Towards Singularity- and Ghost-Free Theories of Gravity

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We present the most general covariant ghost-free gravitational action in a Minkowski vacuum. Apart from the much studied $f(R)$ models, this includes a large class of nonlocal actions with improved UV behavior, which nevertheless recover Einstein's general relativity in the IR.

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The theory of General Relativity (GR) has an ultraviolet (UV) problem which is typically manifested in cosmological or black-hole type singularities. Any resolution to this problem requires a theory which is well behaved in the UV and reduces suitably to Einstein's gravity in the infrared (IR). (In the light of current cosmic acceleration observations, there have been efforts to modify gravity at large distances, see [1] for a review, but we do not discuss these models here.) In this Letter, our aim is to investigate whether the typical divergences at short distances can be ameliorated in higher derivative covariant generalizations of GR.

Higher derivative theories of gravity are generally better behaved in the UV and offer an improved chance to construct a singularity free theory [2]. Furthermore, Ref. [3] demonstrated that fourth order theories of gravity are renormalizable, but inevitably suffer from unphysical ghost states. Therefore, before we address the short-distance behavior of GR, we first enumerate the subset of all possible modifications to Einstein's gravity which are guaranteed to be ghost-free. To the best of our knowledge, a systematic method for this is not presently available.

Generic quadratic action of gravity.—Let us start with the most general covariant action of gravity. We immediately realize that to understand both the asymptotic behavior in the UV and the issue of ghosts, we require only the graviton propagator. In other words, we look at metric fluctuations around the Minkowski background

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{1}$$

and consider terms in the action that are quadratic in $h_{\mu\nu}$. Since in the Minkowski background $R_{\mu\nu\lambda\sigma}$ vanishes, every appearance of the Riemann tensor contributes an $\mathcal{O}(h)$ term in the action. Hence, we consider only terms that are products of at most two curvature terms, and higher ones simply do not play any role in this analysis. The most general relevant action is of the form

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{2} + R_{\mu_1\nu_1\lambda_1\sigma_1}\mathcal{O}^{\mu_1\nu_1\lambda_1\sigma_1}_{\mu_2\nu_2\lambda_2\sigma_2}R^{\mu_2\nu_2\lambda_2\sigma_2}\right], \tag{2}$$

where $\mathcal{O}$ is a differential operator containing covariant derivatives and $\eta_{\mu\nu}$. We note that if there is a differential operator acting on the left Riemann tensor, one can always recast that into the above form by integrating by parts. The most general action is captured by 14 arbitrary functions, the F_i's, which reduce to the 6 we display in Eq. (A1) upon repeated application of the Bianchi identities.

Our next task is to obtain the quadratic (in $h_{\mu\nu}$) free part of this action. Since the curvature vanishes on the Minkowski background, the two h dependent terms must come from the two curvature terms present. This means the covariant derivatives take on their Minkowski values. As is obvious, many of the terms simplify and combine to eventually produce the following action

$$\begin{aligned} S_q = -\int d^4x\Big[&\frac{1}{2}h_{\mu\nu}a(\Box)\Box h^{\mu\nu} + h^{\sigma}_{\mu}b(\Box)\partial_\sigma\partial_\nu h^{\mu\nu} \\ &+ hc(\Box)\partial_\mu\partial_\nu h^{\mu\nu} + \frac{1}{2}hd(\Box)\Box h \\ &+ h^{\lambda\sigma}\frac{f(\Box)}{\Box}\partial_\sigma\partial_\lambda\partial_\mu\partial_\nu h^{\mu\nu}\Big]. \end{aligned} \tag{3}$$

The above can be thought of as a higher derivative generalization of the action considered by van Nieuwenhuizen in Ref. [4]. Here, we have allowed a, b, c, d, and f to be nonlinear functions of the derivative operators that reduce in the appropriate limit to the constants a, b, c, and d of Ref. [4]. The function $f(\Box)$ appears only in higher derivative theories. In the Appendix (A3)–(A7), we have calculated the contribution from the Einstein-Hilbert term and the higher derivative modifications to the action in Eq. (3). From the explicit expressions we observe the following relationships:

$$a + b = 0, \tag{4}$$

$$c + d = 0, \tag{5}$$

$$b + c + f = 0, \tag{6}$$

so that we are left with only two independent arbitrary functions.

The field equations can be derived straightforwardly to yield

$$\begin{aligned} a(\Box)\Box h_{\mu\nu} &+ b(\Box)\partial_\sigma(\partial_\nu h^\sigma_\mu + \partial_\mu h^\sigma_\nu) \\ &+ c(\Box)(\eta_{\mu\nu}\partial_\rho\partial_\sigma h^{\rho\sigma} + \partial_\mu\partial_\nu h) + \eta_{\mu\nu} d(\Box)\Box h \\ &+ f(\Box)\Box^{-1}\partial_\sigma\partial_\lambda\partial_\mu\partial_\nu h^{\lambda\sigma} = -\kappa\tau_{\mu\nu}. \end{aligned} \tag{7}$$

While the matter sector obeys stress energy conservation, the geometric part is also conserved as a consequence of the generalized Bianchi identities:

$$\begin{aligned} -\kappa\tau\nabla_\mu\tau^\mu_\nu = 0 = (a + b)\Box h^\mu_{\nu,\mu} + (c + d)\Box\partial_\nu h \\ + (b + c + f)h^{\alpha\beta}_{,\alpha\beta\nu}. \end{aligned} \tag{8}$$

It is now clear why Eqs. (4)–(6) had to be satisfied.

Propagator and physical poles.—We are now well equipped to calculate the propagator. The above field equations can be written in the form

$$\Pi^{-1\lambda\sigma}_{\mu\nu} h_{\lambda\sigma} = \kappa\tau_{\mu\nu}, \tag{9}$$

where $\Pi^{-1\lambda\sigma}_{\mu\nu}$ is the inverse propagator. One obtains the propagator using the spin projection operators $\{P^2, P^0_s, P^0_w, P^1_m\}$, see Ref. [4]. They correspond to the spin-2, the two scalars, and the vector projections, respectively. These form a complete basis. Considering each sector separately and taking into account the constraints in Eqs. (4)–(6), we eventually arrive at a rather simple result

$$\Pi = \frac{P^2}{ak^2} + \frac{P^0_s}{(a - 3c)k^2}. \tag{10}$$

We note that the vector multiplet and the w scalar have disappeared, and the remaining s scalar has decoupled from the tensorial structure. Further, since we want to recover GR in the IR, we must have

$$a(0) = c(0) = -b(0) = -d(0) = 1, \tag{11}$$

corresponding to the GR values. This also means that as $k^2 \to 0$ we have only the physical graviton propagator:

$$\lim_{k^2\to 0} \Pi^{\mu\nu}_{\lambda\sigma} = (P^2/k^2) - (P^0_s/2k^2). \tag{12}$$

A few remarks are now in order: First, let us point out that although the P_s residue at $k^2 = 0$ is negative, it is a benign ghost. In fact, P^0_s has precisely the coefficient to cancel the unphysical longitudinal degrees of freedom in the spin two part [4]. Thus, we conclude that provided Eq. (11) is satisfied, the $k^2 = 0$ pole just describes the physical graviton state. Second, Eq. (11) essentially means that a and c are nonsingular analytic functions at $k^2 = 0$, and therefore cannot contain nonlocal inverse derivative operators (such as $a(\Box) \sim 1/\Box$).

Let us next scrutinize some of the well known special cases: $f(R)$ *gravity:* they are a subclass of scalar-tensor theories and are studied in great detail both in the context of early Universe cosmology and dark energy phenomenology. Here, only the F_1 appears as a higher derivative contribution (see Appendix). According to our preceding arguments, we obtain the physical states from the R^2 term. Since $a = 1$, it is easy to see that only the s multiplet propagator is modified. It now has two poles: $\Pi \sim -1/2k^2(k^2 - m^2) + \ldots$. The $k^2 = 0$ pole has, as usual, the wrong sign of the residue, while the second pole has the correct sign. This represents an additional scalar degree of freedom confirming the well known fact [5,6]. *Fourth order modification in* $R_{\mu\nu}R^{\mu\nu}$: They have also been considered in the literature. This corresponds to having an F_2 term (see Appendix), which modifies the spin-2 propagator: $\Pi \sim P_2/k^2(k^2 - m^2) + \ldots$. The second pole necessarily has the wrong residue sign and corresponds to the well known Weyl ghost, Refs. [5,6]. In fact, this situation is quite typical: $f(R)$ type models can be ghost-free, but they do not improve UV behavior, while modifications involving $R_{\mu\nu\lambda\sigma}$'s can improve the UV behavior [3] but typically contain the Weyl ghost.

To reconcile the two problems, we now propose first to look at a special class of nonlocal models with $f = 0$ or equivalently $a = c$. The propagator then simplifies to:

$$\Pi^{\mu\nu}_{\lambda\sigma} = \frac{1}{k^2 a(-k^2)}\left(P^2 - \frac{1}{2}P^0_s\right). \tag{13}$$

It is obvious that we are left with only a single arbitrary function $a(\Box)$, since now $a = c = -b = -d$. Most importantly, we now realize that as long as $a(\Box)$ has no zeroes, these theories contain no new states as compared to GR, and only modify the graviton propagator. In particular, by choosing $a(\Box)$ to be a suitable *entire function* we can indeed improve the UV behavior of gravitons without introducing ghosts. This will be discussed below.

Singularity free gravity.—We now analyze the scalar potentials in these nonlocal theories, focussing particularly on the short-distance behavior. As is usual, we solve the linearized modified Einstein's equations (7) for a point source:

$$\tau_{\mu\nu} = \rho\delta^0_\mu\delta^0_\nu = m\delta^3(\vec{r})\delta^0_\mu\delta^0_\nu. \tag{14}$$

Next, we compute the two potentials, $\Phi(r)$, $\Psi(r)$, corresponding to the metric

$$ds^2 = -(1 + 2\Phi)dt^2 + (1 - 2\Psi)dx^2. \tag{15}$$

Because of the Bianchi identities [7,8], we only need to solve the trace and the 00 component of Eq. (7). Since the

Newtonian potentials are static, the trace and 00 equation simplifies considerably to yield

$$(a-3c)\Box h+(4c-2a+f)\partial_\mu\partial_\nu h^{\mu\nu}=\kappa\rho,$$
$$a\Box h_{00}+c\Box h-c\partial_\mu\partial_\nu h^{\mu\nu}=-\kappa\rho, \tag{16}$$

which for the metric Eq. (15) simplify to

$$2(a-3c)[\nabla^2\Phi-4\nabla^2\Psi]=\kappa\rho,$$
$$2(c-a)\nabla^2\Phi-4c\nabla^2\Psi=-\kappa\rho. \tag{17}$$

We are seeking functions $c(\Box)$ and $a(\Box)$, such that there are no ghosts and no $1/r$ divergence at short distances.

For $f=0$, the Newtonian potentials are solved easily:

$$4a(\nabla^2)\nabla^2\Phi=4a(\nabla^2)\nabla^2\Psi=\kappa\rho=\kappa m\delta^3(\vec{r}). \tag{18}$$

Now, we know that in order to avoid the problem of ghosts, $a(\Box)$ must be an entire function. Let us first illustrate the resolution of singularities by considering the following functional dependence [2]:

$$a(\Box)=e^{-\Box/M^2}. \tag{19}$$

Such exponential kinetic operators appear frequently in string theory [9]. In fact, quantum loops in such stringy nonlocal scalar theories remain finite giving rise to interesting physics, such as linear Regge trajectories [10] and thermal duality [11]. We note that there are a wide range of allowed possible energy scales for M, including roughly the range between Λ and M_{pl}.

Taking the Fourier components of Eq. (18), in a straightforward manner one obtains

$$\Phi(r)\sim\frac{m}{M_p^2}\int d^3p\frac{e^{i\vec{p}\vec{r}}}{p^2a(-p^2)}=\frac{4\pi m}{rM_p^2}\int\frac{dp}{p}\frac{\sin pr}{a(-p^2)}. \tag{20}$$

We note that the $1/r$ divergent piece comes from the usual GR action, but now it is ameliorated. For Eq. (19) we have

$$\Phi(r)\sim\frac{m}{M_p^2r}\int\frac{dp}{p}e^{-p^2/M^2}\sin(pr)=\frac{m\pi}{2M_p^2r}\mathrm{erf}\left(\frac{rM}{2}\right), \tag{21}$$

and the same for $\Psi(r)$. We observe that as $r\to\infty$, $\mathrm{erf}(r)\to 1$, and we recover the GR limit. On the other hand, as $r\to 0$, $\mathrm{erf}(r)\to r$, making the Newtonian potential converge to a constant $\sim mM/M_p^2$. Thus, although the matter source has a delta function singularity, the Newtonian potentials remain finite. Further, provided $mM\ll M_p$, our linear approximation can be trusted all the way to $r\to 0$.

Let us next verify the absence of singularities in the spin-2 sector. This will allow us, for example, to derive a singularity free quadrupole potential. We enforce the Lorentz gauge as usual so that the generalized field equations (7) read

$$a\Box h_{\mu\nu}-\frac{f}{2}\partial_\mu\partial_\nu h-\frac{c}{2}\eta_{\mu\nu}\Box h=-\kappa\tau_{\mu\nu}. \tag{22}$$

Again for $f=0$ we have a simple wave equation for the graviton $a(\Box)\Box\bar{h}_{\mu\nu}=-\kappa\tau_{\mu\nu}$. We invert Einstein's equations for $\bar{h}_{\mu\nu}$ to obtain the Greens function, $\bar{G}_{\mu\nu}$, for a pointlike energy-momentum source. In other words, we solve for

$$a(\Box)\Box\bar{G}_{\mu\nu}(x-y)=-\kappa\tau_{\mu\nu}\delta^4(x-y). \tag{23}$$

Under the assumption of slowly varying sources, one has

$$\bar{G}_{\mu\nu}(r)\sim\frac{\kappa}{r}\pi\mathrm{erf}\left[\frac{rM}{2}\right]\tau_{\mu\nu}(r), \tag{24}$$

for $a(\Box)$ given in Eq. (19). We observe that in the limit $r\to 0$, the Greens function remains singularity free. The improved scaling takes effect roughly only for $r<1/M$.

Cosmological singularities.—The very general framework of this Letter allows us to consistently address the singularities in early Universe cosmology. As an example, we note that a solution to Eq. (7) with

$$h\sim\mathrm{diag}(0,A\sin\lambda t,A\sin\lambda t,A\sin\lambda t)\quad\text{with}\quad A\ll 1, \tag{25}$$

describes a Minkowski space-time with small oscillations [12]. This configuration is singularity free. Evaluating the field equations for Eq. (25) gives the constraint $a(-\lambda^2)-3c(-\lambda^2)=0$. Thus, our simple $f=0$ case is not sufficient and we require an additional scalar degree of freedom in the s multiplet. Note that this also explains why a solution such as Eq. (25) is absent in GR. We generalize to $f\neq 0$, but take special care to keep intact our results in Eqs. (11) and (18). The most general ghost-free parametrization for $a\neq c$ is

$$c(\Box)\equiv\frac{a(\Box)}{3}\left[1+2\left(1-\frac{\Box}{m^2}\right)\tilde{c}(\Box)\right], \tag{26}$$

where $\tilde{c}(\Box)$, $a(\Box)$ are entire functions. Note that $m^2\to\infty$ and $\tilde{c}=1$ reproduces the $f=0$ limit. We now find that Eq. (25) is a solution to the vacuum field equations with $\lambda=m$. How the Universe can grow in such models and also how the matter sector can influence the dynamics can possibly be addressed only with knowledge of the full curvature terms. We hope to investigate this in future work, but see Ref. [13,14] for similar considerations.

Generality.—How general are the above arguments leading to a lack of singularities? According to the Weierstrass theorem, any entire function is written as $a(\Box)=e^{-\gamma(\Box)}$, where $\gamma(\Box)$ is an analytic function. For a polynomial $\gamma(\Box)$ it is now easy to see that if $\gamma>0$ as $\Box\to\infty$, the propagator is even more convergent than the exponential case leading to nonsingular UV behavior.

Conclusion.—We have shown that by allowing higher derivative nonlocal operators, we may be able to render gravity singularity free without introducing ghosts or any other pathologies around the Minkowski background. It should be reasonably straightforward to extend the analysis to de Sitter backgrounds by including appropriate cosmological constants. In fact, requiring that the theory remains free from ghosts around different classical vacua may be a way to constrain the higher curvature terms that did not seem to play any role in our analysis. Other ways of constraining or determining the higher curvature terms would be to look for additional symmetries or to try to extend Stelle's renormalizability arguments to these non-local theories. Efforts in this direction have been made [15]. Finally, it is known that one can obtain GR starting from the free quadratic theory for $h_{\mu\nu}$ by consistently coupling to its own stress energy tensor. Similarly, can one obtain *unique* consistent covariant extensions of the higher derivative quadratic actions that we have considered? We leave these questions for future investigations.

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Appendix.—The quadratic action in curvature reads

$$\begin{aligned}S_q = \int d^4x\sqrt{-g}[&RF_1(\Box)R + R_{\mu\nu}F_2(\Box)R^{\mu\nu}\\ &+ R_{\mu\nu\lambda\sigma}F_3(\Box)R^{\mu\nu\lambda\sigma} + RF_4(\Box)\nabla_\mu\nabla_\nu\nabla_\lambda\nabla_\sigma R^{\mu\nu\lambda\sigma}\\ &+ R_\mu^{\nu_1\rho_1\sigma_1}F_5(\Box)\nabla_{\rho_1}\nabla_{\sigma_1}\nabla_{\nu_1}\nabla_\nu\nabla_\rho\nabla_\sigma R^{\mu\nu\lambda\sigma}\\ &+ R^{\mu_1\nu_1\rho_1\sigma_1}F_6(\Box)\nabla_{\rho_1}\nabla_{\sigma_1}\nabla_{\nu_1}\nabla_{\mu_1}\nabla_\mu\nabla_\nu\nabla_\rho\nabla_\sigma R^{\mu\nu\lambda\sigma}],\end{aligned} \tag{A1}$$

where we have used the Bianchi identities:

$$\nabla_\sigma R_{\mu\nu\lambda\rho} + \nabla_\rho R_{\mu\nu\sigma\lambda} + \nabla_\lambda R_{\mu\nu\rho\sigma} = 0, \tag{A2}$$

to absorb all the other covariant terms into the above six. Further, in the F_4, F_5, and F_6 terms, one ends up with anticommutator of the covariant derivatives due to the antisymmetric properties of the Reimann tensor, but these anticommutators produce a third curvature term, and therefore these terms are at least $\mathcal{O}(h^3)$. Thus, the coefficients of the free theory (3) in terms of the F's are given by

$$a(\Box) = 1 - \frac{1}{2}F_2(\Box)\Box - 2F_3(\Box)\Box, \tag{A3}$$

$$b(\Box) = -1 + \frac{1}{2}F_2(\Box)\Box + 2F_3(\Box)\Box, \tag{A4}$$

$$c(\Box) = 1 + 2F_1(\Box) + \frac{1}{2}F_2(\Box)\Box, \tag{A5}$$

$$d(\Box) = -1 - 2F_1(\Box)\Box - \frac{1}{2}F_2(\Box), \tag{A6}$$

$$f(\Box) = -2F_1(\Box)\Box - F_2(\Box)\Box - 2F_3(\Box)\Box. \tag{A7}$$

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