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Sad-Loser contests

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ABSTRACT

We consider asymmetric winner-reimbursed contests. It turns out that such contests (Sad-Loser) have multiple internal pure-strategy equilibria (where at least two players are active). We describe all equilibria and discuss their properties. In particular, we find (1) that an active player is indifferent among all her non-negative choices and her expected payoff is zero in any internal equilibrium, (2) that a higher-value (stronger) player *always* spends less than a lower-value (weaker) player and therefore *always* has a lower chance to win a Sad-Loser contest in *any* internal equilibrium, and (3) a sufficient condition for a net total spending to be higher in a Sad-Loser contest than in the corresponding asymmetric contest.

1. Introduction

A contest is an allocation mechanism where players strategically allocate costly efforts in order to win a prize. In this paper, we consider asymmetric winner-reimbursed contests. In fact, these types of contests (which we call Sad-Loser contests[1]) can be found in politics, R&D, and industrial organization (see Kaplan et al., 2002, Matros and Armanios, 2009). For example, Research & Development (R&D) Tax Credits (signed into law by President Bush on October 3, 2008[2]) require that "*The activity must result in a new or improved ... product ...*" which means that an R&D winner not only obtains a new product (wins the prize) but also gets R&D Tax Credits (gets reimbursed for her expenses). It is a common practice in elections that a qualified candidate is entitled to at least a partial reimbursement of election expenses. In order to qualify for a reimbursement, a candidate must either be elected, or, if not elected, must exceed a pre-specified number of valid votes. For example, Germany, Canada, and Ireland provide reimbursement of election campaign costs.

The paper proceeds as follows. First, we analyze the existence and uniqueness of equilibria in Sad-Loser contests. It turns out that a player's best-reply correspondence is not continuous even for positive opponents' spending. Therefore, fixed-point theorems cannot help to establish equilibrium existence in our model. However, using the best-reply correspondences, we find *all* pure-strategy equilibria in Sad-Loser contests. We show that the model always has multiple equilibria. Such an observation is rare in the contest literature. The few exceptions are Cornes and Hartley (2005), Cohen and Sela (2005), Yamazaki (2008), and Chowdhury and Sheremeta (2011). Cornes and Hartley (2005) and Yamazaki (2008) find multiple equilibria in contests when the contest success function is determined by a production function with increasing returns for each contestant. Cohen and Sela (2005) and Chowdhury and Sheremeta (2011) discuss multiplicity of equilibria in contests with reimbursements. We show that equilibria in our model can be of two types: an i-type and an internal type. i-type equilibria are such that only player i spends a positive amount and all other players spend nothing. Internal-type equilibria are such that at least two players are active (spend positive amounts).[3] Moreover, the number of internal equilibria increases with the number of players in Sad-Loser contests.[4]

Second, we discuss the properties of internal equilibria. It turns out that equilibrium behavior in a Sad-Loser contest is drastically different from that in an asymmetric contest.[5] We demonstrate that an indifference property has to be satisfied in any internal

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1 The name comes from Riley and Samuelson's (1981) example of *Sad-Loser Auction*.

2 See http://www.freedmaxick.com/research_development_r_d_tax_credits.php.

3 Some Sad-Loser contests can have an internal equilibrium where *all* players are active.

4 The number of equilibria can also increase with the number of players in all-pay auctions. See Baye et al. (1996).

5 The classic Tullock's (1980) contest where all players have the same valuations is called *standard* in the contest literature. A contest where players have different prize valuations is typically called *asymmetric*. Players can have different prize valuations in a Sad-Loser contest, but to simplify our notation we will call them Sad-Loser contests instead of asymmetric Sad-Loser contests.

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equilibrium: an active player is indifferent among all her non-negative choices. Since an active player is indifferent between zero and a positive choice, her expected equilibrium payoff has to be zero in an internal equilibrium. This observation is in contrast to the standard result in the contest literature, in which expected individual payoffs are positive in an equilibrium. We also demonstrate a counter intuitive result that the reverse monotonic property holds for the active players: a higher-value (stronger) player *always* spends less than a lower-value (weaker) player and therefore *always* has a lower chance to win the Sad-Loser contest in *an* internal equilibrium. This result can also be explained by the indifference property. An active player is indifferent among all her non-negative choices if her opponents spend exactly her prize value. In an internal equilibrium, active players exert effort in such a way as to keep other players indifferent. Therefore, it requires higher spending to keep a higher-value player indifferent and lower spending to keep a lower-value player indifferent. Since active players are indifferent among their choices in an internal equilibrium, lower-value active players have to spend more than higher-value active players.[6]

Third, we find conditions under which the expected net total spending in an internal equilibrium increases or decreases with the addition of another active player. This finding helps to rank internal equilibria in terms of the expected profit for the designer. We demonstrate that the expected profit is higher with a higher mean of the prize values of active players and with a lower variance of these prize values. Therefore, the highest expected profit is achieved in the internal equilibrium when the set of active players is limited to the two highest-value players. This finding is consistent with numerous examples of contest designers selecting a small – typically two – number of competitors. (See Taylor, 1995, Fullerton and McAfee, 1999, Che and Gale, 2003, Menicucci, 2006.) Moreover, the highest expected profit is higher than the total spending in the standard asymmetric contest. In addition, we discover a sufficient condition for the expected designer profit (net total spending) in *any* internal equilibrium to be higher than total spending in an asymmetric contest. This condition is simple: if *all* players are active in an asymmetric contest (see Hillman and Riley, 1989, Stein, 2002, Matros, 2006), then the expected profit in *any internal equilibrium* in the Sad-Loser contest is higher than total spending (designer profit) in the asymmetric contest.[7] This result can be important for different applications of the Sad-Loser contest, as it is for public goods provision, for example.

The Sad-Loser contest is easy to imagine and implement. The Sad-Loser contest designer has to announce the following rules:

1. Each player is eligible to submit as his contest bid any positive real number.
2. The contest winner receives the prize and *retains her bid*.[8]
3. All other players lose their bids.

There are a few papers on contests with reimbursements. Matros and Armanios (2009) analyze contests with reimbursements where all prize values are *the same* $V_1 = \cdots = V_n$. In this paper, we take the next step and consider asymmetric n-player Sad-Loser contests. Our paper is also a generalization of Cohen and Sela's (2005) two-player asymmetric Sad-Loser contest to n-player case. They show that in the two-player case there exists a unique internal equilibrium where the weak contestant wins with higher probability than the stronger one. Cohen and Sela (2005) analyze the general n-player case by means of the $n = 3$ case. They describe only internal equilibria with two active players.

Baye et al. (2005) is an important complement to our paper. They use an all-pay auction framework in order to compare different litigation systems. There are three main differences between our models. First, they consider a game with incomplete information: their players have private values. Second, they examine an all-pay auction. Finally, the reimbursements are made by players in their model.

The rest of the paper is organized as follows. Section 2 presents the Model. Properties of internal equilibria are studied in Section 3. Section 4 makes a comparison between the Sad-Loser contests and the standard asymmetric contests. Concluding remarks are given in Section 5.

2. The model

Consider an n-player winner-take-all contest where the winner gets reimbursed for her effort. We assume that n risk-neutral players spend resources simultaneously in order to win one prize. Player i's valuation for the prize is V_i. Players' valuations are common knowledge and:

$$V_1 \geq V_2 \geq \cdots \geq V_n > 0. \tag{1}$$

Formally, player i exerts effort x_i in order to maximize the following expression:

$$\max_{x_i \geq 0} \frac{x_i}{\sum_{j=1}^{n} x_j} (V_i + x_i) - x_i, \tag{2}$$

where the first term in (2) is the probability of winning the contest, $\frac{x_i}{\sum_{j=1}^{n} x_j} \geq 0$, times the prize for player i, V_i, and the winner's reimbursement, x_i, and the last term is the cost of effort.[9] The maximization problem (2) can be rewritten as

$$\max_{x_i \geq 0} \left(V_i - \sum_{j \neq i} x_j \right) \frac{x_i}{\sum_{j=1}^{n} x_j}. \tag{3}$$

Since $\frac{x_i}{\sum_{j=1}^{n} x_j}$ is a monotonically increasing function of x_i, the optimal x_i depends on the sign of the expression $\left(V_i - \sum_{j \neq i} x_j\right)$. If $\left(V_i - \sum_{j \neq i} x_j\right) > 0$, then the optimal x_i is not defined (unless $\sum_{j \neq i} x_j = 0$) because there is no budget constraint in the problem and player i wants to spend as much as possible. If the total spending of all other players is zero, $\sum_{j \neq i} x_j = 0$, then player i maximizes her utility by choosing any positive $x_i > 0$, because her effort as the winner is reimbursed and any positive spending makes player i the winner. If $\left(V_i - \sum_{j \neq i} x_j\right) < 0$, then the optimal $x_i = 0$. Finally, if $\left(V_i - \sum_{j \neq i} x_j\right) = 0$, then player i is indifferent

6 In the literature on electoral competition a number of studies have consistently found at least some evidence of a positive relationship between candidate spending and electoral performance (see Nice, 1987, Pattie et al., 1995). We can speculate that an incumbent is more likely to value the office higher than a challenger. However, most papers in this literature (see, for example, Moon, 2006), support the Paradox of Less Effective Incumbent spending. This paradox has a flavor of our reverse monotonic result.

7 Hillman and Riley (1989) show that the asymmetric contest has a set of active high-value players in the unique equilibrium. Stein (2002) describes equilibrium spending of the active players. Fang (2002) proves the uniqueness of this equilibrium.

8 Since the winner has to be reimbursed by the designer, we assume that the designer can observe individual effort. This assumption is not required in standard contests.

9 We assume that if $x_1 = \cdots = x_n = 0$, then nobody wins the prize.

among all her choices. Formally, the first order conditions of the maximization problem (3) are:

$$\left(V_i - \sum_{j\neq i} x_j\right) \frac{\left(\sum_{j\neq i} x_j\right)}{\left(\sum_{j=1}^{n} x_j\right)^2} \leq 0. \tag{4}$$

Note that (4) is equal to zero only if (i) $\sum_{j\neq i} x_j = 0$, or (ii) $\left(V_i - \sum_{j\neq i} x_j\right) = 0$. These two cases correspond to two types of equilibria: case (i) describes all i-equilibria where only player i is active and case (ii) characterizes all internal equilibria where at least two players are active. We will call player i active if $x_i > 0$. Suppose that player i is active in an equilibrium. Then either condition (i) or condition (ii) holds. If condition (i) is satisfied, then the *strong indifference property* holds: an active player is indifferent among all her *positive* choices in an i-equilibrium. If condition (ii) holds, then the *indifference property* is satisfied: an active player is indifferent among all her non-negative choices in an internal equilibrium.

Given the opponents' spending $\sum_{j\neq i} x_j \geq 0$, it is easy to find the best-reply correspondence for player i:

$$BR_i(x_1, \ldots, x_n) = \begin{cases} (0, +\infty), & \text{if } \sum_{j\neq i} x_j = 0, \\ +\infty, & \text{if } 0 < \sum_{j\neq i} x_j < V_i, \\ [0, +\infty), & \text{if } \sum_{j\neq i} x_j = V_i, \\ 0, & \text{if } \sum_{j\neq i} x_j > V_i. \end{cases} \tag{5}$$

The best-reply correspondence summarizes intuitive observations. If opponents of player i spend zero resources, she wins with any positive effort and her effort is reimbursed. If the total opponents' effort is positive but less than the prize value of player i, then the best-reply correspondence is not defined because player i wants to spend as much as possible: her utility is a monotonically increasing function in this case. We use the symbol $+\infty$ to emphasize that. If the total opponents' effort is equal to the prize value of player i, then she is indifferent among *all* her choices: her expected payoff is always zero. Finally, if the total opponents' effort exceeds the prize value for player i, her unique best reply is zero, because otherwise she obtains a negative expected payoff. The best-reply correspondence (5) will help to find all Nash equilibria in the model. In order to illustrate our approach, we start from the $n = 2$ player case.

2.1. $n = 2$

If $n = 2$, then the best-reply correspondences (5) become

$$BR_1(x_1, x_2) = \begin{cases} (0, +\infty), & \text{if } x_2 = 0, \\ +\infty, & \text{if } 0 < x_2 < V_1, \\ [0, +\infty), & \text{if } x_2 = V_1, \\ 0, & \text{if } x_2 > V_1, \end{cases} \tag{6}$$

and

$$BR_2(x_1, x_2) = \begin{cases} (0, +\infty), & \text{if } x_1 = 0, \\ +\infty, & \text{if } 0 < x_1 < V_2, \\ [0, +\infty), & \text{if } x_1 = V_2, \\ 0, & \text{if } x_1 > V_2. \end{cases} \tag{7}$$

Fig. 1 illustrates the best-reply correspondences (6) and (7).

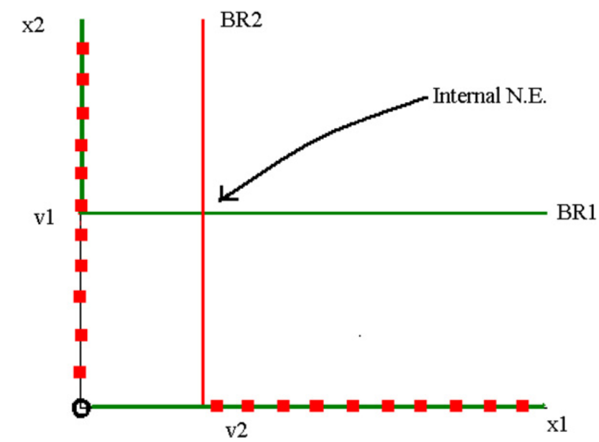

Fig. 1. Best-reply correspondences. Green (solid) – player 1. Red (dots) – player 2. This illustrates all Nash equilibria. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Fig. 1 shows that there are two types of equilibria: with one or two active players. If only player i is active, then she wins with any amount of positive effort and gets reimbursed. In order to keep the other player j inactive, the active player i has to be *very aggressive*: her effort should be at least equal to the prize value of her opponent, $x_i \geq V_j$. Such high effort of player i leads to a non-positive expected payoff for player j and she stays out of the contest. There is only one possibility to have two active players in the equilibrium: each player has to make her opponent indifferent among all her choices: it must be that $x_i = V_j$ in this case. Of course, such an equilibrium is unique. This equilibrium has a property of a mixed-strategy equilibrium: each player makes her opponent indifferent. Cohen and Sela (2005) find the described equilibria analytically.

2.2. General case, $n > 2$

Case $n = 2$ provides the intuition and illustration for the general case, $n > 2$. Again we can have two types of equilibria: with one or several active players. We will call equilibria with one active player i – i-equilibria and with several active players – internal equilibria.

If only player i is active, then she wins with any positive effort and gets reimbursed. In order to keep the other players inactive, the active player i has to be *very aggressive*: her effort should be at least equal to the prize value of her highest-value opponent, $x_i \geq \max\{V_1, \ldots, V_{i-1}, V_{i+1}, \ldots, V_n\}$, which follows from the best-reply correspondence (5). Such high effort of player i leads to a non-positive expected payoff for any other player and as a result, all other players stay out of the contest.

Proposition 1. $(0, \ldots, 0, x_i, 0, \ldots, 0)$ *is an i-equilibrium, if*

$$x_i \geq \begin{cases} V_1, & \text{if } i > 1, \\ V_2, & \text{if } i = 1. \end{cases} \tag{8}$$

In i-type equilibria, only one player has (very) high spending and the other players have zero spending. Since the winner gets reimbursed, player i spends so much that she discourages all other players from participation in the Sad-Loser contest, because they have negative expected payoffs for any positive spending level. i-type equilibria are similar to i-type equilibria in the second-price seal-bid auctions where just one bidder – bidder i – places a (very) high bid and all other bidders bid zero.

There are multiple possibilities to have several active players in an internal equilibrium. In each equilibrium with several active players the indifference property must hold: each active player i has to be indifferent among all her choices, or it must be that

$$\sum_{j\neq i} x_j = V_i, \tag{9}$$

for any active player i. Summing (9) over $k \geq 2$ active players,

$$\sum_{i=i_1}^{i_k} \sum_{j\neq i} x_j = \sum_{i=i_1}^{i_k} V_i.$$

Denote total spending of k active players $i_1, \ldots, i_k$ by

$$s(i_1, \ldots, i_k) = \sum_{i=i_1}^{i_k} x_i.$$

Then, total spending of active players $i_1, \ldots, i_k$ is

$$s(i_1, \ldots, i_k) = \frac{1}{(k-1)} \sum_{j=i_1}^{i_k} V_j. \tag{10}$$

From (9) and (10), we get

$$x_i = s(i_1, \ldots, i_k) - V_i = \frac{1}{(k-1)} \sum_{i=i_1}^{i_k} V_i - V_i. \tag{11}$$

Expression (11) describes all internal equilibria.

Proposition 2. *Suppose that there exists an internal equilibrium $(x_1, \ldots, x_n)$ with k active players, $i_1, \ldots, i_k$. Then there exists a unique internal equilibrium with active players $i_1, \ldots, i_k$ and each active player exerts effort according to formula (11).*

In each internal equilibrium there exists a set of active players. Active players' spending is uniquely described by expression (11), which means that there exists a unique internal equilibrium with a particular set of active players. Moreover, all players might be active in an internal equilibrium. If an equilibrium with all active players exists, then it is unique. Propositions 1 and 2 characterize all equilibria in the Sad-Loser contest. The following example illustrates the $n = 3$ case.

Example 1. Suppose that $n = 3$ and $V_1 \geq V_2 \geq V_3 > 0$. Then, from Proposition 1, there are i-type equilibria in pure strategies:

- 1-type: $(x_1, 0, 0)$, where $x_1 \geq V_2$;
- 2-type: $(0, x_2, 0)$, where $x_2 \geq V_1$;
- 3-type: $(0, 0, x_3)$, where $x_3 \geq V_1$.

From Proposition 2, there are at least 2 internal equilibria in pure strategies:

- $(V_2, V_1, 0)$;
- $(V_3, 0, V_1)$.

If $V_1 \leq V_2 + V_3$, then from Proposition 2 there are 2 other internal equilibria: One with two active players:

- $(0, V_3, V_2)$

and another with all three active players:

- $\left(\frac{1}{2}[V_2 + V_3 - V_1], \frac{1}{2}[V_1 + V_3 - V_2], \frac{1}{2}[V_1 + V_2 - V_3]\right)$.

Cohen and Sela (2005) analyze the general $n > 2$ case by means of an example when $n = 3$. They find i-type equilibria and only mention internal equilibria with two active players.

2.3. Expected equilibrium payoffs

What can be said about players' expected equilibrium payoffs? It is obvious that the expected equilibrium payoff of any non-active player is zero. We also know that the indifference property must hold for all active players in an internal equilibrium. Since players are indifferent among all non-negative choices including zero, it is intuitive that each active player should also have expected zero payoff in an internal equilibrium.

Proposition 3. *The expected payoff of*

(i) *active player i in an i-equilibrium is V_i;*
(ii) *each player in any internal equilibrium is zero.*

Proposition 3 gives intuition for why the Sad-Loser contest can generate a higher net total spending than the standard contest: the players' expected payoffs are zero; hence, the designer should obtain all expected profit. This result is in contrast to the standard observation in the contest literature where the expected individual payoffs are usually positive (see for example, Tullock, 1980, Nitzan, 1994, Congleton et al., 2008, Konrad, 2009).

3. Internal equilibria

As we saw above there are multiple internal equilibria if $n > 2$. We discuss the properties of internal equilibria in this section. First, we show that a higher-value (stronger) active player *always* exerts less effort than a lower-value (weaker) active player and therefore has a lower chance to win the Sad-Loser contest in *an* internal equilibrium. Next, the expected net total spending is analyzed.

3.1. Weaker players win more often

Now we show how players' prize valuations affect their spending in an internal equilibrium. Denote p_{i_l} as the probability that player i_l wins the Sad-Loser contest.

Theorem 1. *Consider an internal equilibrium with active players $i_1, \ldots, i_k$. Then,*

$$x_{i_1} \leq \cdots \leq x_{i_k}$$

and

$$p_{i_1} \leq \cdots \leq p_{i_k}$$

if and only if

$$V_{i_1} \geq \cdots \geq V_{i_k}.$$

Theorem 1 leads to the following surprising conclusion: in an internal equilibrium, an active higher-value player *always* has a lower chance to win than a lower-value active player.

Corollary 1. *Suppose that $V_i > V_j$. Consider an internal equilibrium where both players i and j are active. Then $p_i < p_j$.*

So far, a monotonic relationship has been identified in the contest literature: higher value leads to active participation and more aggressive equilibrium spending which, as a result, leads to higher winning chances (see, for example, Hillman and Riley, 1989, Nti, 1999). The Sad-Loser contest gives the opposite result. Why does such a contrast result arise?[10] Consider again the $n = 2$ case.[11] In the unique internal equilibrium each player makes the opponent indifferent among all her choices: the indifference property holds. In order to make player i indifferent, her opponent has to spend V_i. This means that the lower-value player has to spend more than the higher-value player in order to keep the higher-value player indifferent. The lower-value player can do it because she is also indifferent among all her non-negative choices. It is therefore not surprising that both players obtain zero expected payoff in an internal equilibrium.

10 Cohen and Sela (2008) show that the designer can influence contestants' winning probabilities by choosing the number of prizes in the all-pay auctions.

11 Cohen and Sela (2005) notice this effect in the case of two players. They also point out (Proposition 2) that "*in the n player contest …underdogs* **may** *win with the highest probability*". We prove that this effect holds in *any internal equilibrium*.

3.2. Expected net total spending

Consider the expected net total spending in an internal equilibrium with $i_1, \ldots, i_k$ active players. Since net total spending is the designer's profit, we will call the expected net total spending the expected profit. The expected profit is then

$$\pi(i_1, \ldots, i_k) = \frac{x_{i_1}}{s(i_1, \ldots, i_k)}\left[s(i_1, \ldots, i_k) - x_{i_1}\right] + \cdots + \frac{x_{i_k}}{s(i_1, \ldots, i_k)}\left[s(i_1, \ldots, i_k) - x_{i_k}\right]. \quad (12)$$

From (11),

$$\pi(i_1, \ldots, i_k) = \frac{s(i_1, \ldots, i_k) - V_{i_1}}{s(i_1, \ldots, i_k)} V_{i_1} + \cdots + \frac{s(i_1, \ldots, i_k) - V_{i_k}}{s(i_1, \ldots, i_k)} V_{i_k},$$

or

$$\pi(i_1, \ldots, i_k) = \sum_{i=i_1}^{i_k} V_i - (k-1)\frac{\sum_{i=i_1}^{i_k} V_i^2}{\sum_{i=i_1}^{i_k} V_i}. \quad (13)$$

Eq. (13) can be rewritten as

$$\pi(i_1, \ldots, i_k) = \overline{V^k} - \frac{\operatorname{Var} V^k}{\overline{V^k}}, \quad (14)$$

where

$$\overline{V^k} \equiv \frac{1}{(k-1)} \sum_{j=i_1}^{i_k} V_j, \qquad \operatorname{Var} V^k = \sum_{j=i_1}^{i_k} \left(\overline{V^k} - V_j\right)^2.$$

It is easy to see from (14) that $\pi(i_1, \ldots, i_k)$ is increasing in $\overline{V^k}$ and decreasing in $\operatorname{Var} V^k$. In other words, the expected profit is higher with a higher mean of the prize values of active players and with a lower variance of these prize values. Therefore, the highest expected profit should be obtained if only the top two players are active, because it gives the highest mean and the lowest variance. However, if all prize values are the same, the expected profit is *the same in all* internal equilibria.

Proposition 4. *Suppose that*

$$V_1 = V_2 = \cdots = V_n = V.$$

Then, there are $2^n - (n+1)$ internal equilibria. Expected profit is equal to V in any internal equilibrium.

If players have different prize values, then we can rank internal equilibria in terms of their expected profits. Our next result shows how the expected profit changes with the addition of an active player.

Theorem 2. *Consider two internal equilibria with $i_1, \ldots, i_k$ and $i_1, \ldots, i_k, i_{k+1}$ active players. Then,*

$$\begin{aligned} &\pi(i_1, \ldots, i_k) > \pi(i_1, \ldots, i_k, i_{k+1}), \\ &\quad \text{if } \sum_{j=i_1}^{i_k} V_j\left(V_j - V_{i_{k+1}}\right) > 0, \\ &\pi(i_1, \ldots, i_k) < \pi(i_1, \ldots, i_k, i_{k+1}), \\ &\quad \text{if } \sum_{j=i_1}^{i_k} V_j\left(V_j - V_{i_{k+1}}\right) < 0. \end{aligned} \quad (15)$$

It follows from Theorem 2 that addition of the lowest-value player to the set of active players can only reduce the expected profit.

Corollary 2a. *Consider two internal equilibria with $i_1, \ldots, i_k$ and $i_1, \ldots, i_k, i_{k+1}$ active players. Suppose that*

$$\min\left\{V_{i_1}, \ldots, V_{i_k}\right\} > V_{i_{k+1}} > 0. \quad (16)$$

Then,

$$\pi(i_1, \ldots, i_k) > \pi(i_1, \ldots, i_k, i_{k+1}).$$

However, the addition of the highest-value player to the set of active players can only increase the expected profit.

Corollary 2b. *Consider two internal equilibria with $i_1, \ldots, i_k$ and $i_1, \ldots, i_k, i_{k+1}$ active players. Suppose that*

$$V_{i_{k+1}} > \max\left\{V_{i_1}, \ldots, V_{i_k}\right\} > 0. \quad (17)$$

Then,

$$\pi(i_1, \ldots, i_k) < \pi(i_1, \ldots, i_k, i_{k+1}).$$

Using Theorem 2 and Corollaries 2a and 2b we can characterize the range of the expected profits in the Sad-Loser contest. In particular, we find internal equilibria with the lowest and the highest expected profits. The following proposition confirms our observation in (14) that the highest expected profit is reached in the internal equilibrium with the top two active players.

Proposition 5. (i) *The lowest expected profit is achieved in the internal equilibrium with the two lowest-value active players. This expected profit is $\frac{2V_{n-1}V_n}{V_{n-1}+V_n}$.*

(ii) *The highest expected profit is achieved in the internal equilibrium with the two highest-value active players. This expected profit is $\frac{2V_1V_2}{V_1+V_2}$.*

(iii) *The expected profit is at least $\frac{2V_{n-1}V_n}{V_{n-1}+V_n}$ and at most $\frac{2V_1V_2}{V_1+V_2}$ in an internal equilibrium.*

4. Asymmetric vs. Sad-Loser contests

The contest designer is typically assumed to maximize her profit. So, in this section, we compare net total spending (the expected profit) in a Sad-Loser contest with total spending in an asymmetric contest. First, we describe total equilibrium spending in an asymmetric contest. Next, total spending in an asymmetric contest is compared with net total spending in the corresponding Sad-Loser contest. We will consider only internal equilibria in the Sad-Loser contest in this section.

4.1. Asymmetric contest

Hillman and Riley (1989) identify the set of active players in an asymmetric rent-seeking contest and total equilibrium spending. Stein (2002) follows Hillman and Riley (1989) and describes the players' equilibrium strategies.

Consider an asymmetric contest among n risk-neutral players where (1) holds. Players exert effort simultaneously in order to win one prize. In particular, player i spends $b_i \geq 0$ in order to win prize V_i. The players' valuations are commonly known among the players. Player i obtains the prize with probability $\frac{b_i}{\sum_{i=1}^n b_i}$, if $b_i > 0$. Each player i has to solve the following maximization problem:

$$\max_{b_i \geq 0} \frac{b_i}{\sum_{j=1}^n b_j} V_i - b_i.$$

Hillman and Riley (1989) demonstrate that the top $1, 2, \ldots, \bar{\mathbf{n}}$ players are active in an asymmetric contest and total spending in the unique equilibrium[12] is

$$T(n) = \frac{(\bar{\mathbf{n}}-1)}{\bar{\mathbf{n}}}\widehat{V_{\bar{\mathbf{n}}}},$$

where $\widehat{V_{\bar{\mathbf{n}}}}$ is the harmonic mean of the highest $\bar{\mathbf{n}}$ players' prizes

$$\widehat{V_{\bar{\mathbf{n}}}} \equiv \frac{n}{\sum_{j=1}^{n}\frac{1}{V_j}}$$

and a non-active player $(\bar{\mathbf{n}}+1)$ has prize value such that

$$V_{\bar{\mathbf{n}}+1} \leq \frac{(\bar{\mathbf{n}}-1)}{\bar{\mathbf{n}}}\widehat{V_{\bar{\mathbf{n}}}}. \quad (18)$$

Note that higher-value players spend more than lower-value players and have a higher chance to win an asymmetric contest. Moreover, each active player has a positive expected payoff.

4.2. Net total spending

Sad-Loser contests have multiple internal equilibria if $n > 2$. Therefore, we first select an internal equilibrium in a Sad-Loser contest and then compare the expected profit in this equilibrium with total spending in the corresponding asymmetric contest. Proposition 5 ranks internal equilibria in terms of their expected profits. We consider the internal equilibrium with the highest expected profit and compare it with total spending in the corresponding asymmetric contest. It turns out that the expected profit in the internal equilibrium with the two highest-value active players is always higher than total spending in the asymmetric contest.

Proposition 6. *The expected profit in the internal equilibrium* $(V_2, V_1, 0, \ldots, 0)$ *is higher than total spending in an asymmetric contest.*

Theorem 3 provides the sufficient condition for the expected profit in *any* internal equilibrium to be higher than total spending in an asymmetric contest. This condition is very natural: all players have to be active in an asymmetric contest. If all players are active in an asymmetric contest, then total spending must be below the lowest prize value, because all players obtain expected positive payoffs in the unique equilibrium. Consider now the Sad-Loser contest. Proposition 5 describes an internal equilibrium with the lowest profit. There are two lowest-value active players in that equilibrium. Since the players have to keep each other indifferent, their total spending is $V_{n-1} + V_n$ and their net total spending is higher than the lowest prize value, V_n. Therefore, the expected profit in *any* internal equilibrium in the Sad-Loser contest is higher than total spending in an asymmetric contest in this case.

Theorem 3. *Suppose that all players are active in an asymmetric contest, or*

$$V_n > \frac{n-2}{n-1}\widehat{V}_{n-1}. \quad (19)$$

Then, the expected profit in any internal equilibrium in the Sad-Loser contest is higher than total spending in an asymmetric contest.

Theorem 3 suggests when the designer should run the Sad-Loser contest instead of an asymmetric contest. There are several corollaries from this theorem.

Table 1
Sad-Loser contest.

Values	eq'm (1,2)	eq'm (1,3)	eq'm (2,3)	eq'm (1,2,3)
10	10	5	0	2.5
10	10	0	5	2.5
5	0	10	10	7.5
Expected profit	10	6.67	6.67	7

Table 2
Sad-Loser contest.

Values	eq'm (1,2)	eq'm (1,3)	eq'm (2,3)	eq'm (1,2,3)
10	10	1	0	0.5
10	10	0	1	0.5
1	0	10	10	9.5
Expected profit	10	1.818	1.818	1.857

Corollary 3. *If $n = 2$, or all prize values are the same,*

$$V_1 = V_2 = \cdots = V_n > 0, \quad (20)$$

then the expected profit in an internal equilibrium is higher than total spending in an asymmetric contest.

Proof. If $n = 2$, or condition (20) is satisfied, then condition (19) holds.[13] □

The following example shows that Theorem 3 provides a sufficient condition. In this example, condition (19) does not hold, but the expected profits in *all internal* equilibria are higher than total spending in an asymmetric contest.

Example 2. Suppose that $n = 3$ and $V_1 = V_2 = 10$, $V_3 = 5$. Then, total spending in the asymmetric contest (see Hillman and Riley, 1989, Stein, 2002) is $T(3) = 5$. In Example 1, all internal equilibria in the 3-player Sad-Loser contest are calculated. See Table 1.

Table 1 shows that the expected profits in all internal equilibria in the Sad-Loser contest are higher than total spending in the asymmetric contest.

However, if condition (19) does not hold, the expected profits in all but one of the internal equilibria in the Sad-Loser contest can be lower than total spending in an asymmetric contest. This is illustrated in the following example.

Example 3. Suppose that $n = 3$ and $V_1 = V_2 = 10$, $V_3 = 1$. Then, total spending in the asymmetric contest is $T(3) = 5$. Based on Example 1, all internal equilibria in the Sad-Loser contest are calculated in Table 2.

Total spending in the asymmetric contest is higher than the expected profits in all but one internal equilibrium in the Sad-Loser contest.

5. Conclusion

This paper considers Sad-Loser contests. All equilibria in pure strategies are found and their properties are discussed. There are several natural extensions of this paper. It will be interesting to test the results in the experimental laboratory and in the field. We have already started an experimental investigation of the Sad-Loser contests. In particular, the counter-intuitive aggressive spending of weak players in equilibrium will be tested. Another direction is an application of Sad-Loser contests to public goods provision. Since, as Morgan (2000) and Duncan (2002) show, lotteries increase the provision of public goods, a Sad-Loser contest might be an even better tool for determining higher public good provision than a lottery.

12 Fang (2002) shows that there exists a unique equilibrium in the asymmetric contest.

13 We can see the same result in the following way. Proposition 4 shows that the expected profit in any internal equilibrium is V. Tullock (1980) demonstrates that total spending is equal to $\frac{n-1}{n}V$ in the symmetric n-player contest.

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Appendix

Proof of Theorem 1. Consider an internal equilibrium with k active players $i_1, \ldots, i_k$. Suppose that players i_1 and i_2 are active in this equilibrium and

$$V_{i_1} \geq V_{i_2}.$$

It follows from condition (11) that

$$x_{i_1} = s(i_1, \ldots, i_k) - V_{i_1} \leq s(i_1, \ldots, i_k) - V_{i_2} = x_{i_2}.$$

Note that

$$p_{i_1} = \frac{x_{i_1}}{s(i_1, \ldots, i_k)} \leq \frac{x_{i_2}}{s(i_1, \ldots, i_k)} = p_{i_2}. \quad \square$$

Proof of Proposition 4. Note that there are exactly $\frac{n!}{(n-k)!k!}$ possibilities to have k active players in the Sad-Loser contest. Active player equilibrium spending is uniquely determined by expression (11). Therefore, there are at most $\sum_{k=2}^{n} \frac{n!}{(n-k)!k!} = 2^n - (n+1)$ internal equilibria. Since the indifference property holds in any internal equilibrium and all prize values are the same, there are exactly $2^n - (n+1)$ internal equilibria.

Consider an internal equilibrium $(x_1, \ldots, x_n)$ with $i_1, \ldots, i_k$ active players. From (14), we get

$$\pi(i_1, \ldots, i_k) = \frac{k}{k-1}V - \frac{k\left(\frac{k}{k-1}V - V\right)^2}{\frac{k}{k-1}V} = V. \quad \square$$

Proof of Theorem 2. Consider an internal equilibrium $(x_1, \ldots, x_n)$ with $i_1, \ldots, i_k$ active players. Suppose that an active player i_{k+1} is added. Then, from (13),

$$\pi(i_1, \ldots, i_k) - \pi(i_1, \ldots, i_k, i_{k+1})$$
$$= \frac{k\left(\sum_{j=i_1}^{i_k} V_j^2 + V_{i_{k+1}}^2\right)\left(\sum_{j=i_1}^{i_k} V_j\right) - (k-1)\left(\sum_{j=i_1}^{i_k} V_j^2\right)\left(\sum_{j=i_1}^{i_k} V_j + V_{i_{k+1}}\right)}{\left(\sum_{j=i_1}^{i_k} V_j + V_{i_{k+1}}\right)\left(\sum_{j=i_1}^{i_k} V_j\right)}$$
$$- \frac{V_{i_{k+1}}\left(\sum_{j=i_1}^{i_k} V_j + V_{i_{k+1}}\right)\left(\sum_{j=i_1}^{i_k} V_j\right)}{\left(\sum_{j=i_1}^{i_k} V_j + V_{i_{k+1}}\right)\left(\sum_{j=i_1}^{i_k} V_j\right)}.$$

Note that

$$\begin{aligned} & k\left(\sum_{j=i_1}^{i_k} V_j^2 + V_{i_{k+1}}^2\right)\left(\sum_{j=i_1}^{i_k} V_j\right) - (k-1)\left(\sum_{j=i_1}^{i_k} V_j^2\right) \\ & \quad \times \left(\sum_{j=i_1}^{i_k} V_j + V_{i_{k+1}}\right) - V_{i_{k+1}}\left(\sum_{j=i_1}^{i_k} V_j + V_{i_{k+1}}\right)\left(\sum_{j=i_1}^{i_k} V_j\right) \\ & = \left(\sum_{j=i_1}^{i_k} V_j\right)\left(\sum_{j=i_1}^{i_k} V_j\left(V_j - V_{i_{k+1}}\right)\right) \\ & \quad + (k-1)\left[\left(\sum_{j=i_1}^{i_k} V_j^2 + V_{i_{k+1}}^2\right)\left(\sum_{j=i_1}^{i_k} V_j\right) - \left(\sum_{j=i_1}^{i_k} V_j^2\right) \right. \\ & \quad \left. \times \left(\sum_{j=i_1}^{i_k} V_j + V_{i_{k+1}}\right)\right]. \end{aligned} \tag{21}$$

Since

$$\left(\sum_{j=i_1}^{i_k} V_j^2 + V_{i_{k+1}}^2\right)\left(\sum_{j=i_1}^{i_k} V_j\right) - \left(\sum_{j=i_1}^{i_k} V_j^2\right)\left(\sum_{j=i_1}^{i_k} V_j + V_{i_{k+1}}\right)$$
$$= V_{i_{k+1}}\left(\sum_{j=i_1}^{i_k} V_j\left(V_{i_{k+1}} - V_j\right)\right),$$

(21) becomes

$$\begin{aligned} & \left(\sum_{j=i_1}^{i_k} V_j\right)\left(\sum_{j=i_1}^{i_k} V_j\left(V_j - V_{i_{k+1}}\right)\right) \\ & \quad - (k-1)V_{i_{k+1}}\left(\sum_{j=i_1}^{i_k} V_j\left(V_j - V_{i_{k+1}}\right)\right) \\ & = (k-1)\left(\sum_{j=i_1}^{i_k} V_j\left(V_j - V_{i_{k+1}}\right)\right)\left[\frac{1}{k-1}\left(\sum_{j=i_1}^{i_k} V_j\right) - V_{i_{k+1}}\right]. \end{aligned}$$

The statement of the proposition follows from expression (10) and assumption (15). $\square$

Proof of Proposition 5. (i) Consider an internal equilibrium with $i_1, \ldots, i_k$ active players where $2 \leq k \leq n$ and

$$V_{i_1} \geq \cdots \geq V_{i_k}.$$

There are four cases.

Case 1. Suppose that

$$V_{i_1} \geq \cdots \geq V_{i_k} > V_{n-1} \geq V_n.$$

Then, from Theorem 2,

$$\begin{aligned} \pi(n-1, n) &< \pi(i_k, n-1, n) < \cdots < \pi(i_1, \ldots, i_k, n-1, n) \\ &< \pi(i_1, \ldots, i_k, n-1) < \pi(i_1, \ldots, i_k). \end{aligned}$$

Case 2. Suppose that

$$V_{i_1} \geq \cdots \geq V_{i_k} = V_{n-1} \geq V_n.$$

Then, from Theorem 2,

$$\begin{aligned} \pi(n-1, n) &< \pi(i_{k-1}, i_k, n) < \cdots < \pi(i_1, \ldots, i_k, n) \\ &< \pi(i_1, \ldots, i_k). \end{aligned}$$

Case 3. Suppose that

$$V_{i_1} \geq \cdots \geq V_{i_{k-1}} = V_{n-1} \geq V_{i_k} = V_n.$$

Then, from Theorem 2,

$$\pi(n-1, n) < \pi(i_{k-2}, i_{k-1}, i_k) < \cdots < \pi(i_1, \ldots, i_k).$$

Case 4. Suppose that

$$V_{i_1} \geq \cdots \geq V_{i_{k-1}} > V_{n-1} \geq V_{i_k} = V_n.$$

First, we show that

$$\pi(n-1, n) \leq \pi(k, n), \quad \text{for any } 1 \leq k \leq n-1. \tag{22}$$

Note that (22) holds if and only if

$$2\frac{V_{n-1}V_n}{V_{n-1} + V_n} \leq 2\frac{V_k V_n}{V_k + V_n} \iff V_k \geq V_{n-1}.$$

Therefore, from (22) and Theorem 2,

$$\begin{aligned} \pi(n-1, n) &< \pi(i_{k-1}, n) \\ &< \pi(i_{k-2}, i_{k-1}, i_k) < \cdots < \pi(i_1, \ldots, i_k). \end{aligned}$$

Hence, the lowest expected profit is reached in the equilibrium with the two lowest-value players. Players spend V_n and V_{n-1} in this equilibrium. Hence,

$$\min_k \pi(k) = \min_{i_1,i_2} \pi(i_1, i_2) = \frac{2V_{n-1}V_n}{V_{n-1}+V_n}.$$

(ii) can be proven in a way similar to part (i).
(iii) follows from (i) and (ii). □

Proof of Proposition 6. The expected profit in the internal equilibrium $(V_2, V_1, 0, \ldots, 0)$ is $\frac{2V_1V_2}{V_1+V_2} \geq V_2$. Player 2 is always active in an asymmetric contest and obtains positive payoff. This means that $T(n) < V_2$. □

Proof of Theorem 3. Proposition 5 shows that the minimal expected profit is achieved in the internal equilibrium with the two lowest-value active players $(n-1)$ and n. This expected profit is equal to $\frac{2V_{n-1}V_n}{V_{n-1}+V_n} \geq V_n$. The total spending in an asymmetric contest with all n active players must be smaller than a prize value of any player, including the lowest value player n. Therefore, the expected profit in any internal equilibrium is higher than total spending in an asymmetric contest. □

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