that pass through a_0 . This follows from Lemma 2 when $d = 4$. If $d = 3$ and $n \geq 7$ we know from Lemma 10 that $C_{a_0}(\Phi) = C_{a_0}^1$ is integral and has degree 6, so that the result follows from Lemma 3. Finally for the case $(d, n) = (3, 6)$, we apply part (ii) of Lemma 13, and for the case $(d, n) = (3, 5)$ we apply Lemma 14.

In order to estimate the total number of rational lines of height $A_0 A_1$ that are parametrised by $\Phi_{a_0}^*$, we employ the over-counting argument used in (4.5) to deduce that there are

$$\ll \frac{A_0}{A_1} \#\{a_1 \in C_{a_0}(\Phi^*)(\mathbb{Q}) : H(a_1) \leq A_1\} \ll_\varepsilon \frac{A_0}{A_1} A_1^{n-3+\varepsilon} = A_0 A_1^{n-4+\varepsilon}$$

such lines. Moreover Lemma 4 implies that any line of height $A_0 A_1$ contains $O_\varepsilon(B^{2+\varepsilon}/(A_0 A_1))$ points of height at most B , since $A_0 A_1 \ll_\varepsilon B^{1+\varepsilon}$ by (5.3). Putting all of this together we therefore obtain

$$\begin{aligned} M_0(B; A_0, A_1) &\ll_\varepsilon B^{n-1} + \sum_{\substack{a_0 \in (X \setminus Z_1)(\mathbb{Q}) \\ H(a_0) \leq A_0}} A_0 A_1^{n-4+\varepsilon} \cdot \frac{B^{2+\varepsilon}}{A_0 A_1} \\ &\ll_\varepsilon B^{n-1} + A_0^{n-1+\theta} A_1^{n-5} B^{2+\varepsilon}. \end{aligned}$$

Thus (5.3) yields

$$M_0(B; A_0, A_1) \ll_\varepsilon B^{n-1} + A_0^{4+\theta} B^{n-3+\varepsilon} \ll_\varepsilon B^{n-1+\theta/2+\varepsilon},$$

which thereby establishes (5.4) in the case $i = 0$.

We now turn to the proof of (5.4) for $i = 1$. Let A_0, A_1 be real numbers such that (5.3) holds. For any $a_0 \in Z_1$, it follows from (3.3) that $\dim C_{a_0}(\Phi) \leq n-2$, and from (5.1) that Z_1 has dimension at most $n-3$. Recall the definitions (4.2) and (4.3) of σ_n and τ_n , respectively. Then part (i) of Lemma 13 implies that Z_1 contains $O_\varepsilon(A_0^{\sigma_n(n-3)+\varepsilon})$ rational points of height A_0 , and that $C_{a_0}(\Phi)$ contains $O_\varepsilon(A_1^{\sigma_n(n-2)+\varepsilon})$ rational points of height A_1 . We therefore obtain the estimate

$$\begin{aligned} M_1(B; A_0, A_1) &\ll_\varepsilon B^{n-1} + A_0^{\sigma_n(n-3)} \frac{A_0}{A_1} \cdot A_1^{\sigma_n(n-2)} \frac{B^{2+\varepsilon}}{A_0 A_1} \\ &\ll_\varepsilon B^{n-1} + A_0^{\sigma_n(n-3)} A_1^{\sigma_n(n-2)-2} B^{2+\varepsilon}, \end{aligned}$$

for $i \geq 1$. Suppose first that $n = 5$, so that $\sigma_n(n-3) = \sigma_n(n-2) = 3$. Then it follows from (5.3) that

$$M_1(B; A_0, A_1) \ll_\varepsilon B^4 + A_0^3 A_1 B^{2+\varepsilon} \ll_\varepsilon B^{4+\varepsilon},$$

which is satisfactory for (5.4). If however $n \geq 6$, then

$$\sigma_n(n-3) \leq n - 11/4, \quad \sigma_n(n-2) = n - 7/4,$$

and we see that for sufficiently small $\varepsilon > 0$ we have

$$M_1(B; A_0, A_1) \ll_\varepsilon B^{n-1} + A_0^{n-11/4} A_1^{n-15/4} B^{2+\varepsilon} \ll B^{n-1}.$$

This completes the proof of (5.4), and so the proof of Proposition 1 for $m = 1$.

6 Proof of Theorem 1: planes

Next we consider the case $m = 2$ of planes $M_j \subset X_2$, for $(d, n) \in \mathcal{E}$. In particular we may henceforth assume that $n \geq 6$, since $m < (n-1)/2$ by (4.1). We begin by dispatching the case $n = 6$, for which we may assume that $d = 4$. Indeed we have already observed in the context of Conjecture 2 that a non-singular cubic hypersurface $X \subset \mathbb{P}^6$ is not a union of planes, so that in particular $\tilde{F}_2(X)$ is empty. Assuming therefore that $(d, n) = (4, 6)$, and that $\tilde{F}_2(X)$ is non-empty, we proceed by employing a simple version of the CL-method that was outlined in §4. The planes P in which we are interested have height $H(P) = O_\varepsilon(B^{1+\varepsilon})$, so any smallest generator for P must have height $O_\varepsilon(B^{1/3+\varepsilon})$. Now it follows from Lemma 11 that $\tilde{F}_2(X)$ has dimension at most 3. If $\Phi \subseteq \tilde{F}_2(X)$ is any integral component then Lemma 9 yields a stratification $X = Z_0(\Phi) \supset Z_1(\Phi)$, such that $\deg Z_1(\Phi) = O(1)$, $\dim Z_1(\Phi) \leq 3$, and

$$\dim \Phi_y = \dim \Phi - 5 + 2 \leq 0 \tag{6.1}$$

for any $y \in X \setminus Z_1(\Phi)$. We can and will assume that the component Φ is fixed, so that we may write $Z_1 = Z_1(\Phi)$ for convenience.

We shall write $M_0(B)$ for the overall contribution to $N_{X_2}(B)$, arising from the planes whose smallest generator lies in $X \setminus Z_1$. In this setting it follows from (6.1) that each point $a_0 \in X \setminus Z_1$ gives rise to only $O(1)$ planes in Φ_{a_0} , and each such plane contributes $O(B^3)$ by Lemma 2. Thus a further application of Lemma 2 yields

$$M_0(B) \ll \sum_{\substack{a_0 \in X(\mathbb{Q}) \\ H(a_0) \ll_\varepsilon B^{1/3+\varepsilon}}} B^3 \ll_\varepsilon (B^{1/3+\varepsilon})^6 B^3 \ll_\varepsilon B^{5+\varepsilon},$$

which is satisfactory for Proposition 1. Next we tackle the contribution $M_1(B)$ from the planes that have smallest generator $a_0 \in Z_1$. Lemma 4 implies that

$$M_1(B) \ll_\varepsilon \sum_{\substack{a_0 \in Z_1(\mathbb{Q}) \\ H(a_0) \ll_\varepsilon B^{1/3+\varepsilon}}} \sum_R \sum_{\substack{P \in \Phi_{a_0}(\mathbb{Q}) \\ R/2 < H(P) \leq R}} \frac{B^{3+\varepsilon}}{R},$$

where the summation over R is over $O_\varepsilon(B^\varepsilon)$ dyadic intervals for $R \ll_\varepsilon B^{1+\varepsilon}$. We have seen that Z_1 has dimension at most 3, and we claim that Φ_{a_0} has dimension at most 2 for any $a_0 \in Z_1$. To see this we note that if $\Phi_{a_0} \subseteq \Phi$ had dimension 3 for some $a_0 \in X$, then it would follow that $\Phi_{a_0} = \Phi$, since Φ is integral and has dimension at most 3. However $D(\Phi_{a_0})$ is a cone with vertex a_0 , so that $D(\Phi_{a_0})$ cannot be a non-singular hypersurface of degree 2 or more. Since $D(\Phi) = X$ is non-singular we obtain a contradiction, which establishes the claim. In view of part (ii) of Lemma 7 we see that Φ_{a_0} does not contain any linear spaces of dimension 2. Indeed, by [11, p.123], even a line in $F_2(X)$ would produce a collection of planes which spanned a 3-plane, contradicting Lemma 7, part (ii). We also see that Z_1 does not contain any 3-planes. Hence Lemma 3 implies that

$$M_1(B) \ll_\varepsilon \sum_{\substack{a_0 \in Z_1(\mathbb{Q}) \\ H(a_0) \ll_\varepsilon B^{1/3+\varepsilon}}} \sum_{R \ll_\varepsilon B^{1+\varepsilon}} R^{1+\varepsilon} B^{3+\varepsilon} \ll_\varepsilon (B^{1/3+\varepsilon})^{3+\varepsilon} B^{4+\varepsilon} \ll_\varepsilon B^{5+\varepsilon}.$$

This too is satisfactory for Proposition 1, and so completes the proof of this result in the case $m = 2$ and $n = 6$.

For the rest of this section we shall assume that $n \geq 7$ so that together Lemmas 7 and 11 imply that

$$\dim \tilde{F}_2(X) \leq 3n - 16. \quad (6.2)$$

Let $\mathbf{A} = (A_0, A_1, A_2)$, with

$$A_2 \geq A_1 \geq A_0 \geq 1, \quad A_0 A_1 A_2 \ll_\varepsilon B^{1+\varepsilon}. \quad (6.3)$$

Then our objective is to estimate the contribution $N_{X_2}(B; \mathbf{A})$, say, from the planes $P \in \tilde{F}_2(X)$ with height of order $A_0 A_1 A_2$, that are generated by linearly

independent points $a_0, a_1, a_2 \in X(\mathbb{Q})$ such that

$$A_0/2 < H(a_0) \leq A_0, \quad A_1/2 < H(a_1) \leq A_1, \quad A_2/2 < H(a_2) \leq A_2. \quad (6.4)$$

In doing so it will plainly suffice to estimate the contribution from those planes that belong to Φ , for a fixed integral component $\Phi \subseteq \tilde{F}_2(X)$. Our plan will be to apply the CL-method, as described in §4.

Suppose that $a_0 \in X(\mathbb{Q})$ is a fixed point such that $A_0/2 < H(a_0) \leq A_0$, and let Ψ be an integral component of Φ_{a_0} . Then we are interested in the planes that are parametrised by Ψ , the union of which form a cone $D(\Psi)$. We claim that it suffices to assume that

$$\dim D(\Psi) \geq n - 3. \quad (6.5)$$

To see that this is permissible, we suppose for the moment that $D(\Psi)$ has dimension at most $n - 4$, and deduce from part (ii) of Lemma 13 that there are $O_\varepsilon(B^{\tau_n(n-4)+\varepsilon})$ rational points of height at most B contained in $D(\Psi) \setminus E$. Since $n \geq 7$ we have $\tau_n(n-4) \leq n-4+1/4$. Moreover there are $O(A_0^n)$ possible choices for a_0 by Lemma 2. On applying (6.3) we therefore obtain the overall contribution

$$\ll_\varepsilon (B^{1/3+\varepsilon})^n B^{n-4+1/4+\varepsilon} \ll_\varepsilon B^{4n/3-4+1/4+\varepsilon}$$

to $N_{X_2}(B, \mathbf{A})$ from this scenario. Once summed over dyadic intervals for the A_0, A_1, A_2 this is plainly satisfactory for Proposition 1, since $n \leq 8$. In fact, strictly speaking, summation over dyadic values of A_1, A_2 is unnecessary, since the above bound deals with all planes through a_0 . It therefore suffices to assume that (6.5) holds for each $a_0 \in X(\mathbb{Q})$ and each integral component Ψ of Φ_{a_0} .

We now apply Lemma 9 with $Y = D(\Psi)$. This produces a stratification of subvarieties

$$D(\Psi) = W_0(\Psi) \supseteq W_1(\Psi) \supseteq W_2(\Psi) \supseteq \dots,$$

such that $\deg W_i(\Psi) = O(1)$, with

$$\dim W_i(\Psi) \leq \dim D(\Psi) - 1 - i, \quad (i \geq 1), \quad (6.6)$$

and

$$\dim \Psi_y \leq \dim \Phi_{a_0} - \dim D(\Psi) + 2 + i, \quad (i \geq 0), \quad (6.7)$$

for any $y \in W_i(\Psi) \setminus W_{i+1}(\Psi)$. We proceed to estimate the contribution from those planes $P \in \Phi_{a_0}$, with height of order $A_0 A_1 A_2$, which are generated by linearly independent rational points a_0, a_1, a_2 such that (6.4) holds. We shall do this according to the value of $i \geq 0$ for which

$$a_1 \in W_i(\Psi) \setminus W_{i+1}(\Psi).$$

For each fixed value of a_1 , we have $a_2 \in C_{a_1}(\Psi) = D(\Psi_{a_1})$, which we suppose has dimension $\alpha_2 = \alpha_2(i; a_0, a_1, \Psi)$. In particular we must have $\alpha_2 \geq 2$ whenever Ψ_{a_1} is non-empty. Taken together with Lemma 14, part (ii) of Lemma 13 now shows that there are $O_\varepsilon(A_2^{\tau_n(\alpha_2)+\varepsilon})$ relevant points a_2 , where τ_n is given by (4.3).

By the over-counting argument used in (4.6) this produces $O_\varepsilon(A_0A_2^{\tau_n(\alpha_2)-1+\varepsilon})$ planes of height $A_0A_1A_2$ through the line $\langle a_0, a_1 \rangle$. In the special case $\alpha_2 = 2$ we may improve this, since there are $O(1)$ planes in Ψ_{a_1} . We therefore write

$$\mu_n(k) = \begin{cases} \tau_n(k), & k \geq 3, \\ 1, & k \leq 2, \end{cases} \quad (6.8)$$

and deduce that there are $\ll_\varepsilon A_0A_2^{\mu_n(\alpha_2)-1+\varepsilon}$ available planes.

Now write $\alpha_1 = \alpha_1(i; a_0, \Psi)$ for the dimension of $W_i(\Psi)$, so that Lemma 13, part (i), shows there to be $O_\varepsilon(A_1^{\sigma_n(\alpha_1)+\varepsilon})$ available points $a_1 \in W_i(\Psi) \setminus W_{i+1}(\Psi)$, where σ_n is given by (4.2). This time the over-counting argument used in (4.5) shows that there are

$$\ll_\varepsilon \frac{A_0}{A_1} \cdot A_1^{\sigma_n(\alpha_1)+\varepsilon} = A_0A_1^{\sigma_n(\alpha_1)-1+\varepsilon}$$

available lines corresponding to points $a_1 \in W_i(\Psi) \setminus W_{i+1}(\Psi)$. In conclusion we have therefore shown that for fixed $a_0 \in X$, and any fixed integral component $\Psi \subseteq \Phi_{a_0}$, the overall number of planes in Φ_{a_0} that have height of order $A_0A_1A_2$, and whose smallest generators are of order A_0, A_1, A_2 , respectively, is

$$\ll_\varepsilon A_0^2 A_1^{\sigma_n(\alpha_1)-1} A_2^{\mu_n(\alpha_2)-1+\varepsilon}, \quad (6.9)$$

for certain integers $\alpha_1, \alpha_2 \geq 0$.

Before going on to consider the corresponding number of points a_0 , we record some useful inequalities concerning the quantities α_1 and α_2 introduced above. Suppose first that $i \geq 1$ in the stratification. Then it follows from (6.6) and (6.7) that

$$\alpha_1 + \alpha_2 \leq \dim \Phi_{a_0} + 3, \quad (i \geq 1). \quad (6.10)$$

Alternatively, if $i = 0$ then we have $D(\Psi_{a_1}) \subseteq D(\Psi) = W_0(\Psi)$, leading to the inequality $\alpha_2 \leq \alpha_1$. Thus it follows from (6.5) and (6.7) that

$$2 \leq \alpha_2, n - 3 \leq \alpha_1, \quad \alpha_1 + \alpha_2 \leq \dim \Phi_{a_0} + 4, \quad (i = 0), \quad (6.11)$$

in this case. Here, as throughout the remainder of our work, the first set of inequalities is always taken to mean $2 \leq \min\{\alpha_2, n - 3\} \leq \max\{\alpha_2, n - 3\} \leq \alpha_1$. Finally we record the trivial inequalities

$$\alpha_1 \leq n - 2, \quad \alpha_2 \leq n - 3, \quad (i \geq 0), \quad (6.12)$$

that follow from (3.3) and Lemma 8. It is important to note that the inequalities (6.10)–(6.12) are valid for any choice of $a_0 \in X(\mathbb{Q})$. Moreover, there are clearly $O(1)$ possible integral components $\Psi \subseteq \Phi_{a_0}$. Hence we deduce that for fixed $a_0 \in X$ the number of planes in Φ_{a_0} that have height of order $A_0A_1A_2$ is given by (6.9), where $\alpha_1, \alpha_2 \geq 0$ satisfy (6.10) or (6.11), together with (6.12).

We now turn to the problem of estimating the number of possible smallest generators a_0 . Here we shall combine (6.2) with an application of Lemma 9 in the case $Y = X$. Thus there exists a stratification of subvarieties

$$X = Z_0(\Phi) \supseteq Z_1(\Phi) \supseteq Z_2(\Phi) \supseteq \cdots,$$

such that $\deg Z_j(\Phi) = O(1)$, with

$$\dim Z_j(\Phi) \leq n - 2 - j, \quad (j \geq 1), \quad (6.13)$$

and

$$\dim \Phi_y \leq 2n - 13 + j, \quad (j \geq 0), \quad (6.14)$$

for any $y \in Z_j(\Phi) \setminus Z_{j+1}(\Phi)$. We write $Z_j = Z_j(\Phi)$ for $j \geq 0$, for convenience. In keeping with the above, our plan is to estimate the overall contribution from the planes $P \in \Phi$ with height of order $A_0 A_1 A_2$, that are generated by points a_0, a_1, a_2 such that (6.4) holds. We shall classify such planes according to the value of $j \geq 0$ for which $a_0 \in Z_j \setminus Z_{j+1}$. Write $\alpha_0 = \alpha_0(j)$ for the dimension of Z_j , and let

$$\nu_{n,\theta}(k) = \begin{cases} \sigma_n(k), & k \leq n - 2, \\ k + \theta, & k = n - 1, \end{cases} \quad (6.15)$$

for $k \in \mathbb{N}$. Then it follows from part (i) of Lemma 13 and the fact that $\text{PHH}[\theta]$ holds, that the total number of available points a_0 is

$$\ll_{\varepsilon} A_0^{\nu_{n,\theta}(\alpha_0)+\varepsilon}.$$

On combining this with (6.9) we therefore conclude that the total number of planes under consideration is

$$\ll_{\varepsilon} A_0^{\nu_{n,\theta}(\alpha_0)+2} A_1^{\sigma_n(\alpha_1)-1} A_2^{\mu_n(\alpha_2)-1+\varepsilon}, \quad (6.16)$$

for certain integers $\alpha_0, \alpha_1, \alpha_2 \geq 0$.

We now collect together some of the inequalities satisfied by $\alpha_0, \alpha_1, \alpha_2$ as we range over values of $i, j \geq 0$ in our double stratification. Suppose first that $j \geq 1$. Then it follows from (6.13) that $\alpha_0 \leq n - 2 - j$, whence we may combine (6.10), (6.11) and (6.14) to deduce that

$$\alpha_0 + \alpha_1 + \alpha_2 \leq 3n - 12, \quad (i, j \geq 1), \quad (6.17)$$

and

$$2 \leq \alpha_2, n - 3 \leq \alpha_1, \quad \alpha_0 + \alpha_1 + \alpha_2 \leq 3n - 11, \quad (i = 0, j \geq 1). \quad (6.18)$$

Similarly, since $Z_0 = X$, we see that if $j = 0$ then (6.10), (6.11) and (6.14) combine to give

$$\alpha_0 = n - 1, \quad \alpha_0 + \alpha_1 + \alpha_2 \leq 3n - 11, \quad (i \geq 1, j = 0), \quad (6.19)$$

and

$$\alpha_0 = n - 1, \quad 2 \leq \alpha_2, n - 3 \leq \alpha_1, \quad \alpha_0 + \alpha_1 + \alpha_2 \leq 3n - 10, \quad (i = j = 0). \quad (6.20)$$

We are now ready to complete our treatment of the planes in Proposition 1. By Lemma 4 it follows that any plane of height $A_0 A_1 A_2$ contains

$$\ll B^2 + \frac{B^3}{A_0 A_1 A_2} \ll_{\varepsilon} \frac{B^{3+\varepsilon}}{A_0 A_1 A_2}$$

rational points of height at most B . On combining this with (6.16), we therefore conclude the proof of the following result.

Lemma 15. *Suppose that A_0, A_1, A_2 satisfy (6.3). Then there exists a triple $\alpha = (\alpha_0, \alpha_1, \alpha_2)$ of non-negative integers satisfying one of the conditions (6.17)–(6.20), together with (6.12), such that*

$$N_{X_2}(B; \mathbf{A}) \ll_{\varepsilon} A_0^{\nu_{n,\theta}(\alpha_0)+1} A_1^{\sigma_n(\alpha_1)-2} A_2^{\mu_n(\alpha_2)-2} B^{3+\varepsilon}.$$

Here $\nu_{n,\theta}, \sigma_n, \mu_n$ are given by (6.15), (4.2) and (6.8), respectively.

It remains to apply the linear programming result Lemma 1 in order to show that the bound in Lemma 15 is satisfactory for Proposition 1. In view of the inequalities (6.3) satisfied by A_0, A_1, A_2 , we deduce from Lemmas 1 and 15 that

$$N_{X_2}(B; \mathbf{A}) \ll_{\varepsilon} \left(M_1(B) + M_2(B) \right) B^{3+\varepsilon},$$

where

$$M_1(B) = B^{\mu_n(\alpha_2)-2} + B^{(\sigma_n(\alpha_1)+\mu_n(\alpha_2)-4)/2},$$

and

$$M_2(B) = B^{(\nu_{n,\theta}(\alpha_0)+\sigma_n(\alpha_1)+\mu_n(\alpha_2)-3)/3}.$$

On summing over dyadic intervals for the A_0, A_1, A_2 , we see that it suffices to show that

$$M_i(B) = O(B^{n-4+\theta/3}),$$

for $i = 1, 2$, in order to complete the proof of Proposition 1 in the case $m = 2$. We first establish the case $i = 1$ of this estimate. On recalling the definitions (4.2) and (6.8) of σ_n and μ_n , respectively, we may clearly apply (6.12) to deduce that

$$M_1(B) \leq B^{\mu_n(n-3)-2} + B^{(\sigma_n(n-2)+\mu_n(n-3)-4)/2} \leq B^{n-19/4} + B^{n-17/4}.$$

This is plainly satisfactory.

Turning to the estimate for $M_2(B)$, we have four different cases to consider. In each one our task is to show that

$$E_n(\alpha) = \frac{\nu_{n,\theta}(\alpha_0) + \sigma_n(\alpha_1) + \mu_n(\alpha_2)}{3} - 1 \leq n - 4 + \theta/3. \quad (6.21)$$

Suppose firstly that the triple α satisfies (6.17). Then it is trivial to see that $E_n(\alpha) \leq (3n - 9)/3 - 1 = n - 4$, which is satisfactory for (6.21). Next suppose that α satisfies (6.19). Then $\alpha_0 = n - 1$ and $\alpha_1 + \alpha_2 \leq 2n - 10$, from which it follows that $E_n(\alpha) \leq (3n - 9 + \theta)/3 - 1 = n - 4 + \theta/3$. This too is satisfactory for (6.21). In order to handle the remaining two cases in which α satisfies (6.18) or (6.20), it plainly suffices to assume that $\alpha_0 + \alpha_1 + \alpha_2 \geq 3n - 11$. It will be convenient to handle the cases $n = 7$ and $n = 8$ separately.

Let $n = 7$. Then we have

$$\nu_{7,\theta}(k) = \begin{cases} k + 1, & k \leq 3, \\ k + 1/4, & k \geq 4, \\ k + \theta, & k = 6, \end{cases}$$

$$\sigma_7(k) = \begin{cases} k + 1, & k \leq 3, \\ k + 1/4, & k \geq 4, \end{cases}$$

and

$$\mu_7(k) \leq \begin{cases} 1, & k = 2, \\ k + 1/4, & k \geq 3. \end{cases}$$

Beginning with the case (6.18), we see that $2 \leq \max\{\alpha_2, 4\} \leq \alpha_1$, and as indicated above we may assume that $\alpha_0 + \alpha_1 + \alpha_2 = 10$. If $\alpha_2 = 2$ then $\alpha_0 + \alpha_1 = 8$, and it follows that $E_7(\boldsymbol{\alpha}) \leq 3$. This is satisfactory for (6.21). Alternatively we have $\alpha_1 \geq 4$ and $\alpha_2 \geq 3$, so that $E_7(\boldsymbol{\alpha}) \leq 3$ in this case also. The remaining case (6.20) is impossible for $n = 7$, since we would have $\alpha_0 = 6$, $\alpha_1 \geq 4$ and $\alpha_2 \geq 2$, with $\alpha_1 + \alpha_2 \leq 5$.

We now turn to the case $n = 8$, in which setting we have

$$\nu_{8,\theta}(k) = \begin{cases} k + 1, & k \leq 3, \\ k + 1/4, & k \geq 4, \\ k + \theta, & k = 7, \end{cases}$$

and

$$\sigma_8(k) = \begin{cases} k + 1, & k \leq 3, \\ k + 1/4, & k \geq 4. \end{cases}$$

Moreover we see that $\mu_8(k) = \sigma_8(k)$ if $k \geq 3$, and $\mu_8(k) = 1$ otherwise. We begin with the case (6.18), for which we have $2 \leq \max\{\alpha_2, 5\} \leq \alpha_1$ and $\alpha_0 + \alpha_1 + \alpha_2 = 13$. In view of (6.12) we conclude that $\alpha_1 = 5$ or 6 , and at most one of α_0 or α_2 is less than 4. Hence we deduce that $E_8(\boldsymbol{\alpha}) \leq 4$ in this case, which is clearly satisfactory for (6.21). Finally we must handle the case in which (6.20) holds. Then $\alpha_0 = 7$, and we have the inequalities $2 \leq \max\{\alpha_2, 5\} \leq \alpha_1$ and $\alpha_1 + \alpha_2 \leq 7$. Thus either $\alpha_1 + \alpha_2 \leq 6$, in which case it is clear that $E_8(\boldsymbol{\alpha}) \leq 4 + \theta/3$, or else we must have $\boldsymbol{\alpha} = (7, 5, 2)$. One easily deduces that this final possibility is also satisfactory for (6.21). This completes the treatment of the planes.

7 Proof of Theorem 1: 3-planes

Our last task is to consider the case of 3-planes $M_j \subset X_3$, for $(d, n) \in \mathcal{E}$. It follows from (4.1) that we may henceforth assume that $n = 8$. It is worth highlighting that this section would be redundant were we to have a proof of Conjecture 2 in the cases $(d, n) = (3, 8)$ and $(4, 8)$, since then $\tilde{F}_3(X)$ would be empty. In the absence of such a proof there is still work to be done. We shall essentially employ a blend of the CC-method and the CL-method. For any $a \in X$ let $C_a^3 = C_a(\tilde{F}_3(X))$. As usual the idea will be to estimate the contribution from the 3-planes of height $O_\varepsilon(B^{1+\varepsilon})$ according to the value of their smallest generator, which will necessarily have height $O_\varepsilon(B^{1/4+\varepsilon})$. Beginning with the set of smallest generators a for which the corresponding cone C_a^3 has dimension at most 5, we obtain the contribution

$$\ll \sum_{\substack{a \in X(\mathbb{Q}) \\ H(a) \ll_\varepsilon B^{1/4+\varepsilon}}} \#\{x \in C_a^3(\mathbb{Q}) : H(x) \leq B\}$$

to $N_{X_3}(B)$. Since $\deg C_a^3 = O(1)$ for each $a \in X$, an application of part (i) of Lemma 13 yields the contribution

$$\ll_{\varepsilon} (B^{1/4+\varepsilon})^{7+\theta} B^{5+1/4+\varepsilon} \ll_{\varepsilon} B^{7+\theta/4+\varepsilon}.$$

This is clearly satisfactory for Proposition 1.

We now turn to the case in which the cone C_a^3 has dimension at least 6. By (3.3), the dimension is indeed precisely 6. Lemma 12 now shows that $d = 4$, and that the set

$$G_a = \{T \in \tilde{F}_3(X) : a \in T\}$$

has $\dim G_a = 3$. If G_a were to contain a 3-plane, this would produce a 4-plane in X , contradicting Lemma 7. Hence Lemma 3 implies that

$$\#\{T \in G_a(\mathbb{Q}) : H/2 < H(T) \leq H\} \ll_{\varepsilon} H^{3+\varepsilon}.$$

Moreover Lemma 4 implies that if $T \in G_a(\mathbb{Q})$ and $H/2 < H(T) \leq H$, then T contains $O(B^3 + B^4/H)$ rational points of height at most B . Hence for given $a \in X$ such that C_a^3 has dimension 6, there is an overall contribution of

$$\ll_{\varepsilon} \sum_{H \ll_{\varepsilon} B^{1+\varepsilon}} H^{3+\varepsilon} \cdot \frac{B^{4+\varepsilon}}{H} \ll_{\varepsilon} B^{6+\varepsilon}, \quad (7.1)$$

from the 3-planes of height $O_{\varepsilon}(B^{1+\varepsilon})$ that have smallest generator a . It remains to sum this over appropriate values of a .

Let $\Psi \subseteq \tilde{F}_3(X)$ be any integral component, so that in particular $D(\Psi) = X$, and consider the set of lines

$$\Phi = \{L \in F_1(X) : \exists T \in \Psi \text{ such that } L \subset T\}.$$

Then $\Phi \subseteq F_1(X)$ is an integral subvariety that covers X , in the sense that $D(\Phi) = X$. In particular it follows from Lemma 7 that

$$\dim \Phi \leq 9, \quad (7.2)$$

since $(d, n) = (4, 8)$. We then note that $D(\Phi_a) = D(\Psi_a)$ for any $a \in X$. Thus, for fixed sets Ψ and Φ , we are interested in the points $a \in X$ for which $D(\Phi_a)$ has dimension 6. To get a handle on this set we employ the stratification leading to (5.1) and (5.2), in conjunction with (7.2), to conclude that there are subvarieties

$$X = Z_0 \supseteq Z_1 \supseteq Z_2 \supseteq \cdots,$$

such that $\deg Z_i = O(1)$, with

$$\dim Z_i \leq 6 - i, \quad (i \geq 1),$$

and

$$\dim \Phi_a \leq 3 + i, \quad (i \geq 0), \quad (7.3)$$

for any $a \in Z_i \setminus Z_{i+1}$, for $i \geq 0$.

It now follows that if $\dim D(\Phi_a) = 6$ then $\dim \Phi_a = 5$, and hence that $a \in Z_i$ for some $i \geq 2$. Moreover we will have $\dim Z_i \leq 4$. Thus the overall contribution to $N_{X_3}(B)$ corresponding to points with $H(a) \leq A$ will be

$$\ll_{\varepsilon} A^{4+1/4+\varepsilon} B^{6+\varepsilon}$$

in view of part (i) of Lemma 13 and the estimate (7.1). On choosing $A = B^{4/17}$ we see that this gives a satisfactory treatment of 3-planes whose smallest generator has height at most $B^{4/17}$.

It now remains to deal with the case in which the smallest generator a of T has height larger than $B^{4/17}$. Now it may obviously be assumed that none of the 3-planes in which we are interested lie in Z_1 . Indeed Lemma 2 implies that there are at most $O(B^6)$ points of height at most B contained in Z_1 , which is satisfactory for Proposition 1. We have already seen in §4 that any 3-plane T defined over \mathbb{Q} contains linearly independent points $a_0, a_1, a_2, a_3 \in \mathbb{P}^8(\mathbb{Q})$ such that

$$H(a_0) \leq \dots \leq H(a_3), \quad H(T) \ll H(a_0)H(a_1)H(a_2)H(a_3) \ll H(T).$$

Since $H(a_0) \geq B^{4/17}$ and $H(T) \ll_{\varepsilon} B^{1+\varepsilon}$ it follows that $H(a_3) \ll_{\varepsilon} B^{5/17+\varepsilon}$. In particular $H(a_i) \ll B^{1/3}$ for $i = 0, \dots, 3$. We claim that there is a point $b \in T \setminus Z_1$ for which $H(b) \ll B^{1/3}$. Indeed there are $\gg L^4$ points

$$b = \lambda_0 a_0 + \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 \in T, \quad 1 \leq \lambda_i \leq L,$$

of which $O(L^3)$ can lie on the proper subvariety $T \cap Z_1$, by Lemma 2. Thus if L is a sufficiently large constant we may produce at least one point $b \in T \setminus Z_1$, with $H(b) \ll B^{1/3}$.

For each 3-plane T under consideration we now choose an appropriate point b . We then proceed to count points on T according to the corresponding values of b , noting that if $x \in T$ then $x \in C_b(\Psi) = D(\Psi_b)$. Since $b \notin Z_1$ we will have

$$\dim C_b(\Psi) \leq \dim C_b^1 = \dim D(\Phi_b) = 1 + \dim \Phi_b \leq 4,$$

by (7.3). According to part (i) of Lemma 13 the total contribution to $N_{X_3}(B)$ arising in this way is then

$$\leq \sum_{\substack{b \in X(\mathbb{Q}) \\ H(b) \leq B^{1/3}}} \#\{x \in C_b(\Psi)(\mathbb{Q}) : H(x) \leq B\} \ll_{\varepsilon} (B^{1/3})^{7+\theta} B^{4+1/4+\varepsilon},$$

which is satisfactory for Proposition 1. This completes the proof of Proposition 1 in the case $m = 3$.

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Appendix

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Abstract

A smooth, nondegenerate hypersurface in projective space contains no linear subvarieties of greater than half its dimension. It can contain linear subvarieties of half its dimension. This note proves that a smooth hypersurface of degree $d \geq 3$ contains at most finitely many such subvarieties.

Let k be a field. Let $X \subset \mathbb{P}_k^n$ be a hypersurface of degree $d > 1$. For each integer $m > 0$, denote by $F_m(X)$ the *Fano scheme of m -planes in X* , cf. [1]. There are a number of natural questions about $F_m(X)$: is this scheme non-empty, is this scheme connected, is this scheme irreducible, is this scheme reduced, is this scheme smooth, what is the dimension of this scheme, what is the degree of this scheme, etc.? For each of these questions, the answer is uniform for a *generic hypersurface*. More precisely, there is a non-empty open subset U of the parameter space of hypersurfaces, such that for every point in U the answer to the question is the same. For a generic hypersurface, the answer is often easy to find: the total space of the relative Fano scheme of the universal hypersurface is itself a projective bundle over the Grassmannian $\mathbb{G}(m, n)$, so if the question for a generic hypersurface can be reformulated as a question about the total space of the relative Fano scheme, it is easy to answer the question. However, much less is known if X is assumed to be smooth, but *not* generic.

There are a few easy results, such as the following.

Proposition 1. Let $X \subset \mathbb{P}^n$ be a hypersurface of degree $d > 1$ and let m be an integer such that $2m \geq n$. Every m -plane $\Lambda \subset X$ intersects the singular locus of X . In particular, if X is smooth then $F_m(X)$ is empty.

Proof. Choose a system of homogeneous coordinates x_0, \dots, x_n on \mathbb{P}^n such that Λ is given by $x_{m+1} = \dots = x_n = 0$. Let F be a defining equation for X . Because $\Lambda \subset X$, $F(x_0, \dots, x_m, 0, \dots, 0) = 0$. Also,

$$\frac{\partial F}{\partial x_i}(x_0, \dots, x_m, 0, \dots, 0) = 0,$$

for $i = 0, \dots, m$. Because $d > 1$, for $i = 1, \dots, n - m$, the homogeneous polynomial on Λ ,

$$\frac{\partial F}{\partial x_{m+i}}(x_0, \dots, x_m, 0, \dots, 0),$$

is non-constant. Since $n - m \leq m$, these $n - m$ non-constant homogeneous polynomials have a common zero in Λ . By the Jacobian criterion, this is a singular point of X . \square

Remark 1. This also follows easily from the Lefschetz hyperplane theorem.

What happens if $n = 2m + 1$? If $m > 1$ or if $m = 1$ and $d > 3$, then a *generic* hypersurface $X \subset \mathbb{P}^n$ contains no m -plane. However there do exist smooth hypersurfaces containing an m -plane. For instance, if $\text{char}(k)$ does not divide d then the Fermat hypersurface $x_0^d + \cdots + x_n^d = 0$ is smooth and contains many m -planes, e.g., $x_0 + x_1 = x_2 + x_3 = \cdots = x_{n-1} + x_n = 0$ when d is odd. However, if $d \geq 3$, a smooth hypersurface cannot contain a positive-dimensional family of m -planes. This was proved independently by Olivier Debarre, using a different argument.

The setup is as follows. Let $n = 2m + 1$. Let $X \subset \mathbb{P}^n$ be a hypersurface of degree d . Let $\Lambda_1, \Lambda_2 \subset X$ be m -planes. Denote by Z the intersection $\Lambda_1 \cap \Lambda_2$. This is either empty or else an r -plane for some integer r . If Z is empty, define r to be -1 .

Denote by $X_{\text{sm}} \subset X$ the smooth locus of X , i.e., the maximal open subscheme that is smooth. Denote $\Lambda_{i,\text{sm}} = \Lambda_i \cap X_{\text{sm}}$ for $i = 1, 2$. There are Chow classes,

$$[\Lambda_{i,\text{sm}}] \in A_m(X_{\text{sm}}), \quad i = 1, 2.$$

Because X_{sm} is smooth, the intersection product $[\Lambda_{1,\text{sm}}] \cdot [\Lambda_{2,\text{sm}}] \in A_0(X_{\text{sm}})$ is defined.

Lemma 1. If Z is contained in X_{sm} , then the degree of $[\Lambda_{1,\text{sm}}] \cdot [\Lambda_{2,\text{sm}}]$ is $(1 - (1 - d)^{r+1})/d$.

Proof. If $r = -1$, i.e., if Z is empty, this is obvious. Therefore suppose that Z is an r -plane for some $r \geq 0$. By the excess intersection formula, the class $[\Lambda_{1,\text{sm}}] \cdot [\Lambda_{2,\text{sm}}]$ is the pushforward from Z of the refined intersection product, $(\Lambda_{1,\text{sm}} \cdot \Lambda_{2,\text{sm}})^Z$. And by [2, Prop. 9.1.1], the refined intersection product is,

$$(\Lambda_{1,\text{sm}} \cdot \Lambda_{2,\text{sm}})^Z = \{c(N_{\Lambda_{1,\text{sm}}/X_{\text{sm}}})/c(N_{Z/\Lambda_{2,\text{sm}}}) \cap [Z]\}_r.$$

Denote $H = c_1(\mathcal{O}_Z(1))$. The normal bundle of Z in $\Lambda_{1,\text{sm}}$ is $\mathcal{O}_Z(1)^{m-r}$. The restriction to Z of the normal bundle of $\Lambda_{2,\text{sm}}$ in \mathbb{P}^n is $\mathcal{O}_Z(1)^{n-m} = \mathcal{O}_Z(1)^{m+1}$. And the restriction to Z of the normal bundle of X in \mathbb{P}^n is $\mathcal{O}_Z(d)$. Therefore,

$$c(N_{\Lambda_{1,\text{sm}}/X_{\text{sm}}})/c(N_{Z/\Lambda_{2,\text{sm}}}) = \frac{(1 + H)^{m+1}}{(1 + H)^{m-r}(1 + dH)} = \frac{(1 + H)^{r+1}}{1 + dH}.$$

Expanding this out gives,

$$\left(\sum_{i=0}^{r+1} \binom{r+1}{i} H^i \right) \left(\sum_{j=0}^{\infty} (-1)^j d^j H^j \right).$$

In particular, the coefficient of H^r is,

$$\begin{aligned} \sum_{i=0}^r \binom{r+1}{i} (-1)^{r-i} d^{r-i} &= \frac{-1}{d} \sum_{i=0}^r \binom{r+1}{i} (-1)^{r+1-i} d^{r+1-i} \\ &= (1 - (1 - d)^{r+1})/d. \end{aligned}$$

□

Proposition 2. If $d \geq 3$, if Λ_1 and Λ_2 are distinct, and if at least one of Λ_1 , Λ_2 is contained in X_{sm} , then $[\Lambda_{1,\text{sm}}]$ is not numerically equivalent to $[\Lambda_{2,\text{sm}}]$.

Proof. Let Λ_1 be contained in X_{sm} . By Lemma 1,

$$\deg([\Lambda_1] \cdot [\Lambda_1]) = (1 - (1 - d)^{m+1})/d.$$

Also, $Z = \Lambda_1 \cap \Lambda_2$ is contained in X_{sm} . So by Lemma 1,

$$\deg([\Lambda_1] \cdot [\Lambda_{2,\text{sm}}]) = (1 - (1 - d)^{r+1})/d.$$

Because $d - 1 \geq 2$ and $r < m$, we have $(d - 1)^{r+1} < (d - 1)^{m+1}$. Therefore $(1 - (1 - d)^{r+1})/d \neq (1 - (1 - d)^{m+1})/d$, and so $[\Lambda_1]$ is not numerically equivalent to $[\Lambda_{2,\text{sm}}]$. \square

Corollary (Debarre). There are only finitely many m -planes contained in X_{sm} .

Proof. By Proposition 2, distinct m -planes contained in X_{sm} are not algebraically equivalent. Therefore every irreducible component of $F_m(X)$ that contains a point parametrising an m -plane in X_{sm} is just a point. Because $F_m(X)$ is quasi-compact, the number of these irreducible components is finite. \square

Remark 2. Debarre's proof shows more than the statement of the corollary: for any m -plane Λ contained in X_{sm} , $h^0(\Lambda, N_{\Lambda/X}) = 0$. It follows that each such point is a connected component of $F_m(X)$, and that $F_m(X)$ is reduced at this point.

A natural question is, what is the maximal number of m -planes contained in a smooth hypersurface of degree d in \mathbb{P}^{2m+1} ? There is a naive upper bound that grows as $d^{(m+1)^2}$, but this is too large. The Fermat hypersurface contains $C_m d^{m+1}$ distinct m -planes, where $C_m = (2m + 1)(2m - 1) \cdots 3 \cdot 1$. Joe Harris points out that for $m = 1$, the degree of the *flecnodal curve* gives an upper bound of $11d^2 - 24d$.

More generally, define the *flecnodal locus* $P(X) \subset X$ to be the set of points $p \in X$ such that there is a line $L \subset \mathbb{P}^{2m+1}$ that has contact of order $3m + 1$ with X at p . This is the pushforward in X of a subscheme in $\mathbb{P}(T_X)$ that is the zero locus of a section of a locally free sheaf. If X is generic, this section is a regular section. Then a Chern class computation gives that the degree of $P(X)$ is a polynomial $p_m(d)$ of degree $m + 1$ in d whose leading term is,

$$\left(\frac{(3m + 1)!}{2} - 1 \right) d^{m+1}.$$

Therefore, for arbitrary X , $\deg(P_m(X)) \leq p_m(d)$, where $P_m(X)$ is the m -cycle of all m -dimensional irreducible components of $P(X)$ (weighted by multiplicity).

Of course every m -plane Λ is contained in $P(X)$. It is not clear that every m -plane is contained in $P_m(X)$, i.e., that Λ is an irreducible component of $P(X)$.

And, indeed, this fails if $d \leq 3m$. For $d \geq 3m$, it may be true that every m -plane is an irreducible component of $P(X)$.

In the special case that $m = 1$, $P(X)$ is a curve, the *flecnodal curve*, for all $d \geq 3$. Therefore, the number of lines in a smooth surface of degree $d \geq 3$ in \mathbb{P}^3 is at most $11d^2 - 24d$. Note this gives the correct answer for $d = 3$. For $d = 4$ this gives the wrong answer, Segre proved the maximal number of lines on a quartic surface is 64, cf. [3]. In fact, by a more involved analysis, Segre proved that the number of lines on a smooth surface of degree $d \geq 3$ is at most $11d^2 - 28d + 12$. The Fermat surface contains $3d^2$ lines. The true maximum is probably strictly between $3d^2$ and $11d^2 - 28d + 12$.

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