Special Lagrangian submanifolds with isolated conical singularities. II. Moduli spaces

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1 Introduction

Special Lagrangian m-folds (SL m-folds) are a distinguished class of real m-dimensional minimal submanifolds which may be defined in $\mathbb{C}^m$, or in *Calabi–Yau m-folds*, or more generally in *almost Calabi–Yau m-folds* (compact Kähler m-folds with trivial canonical bundle).

This is the second in a series of five papers [12, 13, 14, 15] studying SL m-folds with *isolated conical singularities*. That is, we consider an SL m-fold X in M with singularities at $x_1, \ldots, x_n$ in M, such that for some SL cones C_i in $T_{x_i}M \cong \mathbb{C}^m$ with $C_i \setminus \{0\}$ nonsingular, X approaches C_i near x_i in an asymptotic C^1 sense. Readers are advised to begin with the final paper [15], which surveys the series, and applies the results to prove some conjectures.

Having a good understanding of the singularities of special Lagrangian submanifolds will be essential in clarifying the Strominger–Yau–Zaslow conjecture on the Mirror Symmetry of Calabi–Yau 3-folds [22], and also in resolving conjectures made by the author [6] on defining new invariants of Calabi–Yau 3-folds by counting special Lagrangian homology 3-spheres with weights. The series aims to develop such an understanding for simple singularities of SL m-folds.

In this paper we study the *deformation theory* of compact SL m-folds X with conical singularities $x_1, \ldots, x_n$ with cones $C_1, \ldots, C_n$ in an almost Calabi–Yau m-fold M, extending results of McLean [21] for nonsingular compact SL m-folds. We define the *moduli space* $\mathcal{M}_X$ of deformations of X as an SL m-fold with conical singularities in M, and construct a natural *topology* on $\mathcal{M}_X$.

We prove that $\mathcal{M}_X$ is locally homeomorphic to the zeroes of a smooth map $\Phi : \mathcal{I}_{X'} \to \mathcal{O}_{X'}$, where the *infinitesimal deformation space* $\mathcal{I}_{X'}$ and the *obstruction space* $\mathcal{O}_{X'}$ are finite-dimensional vector spaces. Here $\mathcal{I}_{X'}$ depends only on the topology of X, and $\mathcal{O}_{X'}$ only on the singular cones $C_1, \ldots, C_n$. If $\mathcal{O}_{X'}$ is zero then $\mathcal{M}_X$ is a *smooth manifold*. We also consider deformations of X in a *smooth family* of almost Calabi–Yau m-folds $\{(M, J^s, \omega^s, \Omega^s) : s \in \mathcal{F}\}$.

The first paper [12] laid the foundations for the series, and studied the *regularity* of SL m-folds with conical singularities near their singular points. The sequels [13, 14] will consider *desingularizations* of a compact SL m-fold X with conical singularities $x_1, \ldots, x_n$ with cones $C_1, \ldots, C_n$ in M. We will take nonsingular SL m-folds $L_1, \ldots, L_n$ in $\mathbb{C}^m$ asymptotic to $C_1, \ldots, C_n$ at infinity, and glue them in to X at $x_1, \ldots, x_n$ to get a smooth family of compact, *nonsingular* SL m-folds $\tilde{N}$ in M which converge to X.

We begin in §2 with an introduction to special Lagrangian geometry, and the deformation theory of nonsingular compact SL m-folds. Section 3 discusses *special Lagrangian cones* and *conical singularities* of SL m-folds. The previous paper [12] is reviewed in §4. To keep this paper and [13, 14] to a manageable length we have done quite a lot of work on symplectic geometry and asymptotic analysis in advance in [12], and we just quote the results.

Section 5 defines the moduli space $\mathcal{M}_X$ of SL m-folds and its topology, and explains why this definition of topology is a good one. In §6 we define the *infinitesimal deformation space* $\mathcal{I}_{X'}$ and the *obstruction space* $\mathcal{O}_{X'}$, and prove our first main result, Theorem 6.10, which shows that the moduli space $\mathcal{M}_X$ is locally homeomorphic to the zeroes of a smooth map $\Phi : \mathcal{I}_{X'} \rightarrow \mathcal{O}_{X'}$. Thus, if $\mathcal{O}_{X'}$ is zero then $\mathcal{M}_X$ is a manifold. More generally, if $\mathrm{d}\Phi|_0$ is surjective then $\mathcal{M}_X$ is a manifold near X.

Section 7 extends §5–§6 to *families* $\{(M, J^s, \omega^s, \Omega^s) : s \in \mathcal{F}\}$ of almost Calabi–Yau m-folds. We define a *joint moduli space* $\mathcal{M}_X^{\mathcal{F}}$ with *projection* $\pi^{\mathcal{F}} : \mathcal{M}_X^{\mathcal{F}} \rightarrow \mathcal{F}$ such that $\mathcal{M}_X^s = (\pi^{\mathcal{F}})^{-1}(s)$ is the moduli space of deformations of X in $(M, J^s, \omega^s, \Omega^s)$ for $s \in \mathcal{F}$. Then we show that $\mathcal{M}_X^{\mathcal{F}}$ is locally homeomorphic to the zeroes of a smooth map $\Phi^{\mathcal{F}} : \mathcal{F} \times \mathcal{I}_{X'} \rightarrow \mathcal{O}_{X'}$, where $\mathcal{I}_{X'}, \mathcal{O}_{X'}$ are as before.

Section 8 briefly describes various other extensions of the results to immersions, families of SL cones in $\mathbb{C}^m$, and so on. Finally, §9 considers *genericity* and *transversality* results. We show that for any compact SL m-fold X with conical singularities in (M, J, ω, Ω), we can choose a family of deformations $\{(M, J, \omega^s, \Omega) : s \in \mathcal{F}\}$ such that $\mathcal{M}_X^{\mathcal{F}}$ is a manifold near $(0, X)$, and for small generic $s \in \mathcal{F}$ the deformed moduli space $\mathcal{M}_X^s = (\pi^{\mathcal{F}})^{-1}(s)$ is smooth near $(0, X)$. We conjecture that if the Kähler form ω is chosen *generically in its Kähler class*, then $\mathcal{M}_X$ is smooth.

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2 Special Lagrangian geometry

We now introduce special Lagrangian submanifolds (SL m-folds) in two different geometric contexts. First, in §2.1, we define SL m-folds in $\mathbb{C}^m$. Then §2.2 discusses SL m-folds in *almost Calabi–Yau m-folds*, compact Kähler manifolds with a holomorphic volume form, which generalize Calabi–Yau manifolds. Section 2.3 describes the *deformation theory* of compact SL m-folds. Some references for this section are Harvey and Lawson [4], McLean [21], and the author [11].

2.1 Special Lagrangian submanifolds in $\mathbb{C}^m$

We begin by defining *calibrations* and *calibrated submanifolds*, following Harvey and Lawson [4].

Definition 2.1 Let (M, g) be a Riemannian manifold. An *oriented tangent k-plane* V on M is a vector subspace V of some tangent space $T_x M$ to M with $\dim V = k$, equipped with an orientation. If V is an oriented tangent k-plane on M then $g|_V$ is a Euclidean metric on V, so combining $g|_V$ with the orientation on V gives a natural *volume form* vol_V on V, which is a k-form on V.

Now let φ be a closed k-form on M. We say that φ is a *calibration* on M if for every oriented k-plane V on M we have $\varphi|_V \leqslant \mathrm{vol}_V$. Here $\varphi|_V = \alpha \cdot \mathrm{vol}_V$ for some $\alpha \in \mathbb{R}$, and $\varphi|_V \leqslant \mathrm{vol}_V$ if $\alpha \leqslant 1$. Let N be an oriented submanifold of M with dimension k. Then each tangent space $T_x N$ for $x \in N$ is an oriented tangent k-plane. We say that N is a *calibrated submanifold* if $\varphi|_{T_x N} = \mathrm{vol}_{T_x N}$ for all $x \in N$.

It is easy to show that calibrated submanifolds are automatically *minimal submanifolds* [4, Th. II.4.2]. Here is the definition of special Lagrangian submanifolds in $\mathbb{C}^m$, taken from [4, §III].

Definition 2.2 Let $\mathbb{C}^m$ have complex coordinates $(z_1, \ldots, z_m)$, and define a metric g', a real 2-form ω' and a complex m-form Ω' on $\mathbb{C}^m$ by

$$\begin{gathered} g' = |\mathrm{d}z_1|^2 + \cdots + |\mathrm{d}z_m|^2, \quad \omega' = \frac{i}{2}(\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 + \cdots + \mathrm{d}z_m \wedge \mathrm{d}\bar{z}_m), \\ \text{and} \quad \Omega' = \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_m. \end{gathered} \tag{1}$$

Then $\mathrm{Re}\,\Omega'$ and $\mathrm{Im}\,\Omega'$ are real m-forms on $\mathbb{C}^m$. Let L be an oriented real submanifold of $\mathbb{C}^m$ of real dimension m. We say that L is a *special Lagrangian submanifold* of $\mathbb{C}^m$, or *SL m-fold* for short, if L is calibrated with respect to $\mathrm{Re}\,\Omega'$, in the sense of Definition 2.1.

Harvey and Lawson [4, Cor. III.1.11] give the following alternative characterization of special Lagrangian submanifolds:

Proposition 2.3 *Let L be a real m-dimensional submanifold of $\mathbb{C}^m$. Then L admits an orientation making it into an SL submanifold of $\mathbb{C}^m$ if and only if $\omega'|_L \equiv 0$ and $\mathrm{Im}\,\Omega'|_L \equiv 0$.*

Thus SL m-folds are *Lagrangian submanifolds* in $\mathbb{R}^{2m} \cong \mathbb{C}^m$ satisfying the extra condition that $\mathrm{Im}\,\Omega'|_L \equiv 0$, which is how they get their name.

2.2 Almost Calabi–Yau m-folds and SL m-folds

We shall define special Lagrangian submanifolds not just in Calabi–Yau manifolds, as usual, but in the much larger class of *almost Calabi–Yau manifolds*.

Definition 2.4 Let $m \geqslant 2$. An *almost Calabi–Yau m-fold* is a quadruple (M, J, ω, Ω) such that (M, J) is a compact m-dimensional complex manifold,

ω is the Kähler form of a Kähler metric g on M, and Ω is a non-vanishing holomorphic $(m,0)$-form on M.

We call (M,J,ω,Ω) a *Calabi–Yau m-fold* if in addition ω and Ω satisfy

$$\omega^m/m! = (-1)^{m(m-1)/2}(i/2)^m \Omega\wedge\bar\Omega. \tag{2}$$

Then for each $x\in M$ there exists an isomorphism $T_xM\cong\mathbb{C}^m$ that identifies g_x,ω_x and Ω_x with the flat versions g',ω',Ω' on $\mathbb{C}^m$ in (1). Furthermore, g is Ricci-flat and its holonomy group is a subgroup of $\mathrm{SU}(m)$.

This is not the usual definition of a Calabi–Yau manifold, but is essentially equivalent to it.

Definition 2.5 Let (M,J,ω,Ω) be an almost Calabi–Yau m-fold, and N a real m-dimensional submanifold of M. We call N a *special Lagrangian submanifold*, or *SL m-fold* for short, if $\omega|_N\equiv\operatorname{Im}\Omega|_N\equiv 0$. It easily follows that $\operatorname{Re}\Omega|_N$ is a nonvanishing m-form on N. Thus N is orientable, with a unique orientation in which $\operatorname{Re}\Omega|_N$ is positive.

Again, this is not the usual definition of SL m-fold, but is essentially equivalent to it. Suppose (M,J,ω,Ω) is an almost Calabi–Yau m-fold, with metric g. Let $\psi: M\rightarrow(0,\infty)$ be the unique smooth function such that

$$\psi^{2m}\omega^m/m! = (-1)^{m(m-1)/2}(i/2)^m\Omega\wedge\bar\Omega, \tag{3}$$

and define $\tilde g$ to be the conformally equivalent metric $\psi^2 g$ on M. Then $\operatorname{Re}\Omega$ is a *calibration* on the Riemannian manifold $(M,\tilde g)$, and SL m-folds N in (M,J,ω,Ω) are calibrated with respect to it, so that they are minimal with respect to $\tilde g$.

If M is a Calabi–Yau m-fold then $\psi\equiv 1$ by (2), so $\tilde g = g$, and an m-submanifold N in M is special Lagrangian if and only if it is calibrated w.r.t. $\operatorname{Re}\Omega$ on (M,g), as in Definition 2.2. This recovers the usual definition of special Lagrangian m-folds in Calabi–Yau m-folds.

2.3 Deformations of compact SL m-folds

The *deformation theory* of special Lagrangian submanifolds was studied by McLean [21, §3], who proved the following result in the Calabi–Yau case. The extension to the almost Calabi–Yau case is described in [11, §9.5].

Theorem 2.6 *Let N be a compact SL m-fold in an almost Calabi–Yau m-fold (M,J,ω,Ω). Then the moduli space $\mathcal{M}_X$ of special Lagrangian deformations of N is a smooth manifold of dimension $b^1(N)$, the first Betti number of N.*

We now give a partial proof of Theorem 2.6, glossing over the analytic details, and concentrating on the parts we will use later. We start by recalling some symplectic geometry, which can be found in McDuff and Salamon [19].

Let N be a real m-manifold. Then its tangent bundle T^*N has a *canonical symplectic form* $\hat\omega$, defined as follows. Let $(x_1,\ldots,x_m)$ be local coordinates on N. Extend them to local coordinates $(x_1,\ldots,x_m,y_1,\ldots,y_m)$ on T^*N such that $(x_1, \ldots, y_m)$ represents the 1-form $y_1 \mathrm{d}x_1 + \cdots + y_m \mathrm{d}x_m$ in $T^*_{(x_1,\ldots,x_m)}N$. Then $\hat{\omega} = \mathrm{d}x_1 \wedge \mathrm{d}y_1 + \cdots + \mathrm{d}x_m \wedge \mathrm{d}y_m$.

Identify N with the zero section in T^*N. Then N is a *Lagrangian submanifold* of T^*N. The *Lagrangian Neighbourhood Theorem* [19, Th. 3.33] shows that any compact Lagrangian submanifold N in a symplectic manifold looks locally like the zero section in T^*N.

Theorem 2.7 *Let (M, ω) be a symplectic manifold and $N \subset M$ a compact Lagrangian submanifold. Then there exists an open tubular neighbourhood U of the zero section N in T^*N, and an embedding $\Phi : U \to M$ with $\Phi|_N = \mathrm{id} : N \to N$ and $\Phi^*(\omega) = \hat{\omega}$, where $\hat{\omega}$ is the canonical symplectic structure on T^*N.*

In the situation of Theorem 2.6, let g be the Kähler metric on M, and define $\psi : M \to (0, \infty)$ by (3). Applying Theorem 2.7 gives an open neighbourhood U of N in T^*N and an embedding $\Phi : U \to M$. Let $\pi : U \to N$ be the natural projection. Define an m-form β on U by $\beta = \Phi^*(\mathrm{Im}\,\Omega)$. If α is a 1-form on N let $\Gamma(\alpha)$ be the graph of α in T^*N, and write $C^\infty(U) \subset C^\infty(T^*N)$ for the subset of 1-forms whose graphs lie in U.

Then each submanifold $\tilde{N}$ of M which is C^1-close to N is $\Phi(\Gamma(\alpha))$ for some small $\alpha \in C^\infty(U)$. Here is the condition for $\tilde{N}$ to be special Lagrangian.

Lemma 2.8 *In the situation above, if $\alpha \in C^\infty(U)$ then $\tilde{N} = \Phi\bigl(\Gamma(\alpha)\bigr)$ is a special Lagrangian m-fold in M if and only if $\mathrm{d}\alpha = 0$ and $\pi_*\bigl(\beta|_{\Gamma(\alpha)}\bigr) = 0$.*

Proof. By Definition 2.5, $\tilde{N}$ is an SL m-fold in M if and only if $\omega|_{\tilde{N}} \equiv \mathrm{Im}\,\Omega|_{\tilde{N}} \equiv 0$. Pulling back by Φ and pushing forward by $\pi : \Gamma(\alpha) \to N$, we see that $\tilde{N}$ is special Lagrangian if and only if $\pi_*\bigl(\hat{\omega}|_{\Gamma(\alpha)}\bigr) \equiv \pi_*\bigl(\beta|_{\Gamma(\alpha)}\bigr) \equiv 0$, since $\Phi^*(\omega) = \hat{\omega}$ and $\Phi^*(\mathrm{Im}\,\Omega) = \beta$. But as $\hat{\omega}$ is the natural symplectic structure on $U \subset T^*N$ we have $\pi_*\bigl(\hat{\omega}|_{\Gamma(\alpha)}\bigr) = -\mathrm{d}\alpha$, and the lemma follows. □

We rewrite the condition $\pi_*\bigl(\beta|_{\Gamma(\alpha)}\bigr) = 0$ in terms of a function F.

Definition 2.9 Define $F : C^\infty(U) \to C^\infty(N)$ by $\pi_*\bigl(\beta|_{\Gamma(\alpha)}\bigr) = F(\alpha)\,\mathrm{d}V_g$, where $\mathrm{d}V_g$ is the volume form of $g|_N$ on N. Then Lemma 2.8 shows that if $\alpha \in C^\infty(U)$ then $\Phi\bigl(\Gamma(\alpha)\bigr)$ is special Lagrangian if and only if $\mathrm{d}\alpha = F(\alpha) = 0$.

We compute the expansion of F up to first order in α.

Proposition 2.10 *This function F may be written*

$$F(\alpha)[x] = -\mathrm{d}^*\bigl(\psi^m \alpha\bigr) + Q\bigl(x, \alpha(x), \nabla\alpha(x)\bigr) \quad \textit{for } x \in N, \tag{4}$$

where $Q : \bigl\{(x, y, z) : x \in N,\ y \in T^*_x N \cap U,\ z \in \otimes^2 T^*_x N\bigr\} \to \mathbb{R}$ is smooth and $Q(x, y, z) = O(|y|^2 + |z|^2)$ for small y, z.

Proof. The value of $F(\alpha)$ at $x \in N$ depends on the tangent space $T_{x'}\Gamma(\alpha)$, where $x' \in \Gamma(\alpha)$ with $\pi(x') = x$. But $T_{x'}\Gamma(\alpha)$ depends on both $\alpha|_x$ and $\nabla\alpha|_x$. Hence

$F(\alpha)$ depends pointwise on both α and $\nabla\alpha$, rather than just α. So we may take (4) as a *definition* of Q, and Q is then well-defined on the set of all (x,y,z) realized by $\bigl(x,\alpha(x),\nabla\alpha(x)\bigr)$ for $\alpha\in C^\infty(U)$, which is the domain given for Q.

As F depends smoothly on α we see that Q is a smooth function of its arguments. Therefore Taylor's theorem yields

$$Q(x,y,z)=Q(x,0,0)+y\cdot(\partial_yQ)(x,0,0)+z\cdot(\partial_zQ)(x,0,0)+O(|y|^2+|z|^2)$$

for small y,z. So to prove that $Q(x,y,z)=O(|y|^2+|z|^2)$ we just need to show that $Q(x,0,0)=\partial_yQ(x,0,0)=\partial_zQ(x,0,0)=0$. Now $N=\Phi(\Gamma(0))$ is special Lagrangian, so $\alpha=0$ satisfies $F(\alpha)=0$ by Definition 2.9. Thus (4) gives $Q(x,0,0)\equiv 0$.

To compute $\partial_yQ(x,0,0)$ and $\partial_zQ(x,0,0)$, let $\alpha\in C^\infty(U)$ be small, and let v be the vector field on T^*N with $v\cdot\hat\omega=-\pi^*(\alpha)$. Then v is tangent to the fibres of $\pi:T^*N\rightarrow N$, and $\exp(v)$ maps $T^*N\rightarrow T^*N$ taking $\gamma\mapsto\alpha+\gamma$ for 1-forms γ on N. Identifying N with the zero section of T^*N, the image $\exp(sv)[N]$ of N under $\exp(sv)$ is $\Gamma(s\alpha)$ for $s\in[0,1]$.

Therefore $F(s\alpha)\,\mathrm{d}V_g=\exp(sv)^*(\beta)$ for $s\in[0,1]$. Differentiating gives

$$\begin{aligned}\mathrm{d}F|_0(\alpha)\,\mathrm{d}V_g&=\frac{\mathrm{d}}{\mathrm{d}s}\bigl(F(s\alpha)\bigr)\Big|_{s=0}\mathrm{d}V_g=\frac{\mathrm{d}}{\mathrm{d}s}\bigl(\exp(sv)^*(\beta)\bigr)\Big|_{s=0}\\&=\bigl(\mathcal{L}_v\beta\bigr)\Big|_N=\bigl(\mathrm{d}(v\cdot\beta)+v\cdot(\mathrm{d}\beta)\bigr)\Big|_N=\mathrm{d}\bigl((v\cdot\beta)|_N\bigr),\end{aligned}\tag{5}$$

where $\mathcal{L}_v$ is the Lie derivative, '$\cdot$' contracts together vector fields and forms, and $\mathrm{d}\beta=0$ as Ω is closed and $\beta=\Phi^*(\mathrm{Im}\,\Omega)$.

Calculation at a point $x\in N$ shows that $(v\cdot\beta)|_N=\psi^m*\alpha$, where $*$ is the Hodge star of g on N. As $*\mathrm{d}V_g=1$ and $*\mathrm{d}*=-\mathrm{d}^*$ on 1-forms, (5) gives

$$\mathrm{d}F|_0(\alpha)\,\mathrm{d}V_g=\mathrm{d}(\psi^m*\alpha)=\bigl(*\mathrm{d}*(\psi^m\alpha)\bigr)\,\mathrm{d}V_g=\bigl(-\mathrm{d}^*(\psi^m\alpha)\bigr)\,\mathrm{d}V_g.$$

Comparing this with (4) shows that $\partial_yQ(x,0,0)=\partial_zQ(x,0,0)=0$, which completes the proof. □

We briefly sketch the remainder of the proof of Theorem 2.6. From Definition 2.9 and Proposition 2.10 we see that $\mathcal{M}_X$ is locally approximately isomorphic to the vector space of 1-forms α with $\mathrm{d}\alpha=\mathrm{d}^*(\psi^m\alpha)=0$. But by Hodge theory, this is isomorphic to the de Rham cohomology group $H^1(N,\mathbb{R})$, and is a manifold with dimension $b^1(N)$.

To carry out this last step rigorously requires some technical machinery: one must work with certain *Banach spaces* of sections of Λ^kT^*N for $k=0,1,2$, use *elliptic regularity results* to prove that the map $\alpha\mapsto\bigl(\mathrm{d}\alpha,\mathrm{d}F|_0(\alpha)\bigr)$ is *surjective* upon the appropriate Banach spaces, and then use the *Implicit Mapping Theorem for Banach spaces* to show that the kernel of the map is what we expect.

Finally we extend of Theorem 2.6 to *families* of almost Calabi–Yau m-folds.

Definition 2.11 Let (M,J,ω,Ω) be an almost Calabi–Yau m-fold. A *smooth family of deformations of* (M,J,ω,Ω) is a connected open set $\mathcal{F}\subset\mathbb{R}^d$ for $d\geqslant 0$

with $0 \in \mathcal{F}$ called the *base space*, and a smooth family $\{(M, J^s, \omega^s, \Omega^s) : s \in \mathcal{F}\}$ of almost Calabi–Yau structures on M with $(J^0, \omega^0, \Omega^0) = (J, \omega, \Omega)$.

If N is a compact SL m-fold in (M, J, ω, Ω), the moduli of deformations of N in each $(M, J^s, \omega^s, \Omega^s)$ for $s \in \mathcal{F}$ make up a big moduli space $\mathcal{M}_X^{\mathcal{F}}$.

Definition 2.12 Let $\{(M, J^s, \omega^s, \Omega^s) : s \in \mathcal{F}\}$ be a smooth family of deformations of an almost Calabi–Yau m-fold (M, J, ω, Ω), and N be a compact SL m-fold in (M, J, ω, Ω). Define the *moduli space* $\mathcal{M}_X^{\mathcal{F}}$ *of deformations of* N *in the family* $\mathcal{F}$ to be the set of pairs $(s, \hat{N})$ for which $s \in \mathcal{F}$ and $\hat{N}$ is a compact SL m-fold in $(M, J^s, \omega^s, \Omega^s)$ which is diffeomorphic to N and isotopic to N in M. Define a *projection* $\pi^{\mathcal{F}} : \mathcal{M}_X^{\mathcal{F}} \to \mathcal{F}$ by $\pi^{\mathcal{F}}(s, \hat{N}) = s$. Then $\mathcal{M}_X^{\mathcal{F}}$ has a natural topology, and $\pi^{\mathcal{F}}$ is continuous.

The following result is proved by Marshall [17, Th. 3.2.9], using similar methods to Theorem 2.6.

Theorem 2.13 *Let* $\{(M, J^s, \omega^s, \Omega^s) : s \in \mathcal{F}\}$ *be a smooth family of deformations of an almost Calabi–Yau* m*-fold* (M, J, ω, Ω)*, with base space* $\mathcal{F} \subset \mathbb{R}^d$*. Suppose* N *is a compact SL* m*-fold in* (M, J, ω, Ω) *with* $[\omega^s|_N] = 0$ *in* $H^2(N, \mathbb{R})$ *and* $[\operatorname{Im} \Omega^s|_N] = 0$ *in* $H^m(N, \mathbb{R})$ *for all* $s \in \mathcal{F}$*. Let* $\mathcal{M}_X^{\mathcal{F}}$ *be the moduli space of deformations of* N *in* $\mathcal{F}$*, and* $\pi^{\mathcal{F}} : \mathcal{M}_X^{\mathcal{F}} \to \mathcal{F}$ *the natural projection.*

Then $\mathcal{M}_X^{\mathcal{F}}$ *is a smooth manifold of dimension* $d + b^1(N)$*, and* $\pi^{\mathcal{F}} : \mathcal{M}_X^{\mathcal{F}} \to \mathcal{F}$ *a smooth submersion. For small* $s \in \mathcal{F}$ *the moduli space* $\mathcal{M}_X^s = (\pi^{\mathcal{F}})^{-1}(s)$ *of deformations of* N *in* $(M, J^s, \omega^s, \Omega^s)$ *is a nonempty smooth manifold of dimension* $b^1(N)$*, with* $\mathcal{M}_X^0 = \mathcal{M}_X$*.*

Here a necessary condition for the existence of an SL m-fold $\hat{N}$ isotopic to N in $(M, J^s, \omega^s, \Omega^s)$ is that $[\omega^s|_N] = [\operatorname{Im} \Omega^s|_N] = 0$ in $H^*(N, \mathbb{R})$, since $[\omega^s|_N]$ and $[\omega^s|_{\hat{N}}]$ are identified under the natural isomorphism between $H^2(N, \mathbb{R})$ and $H^2(\hat{N}, \mathbb{R})$, and similarly for $\operatorname{Im} \Omega^s$.

The point of the theorem is that these conditions $[\omega^s|_N] = [\operatorname{Im} \Omega^s|_N] = 0$ are also *sufficient* for the existence of $\hat{N}$ when s is close to 0 in $\mathcal{F}$. That is, the only *obstructions* to existence of compact SL m-folds when we deform the underlying almost Calabi–Yau m-fold are the obvious cohomological ones.

3 SL cones and conical singularities

After some preliminary work in §3.1 on *special Lagrangian cones*, and some examples in §3.2, section 3.3 defines *special Lagrangian* m*-folds with conical singularities* in almost Calabi–Yau manifolds, which are the subject of the paper.

3.1 Preliminaries on special Lagrangian cones

We now give some definitions and results on *special Lagrangian cones*. Some are quoted from [12], and some are new.

Definition 3.1 A (singular) SL m-fold C in $\mathbb{C}^m$ is called a *cone* if $C = tC$ for all $t > 0$, where $tC = \{t\mathbf{x} : \mathbf{x} \in C\}$. Let C be a closed SL cone in $\mathbb{C}^m$ with an isolated singularity at 0. Then $\Sigma = C \cap \mathcal{S}^{2m-1}$ is a compact, nonsingular $(m-1)$-submanifold of $\mathcal{S}^{2m-1}$, not necessarily connected. Let g_Σ be the restriction of g' to Σ, where g' is as in (1).

Set $C' = C \setminus \{0\}$. Define $\iota : \Sigma \times (0,\infty) \to \mathbb{C}^m$ by $\iota(\sigma, r) = r\sigma$. Then ι has image C'. By an abuse of notation, *identify* C' with $\Sigma \times (0,\infty)$ using ι. The *cone metric* on $C' \cong \Sigma \times (0,\infty)$ is $g' = \iota^*(g') = \mathrm{d}r^2 + r^2 g_\Sigma$.

For $\alpha \in \mathbb{R}$, we say that a function $u : C' \to \mathbb{R}$ is *homogeneous of order* α if $u \circ t \equiv t^\alpha u$ for all $t > 0$. Equivalently, u is homogeneous of order α if $u(\sigma, r) \equiv r^\alpha v(\sigma)$ for some function $v : \Sigma \to \mathbb{R}$.

In [12, Lem. 2.3] we study *homogeneous harmonic functions* on C'.

Lemma 3.2 *In the situation of Definition 3.1, let $u(\sigma, r) \equiv r^\alpha v(\sigma)$ be a homogeneous function of order α on $C' = \Sigma \times (0,\infty)$, for $v \in C^2(\Sigma)$. Then*

$$\Delta u(\sigma, r) = r^{\alpha-2}\bigl(\Delta_\Sigma v - \alpha(\alpha + m - 2)v\bigr),$$

where Δ, Δ_Σ are the Laplacians on (C', g') and (Σ, g_Σ). Hence, u is harmonic on C' if and only if v is an eigenfunction of Δ_Σ with eigenvalue $\alpha(\alpha + m - 2)$.

Following [12, Def. 2.5], we define:

Definition 3.3 In the situation of Definition 3.1, suppose $m > 2$ and define

$$\mathcal{D}_\Sigma = \bigl\{\alpha \in \mathbb{R} : \alpha(\alpha + m - 2) \text{ is an eigenvalue of } \Delta_\Sigma\bigr\}. \tag{6}$$

Then $\mathcal{D}_\Sigma$ is a countable, discrete subset of $\mathbb{R}$. By Lemma 3.2, an equivalent definition is that $\mathcal{D}_\Sigma$ is the set of $\alpha \in \mathbb{R}$ for which there exists a nonzero homogeneous harmonic function u of order α on C'.

Define $m_\Sigma : \mathcal{D}_\Sigma \to \mathbb{N}$ by taking $m_\Sigma(\alpha)$ to be the multiplicity of the eigenvalue $\alpha(\alpha + m - 2)$ of Δ_Σ, or equivalently the dimension of the vector space of homogeneous harmonic functions u of order α on C'. Define $N_\Sigma : \mathbb{R} \to \mathbb{Z}$ by

$$N_\Sigma(\delta) = -\sum_{\alpha \in \mathcal{D}_\Sigma \cap (\delta, 0)} m_\Sigma(\alpha) \text{ if } \delta < 0, \text{ and } N_\Sigma(\delta) = \sum_{\alpha \in \mathcal{D}_\Sigma \cap [0,\delta]} m_\Sigma(\alpha) \text{ if } \delta \geqslant 0. \tag{7}$$

Then N_Σ is monotone increasing and upper semicontinuous, and is discontinuous exactly on $\mathcal{D}_\Sigma$, increasing by $m_\Sigma(\alpha)$ at each $\alpha \in \mathcal{D}_\Sigma$. As the eigenvalues of Δ_Σ are nonnegative, we see that $\mathcal{D}_\Sigma \cap (2 - m, 0) = \emptyset$ and $N_\Sigma \equiv 0$ on $(2 - m, 0)$.

We shall show that there automatically exist homogeneous harmonic functions on C' of orders 1 and 2, using the idea of *moment map*. The group of automorphisms of $\mathbb{C}^m$ preserving g', ω' and Ω' is $\mathrm{SU}(m) \ltimes \mathbb{C}^m$, where $\mathbb{C}^m$ acts by translations. Its Lie algebra $\mathfrak{su}(m) \ltimes \mathbb{C}^m$ acts on $\mathbb{C}^m$ by vector fields.

Let v be such a vector field in $\mathfrak{su}(m) \ltimes \mathbb{C}^m$. Then $v \cdot \omega'$ is a closed 1-form on $\mathbb{C}^m$, and we may write $v \cdot \omega' = \mathrm{d}\mu$ for some function $\mu : \mathbb{C}^m \to \mathbb{R}$, which is unique up to addition of constants, and is in fact a real quadratic polynomial. We call μ a *moment map* for v.

Lemma 3.4 *Let L be an SL m-fold in $\mathbb{C}^m$, and let $\mu : \mathbb{C}^m \to \mathbb{R}$ be a moment map for a vector field v in $\mathfrak{su}(m) \ltimes \mathbb{C}^m$. Then $\mu|_L$ is a harmonic function on L, using the obvious metric $g'|_L$.*

Proof. In the proof of Theorem 2.6 we saw that infinitesimal deformations of an SL m-fold L as a submanifold correspond to 1-forms α on L, and infinitesimal deformations as an SL m-fold to closed and coclosed 1-forms α on L.

Now as $\mathrm{SU}(m) \ltimes \mathbb{C}^m$ takes SL m-folds in $\mathbb{C}^m$ to SL m-folds in $\mathbb{C}^m$, the vector field v in $\mathfrak{su}(m) \ltimes \mathbb{C}^m$ gives an infinitesimal deformation of L as an SL m-fold in $\mathbb{C}^m$. It is easy to see that the corresponding 1-form on L is $(v \cdot \omega)|_L$. Therefore $(v \cdot \omega)|_L = \mathrm{d}\mu|_L$ is a closed and coclosed 1-form on L, and thus $\mathrm{d}^*(\mathrm{d}\mu|_L) = 0$, so $\mu|_L$ is harmonic. $\square$

Proposition 3.5 *Let C be an SL cone in $\mathbb{C}^m$ with isolated singularity at 0, and G the Lie subgroup of $\mathrm{SU}(m)$ preserving C. Set $C' = C \setminus \{0\}$ and $\Sigma = C \cap \mathcal{S}^{2m-1}$, and let m_Σ be as in Definition 3.3. Then*

(a) *The restriction of real linear functions on $\mathbb{C}^m$ to C' form a vector space of order 1 homogeneous harmonic functions on C', with dimension $2m$. Hence $m_\Sigma(1) \geqslant 2m$.*

(b) *The restriction of $\mathfrak{su}(m)$ moment maps $\mu : \mathbb{C}^m \to \mathbb{R}$ with $\mu(0) = 0$ to C' form a vector space of order 2 homogeneous harmonic functions on C', with dimension $m^2 - 1 - \dim G$. Hence $m_\Sigma(2) \geqslant m^2 - 1 - \dim G$.*

Proof. Real linear functions on $\mathbb{C}^m$ are moment maps of *translations* on $\mathbb{C}^m$, and so restrict to harmonic maps on SL m-folds L in $\mathbb{C}^m$ by Lemma 3.4. Thus the vector space in (a) is of harmonic functions on C', which are clearly homogeneous of order 1. Now C has a unique singular point at 0, so it cannot be invariant under nontrivial translations. Therefore the moment map of a nontrivial translation cannot vanish on C', and the restriction in (a) is injective. It follows that the vector space has dimension $2m$, proving part (a).

For (b), each $\mathfrak{su}(m)$ vector field has a unique moment map $\mu : \mathbb{C}^m \to \mathbb{R}$ with $\mu(0) = 0$, which is a homogeneous real quadratic polynomial. It follows as for (a) that the vector space in (b) consists of order 2 homogeneous harmonic functions on C'. This vector space is the image of a linear map from $\mathfrak{su}(m)$, and it is easy to show that the kernel of this map is $\mathfrak{g}$, the Lie algebra of G. Hence the dimension of the vector space is $\dim \mathfrak{su}(m) - \dim \mathfrak{g}$ by rank-nullity, and the proposition follows. $\square$

We define the *stability index* of C, and *stable* and *rigid* cones.

Definition 3.6 Let C be an SL cone in $\mathbb{C}^m$ for $m > 2$ with an isolated singularity at 0, let G be the Lie subgroup of $\mathrm{SU}(m)$ preserving C, and use the notation of Definitions 3.1 and 3.3. Then

$$m_\Sigma(0) = b^0(\Sigma), \quad m_\Sigma(1) \geqslant 2m \quad \text{and} \quad m_\Sigma(2) \geqslant m^2 - 1 - \dim G, \tag{8}$$

where the first equation follows as $m_\Sigma(0)$ is the multiplicity of the eigenvalue 0 of Δ_Σ, and the others from Proposition 3.5.

Define the *stability index* s-ind(C) to be

$$\text{s-ind}(C) = N_\Sigma(2) - b^0(\Sigma) - m^2 - 2m + 1 + \dim G. \tag{9}$$

Then s-ind$(C) \geqslant 0$ by (8), as $N_\Sigma(2) \geqslant m_\Sigma(0) + m_\Sigma(1) + m_\Sigma(2)$ by (7). We call C *stable* if s-ind$(C) = 0$.

Following [12, Def. 6.7], we call C *rigid* if $m_\Sigma(2) = m^2 - 1 - \dim G$. As

$$\text{s-ind}(C) \geqslant m_\Sigma(2) - (m^2 - 1 - \dim G) \geqslant 0,$$

we see that *if C is stable, then C is rigid*.

Here is the point of this definition. In deforming SL m-folds X in an almost Calabi–Yau m-fold M with a conical singularity x modelled on C, it will turn out in §6 that x contributes an *obstruction space* of dimension $N_\Sigma(2)$ to deforming X. However, we will be able to *overcome* a subspace of these obstructions with dimension $b^0(\Sigma) + m^2 + 2m - 1 - \dim G$ automatically, by moving x around in M, and changing the identification $\mathbb{C}^m \cong T_x M$. Thus s-ind(C) is the dimension of the *residual obstruction space*, which we cannot get rid of.

If C is *stable* then the deformation problem is *unobstructed*. Rigid (and more generally *Jacobi integrable*) SL cones were discussed in [12, §6]. An SL cone C is *rigid* if all infinitesimal deformations of C as an SL cone come from $\mathfrak{su}(m)$ rotations.

3.2 Examples of special Lagrangian cones

Examples of SL cones are constructed by Harvey and Lawson [4, §III.3], the author [7, 8], and others. We will study a family of special Lagrangian cones in $\mathbb{C}^m$ constructed by Harvey and Lawson [4, §III.3.A]. For $m \geqslant 3$, define

$$C^m_{\rm HL} = \bigl\{(z_1, \ldots, z_m) \in \mathbb{C}^m : |z_1| = \cdots = |z_m|, \quad i^{m+1} z_1 \cdots z_m \in [0, \infty)\bigr\}. \tag{10}$$

Then $C^m_{\rm HL}$ is a special Lagrangian cone in $\mathbb{C}^m$ with an isolated singularity at 0, and $\Sigma^m_{\rm HL} = C^m_{\rm HL} \cap \mathcal{S}^{2m-1}$ is an $(m-1)$-torus T^{m-1} with a flat metric. Also $C^m_{\rm HL}$ and $\Sigma^m_{\rm HL}$ are invariant under the $\mathrm{U}(1)^{m-1}$ subgroup of $\mathrm{SU}(m)$ acting by

$$(z_1, \ldots, z_m) \mapsto (e^{i\theta_1} z_1, \ldots, e^{i\theta_m} z_m) \quad \text{for } \theta_j \in \mathbb{R} \text{ with } \theta_1 + \cdots + \theta_m = 0. \tag{11}$$

In fact $\pm C^m_{\rm HL}$ for m odd, and $C^m_{\rm HL}, iC^m_{\rm HL}$ for m even, are the unique SL cones in $\mathbb{C}^m$ invariant under (11), which is how Harvey and Lawson constructed them.

We shall find the *stability index* s-ind$(C^m_{\rm HL})$ of these cones, and test whether they are *stable* or *rigid*. This was first done by the author [6, §3.2] for $m = 3$ and Marshall [17, §6.3.4] for $3 \leqslant m \leqslant 8$. The metric on $\Sigma^m_{\rm HL} \cong T^{m-1}$ is flat, so it is not difficult to compute the eigenvalues of $\Delta_{\Sigma^m_{\rm HL}}$. There is a 1-1 correspondence between $(n_1, \ldots, n_{m-1}) \in \mathbb{Z}^{m-1}$ and eigenvectors of $\Delta_{\Sigma^m_{\rm HL}}$ with eigenvalue

$$m \sum_{i=1}^{m-1} n_i^2 - \sum_{i,j=1}^{m-1} n_i n_j. \tag{12}$$

Using (12) and a computer we can find the eigenvalues of $\Delta_{\Sigma^m_{\rm HL}}$, and their multiplicities. Thus we can calculate $N_{\Sigma^m_{\rm HL}}(2)$, which is the sum of multiplicities of eigenvalues in $[0,2m]$, and $m_{\Sigma^m_{\rm HL}}(2)$, which is the multiplicity of the eigenvalue $2m$. A table of eigenvalues and multiplicities for $3\leqslant m\leqslant 8$ is given in Marshall [17, Table 6.1]. Now the subgroup G_m of $\mathrm{SU}(m)$ preserving $C^m_{\rm HL}$ is $\mathrm{U}(1)^{m-1}$, with dimension $m-1$. Thus (9) gives $\text{s-ind}(C^m_{\rm HL})=N_{\Sigma^m_{\rm HL}}(2)-m^2-m-1$. Table 1 gives the data $m, N_{\Sigma^m_{\rm HL}}(2), m_{\Sigma^m_{\rm HL}}(2)$ and $\text{s-ind}(C^m_{\rm HL})$ for $3\leqslant m\leqslant 12$.

m	3	4	5	6	7	8	9	10	11	12
$N_{\Sigma^m_{\rm HL}}(2)$	13	27	51	93	169	311	331	201	243	289
$m_{\Sigma^m_{\rm HL}}(2)$	6	12	20	30	42	126	240	90	110	132
$\text{s-ind}(C^m_{\rm HL})$	0	6	20	50	112	238	240	90	110	132

Table 1: Data for $\mathrm{U}(1)^{m-1}$-invariant SL cones $C^m_{\rm HL}$ in $\mathbb{C}^m$

Motivated by Table 1, with some more work one can prove that

$$N_{\Sigma^m_{\rm HL}}(2)=2m^2+1 \;\text{ and }\; m_{\Sigma^m_{\rm HL}}(2)=\text{s-ind}(C^m_{\rm HL})=m^2-m \;\text{ for } m\geqslant 10. \tag{13}$$

As $C^m_{\rm HL}$ is *stable* when $\text{s-ind}(C^m_{\rm HL})=0$ we see from Table 1 and (13) that $C^3_{\rm HL}$ is a *stable* cone in $\mathbb{C}^3$, but $C^m_{\rm HL}$ is *unstable* for $m\geqslant 4$.

Also $C^m_{\rm HL}$ is *rigid* when $m_{\Sigma^m_{\rm HL}}(2)=m^2-m$. Thus $C^m_{\rm HL}$ is *rigid* if and only if $m\neq 8,9$, by Table 1 and (13). It would be interesting to know whether the SL cones $C^8_{\rm HL}$ and $C^9_{\rm HL}$ are *Jacobi integrable* in the sense of [12, §6], as rigid implies Jacobi integrable but not vice versa. The author guesses that $C^8_{\rm HL}, C^9_{\rm HL}$ are not Jacobi integrable.

3.3 Special Lagrangian m-folds with conical singularities

Now we can define *conical singularities* of SL m-folds, following [12, Def. 3.6].

Definition 3.7 Let (M,J,ω,Ω) be an almost Calabi–Yau m-fold for $m>2$, and define $\psi:M\rightarrow(0,\infty)$ as in (3). Suppose X is a compact singular SL m-fold in M with singularities at distinct points $x_1,\ldots,x_n\in X$, and no other singularities.

Fix isomorphisms $\upsilon_i:\mathbb{C}^m\rightarrow T_{x_i}M$ for $i=1,\ldots,n$ such that $\upsilon_i^*(\omega)=\omega'$ and $\upsilon_i^*(\Omega)=\psi(x_i)^m\Omega'$, where ω',Ω' are as in (1). Let $C_1,\ldots,C_n$ be SL cones in $\mathbb{C}^m$ with isolated singularities at 0. For $i=1,\ldots,n$ let $\Sigma_i=C_i\cap\mathcal{S}^{2m-1}$, and let $\mu_i\in(2,3)$ with

$$(2,\mu_i]\cap\mathcal{D}_{\Sigma_i}=\emptyset, \quad \text{where } \mathcal{D}_{\Sigma_i} \text{ is defined in (6).} \tag{14}$$

Then we say that X has a *conical singularity* at x_i, with *rate* μ_i and *cone* C_i for $i=1,\ldots,n$, if the following holds.

By Darboux' Theorem [19, Th. 3.15] there exist embeddings $\Upsilon_i:B_R\rightarrow M$ for $i=1,\ldots,n$ satisfying $\Upsilon_i(0)=x_i$, $\mathrm{d}\Upsilon_i\vert_0=\upsilon_i$ and $\Upsilon_i^*(\omega)=\omega'$, where B_R

is the open ball of radius R about 0 in $\mathbb{C}^m$ for some small $R > 0$. Define $\iota_i : \Sigma_i \times (0, R) \to B_R$ by $\iota_i(\sigma, r) = r\sigma$ for $i = 1, \ldots, n$.

Define $X' = X \setminus \{x_1, \ldots, x_n\}$. Then there should exist a compact subset $K \subset X'$ such that $X' \setminus K$ is a union of open sets $S_1, \ldots, S_n$ with $S_i \subset \Upsilon_i(B_R)$, whose closures $\bar{S}_1, \ldots, \bar{S}_n$ are disjoint in X. For $i = 1, \ldots, n$ and some $R' \in (0, R]$ there should exist a smooth $\phi_i : \Sigma_i \times (0, R') \to B_R$ such that $\Upsilon_i \circ \phi_i : \Sigma_i \times (0, R') \to M$ is a diffeomorphism $\Sigma_i \times (0, R') \to S_i$, and

$$\left|\nabla^k(\phi_i - \iota_i)\right| = O(r^{\mu_i - 1 - k}) \quad \text{as } r \to 0 \text{ for } k = 0, 1. \tag{15}$$

Here $\nabla, |\,.\,|$ are computed using the cone metric $\iota_i^*(g')$ on $\Sigma_i \times (0, R')$.

If the cones $C_1, \ldots, C_n$ are *stable* in the sense of Definition 3.6, then we say that X has *stable conical singularities*.

The reasoning behind this definition was discussed in [12, §3.3]. Here we just make two remarks:

- We suppose $m > 2$ for two reasons. Firstly, the only SL cones in $\mathbb{C}^2$ are finite unions of SL planes $\mathbb{R}^2$ in $\mathbb{C}^2$ intersecting only at 0. Thus any SL 2-fold with conical singularities is actually *nonsingular* as an immersed 2-fold, so there is really no point in studying them. Secondly, $m = 2$ is a special case in the analysis of [12, §2], and it is simpler to exclude it.

  In the rest of the paper we shall assume $m > 2$.

- The purpose of (14) is to reduce to a minimum the obstructions to deforming X as an SL m-fold with conical singularities. If we omitted condition (14) then each $\alpha \in (2, \mu_i] \cap \mathcal{D}_{\Sigma_i}$ would contribute additional obstructions to deforming X in §6.

4 Review of material from [12]

We now review the definitions and results from the preceding paper [12] which we will need later. Throughout we suppose $m > 2$.

4.1 Analysis on SL m-folds with conical singularities

We will need the following tool [12, Def. 2.6], a smoothed out version of the distance from the singular set $\{x_1, \ldots, x_n\}$ in X.

Definition 4.1 Let (M, J, ω, Ω) be an almost Calabi–Yau m-fold and X a compact SL m-fold in M with conical singularities at $x_1, \ldots, x_n$, and use the notation of Definition 3.7. Define a *radius function* ρ on X' to be a smooth function $\rho : X' \to (0, 1]$ such that $\rho \equiv 1$ on K and $\rho(y) = d(x_i, y)$ for $y \in S_i$ close to x_i, where d is the metric on X. Radius functions always exist.

For $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n) \in \mathbb{R}^n$, define a function $\rho^{\boldsymbol{\beta}}$ on X' by $\rho^{\boldsymbol{\beta}}(y) = \rho(y)^{\beta_i}$ on S_i for $i = 1, \ldots, n$ and $\rho^{\boldsymbol{\beta}}(y) = 1$ on K. Then $\rho^{\boldsymbol{\beta}}$ is well-defined and smooth on X', and equals ρ^{β_i} near x_i in X'. If $\boldsymbol{\beta},\boldsymbol{\gamma}\in\mathbb{R}^n$, write $\boldsymbol{\beta}\geqslant\boldsymbol{\gamma}$ if $\beta_i\geqslant\gamma_i$ for $i=1,\ldots,n$. If $\boldsymbol{\beta}\in\mathbb{R}^n$ and $a\in\mathbb{R}$, write $\boldsymbol{\beta}+a=(\beta_1+a,\ldots,\beta_n+a)$ in $\mathbb{R}^n$.

Now we define some Banach spaces of functions on X', [12, Def. 2.7].

Definition 4.2 Let (M,J,ω,Ω) be an almost Calabi–Yau m-fold with metric g, and X a compact SL m-fold in M with conical singularities at $x_1,\ldots,x_n$, and use the notation of Definitions 3.7 and 4.1. Let ρ be a radius function on X'. Regard X' as a Riemannian manifold, with metric g restricted from M.

For $\boldsymbol{\beta}\in\mathbb{R}^n$ and $k\geqslant 0$ define $C^k_{\boldsymbol{\beta}}(X')$ to be the space of continuous functions f on X' with k continuous derivatives, such that $\bigl|\rho^{-\boldsymbol{\beta}+j}\nabla^j f\bigr|$ is bounded on X' for $j=0,\ldots,k$. Define the norm $\|\,.\,\|_{C^k_{\boldsymbol{\beta}}}$ on $C^k_{\boldsymbol{\beta}}(X')$ by

$$\|f\|_{C^k_{\boldsymbol{\beta}}}=\sum_{j=0}^k\sup_{X'}\bigl|\rho^{-\boldsymbol{\beta}+j}\nabla^j f\bigr|. \tag{16}$$

Then $C^k_{\boldsymbol{\beta}}(X')$ is a Banach space. Define $C^\infty_{\boldsymbol{\beta}}(X')=\bigcap_{k\geqslant 0}C^k_{\boldsymbol{\beta}}(X')$.

For $p\geqslant 1$, $\boldsymbol{\beta}\in\mathbb{R}^n$ and $k\geqslant 0$ define the *weighted Sobolev space* $L^p_{k,\boldsymbol{\beta}}(X')$ to be the set of functions f on X' that are locally integrable and k times weakly differentiable, and for which the norm

$$\|f\|_{L^p_{k,\boldsymbol{\beta}}}=\left(\sum_{j=0}^k\int_{X'}\bigl|\rho^{-\boldsymbol{\beta}+j}\nabla^j f\bigr|^p\rho^{-m}\mathrm{d}V_g\right)^{1/p} \tag{17}$$

is finite. Then $L^p_{k,\boldsymbol{\beta}}(X')$ is a Banach space, and $L^2_{k,\boldsymbol{\beta}}(X')$ a Hilbert space.

We call these *weighted Banach spaces* since the norms are locally weighted by a power of ρ. Roughly speaking, if f lies in $L^p_{k,\boldsymbol{\beta}}(X')$ or $C^k_{\boldsymbol{\beta}}(X')$ then f grows at most like ρ^{β_i} near x_i as $\rho\rightarrow 0$, and so the multi-index $\boldsymbol{\beta}=(\beta_1,\ldots,\beta_n)$ should be interpreted as an *order of growth*.

Here is a weighted version of the *Sobolev Embedding Theorem*, [12, Th. 2.9].

Theorem 4.3 *In the situation above, suppose $k>l\geqslant 0$ are integers and $p>1$ with $\frac{1}{p}<\frac{k-l}{m}$, and $\boldsymbol{\beta},\boldsymbol{\gamma}\in\mathbb{R}^n$ with $\boldsymbol{\beta}\geqslant\boldsymbol{\gamma}$. Then $L^p_{k,\boldsymbol{\beta}}(X')\hookrightarrow C^l_{\boldsymbol{\gamma}}(X')$ is a continuous inclusion.*

Here is a Fredholm result for the operator $P:f\mapsto\mathrm{d}^*(\psi^m\mathrm{d}f)$ on weighted Sobolev spaces, [12, Th. 5.3]. Putting $\alpha=\mathrm{d}f$ in (4), we see that P appears in the linearization of the deformation problem for SL m-folds.

Theorem 4.4 *Let (M,J,ω,Ω) be an almost Calabi–Yau m-fold, and define $\psi:M\rightarrow(0,\infty)$ as in (3). Suppose X is a compact SL m-fold in M with conical singularities at $x_1,\ldots,x_n$ with cones C_i. Define $\mathcal{D}_{\Sigma_i},N_{\Sigma_i}$ and $L^p_{k,\boldsymbol{\beta}}(X')$ as in Definitions 3.3 and 4.2. Fix $p>1$ and $k\geqslant 2$, and for $\boldsymbol{\beta}\in\mathbb{R}^n$ define $P_{\boldsymbol{\beta}}:L^p_{k,\boldsymbol{\beta}}(X')\rightarrow L^p_{k-2,\boldsymbol{\beta}-2}(X')$ by $P_{\boldsymbol{\beta}}(f)=\mathrm{d}^*(\psi^m\mathrm{d}f)$. Then*

(a) *$P_{\boldsymbol\beta}$ is Fredholm if and only if $\boldsymbol\beta \in \bigl(\mathbb{R}\setminus\mathcal{D}_{\Sigma_1}\bigr)\times\cdots\times\bigl(\mathbb{R}\setminus\mathcal{D}_{\Sigma_n}\bigr)$, and then*

$$\operatorname{ind}(P_{\boldsymbol\beta}) = -\sum_{i=1}^{n} N_{\Sigma_i}(\beta_i). \tag{18}$$

(b) *If $\beta_i > 0$ for all i then $P_{\boldsymbol\beta}$ is injective.*

4.2 Homology, cohomology and Hodge theory

Next we discuss *homology* and *cohomology* of SL m-folds with conical singularities, following [12, §2.4]. For a general reference, see for instance Bredon [2]. When Y is a manifold, write $H^k(Y,\mathbb{R})$ for the k^{th} *de Rham cohomology group* and $H^k_{\text{cs}}(Y,\mathbb{R})$ for the k^{th} *compactly-supported de Rham cohomology group* of Y. If Y is compact then $H^k(Y,\mathbb{R}) = H^k_{\text{cs}}(Y,\mathbb{R})$.

Let Y be a topological space, and $Z \subset Y$ a subspace. Write $H_k(Y,\mathbb{R})$ for the k^{th} *real singular homology group* of Y, and $H_k(Y;Z,\mathbb{R})$ for the k^{th} *real singular relative homology group* of $(Y;Z)$. When Y is a manifold and Z a submanifold we define $H_k(Y,\mathbb{R})$ and $H_k(Y;Z,\mathbb{R})$ using *smooth* simplices, as in [2, §V.5]. Then the pairing between (singular) homology and (de Rham) cohomology is defined at the chain level by integrating k-forms over k-simplices.

Suppose X is a compact SL m-fold in M with conical singularities $x_1,\ldots,x_n$ and cones $C_1,\ldots,C_n$, and set $X' = X\setminus\{x_1,\ldots,x_n\}$ and $\Sigma_i = C_i\cap\mathcal{S}^{2m-1}$ as above. Then by [12, §2.4] there is a natural long exact sequence

$$\cdots\rightarrow H^k_{\text{cs}}(X',\mathbb{R})\rightarrow H^k(X',\mathbb{R})\rightarrow \bigoplus_{i=1}^{n} H^k(\Sigma_i,\mathbb{R})\rightarrow H^{k+1}_{\text{cs}}(X',\mathbb{R})\rightarrow\cdots, \tag{19}$$

and natural isomorphisms

$$H_k\bigl(X;\{x_1,\ldots,x_n\},\mathbb{R}\bigr)^*\cong H^k_{\text{cs}}(X',\mathbb{R})\cong H_{m-k}(X',\mathbb{R})\cong H^{m-k}(X',\mathbb{R})^* \tag{20}$$

$$\text{and}\quad H^k_{\text{cs}}(X',\mathbb{R})\cong H_k(X,\mathbb{R})^*\quad\text{for all } k>1. \tag{21}$$

The inclusion $\iota : X\rightarrow M$ induces homomorphisms $\iota_* : H_k(X,\mathbb{R})\rightarrow H_k(M,\mathbb{R})$ and $\iota^* : H^k(M,\mathbb{R})\rightarrow H^k(X',\mathbb{R})$.

If (Y,g) is a compact Riemannian manifold, then *Hodge theory* shows that each class in $H^k(Y,\mathbb{R})$ is represented by a unique k-form α with $\mathrm{d}\alpha = \mathrm{d}^*\alpha = 0$. Here is an analogue of this on X' when $k=1$, part of [12, Th. 5.4].

Theorem 4.5 *Let (M,J,ω,Ω) be an almost Calabi–Yau m-fold, and define $\psi : M\rightarrow(0,\infty)$ as in (3). Suppose X is a compact SL m-fold in M with conical singularities at $x_1,\ldots,x_n$. Set $X' = X\setminus\{x_1,\ldots,x_n\}$, and let ρ be a radius function on X', in the sense of Definition 4.1. Define*

$$\begin{aligned} Y_{X'} = \bigl\{\alpha\in C^\infty(T^*X') : \mathrm{d}\alpha = 0,\quad \mathrm{d}^*(\psi^m\alpha) = 0,\\ \lvert\nabla^k\alpha\rvert = O(\rho^{-1-k})\ \text{for } k\geqslant 0\bigr\}. \end{aligned} \tag{22}$$

Then the map $\pi : Y_{X'}\rightarrow H^1(X',\mathbb{R})$ taking $\pi : \alpha\mapsto[\alpha]$ is an isomorphism.

4.3 Lagrangian Neighbourhood Theorems

In [12, §4] we extend the *Lagrangian Neighbourhood Theorem*, Theorem 2.7, to situations involving conical singularities, first to *SL cones*, [12, Th. 4.3].

Theorem 4.6 *Let C be an SL cone in $\mathbb{C}^m$ with isolated singularity at 0, and set $\Sigma = C \cap \mathcal{S}^{2m-1}$. Define $\iota : \Sigma \times (0,\infty) \to \mathbb{C}^m$ by $\iota(\sigma,r) = r\sigma$, with image $C \setminus \{0\}$. For $\sigma \in \Sigma$, $\tau \in T^*_\sigma\Sigma$, $r \in (0,\infty)$ and $u \in \mathbb{R}$, let (σ,r,τ,u) represent the point $\tau + u\,\mathrm{d}r$ in $T^*_{(\sigma,r)}\bigl(\Sigma\times(0,\infty)\bigr)$. Identify $\Sigma \times (0,\infty)$ with the zero section $\tau=u=0$ in $T^*\bigl(\Sigma\times(0,\infty)\bigr)$. Define an action of $(0,\infty)$ on $T^*\bigl(\Sigma\times(0,\infty)\bigr)$ by*

$$t : (\sigma,r,\tau,u) \longmapsto (\sigma,tr,t^2\tau,tu) \quad \text{for } t \in (0,\infty), \tag{23}$$

so that $t^*(\hat\omega)=t^2\hat\omega$, for $\hat\omega$ the canonical symplectic structure on $T^*\bigl(\Sigma\times(0,\infty)\bigr)$.

Then there exists an open neighbourhood U_C of $\Sigma\times(0,\infty)$ in $T^*\bigl(\Sigma\times(0,\infty)\bigr)$ invariant under (23) given by

$$U_C = \bigl\{(\sigma,r,\tau,u) \in T^*\bigl(\Sigma\times(0,\infty)\bigr) : \bigl|(\tau,u)\bigr| < 2\zeta r\bigr\} \quad \text{for some } \zeta > 0, \tag{24}$$

where $|\,.\,|$ is calculated using the cone metric $\iota^*(g')$ on $\Sigma\times(0,\infty)$, and an embedding $\Phi_C : U_C \to \mathbb{C}^m$ with $\Phi_C|_{\Sigma\times(0,\infty)} = \iota$, $\Phi_C^*(\omega') = \hat\omega$ and $\Phi_C \circ t = t\,\Phi_C$ for all $t > 0$, where t acts on U_C as in (23) and on $\mathbb{C}^m$ by multiplication.

In [12, Th. 4.4] we construct a particular choice of ϕ_i in Definition 3.7.

Theorem 4.7 *Let (M,J,ω,Ω), $\psi, X, n, x_i, \upsilon_i, C_i, \Sigma_i, \mu_i, R, \Upsilon_i$ and ι_i be as in Definition 3.7. Theorem 4.6 gives $\zeta > 0$, neighbourhoods U_{C_i} of $\Sigma_i \times (0,\infty)$ in $T^*\bigl(\Sigma_i\times(0,\infty)\bigr)$ and embeddings $\Phi_{C_i} : U_{C_i} \to \mathbb{C}^m$ for $i = 1,\dots,n$.*

Then for sufficiently small $R' \in (0,R]$ there exist unique closed 1-forms η_i on $\Sigma_i \times (0,R')$ for $i = 1,\dots,n$ written $\eta_i(\sigma,r) = \eta_i^1(\sigma,r) + \eta_i^2(\sigma,r)\mathrm{d}r$ for $\eta_i^1(\sigma,r) \in T^*_\sigma\Sigma_i$ and $\eta_i^2(\sigma,r) \in \mathbb{R}$, and satisfying $|\eta_i(\sigma,r)| < \zeta r$ and

$$\bigl|\nabla^k\eta_i\bigr| = O(r^{\mu_i-1-k}) \quad \text{as } r \to 0 \text{ for } k = 0,1, \tag{25}$$

computing $\nabla, |\,.\,|$ using the cone metric $\iota_i^*(g')$, such that the following holds.

Define $\phi_i : \Sigma_i\times(0,R') \to B_R$ by $\phi_i(\sigma,r) = \Phi_{C_i}\bigl(\sigma,r,\eta_i^1(\sigma,r),\eta_i^2(\sigma,r)\bigr)$. Then $\Upsilon_i \circ \phi_i : \Sigma_i \times (0,R') \to M$ is a diffeomorphism $\Sigma_i \times (0,R') \to S_i$ for open sets $S_1,\dots,S_n$ in X' with $\bar S_1,\dots,\bar S_n$ disjoint, and $K = X' \setminus (S_1 \cup \dots \cup S_n)$ is compact. Also ϕ_i satisfies (15), so that R', ϕ_i, S_i, K satisfy Definition 3.7.

Next we extend Theorem 2.7 to SL m-folds with conical singularities [12, Th. 4.6], in a way compatible with Theorems 4.6 and 4.7.

Theorem 4.8 *Suppose (M,J,ω,Ω) is an almost Calabi–Yau m-fold and X a compact SL m-fold in M with conical singularities at $x_1,\dots,x_n$. Let the notation $\psi, \upsilon_i, C_i, \Sigma_i, \mu_i, R, \Upsilon_i$ and ι_i be as in Definition 3.7, and let $\zeta, U_{C_i}, \Phi_{C_i}, R'$, $\eta_i, \eta_i^1, \eta_i^2, \phi_i, S_i$ and K be as in Theorem 4.7.*

Then making R' smaller if necessary, there exists an open tubular neighbourhood $U_{X'} \subset T^*X'$ of the zero section X' in T^*X', such that under $\mathrm{d}(\Upsilon_i \circ \phi_i) : T^*\bigl(\Sigma_i \times (0,R')\bigr) \to T^*X'$ for $i=1,\ldots,n$ we have

$$\bigl(\mathrm{d}(\Upsilon_i \circ \phi_i)\bigr)^*(U_{X'}) = \bigl\{(\sigma,r,\tau,u) \in T^*\bigl(\Sigma_i \times (0,R')\bigr) : \bigl|(\tau,u)\bigr| < \zeta r\bigr\}, \qquad (26)$$

and there exists an embedding $\Phi_{X'} : U_{X'} \to M$ with $\Phi_{X'}|_{X'} = \mathrm{id} : X' \to X'$ and $\Phi_{X'}^*(\omega) = \hat\omega$, where $\hat\omega$ is the canonical symplectic structure on T^*X', such that

$$\Phi_{X'} \circ \mathrm{d}(\Upsilon_i \circ \phi_i)(\sigma,r,\tau,u) \equiv \Upsilon_i \circ \Phi_{C_i}\bigl(\sigma,r,\tau+\eta_i^1(\sigma,r),u+\eta_i^2(\sigma,r)\bigr) \qquad (27)$$

for all $i=1,\ldots,n$ and $(\sigma,r,\tau,u) \in T^*\bigl(\Sigma_i \times (0,R')\bigr)$ with $\bigl|(\tau,u)\bigr| < \zeta r$. Here $|(\tau,u)|$ is computed using the cone metric $\iota_i^*(g')$ on $\Sigma_i \times (0,R')$.

Here is an extension of Theorem 4.8 to *families* of almost Calabi–Yau m-folds $(M,J^s,\omega^s,\Omega^s)$ for $s \in \mathcal{F}$, deduced from [12, Th. 4.8 & Th. 4.9].

Theorem 4.9 *Let (M,J,ω,Ω) be an almost Calabi–Yau m-fold and X a compact SL m-fold in M with conical singularities at $x_1,\ldots,x_n$, with identifications υ_i and cones C_i. Let the notation $R,\Upsilon_i,\zeta,\Phi_{C_i},R',\eta_i,\eta_i^1,\eta_i^2,\phi_i,S_i,K$ be as in Theorem 4.7, and let $U_{X'},\Phi_{X'}$ be as in Theorem 4.8.*

Suppose $\bigl\{(M,J^s,\omega^s,\Omega^s) : s \in \mathcal{F}\bigr\}$ is a smooth family of deformations of (M,J,ω,Ω), in the sense of Definition 2.11, such that $\iota_*(\gamma)\cdot[\omega^s]=0$ for all $\gamma \in H_2(X,\mathbb{R})$ and $s \in \mathcal{F}$, where $\iota : X \to M$ is the inclusion and $\iota_* : H_2(X,\mathbb{R}) \to H_2(M,\mathbb{R})$ the induced homomorphism. Define $\psi^s : M \to (0,\infty)$ for $s \in \mathcal{F}$ as in (3), but using ω^s,Ω^s.

Then making R,R' and $U_{X'}$ smaller if necessary, for some connected open $\mathcal{F}' \subseteq \mathcal{F}$ with $0 \in \mathcal{F}'$ and all $s \in \mathcal{F}'$ there exist

- (a) *isomorphisms $\upsilon_i^s : \mathbb{C}^m \to T_{x_i}M$ for $i=1,\ldots,n$ with $\upsilon_i^0=\upsilon_i$, $(\upsilon_i^s)^*(\omega^s)=\omega'$ and $(\upsilon_i^s)^*(\Omega)=\psi^s(x_i)^m\Omega'$,*
- (b) *embeddings $\Upsilon_i^s : B_R \to M$ for $i=1,\ldots,n$ with $\Upsilon_i^0=\Upsilon_i$, $\Upsilon_i^s(0)=x_i$, $\mathrm{d}\Upsilon_i^s|_0=\upsilon_i^s$, $(\Upsilon_i^s)^*(\omega^s)=\omega'$, and*
- (c) *an embedding $\Phi_{X'}^s : U_{X'} \to M$ with $\Phi_{X'}^0=\Phi_{X'}$ and $(\Phi_{X'}^s)^*(\omega^s)=\hat\omega$,*

all depending smoothly on $s \in \mathcal{F}'$ with

$$\Phi_{X'}^s \circ \mathrm{d}(\Upsilon_i \circ \phi_i)(\sigma,r,\tau,u) \equiv \Upsilon_i^s \circ \Phi_{C_i}\bigl(\sigma,r,\tau+\eta_i^1(\sigma,r),u+\eta_i^2(\sigma,r)\bigr) \qquad (28)$$

for all $s \in \mathcal{F}'$, $i=1,\ldots,n$ and $(\sigma,r,\tau,u) \in T^*\bigl(\Sigma_i \times (0,R')\bigr)$ with $\bigl|(\tau,u)\bigr| < \zeta r$.

The condition that $\iota_*(\gamma)\cdot[\omega^s]=0$ for all $\gamma \in H_2(X,\mathbb{R})$ essentially says that $\iota^*\bigl([\omega^s]\bigr)=0$ in $H^2(X,\mathbb{R})$. However, we have not put it like this as we have not defined de Rham cohomology on the singular manifold X. We could make sense of this by, for instance, interpreting $[\omega^s]$ as a Čech cohomology class on M using the equivalence of de Rham and Čech cohomology, and pulling back to the Čech cohomology of X.

4.4 Regularity of X near x_i

In [12, §5] we study the asymptotic behaviour of the maps ϕ_i of Theorem 4.7, using the elliptic regularity of the special Lagrangian condition. Combining [12, Th. 5.1], [12, Lem. 4.5] and [12, Th. 5.5] proves:

Theorem 4.10 *In the situation of Theorem 4.7 we have $\eta_i = \mathrm{d}A_i$ for $i = 1,\ldots,n$, where $A_i : \Sigma_i \times (0,R') \rightarrow \mathbb{R}$ is given by $A_i(\sigma,r) = \int_0^r \eta_i^2(\sigma,s)\mathrm{d}s$. Suppose $\mu_i' \in (2,3)$ with $(2,\mu_i'] \cap \mathcal{D}_{\Sigma_i} = \emptyset$ for $i = 1,\ldots,n$. Then*

$$\begin{gathered} \left|\nabla^k(\phi_i - \iota_i)\right| = O(r^{\mu_i'-1-k}), \quad \left|\nabla^k \eta_i\right| = O(r^{\mu_i'-1-k}) \quad \textit{and} \\ \left|\nabla^k A_i\right| = O(r^{\mu_i'-k}) \quad \textit{as } r \rightarrow 0 \textit{ for all } k \geqslant 0 \textit{ and } i = 1,\ldots,n. \end{gathered} \tag{29}$$

Hence X has conical singularities at x_i with cone C_i and rate μ_i', for all possible rates μ_i' allowed by Definition 3.7. Therefore, the definition of conical singularities is essentially independent of the choice of rate μ_i.

Theorem 4.10 in effect *strengthens* the definition of SL m-folds with conical singularities, Definition 3.7, as it shows that (15) actually implies the much stronger condition (29) on all derivatives. In [12, Th. 6.8] we use *Geometric Measure Theory* to prove a *weakening* of Definition 3.7 for *rigid* cones C.

Theorem 4.11 *Let (M,J,ω,Ω) be an almost Calabi–Yau m-fold and define $\psi : M \rightarrow (0,\infty)$ as in (3). Let $x \in M$ and fix an isomorphism $\upsilon : \mathbb{C}^m \rightarrow T_xM$ with $\upsilon^*(\omega) = \omega'$ and $\upsilon^*(\Omega) = \psi(x)^m\Omega'$, where ω', Ω' are as in (1).*

Suppose that T is a special Lagrangian integral current in M with $x \in T^\circ$, and that $\upsilon_*(C)$ is a multiplicity 1 tangent cone to T at x, where C is a rigid special Lagrangian cone in $\mathbb{C}^m$ in the sense of Definition 3.6. Then T has a conical singularity at x, in the sense of Definition 3.7.

Here *integral currents*, *tangent cones* and *multiplicity* are technical terms from Geometric Measure Theory which are explained in [12, §6]. In fact [12, Th. 6.8] applies to the larger class of *Jacobi integrable* SL cones C, for which all special Lagrangian Jacobi fields are integrable.

Basically, Theorem 4.11 shows that if a singular SL m-fold T in M is locally modelled on a rigid SL cone C in only a very weak sense, then it necessarily satisfies Definition 3.7. One moral of Theorems 4.10 and 4.11 is that, at least for rigid SL cones C, more-or-less any sensible definition of SL m-folds with conical singularities is equivalent to Definition 3.7.

5 Moduli of SL m-folds with conical singularities

The rest of the paper studies *moduli spaces* $\mathcal{M}_X$ of compact SL m-folds X with conical singularities in an almost Calabi–Yau manifold M. This section sets up the notation needed to do this, and defines the moduli space $\mathcal{M}_X$ as a topological space, paying particular attention to the rôle of asymptotic conditions at the singular points in defining the topology on $\mathcal{M}_X$. We continue to suppose $m > 2$.

5.1 Notation to vary the x_i, υ_i

We are interested in deformations of X in M that are allowed to move the singular points $x_1, \ldots, x_n$ and the identifications $\upsilon_i : \mathbb{C}^m \to T_{x_i}M$. We begin by setting up some notation to allow us to do this.

Definition 5.1 Let (M, J, ω, Ω) be an almost Calabi–Yau m-fold and X a compact SL m-fold in M with conical singularities at $x_1, \ldots, x_n$ with identifications $\upsilon_i : \mathbb{C}^m \to T_{x_i}M$ and cones $C_1, \ldots, C_n$, and use the notation of §3.3. Define

$$\begin{aligned} P = \big\{(x, \upsilon) : x \in M,\ \upsilon : \mathbb{C}^m \to T_xM \text{ is a real isomorphism,} \\ \upsilon^*(\omega) = \omega', \quad \upsilon^*(\Omega) = \psi(x)^m \Omega'\big\}, \end{aligned} \tag{30}$$

where ω', Ω' are as in (1). Then $(x_i, \upsilon_i) \in P$ for $i = 1, \ldots, n$, and P is the family of all possible alternative choices of x_i, υ_i, by Definition 3.7.

Regard each matrix $B \in \mathrm{SU}(m)$ as a map $\mathbb{C}^m \to \mathbb{C}^m$. Then if $(x, \upsilon) \in P$ and $B \in \mathrm{SU}(m)$ then $(x, \upsilon \circ B) \in P$ as ω', Ω' are $\mathrm{SU}(m)$-invariant. Define a smooth, free action of $\mathrm{SU}(m)$ on P by $B : (x, \upsilon) \mapsto (x, \upsilon \circ B^{-1})$. If $(x, \upsilon), (x, \hat\upsilon) \in P$ then $B = \hat\upsilon^{-1} \circ \upsilon \in \mathrm{SU}(m)$ and $B(x, \upsilon) = (x, \hat\upsilon)$. Hence the $\mathrm{SU}(m)$-orbits in P correspond to points $x \in M$, and P is a principal $\mathrm{SU}(m)$-bundle over M. Thus $\dim P = m^2 + 2m - 1$.

Let G_i be the Lie subgroup of $\mathrm{SU}(m)$ preserving the cone C_i in $\mathbb{C}^m$ for $i = 1, \ldots, n$. Then G_i acts on P. If (x, υ) and $(x, \hat\upsilon)$ lie in the same G_i-orbit then they define *equivalent* alternative choices for (x_i, υ_i), since $\upsilon(C_i)$ and $\hat\upsilon(C_i)$ are the same SL cone in T_xM. Therefore if we use P to parametrize alternative choices for (x_i, υ_i) we will have redundant parameters when $\dim G_i > 0$, since each cone $\upsilon(C_i)$ in T_xM is represented not by a point in P but by a submanifold isomorphic to G_i.

To avoid this, let $\mathcal{E}_i$ be a small open ball of dimension $\dim P - \dim G_i$ in P containing (x_i, υ_i) and transverse to the orbits of G_i for $i = 1, \ldots, n$. Then $G_i \cdot \mathcal{E}_i$ is a small open neighbourhood of the G_i-orbit of (x_i, υ_i) in P. Define $\mathcal{E} = \mathcal{E}_1 \times \cdots \times \mathcal{E}_n$ and $e = (x_1, \upsilon_1, \ldots, x_n, \upsilon_n) \in \mathcal{E}$. Write a general element of $\mathcal{E}$ as $\hat e = (\hat x_1, \hat\upsilon_1, \ldots, \hat x_n, \hat\upsilon_n)$. Then $\mathcal{E}$ is a family of alternative choices $\hat x_i, \hat\upsilon_i$ of the x_i, υ_i, which represent all nearby alternative choices exactly once up to equivalence, and

$$\begin{aligned} \dim \mathcal{E}_i = m^2 + 2m - 1 - \dim G_i \\ \text{and} \quad \dim \mathcal{E} = n(m^2 + 2m - 1) - \textstyle\sum_{i=1}^n \dim G_i. \end{aligned} \tag{31}$$

The metric g on M induces a Riemannian metric on P which restricts to $\mathcal{E}_i$. Let d_ε be the metric induced on $\mathcal{E} = \mathcal{E}_1 \times \cdots \times \mathcal{E}_n$ by the product Riemannian metric, so that $(\mathcal{E}, d_\varepsilon)$ is a metric space.

The following result, modelled loosely on Theorem 4.9, extends X to a family of Lagrangian m-folds $\hat X$ with conical singularities at $\hat x_i$ and identifications $\hat\upsilon_i$ for $\hat e = (\hat x_1, \hat\upsilon_1, \ldots, \hat x_n, \hat\upsilon_n)$ in an open neighbourhood $\tilde{\mathcal{E}}$ of e in $\mathcal{E}$, and also defines Lagrangian neighbourhoods $\Phi^{\hat e}_{X'}$ for $\hat X$.

Theorem 5.2 *Suppose (M,J,ω,Ω) is an almost Calabi–Yau m-fold and X a compact SL m-fold in M with conical singularities at $x_1,\ldots,x_n$. Use the notation of Theorem 4.7, let $U_{X'},\Phi_{X'}$ be as in Theorem 4.8, and $e,\mathcal{E}$ as in Definition 5.1. Then for some connected open $\tilde{\mathcal{E}}\subseteq\mathcal{E}$ with $e\in\tilde{\mathcal{E}}$ and all $\hat e=(\hat x_1,\hat\upsilon_1,\ldots,\hat x_n,\hat\upsilon_n)$ in $\tilde{\mathcal{E}}$ there exist*

(a) *embeddings $\Upsilon_i^{\hat e}:B_R\rightarrow M$ for $i=1,\ldots,n$ with*

$$\Upsilon_i^e=\Upsilon_i,\quad (\Upsilon_i^{\hat e})^*(\omega)=\omega',\quad \Upsilon_i^{\hat e}(0)=\hat x_i\quad\textit{and}\quad \mathrm{d}\Upsilon_i^{\hat e}|_0=\hat\upsilon_i, \tag{32}$$

(b) *an embedding $\Phi_{X'}^{\hat e}:U_{X'}\rightarrow M$ with $\Phi_{X'}^e=\Phi_{X'}$ and $(\Phi_{X'}^{\hat e})^*(\omega)=\hat\omega$, such that $\Phi_{X'}^{\hat e}\equiv\Phi_{X'}$ on $\pi^*(K)\subset U_{X'}$,*

all depending smoothly on $\hat e\in\tilde{\mathcal{E}}$, with

$$\Phi_{X'}^{\hat e}\circ\mathrm{d}(\Upsilon_i\circ\phi_i)(\sigma,r,\tau,u)\equiv\Upsilon_i^{\hat e}\circ\Phi_{C_i}\bigl(\sigma,r,\tau+\eta_i^1(\sigma,r),u+\eta_i^2(\sigma,r)\bigr) \tag{33}$$

for all $\hat e\in\tilde{\mathcal{E}}$, $i=1,\ldots,n$ and $(\sigma,r,\tau,u)\in T^*\bigl(\Sigma_i\times(0,R')\bigr)$ with $\bigl|(\tau,u)\bigr|<\zeta r$.

Proof. We shall define $\Upsilon_i^{\hat e}$ and $\Phi_{X'}^{\hat e}$ by modifying $\Upsilon_i,\Phi_{X'}$ near $x_i\in M$ using a symplectomorphism of $B_R\subset\mathbb{C}^m$. Let $R''\in(0,\frac12R)$ satisfy conditions we will specify at the end of the proof, and let $B_{R''},B_{2R''}\subset B_R$ be the open balls of radius $R'',2R''$ about 0 in $\mathbb{C}^m$. Choose a connected open neighbourhood $\tilde{\mathcal{E}}$ of e in $\mathcal{E}$ such that for all $\hat e=(\hat x_1,\hat\upsilon_1,\ldots,\hat x_n,\hat\upsilon_n)$ in $\tilde{\mathcal{E}}$ we have $\hat x_i\in\Upsilon_i(B_{R''})$ for $i=1,\ldots,n$. Clearly this is possible.

Next, choose diffeomorphisms $\Xi_i^{\hat e}:B_R\rightarrow B_R$ for $i=1,\ldots,n$ and $\hat e\in\tilde{\mathcal{E}}$ depending smoothly on $\hat e$, such that

(i) Ξ_i^e is the identity on B_R for $i=1,\ldots,n$,

(ii) $(\Xi_i^{\hat e})^*(\omega')=\omega'$ for $\hat e\in\tilde{\mathcal{E}}$ and $i=1,\ldots,n$,

(iii) $\Upsilon_i\circ\Xi_i^{\hat e}(0)=\hat x_i$ and $\mathrm{d}(\Upsilon_i\circ\Xi_i^{\hat e})|_0=\hat\upsilon_i$ for $\hat e=(\hat x_1,\hat\upsilon_1,\ldots,\hat x_n,\hat\upsilon_n)\in\tilde{\mathcal{E}}$ and $i=1,\ldots,n$, and

(iv) $\Xi_i^{\hat e}$ is the identity outside $B_{2R''}\subset B_R$ for $\hat e\in\tilde{\mathcal{E}}$ and $i=1,\ldots,n$.

Making $\tilde{\mathcal{E}}$ smaller if necessary, one can do this explicitly using standard but messy symplectic geometry techniques, and we leave it as an exercise.

Now define an embedding $\Upsilon_i^{\hat e}=\Upsilon_i\circ\Xi_i^{\hat e}:B_R\rightarrow M$ for $i=1,\ldots,n$ and $\hat e\in\tilde{\mathcal{E}}$. Then $\Upsilon_i^{\hat e}$ depends smoothly on $\hat e$ as $\Xi_i^{\hat e}$ does, and (32) follows immediately from $\Upsilon_i^*(\omega)=\omega'$ and parts (i)–(iii) above. Regard (33) as a *definition* of $\Phi_{X'}^{\hat e}$ on $\pi^*(S_i)\subset U_{X'}$ for $i=1,\ldots,n$, and define $\Phi_{X'}^{\hat e}\equiv\Phi_{X'}$ on $\pi^*(K)\subset U_{X'}$. Then $\Phi_{X'}^{\hat e}:U_{X'}\rightarrow M$ is well-defined, and satisfies (33).

To see that $\Phi_{X'}^{\hat e}$ is smooth, we need to show that its definitions on $\pi^*(S_i)$ and $\pi^*(K)$ join together smoothly on $\pi^*(\partial K)$. This follows from part (iv) above provided $\Phi_{X'}\bigl(\pi^*(\partial K)\bigr)$ does not intersect $\Upsilon_i(B_{2R''})$, since then when r is close to R' in (33) we have $\Upsilon_i^{\hat e}=\Upsilon_i$, and thus $\Phi_{X'}^{\hat e}=\Phi_{X'}$ near the boundary of $\pi^*(S_i)$ where it joins onto $\pi^*(K)$.

Hence, choosing $R'' \in (0, \frac{1}{2}R)$ such that $\Phi_{X'}\big(\pi^*(\partial K)\big)$ does not intersect $\Upsilon_i(B_{2R''})$ for $i = 1, \ldots, n$ ensures that $\Phi_{X'}^{\hat{e}}$ is smooth for all $\hat{e} \in \tilde{\mathcal{E}}$, and making $\tilde{\mathcal{E}}$ smaller if necessary we can assume it is an embedding. As $\Phi_{X'}^*(\omega) = \hat{\omega}$ we see that $(\Phi_{X'}^{\hat{e}})^*(\omega) = \hat{\omega}$ on $\pi^*(K)$, and $(\Phi_{X'}^{\hat{e}})^*(\omega) = \hat{\omega}$ on $\pi^*(S_i)$ follows from (33) since $(\Upsilon_i^{\hat{e}})^*(\omega) = \omega'$. Finally, $\Phi_{X'}^{e} = \Phi_{X'}$ as $\Upsilon_i^e = \Upsilon_i$ for $i = 1, \ldots, n$. □

In the situation of the theorem, fix $\hat{e} \in \tilde{\mathcal{E}}$ and define $\hat{X}' = \Phi_{X'}^{\hat{e}}(X')$, where $X' \subset U_{X'} \subset T^*X'$ is the zero section, and set $\hat{X} = \hat{X}' \cup \{\hat{x}_1, \ldots, \hat{x}_n\}$. As $(\Phi_{X'}^{\hat{e}})^*(\omega) = \hat{\omega}$ it follows that $\hat{X}'$ is a Lagrangian submanifold of M, and thus $\hat{X}$ is a compact *Lagrangian* m-fold in M with conical singularities at $\hat{x}_1, \ldots, \hat{x}_n$, identifications $\hat{\upsilon}_1, \ldots, \hat{\upsilon}_n$ and cones $C_1, \ldots, C_n$, generalizing Definition 3.7 in the obvious way.

Thus we have extended X to a smooth family of Lagrangian m-folds $\hat{X}$ with conical singularities, which realize all nearby alternative choices of x_i, υ_i exactly once up to equivalence. When $\hat{e}$ is close to e, $\hat{X}$ will be approximately special Lagrangian, and so we can try to deform it to an exactly special Lagrangian m-fold with the same $\hat{x}_i, \hat{\upsilon}_i$.

5.2 Small deformations of X and moduli spaces

Suppose that (M, J, ω, Ω) is an almost Calabi–Yau m-fold and that $X, \hat{X}$ are compact SL m-folds in M which both have n conical singular points $x_1, \ldots, x_n$ and $\hat{x}_1, \ldots, \hat{x}_n$ respectively, with the same cones $C_1, \ldots, C_n$ and rates $\mu_1, \ldots, \mu_n$. When $X, \hat{X}$ are 'sufficiently close' in a C^1 sense we shall write $\hat{X}$ in terms of a small closed 1-form α on X' with prescribed decay, using the Lagrangian neighbourhood $\Phi_{X'}^{\hat{e}}$ of Theorem 5.2. Thus we shall define a topology on the set of compact SL m-folds in M with conical singularities.

Theorem 5.3 *Suppose (M, J, ω, Ω) is an almost Calabi–Yau m-fold and X a compact SL m-fold in M with conical singularities at $x_1, \ldots, x_n$, with identifications υ_i, cones C_i and rates μ_i. Let $e, \mathcal{E}$ be as in Definition 5.1, and $U_{X'}, \Phi_{X'}, \tilde{\mathcal{E}}, \Upsilon_i^{\hat{e}}$ and $\Phi_{X'}^{\hat{e}}$ be as in Theorem 5.2.*

Let $\hat{e} = (\hat{x}_1, \hat{\upsilon}_1, \ldots, \hat{x}_n, \hat{\upsilon}_n) \in \tilde{\mathcal{E}}$, and suppose $\hat{X}$ is a compact SL m-fold in M with conical singularities at $\hat{x}_1, \ldots, \hat{x}_n$, with identifications $\hat{\upsilon}_i$, cones C_i and rates μ_i. Then if $\hat{e}, e$ are sufficiently close in $\tilde{\mathcal{E}}$ and $X', \hat{X}'$ are sufficiently close as submanifolds in a C^1 sense away from $x_1, \ldots, x_n$, there exists a closed 1-form α on X' such that the graph $\Gamma(\alpha)$ lies in $U_{X'} \subset T^*X'$, and $\hat{X}' = \Phi_{X'}^{\hat{e}}\big(\Gamma(\alpha)\big)$. Furthermore we may write $\alpha = \beta + \mathrm{d}f$, where β is a closed 1-form supported in K and $f \in C^\infty_{\boldsymbol{\mu}}(X')$.

Proof. Apply Theorem 4.7 to X and $\hat{X}$, using $\Upsilon_i^e = \Upsilon_i$ for X and $\Upsilon_i^{\hat{e}}$ for $\hat{X}$, and the same R, ζ, U_{C_i} and Φ_{C_i} for both. Theorem 4.7 then gives $R', \hat{R}' \in (0, R]$ and closed 1-forms η_i on $\Sigma_i \times (0, R')$ and $\hat{\eta}_i$ on $\Sigma_i \times (0, \hat{R}')$ for $i = 1, \ldots, n$ such that $X', \hat{X}'$ are parametrized on $S_i, \hat{S}_i$ using maps $\phi_i : \Sigma_i \times (0, R') \to B_R$ and $\hat{\phi}_i : \Sigma_i \times (0, \hat{R}') \to B_R$ defined using $\eta_i, \hat{\eta}_i$ in the usual way.

Theorem 4.10 defines real functions A_i on $\Sigma_i \times (0,R')$ and $\hat A_i$ on $\Sigma_i \times (0,\hat R')$ with $\eta_i = \mathrm{d}A_i$ and $\hat\eta_i = \mathrm{d}\hat A_i$, and proves results on the decay of ϕ_i, η_i, A_i and $\hat\phi_i, \hat\eta_i, \hat A_i$ and their derivatives. Using (25) and $\mu_i > 2$ we see that $\eta_i, \hat\eta_i = o(r)$ for small r. Therefore we may choose $R'' \in \bigl(0, \min(R', \hat R')\bigr]$ such that $|\hat\eta_i - \eta_i| < \zeta r$ on $\Sigma_i \times (0,R'')$ for all $i = 1,\dots,n$.

Let $S_i' = \Upsilon_i \circ \phi_i\bigl(\Sigma_i \times (0,R'')\bigr)$ and $\hat S_i' = \Upsilon_i^{\hat e} \circ \hat\phi_i\bigl(\Sigma_i \times (0,R'')\bigr)$ for $i = 1,\dots,n$, so that $S_i' \subseteq S_i \subset X'$ and $\hat S_i' \subseteq \hat S_i \subset \hat X'$. Define a 1-form α on S_i' by $\alpha = (\Upsilon_i \circ \phi_i)_*(\hat\eta_i - \eta_i)$ for $i = 1,\dots,n$. Now as $\hat\phi_i(\sigma,r) = \Phi_{C_i}\bigl(\sigma, r, \hat\eta_i^1(\sigma,r), \hat\eta_i^2(\sigma,r)\bigr)$ by Theorem 4.7, we see from (33) that if $(\sigma,r) \in \Sigma_i \times (0,R'')$ and $(\tau,u) = (\hat\eta_i^1 - \eta_i^1, \hat\eta_i^2 - \eta_i^2)(\sigma,r)$ then

$$\Phi_{X'}^{\hat e}\bigl[\alpha\bigl(\Upsilon_i \circ \phi_i(\sigma,r)\bigr)\bigr] = \Phi_{X'}^{\hat e} \circ \mathrm{d}(\Upsilon_i \circ \phi_i)(\sigma,r,\tau,u) = \Upsilon_i^{\hat e} \circ \hat\phi_i(\sigma,r) \in \hat S_i' \subset \hat X'.$$

Thus the subsets $\hat S_i'$ in $\hat X'$ coincide with $\Phi_{X'}^{\hat e}\bigl(\Gamma(\alpha)\bigr)$ on the subsets S_i' in X' where α is defined so far. To show that $\hat X' = \Phi_{X'}^{\hat e}\bigl(\Gamma(\alpha)\bigr)$ for some 1-form α defined on the whole of X', we need that

(a) $\hat X'$ should lie in $\Phi_{X'}^{\hat e}(U_{X'})$, and

(b) $\hat X'$ should intersect the image under $\Phi_{X'}^{\hat e}$ of each fibre of $\pi : U_{X'} \to X'$ transversely exactly once.

We have already shown that (a) and (b) hold on the subsets $\hat S_i'$.

Under the assumptions of the theorem $\hat e, e$ are close in $\tilde{\mathcal{E}}$ and $\Phi_{X'}^{\hat e}$ and $\Phi_{X'}^{e} = \Phi_{X'}$ are close on the complement of the S_i'. Also $X', \hat X'$ are close as submanifolds in a C^1 sense away from $x_1,\dots,x_n$, and thus on the complement of the subsets S_i' in X' and $\hat S_i'$ in $\hat X'$ for $i = 1,\dots,n$. Therefore $\hat X'$ satisfies (a) and (b) on the complement of the $\hat S_i'$, and α exists. Since $\hat X'$ is Lagrangian and $(\Phi_{X'}^{\hat e})^*(\omega) = \hat\omega$, the usual argument shows that α is closed.

Define a smooth real function f on S_i' by $f = (\Upsilon_i \circ \phi_i)_*(\hat A_i - A_i)$ for $i = 1,\dots,n$. Then $\alpha = \mathrm{d}f$ on S_i', as $\eta_i = \mathrm{d}A_i$ and $\hat\eta_i = \mathrm{d}\hat A_i$. As α is closed and $S_i' \subseteq S_i$ are homotopy equivalent we can extend f uniquely to S_i with $\alpha = \mathrm{d}f$. Then extend f smoothly over K. This defines a smooth function f on X' with $\alpha = \mathrm{d}f$ on S_i for $i = 1,\dots,n$. Let $\beta = \alpha - \mathrm{d}f$. Then $\alpha = \beta + \mathrm{d}f$ and β is a closed 1-form supported in $K = X' \setminus (S_1 \cup \cdots \cup S_n)$, as we have to prove. Finally, (29) for $A_i, \hat A_i$ with $\mu_i' = \mu_i$ gives $f \in C^\infty_{\boldsymbol{\mu}}(X')$. □

We define the *moduli space* $\mathcal{M}_X$ of SL m-folds $\hat X$ with conical singularities in M, which are isotopic to X in M and have the same cones $C_1,\dots,C_n$.

Definition 5.4 Let (M,J,ω,Ω) be an almost Calabi–Yau m-fold and X a compact SL m-fold in M with conical singularities at $x_1,\dots,x_n$ with identifications $\upsilon_i : \mathbb{C}^m \to T_{x_i}M$ and cones $C_1,\dots,C_n$. Define the *moduli space* $\mathcal{M}_X$ *of deformations of* X to be the set of $\hat X$ such that

(i) $\hat X$ is a compact SL m-fold in M with conical singularities at $\hat x_1,\dots,\hat x_n$ with cones $C_1,\dots,C_n$, for some $\hat x_i$ and identifications $\hat\upsilon_i : \mathbb{C}^m \to T_{\hat x_i}M$.

(ii) There exists a homeomorphism $\hat{\imath} : X \to \hat{X}$ with $\hat{\imath}(x_i) = \hat{x}_i$ for $i = 1, \ldots, n$ such that $\hat{\imath}|_{X'} : X' \to \hat{X}'$ is a diffeomorphism and $\hat{\imath}$ and ι are isotopic as continuous maps $X \to M$, where $\iota : X \to M$ is the inclusion.

Note that by Theorem 4.10 the definition of $\hat{X}$ is independent of choice of rates μ_i, so there is no need to include the μ_i in (i).

Let $\mathcal{V}_X$ be the subset of $\hat{X} \in \mathcal{M}_X$ such that for some $\hat{e} = (\hat{x}_1, \hat{\upsilon}_1, \ldots, \hat{x}_n, \hat{\upsilon}_n)$ in $\tilde{\mathcal{E}}$ and some 1-form α on X' whose graph $\Gamma(\alpha)$ lies in $U_{X'} \subset T^*X'$ we have $\hat{X}' = \Phi^{\hat{e}}_{X'}\bigl(\Gamma(\alpha)\bigr)$, as in Theorem 5.3. Note that if $\tilde{X} \in \mathcal{M}_X$ then $\mathcal{M}_{\tilde{X}} = \mathcal{M}_X$. Thus, for each $\tilde{X} \in \mathcal{M}_X$ we have $\tilde{X} \in \mathcal{V}_{\tilde{X}} \subset \mathcal{M}_X$.

The construction of $\mathcal{V}_X$ above gives a 1-1 correspondence between $\mathcal{V}_X \subseteq \mathcal{M}_X$ and a set of pairs $(\hat{e}, \alpha)$ for $\hat{e} \in \tilde{\mathcal{E}}$ and α a smooth 1-form on X' with prescribed decay. Using the given topology on $\tilde{\mathcal{E}}$ and a suitable choice of topology on the 1-forms α, this 1-1 correspondence induces a *topology* on $\mathcal{V}_X$.

To define the α topology, choose some $\boldsymbol{\mu}$ as in Definition 3.7, and let the $C^k_{\boldsymbol{\mu}-1}$ topology on α be induced by the norm

$$\|\alpha\|_{C^k_{\boldsymbol{\mu}-1}} = \sum_{j=0}^{k} \sup_{X'} \bigl|\rho^{-\boldsymbol{\mu}+1-j} \nabla^j \alpha\bigr|,$$

and the $C^\infty_{\boldsymbol{\mu}-1}$ topology on α be induced by the $C^k_{\boldsymbol{\mu}-1}$ topologies for all $k \geqslant 0$.

Proposition 5.5 *The $C^1_{\boldsymbol{\mu}-1}$ and $C^\infty_{\boldsymbol{\mu}-1}$ topologies on α induce the same topology on $\mathcal{V}_X$, which is also independent of the choice of rates $\boldsymbol{\mu}$.*

Proof. This is implicit in the proofs of Theorems 4.10 and 5.3. In particular, Theorem 4.10 in effect shows that an a priori estimate for the $C^1_{\boldsymbol{\mu}-1}$ norm of α implies a priori estimates for the $C^k_{\boldsymbol{\mu}-1}$ norms for all $k \geqslant 1$, and so the $C^1_{\boldsymbol{\mu}-1}$ and $C^\infty_{\boldsymbol{\mu}-1}$ topologies on α induce the same topology on $\mathcal{V}_X$. It also proves independence of the choice of $\boldsymbol{\mu}$. □

We can now define a *topology* on $\mathcal{M}_X$.

Definition 5.6 For each $\tilde{X} \in \mathcal{M}_X$, use the 1-1 correspondence between $\mathcal{V}_{\tilde{X}}$ and pairs $(\hat{e}, \alpha)$ to define a topology on $\mathcal{V}_{\tilde{X}}$ as in Proposition 5.5. We get the same topology using the $C^1_{\boldsymbol{\mu}-1}$ or $C^\infty_{\boldsymbol{\mu}-1}$ topologies on α for any choice of $\boldsymbol{\mu}$, so there is no ambiguity. One can show that overlaps $\mathcal{V}_{X_1} \cap \mathcal{V}_{X_2}$ are open in $\mathcal{V}_{X_j}$ and the $\mathcal{V}_{X_j}$ topologies agree on the overlaps. Piecing the topologies together therefore defines a unique topology on $\mathcal{M}_X$.

Remarks. Basically, $\mathcal{M}_X$ is the family of compact SL m-folds $\hat{X}$ in M with conical singularities which are deformation equivalent to X in a loose sense. Note that $\mathcal{M}_X$ may not be *connected*, as the isotopies in part (ii) of Definition 5.4 need not be through special Lagrangian embeddings.

In Theorem 5.3 we assumed only that $\hat{e}, e$ are close in $\tilde{\mathcal{E}}$ and that $X', \hat{X}'$ are 'sufficiently close as submanifolds in a C^1 sense away from $x_1, \ldots, x_n$'. These closeness assumptions are actually *very weak*, in that we have imposed *no asymptotic conditions* on how $X', \hat{X}'$ converge to x_i and $\hat{x}_i$, but instead required only C^1 closeness on large compact subsets of $X', \hat{X}'$.

Because of this, we can be confident that the topology defined on $\mathcal{M}_X$ above is a sensible choice. In particular, Theorem 5.3 effectively shows that if $X, \hat{X}$ are close in a very weak sense, then they are close in the $\mathcal{M}_X$ topology. Theorem 6.14 below gives another way of seeing the naturality of the topology on $\mathcal{M}_X$.

Definitions 5.4 and 5.6 don't actually need X to be special Lagrangian in (M, J, ω, Ω), except to ensure that $X \in \mathcal{M}_X$. We are simply using X to fix the topological type, isotopy class and singular cones C_i of $\hat{X} \in \mathcal{M}_X$. In particular, given a family $\{(M, J^s, \omega^s, \Omega^s) : s \in \mathcal{F}\}$ of almost Calabi–Yau structures on M with X special Lagrangian in $(M, J^0, \omega^0, \Omega^0)$, we can define a moduli space $\mathcal{M}_X^s$ of special Lagrangian deformations of X in $(M, J^s, \omega^s, \Omega^s)$, for each $s \in \mathcal{F}$.

6 Deformations, obstructions, and smoothness

We can now prove the first main result of the paper, Theorem 6.10 below, which is an analogue of McLean's Theorem, Theorem 2.6, for compact SL m-folds X with conical singularities $x_1, \ldots, x_n$ in a single almost Calabi–Yau m-fold (M, J, ω, Ω). An important difference with the nonsingular case is that there may be *obstructions* to deforming X, which means that the moduli space $\mathcal{M}_X$ may be singular.

Instead, $\mathcal{M}_X$ is locally homeomorphic by a map Ξ to the zeroes of a smooth map $\Phi : \mathcal{I}_{X'} \to \mathcal{O}_{X'}$ between finite-dimensional vector spaces $\mathcal{I}_{X'}$, the *infinitesimal deformation space*, and $\mathcal{O}_{X'}$, the *obstruction space*. Here $\mathcal{I}_{X'}$ is isomorphic to the image of $H^1_{\rm cs}(X', \mathbb{R})$ in $H^1(X', \mathbb{R})$, and $\mathcal{O}_{X'}$ is a direct sum of subspaces depending on the SL cones $C_1, \ldots, C_n$ of X at $x_1, \ldots, x_n$.

We set up the problem in §6.1, and define $\mathcal{O}_{X'}$ in §6.2. The main theorem is proved in §6.3, with some corollaries on cases when $\mathcal{M}_X$ is smooth. Section 6.4 discusses the naturality (independence of choices) of $\mathcal{I}_{X'}, \mathcal{O}_{X'}, \Phi$ and Ξ, and §6.5 another way to define $\mathcal{I}_{X'}$ and $\mathcal{O}_{X'}$.

6.1 Setting up the deformation problem

We shall parametrize the moduli space $\mathcal{M}_X$ locally in terms of the zeroes of a map F between Banach spaces.

Definition 6.1 Let (M, J, ω, Ω) be an almost Calabi–Yau m-fold and X a compact SL m-fold in M with conical singularities at $x_1, \ldots, x_n$ with identifications $\upsilon_i : \mathbb{C}^m \to T_{x_i}M$ and cones $C_1, \ldots, C_n$. Let $U_{X'}, \Phi_{X'}$ be as in Theorem 4.8, and $e, \mathcal{E}$ as in Definition 5.1, and $\tilde{\mathcal{E}}$, $\Upsilon_i^{\hat{e}}$ and $\Phi_{X'}^{\hat{e}}$ as in Theorem 5.2.

Choose a vector space $\mathcal{H}_{X'}$ of closed 1-forms on X' supported in K, such that the map $\mathcal{H}_{X'} \to H^1_{\rm cs}(X', \mathbb{R})$ given by $\beta \mapsto [\beta]$ is an isomorphism. Since X' retracts onto K, this is clearly possible. Now the subspace of $\mathcal{H}_{X'}$ corresponding to the kernel of the map $H^1_{\rm cs}(X', \mathbb{R}) \to H^1(X', \mathbb{R})$ in (19) consists of *exact* 1-forms on X', so each such 1-form may be written ${\rm d}v$ for some $v \in C^\infty(X')$.

Let the connected components of $S_i \cong \Sigma_i \times (0, R')$ be S_i^j for $j = 1, \ldots, b^0(\Sigma_i)$. As $\mathrm{d}v = 0$ on S_i we see that $v = a_i^j$ on S_i^j for some constants a_i^j. Since v is defined up to addition of a constant, we specify v uniquely by requiring that $\sum_{i,j} a_i^j = 0$.

Define $\mathcal{K}_{X'}$ to be the vector space of all such functions v. Then $\mathrm{d}\mathcal{K}_{X'} = \{\mathrm{d}v : v \in \mathcal{K}_{X'}\}$ is a subspace of $\mathcal{H}_{X'}$, and $\mathrm{d} : \mathcal{K}_{X'} \to \mathcal{H}_{X'}$ is injective. Also $\mathcal{K}_{X'}$ is isomorphic to the kernel of $H^1_{\mathrm{cs}}(X', \mathbb{R}) \to H^1(X', \mathbb{R})$ in (19). Thus by (19) we have an exact sequence

$$0 \to H^0(X', \mathbb{R}) \to \bigoplus_{i=1}^n H^0(\Sigma_i, \mathbb{R}) \to \mathcal{K}_{X'} \to 0,$$

so as X' is connected we see that

$$\dim \mathcal{K}_{X'} = \sum_{i=1}^n b^0(\Sigma_i) - 1. \tag{34}$$

Let the *infinitesimal deformation space* $\mathcal{I}_{X'}$ be a vector subspace of $\mathcal{H}_{X'}$ with

$$\mathcal{H}_{X'} = \mathcal{I}_{X'} \oplus \mathrm{d}\mathcal{K}_{X'}. \tag{35}$$

As $\mathrm{d}\mathcal{K}_{X'}$ corresponds to the kernel of $H^1_{\mathrm{cs}}(X', \mathbb{R}) \to H^1(X', \mathbb{R})$ in (19) and $\mathcal{I}_{X'} \cong \mathcal{H}_{X'}/\mathrm{d}\mathcal{K}_{X'}$, we see that the map $\mathcal{I}_{X'} \to H^1(X', \mathbb{R})$ given by $\beta \mapsto [\beta]$ is an isomorphism between $\mathcal{I}_{X'}$ and the image of $H^1_{\mathrm{cs}}(X', \mathbb{R})$ in $H^1(X', \mathbb{R})$.

Let $k > 2$, $p > m$, and $\boldsymbol{\mu}$ be as in Definition 3.7. Then $L^p_{k,\boldsymbol{\mu}}(X')$ is continuously included in $C^2_{\boldsymbol{\mu}}(X')$ by Theorem 4.3. Define

$$\mathcal{D}_{X'} = \big\{(\beta, f) \in \mathcal{H}_{X'} \times L^p_{k,\boldsymbol{\mu}}(X') : \text{the graph of } \beta + \mathrm{d}f \text{ lies in } U_{X'}\big\}. \tag{36}$$

Then $\mathcal{D}_{X'}$ is an open subset of $\mathcal{H}_{X'} \times L^p_{k,\boldsymbol{\mu}}(X')$ containing $(0, 0)$. Here we use the fact that f is C^1 to make sense of the graph of $\beta + \mathrm{d}f$.

Define a map $F : \tilde{\mathcal{E}} \times \mathcal{D}_{X'} \to C^0(X')$ by

$$\pi_*\big((\Phi^{\hat{e}}_{X'})^*(\operatorname{Im}\Omega)|_{\Gamma(\beta+\mathrm{d}f)}\big) = F(\hat{e}, \beta, f)\,\mathrm{d}V_g, \tag{37}$$

where $\Gamma(\beta+\mathrm{d}f)$ is the graph of $\beta+\mathrm{d}f$ in $U_{X'}$, and $\pi : \Gamma(\beta+\mathrm{d}f) \to X'$ the natural projection, and $\mathrm{d}V_g$ the volume form of the metric g on X'. Since f is C^2 we see that $\Gamma(\beta + \mathrm{d}f)$ is a C^1-submanifold of $U_{X'}$, and so $(\Phi^{\hat{e}}_{X'})^*(\operatorname{Im}\Omega)|_{\Gamma(\beta+\mathrm{d}f)}$ makes sense and its image under π is continuous. Hence $F(\hat{e}, \beta, f)$ lies in $C^0(X')$, the vector space of continuous functions on X'.

The point of the definition is given in the following proposition.

Proposition 6.2 *In the situation of Definition 6.1, suppose $(\hat{e}, \beta, f) \in \tilde{\mathcal{E}} \times \mathcal{D}_{X'}$ with $F(\hat{e}, \beta, f) = 0$. Set $\hat{X}' = \Phi^{\hat{e}}_{X'}\big(\Gamma(\beta + \mathrm{d}f)\big)$ and $\hat{X} = \hat{X}' \cup \{\hat{x}_1, \ldots, \hat{x}_n\}$, where $\hat{e} = (\hat{x}_1, \ldots, \hat{\upsilon}_n)$. Then $f \in C^\infty_{\boldsymbol{\mu}}(X')$ and $\hat{X}$ is a compact SL m-fold in M with conical singularities at $\hat{x}_i$ with identifications $\hat{\upsilon}_i$, cones C_i and rates μ_i.*

Thus $\hat{X}$ lies in $\mathcal{V}_X \subseteq \mathcal{M}_X$ in Definition 5.4. Conversely, each $\hat{X}$ in $\mathcal{V}_X$ comes from a unique $(\hat{e}, \beta, f) \in \tilde{\mathcal{E}} \times \mathcal{D}_{X'}$ with $F(\hat{e}, \beta, f) = 0$. Write $\Psi(\hat{e}, \beta, f) = \hat{X}$. Then $\Psi : F^{-1}(0) \to \mathcal{V}_X$ is a homeomorphism, with $\Psi(e, 0, 0) = X$.

Proof. Suppose $F(\hat{e},\beta,f)=0$. Then $f\in C^2_{\boldsymbol\mu}(X')$ from above, so f is locally C^2 and $\hat{X}'$ is a C^1 submanifold of M. As $(\Phi^{\hat{e}}_{X'})^*(\omega)=\hat\omega$ and $\beta+\mathrm{d}f$ is a closed C^1 1-form, we see that $\omega|_{\hat{X}'}\equiv 0$ by the usual argument. Also (37) implies that $\operatorname{Im}\Omega|_{\hat{X}'}\equiv 0$. Therefore, if we can prove that $\hat{X}'$ is a C^∞ submanifold of M then $\hat{X}'$ is special Lagrangian, by Definition 2.5.

With $\hat{e},\beta$ fixed $F(\hat{e},\beta,f)$ depends pointwise on $\mathrm{d}f,\nabla^2 f$ by (37), so

$$F(\hat{e},\beta,f)[x]=F'\bigl(x,\mathrm{d}f(x),\nabla^2 f(x)\bigr)=0, \tag{38}$$

where F' is a smooth, nonlinear function of its arguments defined on some domain. Now (38) is a second-order nonlinear p.d.e., and using the ideas of §2.3 one can show that it is *elliptic*. Aubin [1, Th. 3.56] gives an elliptic regularity result for such equations which shows that if f is locally C^2 then f is locally C^∞. Thus f is smooth, so $\hat{X}'$ is C^∞ and thus special Lagrangian.

Recall that A_i is a function and $\eta_i=\mathrm{d}A_i$ a 1-form on $\Sigma_i\times(0,R')$ for $i=1,\ldots,n$, defined in Theorems 4.7 and 4.10, and that $\Upsilon_i\circ\phi_i:\Sigma_i\times(0,R')\rightarrow S_i\subset X'$ is a diffeomorphism. Define $\hat{A}_i$ and $\hat\eta_i$ on $\Sigma_i\times(0,R')$ by

$$\hat{A}_i=f\circ\Upsilon_i\circ\phi_i+A_i \quad\text{and}\quad \hat\eta_i=\mathrm{d}\hat{A}_i=\mathrm{d}(f\circ\Upsilon_i\circ\phi_i)+\eta_i. \tag{39}$$

Let $\hat\eta_i^1,\hat\eta_i^2$ be the components of $\hat\eta_i$ as in Theorem 4.7, and define

$$\hat\phi_i:\Sigma_i\times(0,R')\rightarrow B_R \quad\text{by}\quad \hat\phi_i(\sigma,r)=\Phi_{C_i}\bigl(\sigma,r,\hat\eta_i^1(\sigma,r),\hat\eta_i^2(\sigma,r)\bigr). \tag{40}$$

Combining (25), (29), (40) and $f\in C^2_{\boldsymbol\mu}(X')$ from above, we prove that

$$\begin{gathered}\bigl|\nabla^k(\hat\phi_i-\iota_i)\bigr|=O(r^{\mu_i-1-k}) \quad\text{and}\quad \bigl|\nabla^k\hat\eta_i\bigr|=O(r^{\mu_i-1-k}) \quad\text{for } k=0,1\\ \text{and}\quad \bigl|\nabla^k\hat{A}_i\bigr|=O(r^{\mu_i-k}) \quad\text{for } k=0,1,2,\text{ as } r\rightarrow 0 \text{ for } i=1,\ldots,n.\end{gathered} \tag{41}$$

Using (33) and the facts that $\hat{X}'=\Phi^{\hat{e}}_{X'}\bigl(\Gamma(\beta+\mathrm{d}f)\bigr)$ and $\beta=0$ in S_i, we find that $\Upsilon^{\hat{e}}_i\circ\hat\phi_i:\Sigma_i\times(0,R')\rightarrow M$ maps into $\hat{X}'$, and defines a diffeomorphism $\Sigma_i\times(0,R')\rightarrow\hat{S}_i$ with an open subset $\hat{S}_i$ of $\hat{X}'$. Also the natural diffeomorphism $X'\rightarrow\hat{X}'$ identifies S_i and $\hat{S}_i$, and thus $\hat{K}=\hat{X}'\setminus(\hat{S}_1\cup\cdots\cup\hat{S}_n)$ is compact.

Therefore all the conditions of Definition 3.7 are satisfied, and so $\hat{X}$ is a compact SL m-fold in M with conical singularities at $\hat{x}_1,\ldots,\hat{x}_n$, with identifications $\hat\upsilon_i$, cones C_i and rates μ_i, as we have to prove. Applying Theorem 4.10 to X and $\hat{X}$ then shows that $|\nabla^k A_i|=O(r^{\mu_i-k})$ and $|\nabla^k\hat{A}_i|=O(r^{\mu_i-k})$ for all $k\geqslant 0$. Thus (39) gives $|\nabla^k f|=O(\rho^{\mu_i-k})$ on S_i for all $k\geqslant 0$ and $i=1,\ldots,n$. Since f is smooth this implies that $f\in C^\infty_{\boldsymbol\mu}(X')$, as we have to prove.

Definition 5.4 now shows that $\hat{X}\in\mathcal{V}_X$. Conversely, if $\hat{X}\in\mathcal{V}_X$ then Definition 5.4 gives $\hat{X}'=\Phi^{\hat{e}}_{X'}\bigl(\Gamma(\alpha)\bigr)$ for some $\hat{e}\in\tilde{\mathcal{E}}$ and 1-form α on X' whose graph $\Gamma(\alpha)$ lies in $U_{X'}$. The proof of Theorem 5.3 then shows that $\alpha=\tilde\beta+\mathrm{d}\tilde{f}$, where $\tilde\beta$ is a closed 1-form supported in K and $\tilde{f}\in C^\infty_{\boldsymbol\mu}(X')$.

Let β be the unique element of $\mathcal{H}_{X'}$ with $[\beta]=[\tilde\beta]$ in $H^1_{\mathrm{cs}}(X',\mathbb{R})$, where $\mathcal{H}_{X'}$ is as in Definition 6.1. Then $\tilde\beta-\beta=\mathrm{d}\gamma$, for $\gamma\in C^\infty_{\mathrm{cs}}(X')$. Set $f=\tilde{f}+\gamma$. Then

$f \in C^\infty_{\boldsymbol{\mu}}(X')$ with $\alpha = \beta + \mathrm{d}f$. Theorem 4.10 shows that we can improve the rates μ_i of the singularities $\hat{x}_i$ of $\hat{X}$ to some rates $\mu'_i > \mu_i$ for $i = 1, \ldots, n$. It follows that $f \in C^\infty_{\boldsymbol{\mu}'}(X')$, and therefore $f \in L^p_{k,\boldsymbol{\mu}}(X')$ as $C^\infty_{\boldsymbol{\mu}'}(X') \subset L^p_{k,\boldsymbol{\mu}}(X')$. Therefore $(\beta, f) \in \mathcal{D}_{X'}$ by (36).

As $\hat{X}'$ is special Lagrangian $\mathrm{Im}\,\Omega|_{\hat{X}'} \equiv 0$, and it follows from (37) that $F(\hat{e}, \beta, f) = 0$. Thus each $\hat{X}$ in $\mathcal{V}_X$ comes from some $(\hat{e}, \beta, f) \in \tilde{\mathcal{E}} \times \mathcal{D}_{X'}$ with $F(\hat{e}, \beta, f) = 0$. Since there are no nontrivial $G_1 \times \cdots \times G_n$ equivalences in $\tilde{\mathcal{E}}$ by construction, $\hat{X}$ determines $\hat{e}$ uniquely, and $\hat{X}, \hat{e}$ then determine α and so β, f uniquely. Thus $(\hat{e}, \beta, f)$ is unique.

Thus writing $\Psi(\hat{e}, \beta, f) = \hat{X}$ defines a bijection $\Psi : F^{-1}(0) \to \mathcal{V}_X$ with $\Psi(e, 0, 0) = X$. We must show that Ψ is a *homeomorphism*. The topology on $\mathcal{V}_X$ is defined using pairs $(\hat{e}, \alpha)$, where $\hat{e}$ has the $\tilde{\mathcal{E}}$ topology and α either the $C^1_{\boldsymbol{\mu}-1}$ or the $C^\infty_{\boldsymbol{\mu}-1}$ topology on 1-forms for any choice of $\boldsymbol{\mu}$, and Ψ takes $(\hat{e}, \beta, f) \mapsto (\hat{e}, \beta + \mathrm{d}f)$.

Now f has the $L^p_{k,\boldsymbol{\mu}}$ topology, so $\mathrm{d}f$ has the $L^p_{k-1,\boldsymbol{\mu}-1}$ topology. This is intermediate between the $C^1_{\boldsymbol{\mu}-1}$ and $C^\infty_{\boldsymbol{\mu}'-1}$ topologies on α for $\mu'_i > \mu_i$ as above, as $C^\infty_{\boldsymbol{\mu}'-1}(T^*X') \subset L^p_{k-1,\boldsymbol{\mu}-1}(T^*X') \subset C^1_{\boldsymbol{\mu}-1}(T^*X')$ by Theorem 4.3. But the $C^1_{\boldsymbol{\mu}-1}$ and $C^\infty_{\boldsymbol{\mu}'-1}$ topologies on α induce the same topology on $\mathcal{V}_X$ by Proposition 5.5. Thus the $L^p_{k-1,\boldsymbol{\mu}-1}$ topology on $\mathrm{d}f$ also induces the same topology on $\mathcal{V}_X$, and it follows quickly that Ψ is a homeomorphism. $\square$

Here is an analogue of Proposition 2.10 for F.

Proposition 6.3 *In the situation above, for $x \in X'$ we may write*

$$F(\hat{e},\beta,f)[x] = -\mathrm{d}^*\big(\psi^m(\beta+\mathrm{d}f)\big)[x] + Q\big(\hat{e}, x, (\beta+\mathrm{d}f)(x), (\nabla\beta+\nabla^2 f)(x)\big), \quad (42)$$

where $Q : \big\{(\hat{e}, x, y, z) : \hat{e} \in \tilde{\mathcal{E}},\ x \in X',\ y \in T^*_x X' \cap U_{X'},\ z \in \otimes^2 T^*_x X'\big\} \to \mathbb{R}$ is smooth, and for $\rho(x)^{-1}|y|$, $|z|$ and $d_\varepsilon(\hat{e}, e)$ small we have

$$Q(\hat{e}, x, y, z) = O\big(\rho(x)^{-2}|y|^2 + |z|^2 + \rho(x) d_\varepsilon(\hat{e}, e)\big), \quad (43)$$

and more generally for $\rho(x)^{-1}|y|$, $|z|$ and $d_\varepsilon(\hat{e}, e)$ small and $a, b, c \geqslant 0$ we have

$$\begin{aligned}(\nabla_x)^a(\partial_y)^b(\partial_z)^c Q(\hat{e}, x, y, z) = O\big(&\rho(x)^{-a-\max(2,b)}|y|^{\max(0,2-b)} \\ &+ \rho(x)^{-a}|z|^{\max(0,2-c)} + \rho(x)^{1-a-b} d_\varepsilon(\hat{e}, e)\big),\end{aligned} \quad (44)$$

where $\nabla_x, \partial_y, \partial_z$ are the partial derivatives of Q in the x, y, z variables, using the Levi-Civita connection ∇ of g to form ∇_x.

Proof. The value of $F(\hat{e}, \beta, f)$ at $x \in X'$ depends on $\hat{e}$, via $(\Phi^{\hat{e}}_{X'})^*(\mathrm{Im}\,\Omega)$, and on the tangent space to $\Gamma(\beta + \mathrm{d}f)$ at x', where $x' \in \Gamma(\alpha)$ with $\pi(x') = x$. But $T_{x'}\Gamma(\beta+\mathrm{d}f)$ depends on both $(\beta+\mathrm{d}f)|_x$ and $(\nabla\beta+\nabla^2 f)|_x$. Therefore $F(\hat{e}, \beta, f)$ depends pointwise on the arguments of Q in (42).

As in the proof of Proposition 2.10 we may take (42) as a *definition* of Q, and Q is then well-defined on the given domain, which is the set of all $\hat{e}, x, y, z$

realized by $\hat{e}, \beta, f$ in the domain of F. As $\pi, \psi, \operatorname{Im}\Omega, \mathrm{d}V_g$ are smooth and $\Phi^{\hat{e}}_{X'}$ is smooth and depends smoothly on $\hat{e}$, we see that Q is a smooth function of its arguments.

Since $\Phi^{e}_{X'} = \Phi_{X'}$ and $\Phi_{X'}$ is the identity on $X' = \Gamma(0) \subset U_{X'}$ we see that $F(e,0,0)\,\mathrm{d}V_g = \operatorname{Im}\Omega|_{X'} = 0$ as X' is special Lagrangian. Thus $F(e,0,0) = 0$, and so $Q(e,x,0,0) = 0$. Following the proof of Proposition 2.10 we can also show that $\partial_y Q(e,x,0,0) = \partial_z Q(e,x,0,0) = 0$.

Therefore by Taylor expansion of $Q(\hat{e},x,y,z)$ about $\hat{e} = e$, $y = z = 0$ we see that *for fixed x in X'* and small $|y|, |z|, d_{\mathcal{E}}(\hat{e},e)$, we have

$$Q(\hat{e},x,y,z) = O\bigl(|y|^2 + |z|^2 + d_{\mathcal{E}}(\hat{e},e)\bigr), \tag{45}$$

and more generally for fixed x, small $|y|, |z|, d_{\mathcal{E}}(\hat{e},e)$, and $a,b,c \geqslant 0$ we have

$$(\nabla_x)^a(\partial_y)^b(\partial_z)^c Q(\hat{e},x,y,z) = O\bigl(|y|^{\max(0,2-b)} + |z|^{\max(0,2-c)} + d_{\mathcal{E}}(\hat{e},e)\bigr). \tag{46}$$

To prove (43) and (44) we have to extend (45) and (46) to hold uniformly for $x \in X'$ by inserting appropriate functions of x as multipliers. Careful consideration of the asymptotic behaviour of F and Q and their derivatives near x_i for $i = 1,\ldots,n$ shows that the powers of ρ given in (43) and (44) suffice. These powers are independent of $\boldsymbol{\mu}$ as the inequalities $\mu_i > 2$ imply that the terms given dominate other error terms involving the μ_i. □

We can also refine the image of F in $C^0(X')$.

Proposition 6.4 *In the situation above, F maps*

$$F : \tilde{\mathcal{E}} \times \mathcal{D}_{X'} \to \bigl\{u \in L^p_{k-2,\boldsymbol{\mu}-2}(X') : \textstyle\int_{X'} u \,\mathrm{d}V_g = 0\bigr\}, \tag{47}$$

and this is a smooth map of Banach manifolds.

Proof. If $(\hat{e},\beta,f) \in \tilde{\mathcal{E}} \times \mathcal{D}_{X'}$ then β is smooth and compactly-supported and $f \in L^p_{k,\boldsymbol{\mu}}(X')$, so $-\mathrm{d}^*\bigl(\psi^m(\beta + \mathrm{d}f)\bigr) \in L^p_{k-2,\boldsymbol{\mu}-2}(X')$. Hence we must show that the Q term in (42) also lies in $L^p_{k-2,\boldsymbol{\mu}-2}(X')$. For $x \in X'$, write

$$y(x) = (\beta + \mathrm{d}f)(x), \quad z(x) = (\nabla\beta + \nabla^2 f)(x) \quad \text{and} \quad v(x) = Q\bigl(\hat{e},x,y(x),z(x)\bigr).$$

Then we must show that $v \in L^p_{k-2,\boldsymbol{\mu}-2}(X')$.

As $L^p_{k,\boldsymbol{\mu}}(X') \subset C^2_{\boldsymbol{\mu}}(X')$ by Theorem 4.3, we have $|y| = O(\rho^{\boldsymbol{\mu}-1})$ and $|z| = O(\rho^{\boldsymbol{\mu}-2})$. Equation (43) then gives

$$v = Q\bigl(\hat{e},x,y(x),z(x)\bigr) = O(\rho^{2\boldsymbol{\mu}-4}) + O(\rho^{2\boldsymbol{\mu}-4}) + O\bigl(\rho(x) d_{\mathcal{E}}(\hat{e},e)\bigr).$$

Now $2\mu_i - 4 > \mu_i - 2$ and $1 > \mu_i - 2$ as $2 < \mu_i < 3$, so v decays faster than $\rho^{\boldsymbol{\mu}-2}$ near x_i, and it follows that $v \in L^p_{0,\boldsymbol{\mu}-2}(X')$.

For the derivatives of v, by the chain rule we have

$$\begin{aligned}\bigl|\nabla^j v\bigr| \leqslant j! \sum_{\substack{a,b,c\geqslant 0\\ a+b+c\leqslant j}} &\Bigl|(\nabla_x)^a(\partial_y)^b(\partial_z)^c Q\bigl(\hat{e},x,y(x),z(x)\bigr)\Bigr| \\ &\times \sum_{\substack{m_1,\ldots,m_b,n_1,\ldots,n_c\geqslant 1\\ a+m_1+\cdots+m_b+n_1+\cdots+n_c=j}} \prod_{i=1}^{b}\bigl|\nabla^{m_i}y(x)\bigr|\cdot\prod_{i=1}^{c}\bigl|\nabla^{n_i}z(x)\bigr|.\end{aligned}$$

Using (46) to estimate $\bigl|(\nabla_x)^a(\partial_y)^b(\partial_z)^c Q\bigl(\hat{e},x,y(x),z(x)\bigr)\bigr|$ and noting that $y\in L^p_{k-1,\boldsymbol{\mu}-1}(X')$ and $z\in L^p_{k-2,\boldsymbol{\mu}-2}(X')$, after some calculations using Theorem 4.3 and Hölder's inequality we can show that $|\nabla^j v|\in L^p_{0,\boldsymbol{\mu}-2-j}(X')$ for $j=0,\ldots,k-2$, so that $v\in L^p_{k-2,\boldsymbol{\mu}-2}(X')$.

Therefore F maps $\tilde{\mathcal{E}}\times\mathcal{D}_{X'}\rightarrow L^p_{k-2,\boldsymbol{\mu}-2}(X')$. As in Proposition 6.2, each $(\hat{e},\beta,f)\in\tilde{\mathcal{E}}\times\mathcal{D}_{X'}$ defines a compact C^1 Lagrangian m-fold $\hat{X}$ in M with conical singularities. Regard $\hat{X},X$ as m-chains in homology. Then $[\hat{X}]=[X]\in H_m(M,\mathbb{Z})$ as $\hat{X},X$ are isotopic. So using (37) we see that

$$\int_{X'}F(\hat{e},\beta,f)\,\mathrm{d}V_g=\int_{\hat{X}'}\operatorname{Im}\Omega=[\hat{X}]\cdot[\operatorname{Im}\Omega]=[X]\cdot[\operatorname{Im}\Omega]=\int_{\hat{X}'}\operatorname{Im}\Omega=0,$$

as $\operatorname{Im}\Omega$ is closed and X' is special Lagrangian. Thus F maps to the r.h.s. of (47), as we have to prove. The smoothness of F as a map between Banach manifolds easily follows from the smoothness of Q and general limiting arguments. □

6.2 The obstruction space

We shall determine the derivative $\mathrm{d}F|_{(e,0,0)}$ of F at $(e,0,0)$.

Proposition 6.5 *There exists a unique linear map $\chi:T_e\tilde{\mathcal{E}}\rightarrow C^\infty_{\mathbf{0}}(X')$, where $\mathbf{0}=(0,\ldots,0)\in\mathbb{R}^n$ and $\chi(y)\equiv 0$ on K for all $y\in T_e\tilde{\mathcal{E}}$, such that $\mathrm{d}F|_{(e,0,0)}:T_e\tilde{\mathcal{E}}\times\mathcal{H}_{X'}\times L^p_{k,\boldsymbol{\mu}}(X')\rightarrow L^p_{k-2,\boldsymbol{\mu}-2}(X')$ is given by*

$$\mathrm{d}F|_{(e,0,0)}:(y,\beta,f)\mapsto \mathrm{d}^*\bigl(\psi^m(\mathrm{d}[\chi(y)]-\beta-\mathrm{d}f)\bigr). \tag{48}$$

Proof. As F is smooth by Proposition 6.4, $\mathrm{d}F|_{(e,0,0)}$ is well-defined. Equation (42) then shows that $\mathrm{d}F|_{(e,0,0)}$ maps $(0,\beta,f)\mapsto -\mathrm{d}^*(\psi^m(\beta+\mathrm{d}f))$, since (43) implies that the Q term in (42) can only have derivative 0 in β,f at $(0,0)$. This gives the final two terms in (48).

Let $y\in T_e\tilde{\mathcal{E}}$, and differentiate $\Phi^{\hat{e}}_{X'}$ w.r.t. $\hat{e}$ in the direction of y at $\hat{e}=e$. This gives $\partial_y\Phi^{\hat{e}}_{X'}|_{\hat{e}=e}$, which is a section of the vector bundle $(\Phi^e_{X'})^*(TM)$ over $U_{X'}$. Now $\mathrm{d}\Phi^e_{X'}$ induces an isomorphism of $TU_{X'}$ and $(\Phi^e_{X'})^*(TM)$ as vector bundles over $U_{X'}$. Therefore $v=(\mathrm{d}\Phi^e_{X'})^*\bigl(\partial_y\Phi^{\hat{e}}_{X'}|_{\hat{e}=e}\bigr)$ is a section of $TU_{X'}$, that is, a vector field on $U_{X'}$, which depends linearly on y.

Differentiating $(\Phi^{\hat{e}}_{X'})^*(\operatorname{Im}\Omega)$ w.r.t. $\hat{e}$ in the direction of y, we find that

$$\partial_y(\Phi^{\hat{e}}_{X'})^*(\operatorname{Im}\Omega)|_{\hat{e}=e} = \mathcal{L}_v(\Phi^{e}_{X'})^*(\operatorname{Im}\Omega),$$

where $\mathcal{L}_v$ is the Lie derivative. But restricting to $X' \subset U_{X'}$ we have

$$(\Phi^{\hat{e}}_{X'})^*(\operatorname{Im}\Omega)|_{X'} = F(\hat{e},0,0)\,\mathrm{d}V_g,$$

by (37). Combining the last two equations gives

$$\partial_y F(e,0,0)\,\mathrm{d}V_g = \big(\mathcal{L}_v(\Phi^{e}_{X'})^*(\operatorname{Im}\Omega)\big)|_{X'}. \tag{49}$$

Define a 1-form α on $U_{X'}$ by $\alpha = v\cdot\hat{\omega}$. Then from (49) and the proof of Proposition 2.10 we find that

$$\partial_y F(e,0,0) = \mathrm{d}^*(\psi^m \alpha|_{X'}). \tag{50}$$

Since $(\Phi^{\hat{e}}_{X'})^*(\omega) = \hat{\omega}$ for all $\hat{e}\in\tilde{\mathcal{E}}$, it follows that $\mathcal{L}_v\hat{\omega}\equiv 0$, and hence α is a *closed* 1-form on $U_{X'}$. Also $v=\alpha=0$ on $\pi^*(K)$ as $\Phi^{\hat{e}}_{X'}\equiv\Phi_{X'}$ on $\pi^*(K)\subset U_{X'}$, by Theorem 5.2.

Thus $\alpha|_{X'}$ is a closed 1-form on X' which is zero on K. Since X' retracts onto K there exists a unique smooth function $\chi(y): X'\rightarrow\mathbb{R}$ with $\alpha|_{X'} = \mathrm{d}[\chi(y)]$ and $\chi(y)\equiv 0$ on K. Clearly $\chi(y)$ is linear in y, and (50) gives

$$\mathrm{d}F|_{(e,0,0)}\big((y,0,0)\big) = \partial_y F(e,0,0) = \mathrm{d}^*\big(\psi^m\mathrm{d}[\chi(y)]\big).$$

This completes the proof of (48).

It remains to show that χ maps $T_e\tilde{\mathcal{E}}\rightarrow C^\infty_0(X')$. As $\Phi^{\hat{e}}_{X'}$ satisfies (33), one can show that v and α on $\pi^*(S_i)\subset U_{X'}$ are the pull-backs under $\Phi^{e}_{X'}$ of a smooth vector field v' and a smooth closed 1-form α' on $\Upsilon^e_i(B_R)$, where $\Upsilon^*_i(v') = \partial_y\Upsilon^{\hat{e}}_i|_{\hat{e}=e}$ and $\alpha' = v'\cdot\omega$. This implies estimates on the decay of α and its derivatives on S_i for $i=1,\ldots,n$, which imply that $\chi(y)\in C^\infty_{\mathbf{0}}(X')$, as we want. □

To apply the Implicit Mapping Theorem to F in §6.3, we will need to know how close $\mathrm{d}F|_{(e,0,0)}$ is to being injective and surjective. First we show that $\mathrm{d}F|_{(e,0,0)}$ is injective on a large subspace of its domain.

Proposition 6.6 *The restriction of* $\mathrm{d}F|_{(e,0,0)}$ *to* $T_e\tilde{\mathcal{E}}\times\mathrm{d}\mathcal{K}_{X'}\times L^p_{k,\boldsymbol{\mu}}(X')$ *is injective, where* $\mathrm{d}\mathcal{K}_{X'}\leqslant\mathcal{H}_{X'}$ *as in Definition 6.1.*

Proof. Let $(y,\mathrm{d}v,f)\in T_e\tilde{\mathcal{E}}\times\mathrm{d}\mathcal{K}_{X'}\times L^p_{k,\boldsymbol{\mu}}(X')$ with $\mathrm{d}F|_{(e,0,0)}(y,\mathrm{d}v,f)=0$. Then

$$\mathrm{d}^*\big(\psi^m\mathrm{d}[\chi(y)-v-f]\big)=0$$

by (48). Multiplying this equation by $\chi(y)-v-f$ and integrating over X' by parts, we find

$$\int_{X'}\psi^m\big|\mathrm{d}[\chi(y)-v-f]\big|^2\mathrm{d}V_g = 0.$$

This holds even though X' is noncompact, because of the asymptotic behaviour of $\chi(y) - v - f$ and its derivatives near x_i, and may be proved rigorously using [12, Lem. 2.13]. Thus $\mathrm{d}[\chi(y) - v - f] = 0$.

Now $(y, \mathrm{d}v, f)$ corresponds to an infinitesimal deformation of X as a Lagrangian m-fold in M with conical singularities, locally the graph of $\mathrm{d}[\chi(y) - v - f] = 0$. As $\mathrm{d}[\chi(y) - v - f] = 0$ this infinitesimal deformation is trivial, and so cannot change the singular points x_i or identifications υ_i. Therefore $y = 0$, as $\tilde{\mathcal{E}}$ parametrizes nonequivalent choices of x_i, υ_i by definition.

Hence $\mathrm{d}(v + f) = 0$, so $v + f \equiv c \in \mathbb{R}$. As $f \in C^0_{\boldsymbol{\mu}}(X')$ by Theorem 4.3 we have $f(x) \to 0$ as $x \to x_i$ in X'. But $v \equiv a_i^j$ on S_i^j and $\sum_{i,j} a_i^j = 0$, by Definition 6.1. Taking $x \to x_i$ shows that $a_i^j = c$ for all i, j, and thus $c = 0$ as $\sum_{i,j} a_i^j = 0$. Hence $v = 0$ on S_i for all i, and v is *compactly-supported*, so that $[\mathrm{d}v] = 0$ in $H^1_{\mathrm{cs}}(X', \mathbb{R})$. Since the map $\mathcal{K}_{X'} \to H^1_{\mathrm{cs}}(X', \mathbb{R})$ given by $v \mapsto [\mathrm{d}v]$ is injective, by Definition 6.1, we see that $v = 0$, and hence $f = 0$. Therefore $\mathrm{d}F|_{(e,0,0)}$ is injective on $T_e\tilde{\mathcal{E}} \times \mathrm{d}\mathcal{K}_{X'} \times L^p_{k,\boldsymbol{\mu}}(X')$. □

Next we in effect measure how close $\mathrm{d}F|_{(e,0,0)}$ is to being surjective.

Proposition 6.7 *In the situation above, the map* $L^p_{k,\boldsymbol{\mu}}(X') \to L^p_{k-2,\boldsymbol{\mu}-2}(X')$ *given by* $f \mapsto \mathrm{d}^*(\psi^m \mathrm{d}f)$ *is Fredholm with cokernel of dimension* $\sum_{i=1}^n N_{\Sigma_i}(2)$.

Proof. This is just the map $P_{\boldsymbol{\mu}} : L^p_{k,\boldsymbol{\mu}}(X') \to L^p_{k-2,\boldsymbol{\mu}-2}(X')$ of Theorem 4.4. Thus part (b) of Theorem 4.4 shows that $P_{\boldsymbol{\mu}}$ is injective, and then part (a) proves that $P_{\boldsymbol{\mu}}$ is Fredholm with cokernel of dimension $\sum_{i=1}^n N_{\Sigma_i}(\mu_i)$. But $N_{\Sigma_i}(\mu_i) = N_{\Sigma_i}(2)$ by (14), as N_{Σ_i} is upper semicontinuous and discontinuous exactly on $\mathcal{D}_{\Sigma_i}$ by Definition 3.3. □

Now we can define the *obstruction space* in our problem.

Definition 6.8 Proposition 6.4 shows that

$$\mathrm{d}F|_{(e,0,0)}\bigl(T_e\tilde{\mathcal{E}} \times \mathrm{d}\mathcal{K}_{X'} \times L^p_{k,\boldsymbol{\mu}}(X')\bigr) \subseteq \bigl\{u \in L^p_{k-2,\boldsymbol{\mu}-2}(X') : \textstyle\int_{X'} u \, \mathrm{d}V_g = 0\bigr\},$$

and Propositions 6.5 and 6.7 show that this inclusion is of finite codimension. Choose a finite-dimensional vector subspace $\mathcal{O}_{X'}$ of smooth, compactly-supported functions v on X' with $\int_{X'} v \, \mathrm{d}V_g = 0$, such that

$$\begin{aligned} &\bigl\{u \in L^p_{k-2,\boldsymbol{\mu}-2}(X') : \textstyle\int_{X'} u \, \mathrm{d}V_g = 0\bigr\} = \\ &\qquad\qquad \mathcal{O}_{X'} \oplus \mathrm{d}F|_{(e,0,0)}\bigl(T_e\tilde{\mathcal{E}} \times \mathrm{d}\mathcal{K}_{X'} \times L^p_{k,\boldsymbol{\mu}}(X')\bigr). \end{aligned} \tag{51}$$

This is possible as such functions v are dense in the l.h.s. of (51). We call $\mathcal{O}_{X'}$ the *obstruction space*. Propositions 6.5–6.7 imply that

$$\begin{aligned}\dim \mathcal{O}_{X'} &= \sum_{i=1}^{n} N_{\Sigma_i}(2) - \dim \tilde{\mathcal{E}} - \dim \mathcal{K}_{X'} - 1 \\ &= \sum_{i=1}^{n} N_{\Sigma_i}(2) - n(m^2+2m-1) + \sum_{i=1}^{n} \dim G_i - \sum_{i=1}^{n} b^0(\Sigma_i) \\ &= \sum_{i=1}^{n} \bigl(N_{\Sigma_i}(2) - b^0(\Sigma_i) - m^2 - 2m + 1 + \dim G_i\bigr) \\ &= \sum_{i=1}^{n} \text{s-ind}(C_i),\end{aligned} \tag{52}$$

where $\dim \tilde{\mathcal{E}} = \dim \mathcal{E}$ is given in (31) and $\dim \mathcal{K}_{X'}$ in (34), we use (9) in the last line, and $\text{s-ind}(C_i) \geqslant 0$ is the *stability index* of Definition 3.6.

We may interpret (52) by saying that each singular point x_i contributes an obstruction space of dimension $\text{s-ind}(C_i)$ to deforming X as an SL m-fold with conical singularities, and $\mathcal{O}_{X'}$ is the sum of these obstruction spaces.

6.3 The main result

We are now ready to prove our main results on the moduli space $\mathcal{M}_X$ of compact SL m-folds with conical singularities. The key tool is the *Implicit Mapping Theorem*. The following version may be proved from Lang [16, Th. 2.1, p. 131].

Theorem 6.9 *Let Y, Z and T be Banach spaces, and W an open neighbourhood of $(0,0)$ in $Y \times Z$. Suppose that the function $G : W \rightarrow T$ is a smooth map of Banach manifolds with $G(0,0) = 0$, and that $\mathrm{d}G_{(0,0)}|_Z : Z \rightarrow T$ is an isomorphism of Z, T as vector and topological spaces. Then there exist open neighbourhoods U, V of 0 in Y and Z with $U \times V \subseteq W$ and a smooth map $H : U \rightarrow V$ with $H(0) = 0$ such that if $(u,v) \in U \times V$ then $G(u,v) = 0$ if and only if $v = H(u)$.*

Here is our first main result, describing $\mathcal{M}_X$ near X.

Theorem 6.10 *Suppose (M, J, ω, Ω) is an almost Calabi–Yau m-fold and X a compact SL m-fold in M with conical singularities at $x_1, \ldots, x_n$ and cones $C_1, \ldots, C_n$. Let $\mathcal{M}_X$ be the moduli space of deformations of X as an SL m-fold with conical singularities in M, as in Definition 5.4. Set $X' = X \setminus \{x_1, \ldots, x_n\}$.*

Then there exist natural finite-dimensional vector spaces $\mathcal{I}_{X'}$, $\mathcal{O}_{X'}$ such that $\mathcal{I}_{X'}$ is isomorphic to the image of $H^1_{\mathrm{cs}}(X', \mathbb{R})$ in $H^1(X', \mathbb{R})$ and $\dim \mathcal{O}_{X'} = \sum_{i=1}^{n} \text{s-ind}(C_i)$, where $\text{s-ind}(C_i)$ is the stability index of Definition 3.6. There exists an open neighbourhood U of 0 in $\mathcal{I}_{X'}$, a smooth map $\Phi : U \rightarrow \mathcal{O}_{X'}$ with $\Phi(0) = 0$, and a map $\Xi : \{u \in U : \Phi(u) = 0\} \rightarrow \mathcal{M}_X$ with $\Xi(0) = X$ which is a homeomorphism with an open neighbourhood of X in $\mathcal{M}_X$.

Proof. As $\tilde{\mathcal{E}}$ is an open neighbourhood of e in $\mathcal{E}$, which is an open ball, we can choose a smooth identification of $\tilde{\mathcal{E}}$ with an open neighbourhood of 0 in $T_e\tilde{\mathcal{E}}$ which identifies e with 0 and induces the identity map on $T_e\tilde{\mathcal{E}}$. Define

$$\begin{aligned} &Y = \mathcal{I}_{X'}, \quad Z = \mathcal{O}_{X'} \times T_e\tilde{\mathcal{E}} \times \mathcal{K}_{X'} \times L^p_{k,\boldsymbol{\mu}}(X'), \\ &T = \bigl\{u \in L^p_{k-2,\boldsymbol{\mu}-2}(X') : \textstyle\int_{X'} u \,\mathrm{d}V_g = 0\bigr\} \quad \text{and} \end{aligned} \tag{53}$$

$$W = \bigl\{(\beta,\gamma,\hat{e},v,f) \in Y \times Z : \hat{e} \in \tilde{\mathcal{E}} \subset T_e\tilde{\mathcal{E}}, \quad (\beta + \mathrm{d}v, f) \in \mathcal{D}_{X'}\bigr\}. \tag{54}$$

Then $0 \in Z$ is $(0,e,0,0)$. Choose any norms on the finite-dimensional spaces $\mathcal{I}_{X'}, \mathcal{O}_{X'}, T_e\tilde{\mathcal{E}}, \mathcal{K}_{X'}$, and use the usual norms on $L^p_{k,\boldsymbol{\mu}}(X')$ and T. Then Y, Z, T are Banach spaces, and W is an open neighbourhood of $(0,0)$ in $Y \times Z$, as in Theorem 6.9.

Define a map $G : W \to T$ by $G(\beta,\gamma,\hat{e},v,f) = \gamma + F(\hat{e},\beta+\mathrm{d}v,f)$. This is a smooth map of Banach manifolds, by Proposition 6.4, and $G(0,0) = G(0,0,e,0,0) = 0$ as $F(e,0,0) = 0$. The map $\mathrm{d}G_{(0,0)}|_Z$ is given by

$$\mathrm{d}G_{(0,0)}|_Z : (\gamma,y,v,f) \mapsto \gamma + \mathrm{d}F_{(e,0,0)}(y,\mathrm{d}v,f). \tag{55}$$

Now Proposition 6.6 proves that $(y,v,f) \mapsto \mathrm{d}F_{(e,0,0)}(y,\mathrm{d}v,f)$ is an injective map on $T_e\tilde{\mathcal{E}} \times \mathcal{K}_{X'} \times L^p_{k,\boldsymbol{\mu}}(X')$. Also (51) implies that $\mathcal{O}_{X'}$ intersects the image of $\mathrm{d}F_{(e,0,0)}$ only in 0. Therefore $\mathrm{d}G_{(0,0)} : Z \to T$ is *injective*.

But (51) shows that $\mathrm{d}G_{(0,0)}$ is *surjective*. Thus $\mathrm{d}G_{(0,0)}$ is an isomorphism of Z, T as vector spaces. Since $\mathrm{d}G_{(0,0)}$ is continuous, it is an isomorphism of Z, T as topological spaces by the Open Mapping Theorem. Hence the hypotheses of Theorem 6.9 hold, and the theorem gives open neighbourhoods U of 0 in $\mathcal{I}_{X'}$ and V of 0 in Z and a smooth map $H : U \to V \subset Z$ with $H(0) = 0$.

Since $(\beta, v) \mapsto \beta + \mathrm{d}v$ is a homeomorphism $\mathcal{I}_{X'} \times \mathcal{K}_{X'} \to \mathcal{H}_{X'}$ by (35), we see from (54) that the map

$$\bigl\{(\beta,0,\hat{e},v,f) \in W\bigr\} \to \tilde{\mathcal{E}} \times \mathcal{D}_{X'} \quad \text{given by} \quad (\beta,0,\hat{e},v,f) \mapsto (\hat{e},\beta+\mathrm{d}v,f)$$

is a homeomorphism. Applying Proposition 6.2 we see that

(a) The map $\bigl\{(\beta,0,\hat{e},v,f) \in G^{-1}(0) \subset W\bigr\} \to \mathcal{V}_X$ given by $(\beta,0,\hat{e},v,f) \mapsto \Psi(\hat{e},\beta+\mathrm{d}v,f)$ is a homeomorphism taking $(0,0,e,0,0) \mapsto X$.

Define $\Phi : U \to \mathcal{O}_{X'}$, $H_1 : U \to T_e\tilde{\mathcal{E}}$, $H_2 : U \to \mathcal{K}_{X'}$ and $H_3 : U \to L^p_{k,\boldsymbol{\mu}}(X')$ by $H(u) = \bigl(\Phi(u), H_1(u), H_2(u), H_3(u)\bigr) \in V \subset Z$. Then Φ, H_1, H_2, H_3 are smooth as H is smooth, and $\Phi(0) = 0$, $H_j(0) = 0$ as $H(0) = 0$. By Theorem 6.9, if $(u,v) \in U \times V$ then $G(u,v) = 0$ if and only if $v = H(u)$. That is:

(b) if $(\beta,\gamma,\hat{e},v,f) \in U \times V \subseteq W$ then $G(\beta,\gamma,\hat{e},v,f) = 0$ if and only if $\gamma = \Phi(\beta)$, $\hat{e} = H_1(\beta)$, $v = H_2(\beta)$ and $f = H_3(\beta)$.

Combining (a), (b) proves that $\Xi : \{u \in U : \Phi(u) = 0\} \to \mathcal{V}_X$ given by $\Xi(u) = \Psi\bigl(H_1(u), u + \mathrm{d}H_2(u), H_3(u)\bigr)$ is a homeomorphism from U to an open neighbourhood of X in $\mathcal{V}_X$ with $\Xi(0) = X$. This completes the proof. □

Here are two simple corollaries of Theorem 6.10. Firstly, if X has *stable singularities* in the sense of Definition 3.7 then s-ind$(C_i) = 0$, so $\dim \mathcal{O}_{X'} = 0$, and $\mathcal{M}_X$ is locally homeomorphic to $\mathcal{I}_{X'}$. Thus $\mathcal{M}_X$ is a *manifold* near X.

But all SL m-folds $\hat{X} \in \mathcal{M}_X$ have the same cones C_i, so all $\hat{X} \in \mathcal{M}_X$ have stable singularities, and $\mathcal{M}_X$ is a manifold everywhere. The maps Ξ of Theorem 6.10 provide coordinate charts on $\mathcal{M}_X$. It is easy to see that the transition maps are smooth (this follows for instance from Theorem 6.14 below), so $\mathcal{M}_X$ is a *smooth manifold*. This gives:

Corollary 6.11 *Suppose (M, J, ω, Ω) is an almost Calabi–Yau m-fold and X a compact SL m-fold in M with stable conical singularities, and let $\mathcal{M}_X$ and $\mathcal{I}_{X'}$ be as in Theorem 6.10. Then $\mathcal{M}_X$ is a smooth manifold of dimension* $\dim \mathcal{I}_{X'}$.

Here is another simple condition for $\mathcal{M}_X$ to be a manifold near X.

Definition 6.12 Let (M, J, ω, Ω) be an almost Calabi–Yau m-fold and X a compact SL m-fold in M with conical singularities, and let $\mathcal{I}_{X'}, \mathcal{O}_{X'}, U$ and Φ be as in Theorem 6.10. We call X *transverse* if the linear map $\mathrm{d}\Phi|_0 : \mathcal{I}_{X'} \rightarrow \mathcal{O}_{X'}$ is surjective. It is not difficult to see that this definition is independent of the choices made in defining $\mathcal{I}_{X'}, \mathcal{O}_{X'}, U$ and Φ.

If X is transverse then $\{u \in U : \Phi(u) = 0\}$ is a manifold near 0, so we prove:

Corollary 6.13 *Suppose (M, J, ω, Ω) is an almost Calabi–Yau m-fold and X a transverse compact SL m-fold in M with conical singularities, and let $\mathcal{M}_X, \mathcal{I}_{X'}$ and $\mathcal{O}_{X'}$ be as in Theorem 6.10. Then $\mathcal{M}_X$ is near X a smooth manifold of dimension* $\dim \mathcal{I}_{X'} - \dim \mathcal{O}_{X'}$.

6.4 Naturality of $\mathcal{I}_{X'}, \mathcal{O}_{X'}, \Phi$ and Ξ

In the course of proving Theorem 6.10 we made a considerable number of *arbitrary choices* in §4.3, §5 and §6, including $\Upsilon_i, \zeta, U_{X'}, \Phi_{X'}, \mathcal{E}, \tilde{\mathcal{E}}, \Upsilon_i^{\hat{e}}, \Phi_{X'}^{\hat{e}}, \mathcal{H}_{X'}, \mathcal{I}_{X'}$, $\mathcal{O}_{X'}$ and U. We now consider to what extent the final result depends on these choices, in particular the vector spaces $\mathcal{I}_{X'}, \mathcal{O}_{X'}$ and maps Φ, Ξ.

Now $\mathcal{I}_{X'}$ is naturally isomorphic to the image of $H^1_{\mathrm{cs}}(X', \mathbb{R})$ in $H^1(X', \mathbb{R})$ by §6.1. Thus *as a vector space* $\mathcal{I}_{X'}$ depends only on X', though *as a vector space of 1-forms* it depends on an arbitrary choice. Let us *identify* $\mathcal{I}_{X'}$ with the image of $H^1_{\mathrm{cs}}(X', \mathbb{R})$ in $H^1(X', \mathbb{R})$, so that $\mathcal{I}_{X'}$ is independent of choices.

Then Ξ maps $\Phi^{-1}(0) \subset \mathcal{I}_{X'} \subseteq H^1(X', \mathbb{R})$ to $\mathcal{M}_X$, as a local homeomorphism. In the next theorem we shall construct an *inverse* Θ for Ξ, defined near X in $\mathcal{M}_X$ and mapping into $H^1(X', \mathbb{R})$, which is independent of all arbitrary choices. This proves that both Ξ and its domain $\{u \in U : \Phi(u) = 0\} \subset \mathcal{I}_{X'}$ are independent of arbitrary choices near 0 in $\mathcal{I}_{X'}$.

In §6.5 we will explain an alternative construction of $\mathcal{O}_{X'}$ *as a vector space* which is independent of choices. The author does not know to what extent Φ is natural where it is nonzero, but this does not seem a very important question. The theorem is based on the construction of *natural coordinates* on moduli spaces $\mathcal{M}_X$ of compact, nonsingular SL m-folds, which is described by Hitchin [5, §4] and the author [11, §9.4].

Theorem 6.14 *Let $(M,J,\omega,\Omega), X, X', \mathcal{M}_X, U, \Xi$ and Φ be as in Theorem 6.10, and let V be a path-connected, simply-connected open neighbourhood of X in $\mathcal{M}_X$. Then there exists a natural, continuous map $\Theta: V \to H^1(X',\mathbb{R})$ depending only on M,ω,X and V, such that Θ,Ξ are inverse maps on the connected component of $V\cap\Xi(U)$ containing X.*

Proof. Let $\hat X \in V$. As V is path-connected and simply-connected there is a unique isotopy class of continuous paths $\gamma:[0,1]\to V$ with $\gamma(0)=X$ and $\gamma(1)=\hat X$. This determines a unique isotopy class of continuous maps $\Pi: [0,1]\times X'\to M$ with $\Pi\bigl(\{0\}\times X'\bigr)=X'$ and $\Pi\bigl(\{1\}\times X'\bigr)=\hat X'$. Let Π be a smooth map in this isotopy class. Then $\Pi^*(\omega)$ is a closed 2-form on $[0,1]\times X'$ vanishing on $\{0,1\}\times X'$, since $X',\hat X'$ are Lagrangian.

Thus $[\Pi^*(\omega)]$ defines a class in $H^2\bigl([0,1]\times X';\{0,1\}\times X',\mathbb{R}\bigr)$, the *relative de Rham cohomology group*, which depends only on M,ω,V,X and $\hat X$. Define $\Theta(\hat X)$ to be the class in $H^1(X',\mathbb{R})$ corresponding to $[\Pi^*(\omega)]$ under the natural isomorphism $H^1(X',\mathbb{R})\cong H^2\bigl([0,1]\times X';\{0,1\}\times X',\mathbb{R}\bigr)$. Then $\Theta: V\to H^1(X',\mathbb{R})$ depends only on M,ω,X and V, and is clearly continuous.

We must show that Θ,Ξ are inverse near X. Let $\hat X$ lie in the connected component of $V\cap\Xi(U)$ containing X. From §6.1–§6.3 we find that $\hat X' = \Phi^{\hat e}_{X'}\bigl(\Gamma(\beta+\mathrm{d}f)\bigr)$ for some $(\hat e,\beta,f)\in\tilde{\mathcal{E}}\times\mathcal{D}_{X'}$, and that $[\beta]\in H^1(X',\mathbb{R})$ lies in $U\subset\mathcal{I}_{X'}\subset H^1(X',\mathbb{R})$ with $\Phi\bigl([\beta]\bigr)=0$ and $\Xi\bigl([\beta]\bigr)=\hat X$.

Now $\Phi^{\hat e}_{X'}\equiv\Phi_{X'}$ on $\pi^*(K)$. Assuming the fibres of $\pi: U_{X'}\to X'$ are convex for simplicity, we may take $\Pi|_{[0,1]\times K}$ above to be $\Pi(t,x)=\Phi_{X'}\bigl(t(\beta+\mathrm{d}f)|_x\bigr)$. This has the correct isotopy class as $X,\hat X$ lie in the same component of $V\cap\Xi(U)$. Since $\Phi^*_{X'}(\omega)=\hat\omega$, a short calculation then shows that $\Pi^*(\omega)=(\beta+\mathrm{d}f)\wedge\mathrm{d}t$ on $[0,1]\times K$. As X' retracts onto K, we find that $\Theta(\hat X)$ is $[\beta+\mathrm{d}f]=[\beta]\in H^1(X',\mathbb{R})$. But $\Xi\bigl([\beta]\bigr)=\hat X$, so Θ,Ξ are inverse. □

The theorem implies that the topology on $\mathcal{M}_X$ is locally induced from the Euclidean topology on $H^1(X',\mathbb{R})$ via Θ. This gives another way of seeing the naturality of the topology on $\mathcal{M}_X$.

6.5 Another way of thinking about $\mathcal{I}_{X'}, \mathcal{O}_{X'}$

In §2.3 we saw that for a compact, nonsingular SL m-fold N in an almost Calabi–Yau m-fold M, the infinitesimal deformations correspond to 1-forms α on N with $\mathrm{d}\alpha=\mathrm{d}^*(\psi^m\alpha)=0$, which form a vector space naturally isomorphic to $H^1(N,\mathbb{R})$. To extend this to SL m-folds X with conical singularities $x_1,\dots,x_n$ with rates $\mu_1,\dots,\mu_n$, we need to regard α as a 1-form on X' with *asymptotic conditions* on α and its derivatives.

We saw in Theorem 4.5 that the most natural asymptotic condition on α from the point of view of *Hodge theory* is $|\nabla^k\alpha|=O(\rho^{-1-k})$ for all $k\geqslant 0$. The vector space $Y_{X'}$ of such α is isomorphic to $H^1(X',\mathbb{R})$. Consider for the moment only deformations of X that fix the x_i and υ_i. Then the most natural asymptotic condition on α for the *deformation theory* of X is $|\nabla^k\alpha| = O(\rho^{\boldsymbol{\mu}-1-k})$ for all $k \geqslant 0$.

Clearly if $|\nabla^k\alpha| = O(\rho^{\boldsymbol{\mu}-1-k})$ then $|\nabla^k\alpha| = O(\rho^{-1-k})$. So define

$$Z_{X'} = \bigl\{\alpha \in Y_{X'} : |\nabla^k\alpha| = O(\rho^{\boldsymbol{\mu}-1-k}) \text{ for all } k \geqslant 0\bigr\}.$$

This is an obvious candidate for the infinitesimal deformations of X which fix the x_i, υ_i. Therefore we ask: *how big a subspace of* $Y_{X'} \cong H^1(X',\mathbb{R})$ *is* $Z_{X'}$?

First note that if the image of $[\alpha] \in H^1(X',\mathbb{R})$ under the map $H^1(X',\mathbb{R}) \to \bigoplus_{i=1}^n H^1(\Sigma_i,\mathbb{R})$ of (19) is nonzero, then one can easily see from the proof of Theorem 4.5 in [12, §2.5] that α decays exactly at rate $O(\rho^{-1})$ near some x_i, and thus $\alpha \notin Z_{X'}$. Hence $Z_{X'}$ corresponds to a subspace of the kernel of $H^1(X',\mathbb{R}) \to \bigoplus_{i=1}^n H^1(\Sigma_i,\mathbb{R})$, that is, to a subspace of the image of $H^1_{\rm cs}(X',\mathbb{R}) \to H^1(X',\mathbb{R})$ in (19), which is isomorphic to $\mathcal{I}_{X'}$.

Define G_i to be the space of *germs of smooth* 1*-forms on* X' *near* x_i, that is, smooth 1-forms ξ defined on $U_i \setminus \{x_i\}$ for some small open neighbourhood U_i of x_i in X, where two such 1-forms are equivalent if they agree on the intersection of their domains. For $i = 1,\ldots,n$ define

$$\mathcal{O}_i = \frac{\bigl\{\xi \in G_i : \xi \text{ is exact, } {\rm d}^*(\psi^m\xi) = 0,\ |\nabla^k\xi| = O(\rho^{-1-k}) \text{ for all } k \geqslant 0\bigr\}}{\bigl\{\xi \in G_i : \xi \text{ is exact, } {\rm d}^*(\psi^m\xi) = 0,\ |\nabla^k\xi| = O(\rho^{\mu_i-1-k}) \text{ for all } k \geqslant 0\bigr\}}.$$

Then one can show that $\mathcal{O}_i$ is a vector space of dimension $N_{\Sigma_i}(2) - b^0(\Sigma_i)$, an *obstruction space*. Each ξ in the subspace of $Y_{X'}$ corresponding to $\mathcal{I}_{X'}$ has a natural projection to $\mathcal{O}_i$ for $i = 1,\ldots,n$, and $\xi \in Z_{X'}$ if and only if all of these projections are zero. Thus the infinitesimal deformation space $Z_{X'}$ is the kernel of a linear map $\mathcal{I}_{X'} \to \bigoplus_{i=1}^n \mathcal{O}_i$, and each obstruction space $\mathcal{O}_i$ depends only on the germ of X at x_i, and essentially only on the cone C_i.

In fact $\bigoplus_{i=1}^n \mathcal{O}_i$ does not correspond exactly to the obstruction space $\mathcal{O}_{X'}$ of §6.2, as $\mathcal{O}_{X'}$ is the obstructions to deformations which can vary x_i, υ_i. Each $\mathcal{O}_i$ contains a vector subspace $\mathcal{P}_i$ isomorphic to $T_{(x_i,\upsilon_i)}\mathcal{E}_i$, corresponding to infinitesimal deformations ξ which vary x_i, υ_i. It can be shown that there is a natural isomorphism $\mathcal{O}_{X'} \cong \bigoplus_{i=1}^n \mathcal{O}_i/\mathcal{P}_i$. The corresponding linear map $\mathcal{I}_{X'} \to \mathcal{O}_{X'}$ is ${\rm d}\Phi|_0$, in the notation of §6.3.

This way of thinking about the infinitesimal deformation and obstruction spaces $\mathcal{I}_{X'}, \mathcal{O}_{X'}$ has the advantages of being closer to McLean's method, and of presenting $\mathcal{O}_{X'}$ as a direct sum of contributions from each singular point x_i, in a way that was implicit in (52) but was not brought out in §6.2. However, the author did not find it helpful in actually writing down a proof.

7 Extension to families $\bigl\{(M,J^s,\omega^s,\Omega^s) : s \in \mathcal{F}\bigr\}$

We now extend the material of §5 and §6 from a single almost Calabi–Yau m-fold (M,J,ω,Ω) to a *smooth family of deformations* $\bigl\{(M,J^s,\omega^s,\Omega^s) : s \in \mathcal{F}\bigr\}$

of (M, J, ω, Ω), as in Definition 2.11. The basic idea is that we consider deformations $\hat{X}$ of a compact SL m-fold X in (M, J, ω, Ω) with conical singularities, not just in (M, J, ω, Ω) but in $(M, J^s, \omega^s, \Omega^s)$ for $s \in \mathcal{F}$.

We collect these deformations $(s, \hat{X})$ into a big moduli space $\mathcal{M}_X^{\mathcal{F}}$ with a natural topology and a continuous projection $\pi^{\mathcal{F}} : \mathcal{M}_X^{\mathcal{F}} \to \mathcal{F}$, generalizing §5. Then we show that $\mathcal{M}_X^{\mathcal{F}}$ is homeomorphic near $(0, X)$ to the zeroes of a smooth map $\Phi^{\mathcal{F}} : \mathcal{F} \times \mathcal{I}_{X'} \to \mathcal{O}_{X'}$ between finite-dimensional spaces, generalizing §6.

7.1 Moduli spaces of SL m-folds in families $(M, J^s, \omega^s, \Omega^s)$

We first explain how to extend §5 to *families* $\{(M, J^s, \omega^s, \Omega^s) : s \in \mathcal{F}\}$ of almost Calabi–Yau m-folds, as in Definition 2.11. In fact this is not very much work, as we are *already* dealing with families $\mathcal{E}$ of choices of x_i, υ_i, so we simply have to enlarge these families to include $\mathcal{F}$, and make appropriate changes. Consider the following situation.

Definition 7.1 Let (M, J, ω, Ω) be an almost Calabi–Yau m-fold and X a compact SL m-fold in M with conical singularities at $x_1, \ldots, x_n$ with identifications $\upsilon_i : \mathbb{C}^m \to T_{x_i}M$, cones $C_1, \ldots, C_n$ and rates μ_i. Suppose $\{(M, J^s, \omega^s, \Omega^s) : s \in \mathcal{F}\}$ is a smooth family of deformations of (M, J, ω, Ω), where $\mathcal{F} \subset \mathbb{R}^d$ is the *base space*, such that $\iota_*(\gamma) \cdot [\omega^s] = 0$ for all $\gamma \in H_2(X, \mathbb{R})$ and $s \in \mathcal{F}$ and $[X] \cdot [\operatorname{Im} \Omega^s] = 0$ for all $s \in \mathcal{F}$. Here $\iota : X \to M$ is the inclusion, $\iota_* : H_2(X, \mathbb{R}) \to H_2(M, \mathbb{R})$ the induced map, $[\omega^s] \in H^2(M, \mathbb{R})$, $[X] \in H_m(M, \mathbb{R})$ and $[\operatorname{Im} \Omega^s] \in H^m(M, \mathbb{R})$.

The point of this definition is that $\iota_*(\gamma) \cdot [\omega^s] = 0$ for all γ and $[X] \cdot [\operatorname{Im} \Omega^s] = 0$ are *necessary conditions* for there to exist an SL m-fold $\hat{X}$ in $(M, J^s, \omega^s, \Omega^s)$ with conical singularities, isotopic to X in M. For if $\hat{\iota} : \hat{X} \to \mathbb{R}$ is the inclusion then by isotopy $\hat{\iota}_*(\gamma) = \iota_*(\gamma)$ under the natural isomorphism $H_2(\hat{X}, \mathbb{R}) \cong H_2(X, \mathbb{R})$ and $[\hat{X}] = [X]$. But clearly $\hat{\iota}_*(\gamma) \cdot [\omega^s] = 0$ for all $\gamma \in H_2(\hat{X}, \mathbb{R})$ and $[\hat{X}] \cdot [\operatorname{Im} \Omega^s] = 0$, since $\omega^s|_{\hat{X}'} \equiv \operatorname{Im} \Omega^s|_{\hat{X}'} \equiv 0$.

We have written these conditions in an odd way. In effect $\iota_*(\gamma) \cdot [\omega^s] = 0$ for all γ and $[X] \cdot [\operatorname{Im} \Omega^s] = 0$ simply mean that $[\omega^s|_X] = [\operatorname{Im} \Omega^s|_X] = 0$ in $H^*(X, \mathbb{R})$. However, we have not defined the de Rham cohomology $H^*(X, \mathbb{R})$ of the singular manifold X, so this does not make sense. The conditions $[\omega^s|_{X'}] = [\operatorname{Im} \Omega^s|_{X'}] = 0$ in $H^*(X', \mathbb{R})$ do make sense, but are not strong enough.

Here are the analogues of Definition 5.1 and Theorems 5.2 and 5.3.

Definition 7.2 In the situation of Definition 7.1, for $s \in \mathcal{F}$ define $\psi^s : M \to (0, \infty)$ as in (3), but using ω^s, Ω^s. Extending (30), define

$$P^{\mathcal{F}} = \big\{(s, x, \upsilon) : s \in \mathcal{F},\, x \in M,\, \upsilon : \mathbb{C}^m \to T_xM \text{ is a real isomorphism,} \\ \upsilon^*(\omega^s) = \omega', \quad \upsilon^*(\Omega^s) = \psi^s(x)^m \Omega'\big\}, \tag{56}$$

where ω', Ω' are as in (1). Define $\pi^{\mathcal{F}} : P^{\mathcal{F}} \to \mathcal{F}$ by $\pi^{\mathcal{F}} : (s, x, \upsilon) \mapsto s$. Define a free $\mathrm{SU}(m)$-action on $P^{\mathcal{F}}$ by $B : (s, x, \upsilon) \mapsto (s, x, \upsilon \circ B^{-1})$. Then $P^{\mathcal{F}}$ is a principal $\mathrm{SU}(m)$-bundle over $\mathcal{F} \times M$.

Let G_i be the Lie subgroup of $\mathrm{SU}(m)$ preserving C_i. Let $0 \in \mathcal{F}' \subseteq \mathcal{F}$ and $\upsilon_i^s : \mathbb{C}^m \to T_{x_i}M$ for $i = 1,\dots,n$ and $s \in \mathcal{F}'$ be as in Theorem 4.9. Then $(s, x_i, \upsilon_i^s) \in P^{\mathcal{F}}$ for $i = 1,\dots,n$ and $s \in \mathcal{F}'$. Let $\mathcal{E}_i, \mathcal{E}$ be as in Definition 5.1.

For $i = 1,\dots,n$ let $\mathcal{E}_i^{\mathcal{F}'}$ be a submanifold of dimension $\dim P^{\mathcal{F}} - \dim G_i$ in $(\pi^{\mathcal{F}})^*(\mathcal{F}') \subseteq P^{\mathcal{F}}$ such that $\pi^{\mathcal{F}} : \mathcal{E}_i^{\mathcal{F}'} \to \mathcal{F}'$ is a submersion, $(\pi^{\mathcal{F}})^{-1}(s)$ is a small ball containing (s, x_i, υ_i^s) for $s \in \mathcal{F}'$ which is transverse to the orbits of G_i, and $(\pi^{\mathcal{F}})^{-1}(0) = \{0\} \times \mathcal{E}_i$. Making $\mathcal{F}'$ smaller if necessary, such $\mathcal{E}_i^{\mathcal{F}'}$ exist. Define

$$\mathcal{E}^{\mathcal{F}'} = \bigl\{(s, \hat{x}_1, \hat{\upsilon}_1, \dots, \hat{x}_n, \hat{\upsilon}_n) : (s, \hat{x}_i, \hat{\upsilon}_i) \in \mathcal{E}_i^{\mathcal{F}'} \text{ for } i = 1,\dots,n\bigr\}. \tag{57}$$

Write a general element of $\mathcal{E}^{\mathcal{F}'}$ as $(s, \hat{e})$ for $s \in \mathcal{F}'$ and $\hat{e} = (\hat{x}_1, \hat{\upsilon}_1, \dots, \hat{x}_n, \hat{\upsilon}_n)$ as in §5.1, and let $e^s = (x_1, \upsilon_1^s, \dots, x_n, \upsilon_n^s)$, so that $(s, e^s) \in \mathcal{E}^{\mathcal{F}'}$ for all $s \in \mathcal{F}'$. Define $\pi^{\mathcal{F}} : \mathcal{E}^{\mathcal{F}'} \to \mathcal{F}'$ by $\pi^{\mathcal{F}} : (s, \hat{x}_1, \dots, \hat{\upsilon}_n) \mapsto s$. Then $(\pi^{\mathcal{F}})^{-1}(0) = \{0\} \times \mathcal{E}$.

This $\mathcal{E}^{\mathcal{F}'}$ is a family of $(s, \hat{x}_i, \hat{\upsilon}_i)$ such that $\hat{x}_i, \hat{\upsilon}_i$ are close to x_i, υ_i, and are valid alternative choices of x_i, υ_i in $(M, J^s, \omega^s, \Omega^s)$, noting that $\hat{\upsilon}_i : \mathbb{C}^m \to T_{\hat{x}_i}M$ has to be compatible with ω^s, Ω^s as in §3.3. Each $G_1 \times \cdots \times G_n$ equivalence class of choices of $s, \hat{x}_i, \hat{\upsilon}_i$ close to $0, x_i, \upsilon_i$ is represented *exactly once* in $\mathcal{E}^{\mathcal{F}'}$.

Theorem 7.3 *In the situation above, use the notation of Theorem 4.7, let $U_{X'}, \Phi_{X'}$ be as in Theorem 4.8, let $0 \in \mathcal{F}' \subseteq \mathcal{F}$ and $\upsilon_i^s, \Upsilon_i^s, \Phi_{X'}^s$ for $s \in \mathcal{F}'$ be as in Theorem 4.9, and let $\tilde{\mathcal{E}}, \Upsilon_i^{\hat{e}}$ and $\Phi_{X'}^{\hat{e}}$ be as in Theorem 5.2.*

Then making $\mathcal{F}'$ smaller if necessary, there exists a connected open subset $\tilde{\mathcal{E}}^{\mathcal{F}'} \subseteq \mathcal{E}^{\mathcal{F}'}$ with $(s, e^s) \in \tilde{\mathcal{E}}^{\mathcal{F}'}$ for all $s \in \mathcal{F}'$ and $(\pi^{\mathcal{F}})^{-1}(0) \cap \tilde{\mathcal{E}}^{\mathcal{F}'} = \{0\} \times \tilde{\mathcal{E}}$, and for all $(s, \hat{e}) = (s, \hat{x}_1, \hat{\upsilon}_1, \dots, \hat{x}_n, \hat{\upsilon}_n)$ in $\tilde{\mathcal{E}}^{\mathcal{F}'}$ there exist

(a) *embeddings $\Upsilon_i^{s,\hat{e}} : B_R \to M$ for $i = 1,\dots,n$ with*

$$\Upsilon_i^{s,e^s} = \Upsilon_i^s, \quad \Upsilon_i^{s,\hat{e}}(0) = \hat{x}_i, \quad \mathrm{d}\Upsilon_i^{s,\hat{e}}|_0 = \hat{\upsilon}_i \quad \textit{and} \quad (\Upsilon_i^{s,\hat{e}})^*(\omega^s) = \omega', \tag{58}$$

(b) *an embedding $\Phi_{X'}^{s,\hat{e}} : U_{X'} \to M$ with $\Phi_{X'}^{s,e^s} = \Phi_{X'}^s$ and $(\Phi_{X'}^{s,\hat{e}})^*(\omega^s) = \hat{\omega}$, such that $\Phi_{X'}^{s,\hat{e}} \equiv \Phi_{X'}^s$ on $\pi^*(K) \subset U_{X'}$,*

all depending smoothly on $(s, \hat{e}) \in \tilde{\mathcal{E}}^{\mathcal{F}'}$, with $\Upsilon_i^{0,\hat{e}} = \Upsilon_i^{\hat{e}}$ and $\Phi_{X'}^{0,\hat{e}} = \Phi_{X'}^{\hat{e}}$ for all $\hat{e} \in \tilde{\mathcal{E}}$ and

$$\Phi_{X'}^{s,\hat{e}} \circ \mathrm{d}(\Upsilon_i \circ \phi_i)(\sigma, r, \tau, u) \equiv \Upsilon_i^{s,\hat{e}} \circ \Phi_{C_i}\bigl(\sigma, r, \tau + \eta_i^1(\sigma, r), u + \eta_i^2(\sigma, r)\bigr) \tag{59}$$

for all $(s, \hat{e}) \in \tilde{\mathcal{E}}^{\mathcal{F}'}$, $1 \leqslant i \leqslant n$ and $(\sigma, r, \tau, u) \in T^*\bigl(\Sigma_i \times (0, R')\bigr)$ with $\bigl|(\tau, u)\bigr| < \zeta r$.

Theorem 7.4 *In the situation above, let $(s, \hat{e}) = (s, \hat{x}_1, \hat{\upsilon}_1, \dots, \hat{x}_n, \hat{\upsilon}_n) \in \tilde{\mathcal{E}}^{\mathcal{F}'}$, and suppose $\hat{X}$ is a compact SL m-fold in $(M, J^s, \omega^s, \Omega^s)$ with conical singularities at $\hat{x}_1, \dots, \hat{x}_n$, with identifications $\hat{\upsilon}_i$, cones C_i and rates μ_i. Then if $(s, \hat{e}), (0, e)$ are sufficiently close in $\tilde{\mathcal{E}}^{\mathcal{F}'}$ and $X', \hat{X}'$ are sufficiently close as submanifolds in a C^1 sense away from $x_1, \dots, x_n$, there exists a closed 1-form α on X' such that the graph $\Gamma(\alpha)$ lies in $U_{X'} \subset T^*X'$, and $\hat{X}' = \Phi_{X'}^{s,\hat{e}}\bigl(\Gamma(\alpha)\bigr)$. Furthermore we may write $\alpha = \beta + \mathrm{d}f$, where β is a closed 1-form supported in K and $f \in C^\infty_{\boldsymbol{\mu}}(X')$.*

The proofs are straightforward modifications of §5.1, replacing Theorem 4.8 by Theorem 4.9. Here is the analogue of Definitions 5.4 and 5.6.

Definition 7.5 In the situation above, define the *moduli space* $\mathcal{M}_X^{\mathcal{F}}$ *of deformations of* X *in the family* $\mathcal{F}$ to be the set of pairs $(s,\hat X)$ such that

(i) $s\in\mathcal{F}$ and $\hat X$ is a compact SL m-fold in $(M,J^s,\omega^s,\Omega^s)$ with conical singularities at $\hat x_1,\ldots,\hat x_n$ with cones $C_1,\ldots,C_n$, for some $\hat x_i$.

(ii) There exists a homeomorphism $\hat\iota:X\rightarrow\hat X$ with $\hat\iota(x_i)=\hat x_i$ for $i=1,\ldots,n$ such that $\hat\iota|_{X'}:X'\rightarrow\hat X'$ is a diffeomorphism and $\hat\iota$ and ι are isotopic as continuous maps $X\rightarrow M$, where $\iota:X\rightarrow M$ is the inclusion.

Define $\pi^{\mathcal{F}}:\mathcal{M}_X^{\mathcal{F}}\rightarrow\mathcal{F}$ by $\pi^{\mathcal{F}}:(s,\hat X)\mapsto s$. Let $\mathcal{V}_{0,X}^{\mathcal{F}}$ be the subset of $(s,\hat X)\in\mathcal{M}_X^{\mathcal{F}}$ such that for some $(s,\hat e)\in\tilde{\mathcal{E}}^{\mathcal{F}\prime}$ and some 1-form α on X' whose graph $\Gamma(\alpha)$ lies in $U_{X'}\subset T^*X'$ we have $\hat X'=\Phi_{X'}^{s,\hat e}\bigl(\Gamma(\alpha)\bigr)$, as in Theorem 7.4.

This gives a 1-1 correspondence between $\mathcal{V}_{0,X}^{\mathcal{F}}$ and a set of triples $(s,\hat e,\alpha)$ for $(s,\hat e)\in\tilde{\mathcal{E}}^{\mathcal{F}\prime}$ and α a smooth 1-form on X' with prescribed decay. Also $(\pi^{\mathcal{F}})^{-1}(0)\cap\mathcal{V}_{0,X}^{\mathcal{F}}=\{0\}\times\mathcal{V}_X$, where $\mathcal{V}_X$ is as in Definition 5.4, and the triples $(0,\hat e,\alpha)$ for $(\pi^{\mathcal{F}})^{-1}(0)\cap\mathcal{V}_{0,X}^{\mathcal{F}}$ agree with the pairs $(\hat e,\alpha)$ for $\mathcal{V}_X$ in Definition 5.4.

Use this 1-1 correspondence to define a topology on $\mathcal{V}_{0,X}^{\mathcal{F}}$, using the natural topology on $\tilde{\mathcal{E}}^{\mathcal{F}\prime}$ and either the $C^1_{\boldsymbol{\mu}-1}$ or the $C^\infty_{\boldsymbol{\mu}-1}$ topology on α, defined as in §5.2. The analogue of Proposition 5.5 shows that these yield the same topology on $\mathcal{V}_{0,X}^{\mathcal{F}}$, which is also independent of choice of rates μ_i.

For each $(\tilde s,\tilde X)\in\mathcal{M}_X^{\mathcal{F}}$ we can regard $\bigl\{(M,J^s,\omega^s,\Omega^s):s\in\mathcal{F}\bigr\}$ as a family of deformations of $(M,J^{\tilde s},\omega^{\tilde s},\Omega^{\tilde s})$ rather than of $(M,J^0,\omega^0,\Omega^0)$, and we can redo the whole of this section replacing $0\in\mathcal{F}$ by $\tilde s\in F$ and X by $\tilde X$. In this way we define a subset $\mathcal{V}_{\tilde s,\tilde X}^{\mathcal{F}}$ of $\mathcal{M}_X^{\mathcal{F}}$ containing $(\tilde s,\tilde X)$ with a 1-1 correspondence between $\mathcal{V}_{\tilde s,\tilde X}^{\mathcal{F}}$ and a set of triples $(s,\hat e,\alpha)$, and a topology on $\mathcal{V}_{\tilde s,\tilde X}^{\mathcal{F}}$.

One can show that the topologies on different neighbourhoods $\mathcal{V}_{\tilde s,\tilde X}^{\mathcal{F}}$ agree on the overlaps, and that the overlaps are open in each. Piecing the topologies together therefore defines a unique topology on $\mathcal{M}_X^{\mathcal{F}}$. In this topology $\pi^{\mathcal{F}}:\mathcal{M}_X^{\mathcal{F}}\rightarrow\mathcal{F}$ is continuous, and $\mathcal{V}_{0,X}^{\mathcal{F}}$ is an open neighbourhood of $(0,X)$.

Note that $(\pi^{\mathcal{F}})^{-1}(0)\subset\mathcal{M}_X^{\mathcal{F}}$ is just $\{0\}\times\mathcal{M}_X$ in the notation of §5.2, and the subspace topology on $(\pi^{\mathcal{F}})^{-1}(0)$ agrees with the topology on $\mathcal{M}_X$ in Definition 5.6. More generally, if $(s,\hat X)\in\mathcal{M}_X^{\mathcal{F}}$ then $(\pi^{\mathcal{F}})^{-1}(s)\subset\mathcal{M}_X^{\mathcal{F}}$ is $\{s\}\times\mathcal{M}_{\hat X}$ as a topological space, where $\mathcal{M}_{\hat X}$ is the moduli space of deformations of $\hat X$ in $(M,J^s,\omega^s,\Omega^s)$.

Remarks. Basically, $\mathcal{M}_X^{\mathcal{F}}$ is the family of pairs $(s,\hat X)$ where $s\in\mathcal{F}$ and $\hat X$ is a compact SL m-fold in M with conical singularities, which is deformation equivalent to X in a loose sense. Note that $\mathcal{M}_X^{\mathcal{F}}$ may not be *connected*. The fibres $(\pi^{\mathcal{F}})^{-1}(s)$ of $\pi^{\mathcal{F}}:\mathcal{M}_X^{\mathcal{F}}\rightarrow\mathcal{F}$ are (as topological spaces) moduli spaces of compact SL m-folds in $(M,J^s,\omega^s,\Omega^s)$ with conical singularities, deformation equivalent to X, and with $(\pi^{\mathcal{F}})^{-1}(0)=\mathcal{M}_X$.

The whole point of constructing $\mathcal{M}_X^{\mathcal{F}}$, and its topology, is that we can now make sense of the idea of a continuous family of compact SL m-folds $\hat X$ in M with conical singularities, in which the underlying almost Calabi–Yau structure is allowed to vary. That is, we can continuously deform X not just in (M, J, ω, Ω) but also in $(M, J^s, \omega^s, \Omega^s)$ for $s \in \mathcal{F}$.

7.2 The main result for families $(M, J^s, \omega^s, \Omega^s)$

Next we extend §6 to the families case. Here are the analogues of Definition 6.1 and Proposition 6.2.

Definition 7.6 Let $\mathcal{H}_{X'}, \mathcal{K}_{X'}, \mathcal{I}_{X'}, k, p, \boldsymbol{\mu}, \mathcal{D}_{X'}$ and F be as in Definition 6.1. Define a map $F^{\mathcal{F}} : \tilde{\mathcal{E}}^{\mathcal{F}'} \times \mathcal{D}_{X'} \rightarrow C^0(X')$ by

$$\pi_*\big((\Phi_{X'}^{s,\hat{e}})^*(\operatorname{Im}\Omega)|_{\Gamma(\beta+\mathrm{d}f)}\big) = F^{\mathcal{F}}(s, \hat{e}, \beta, f)\, \mathrm{d}V_g \tag{60}$$

for $(s, \hat{e}) \in \tilde{\mathcal{E}}^{\mathcal{F}'}$ and $(\beta, f) \in \mathcal{D}_{X'}$. Then $F^{\mathcal{F}}(0, \hat{e}, \beta, f) \equiv F(\hat{e}, \beta, f)$ on $\tilde{\mathcal{E}} \times \mathcal{D}_{X'}$.

Proposition 7.7 *In the situation above, suppose $(s, \hat{e}, \beta, f) \in \tilde{\mathcal{E}}^{\mathcal{F}'} \times \mathcal{D}_{X'}$ with $F^{\mathcal{F}}(s, \hat{e}, \beta, f) = 0$. Set $\hat{X}' = \Phi_{X'}^{s,\hat{e}}\big(\Gamma(\beta + \mathrm{d}f)\big)$ and $\hat{X} = \hat{X}' \cup \{\hat{x}_1, \ldots, \hat{x}_n\}$, where $\hat{e} = (\hat{x}_1, \ldots, \hat{\upsilon}_n)$. Then $f \in C^\infty_{\boldsymbol{\mu}}(X')$ and $\hat{X}$ is a compact SL m-fold in $(M, J^s, \omega^s, \Omega^s)$ with conical singularities at $\hat{x}_i$ with identifications $\hat{\upsilon}_i$, cones C_i and rates μ_i. Thus $(s, \hat{X})$ lies in $\mathcal{V}_X^{\mathcal{F}} \subset \mathcal{M}_X^{\mathcal{F}}$ in Definition 7.5. Conversely, each $\hat{X}$ in $\mathcal{V}_X^{\mathcal{F}}$ comes from a unique $(s, \hat{e}, \beta, f) \in \tilde{\mathcal{E}}^{\mathcal{F}'} \times \mathcal{D}_{X'}$ with $F^{\mathcal{F}}(s, \hat{e}, \beta, f) = 0$. Write $\Psi^{\mathcal{F}}(s, \hat{e}, \beta, f) = (s, \hat{X})$. Then $\Psi^{\mathcal{F}} : (F^{\mathcal{F}})^{-1}(0) \rightarrow \mathcal{V}_X^{\mathcal{F}}$ is a homeomorphism, with $\Psi^{\mathcal{F}}(0, e, 0, 0) = (0, X)$.*

The modifications to the proof of Proposition 6.2 are just trivial notational ones. We shall use Proposition 6.3 as it is. The analogue of Proposition 6.4 is

Proposition 7.8 *In the situation above, $F^{\mathcal{F}}$ maps*

$$F^{\mathcal{F}} : \tilde{\mathcal{E}}^{\mathcal{F}'} \times \mathcal{D}_{X'} \rightarrow \big\{u \in L^p_{k-2,\boldsymbol{\mu}-2}(X') : \textstyle\int_{X'} u\, \mathrm{d}V_g = 0\big\}, \tag{61}$$

and this is a smooth map of Banach manifolds.

Again, the modifications to the proof are just trivial changes in notation. We shall use all of §6.2 as it is. The point is that $F^{\mathcal{F}}|_{s=0} \equiv F$, so the calculations in §6.2 about $\mathrm{d}F|_{(e,0,0)}$ immediately tell us about the restriction of $\mathrm{d}F^{\mathcal{F}}|_{(0,e,0,0)}$ to the vector subspace with $s = 0$.

We can now prove the main result of this section, the analogue of Theorem 6.10 for families, which describes $\mathcal{M}_X^{\mathcal{F}}$ near $(0, X)$.

Theorem 7.9 *Suppose (M, J, ω, Ω) is an almost Calabi–Yau m-fold and X a compact SL m-fold in M with conical singularities at $x_1, \ldots, x_n$. Let $\mathcal{M}_X, X'$, $\mathcal{I}_{X'}, \mathcal{O}_{X'}, U, \Phi$ and Ξ be as in Theorem 6.10.*

Suppose $\big\{(M, J^s, \omega^s, \Omega^s) : s \in \mathcal{F}\big\}$ is a smooth family of deformations of (M, J, ω, Ω), in the sense of Definition 2.11, such that $\iota_*(\gamma) \cdot [\omega^s] = 0$ for all $\gamma \in H_2(X, \mathbb{R})$ and $s \in \mathcal{F}$, where $\iota : X \rightarrow M$ is the inclusion, and $[X] \cdot [\operatorname{Im}\Omega^s] = 0$ for all $s \in \mathcal{F}$, where $[X] \in H_m(M, \mathbb{R})$ and $[\operatorname{Im}\Omega^s] \in H^m(M, \mathbb{R})$. Let $\mathcal{M}_X^{\mathcal{F}}$ and $\pi^{\mathcal{F}} : \mathcal{M}_X^{\mathcal{F}} \rightarrow \mathcal{F}$ be as in Definition 7.5.

Then there exists an open neighbourhood $U^{\mathcal{F}}$ of $(0,0)$ in $\mathcal{F}\times U$, a smooth map $\Phi^{\mathcal{F}}: U^{\mathcal{F}}\rightarrow\mathcal{O}_{X'}$ with $\Phi^{\mathcal{F}}(0,u)\equiv\Phi(u)$, and a map $\Xi^{\mathcal{F}}:\{(s,u)\in U^{\mathcal{F}}:\Phi^{\mathcal{F}}(s,u)=0\}\rightarrow\mathcal{M}_X^{\mathcal{F}}$ with $\Xi^{\mathcal{F}}(0,u)\equiv\bigl(0,\Xi(u)\bigr)$ and $\pi^{\mathcal{F}}\circ\Xi^{\mathcal{F}}(s,u)\equiv s$, which is a homeomorphism with an open neighbourhood of $(0,X)$ in $\mathcal{M}_X^{\mathcal{F}}$.

Proof. Recall that $0\in\mathcal{F}'\subseteq\mathcal{F}\subset\mathbb{R}^d$ and $\pi^{\mathcal{F}}:\tilde{\mathcal{E}}^{\mathcal{F}'}\rightarrow\mathcal{F}'$ is a submersion with fibres open balls, and $\tilde{\mathcal{E}}^{\mathcal{F}'}\supset(\pi^{\mathcal{F}})^{-1}(0)=\{0\}\times\tilde{\mathcal{E}}$. Thus we can choose a smooth identification of $\tilde{\mathcal{E}}^{\mathcal{F}'}$ with an open neighbourhood of $(0,0)$ in $\mathcal{F}'\times T_e\tilde{\mathcal{E}}\subset\mathbb{R}^d\times T_e\tilde{\mathcal{E}}$ which identifies the projections $\pi^{\mathcal{F}}:\tilde{\mathcal{E}}^{\mathcal{F}'}\rightarrow\mathcal{F}'$ and $\pi^{\mathcal{F}}:\mathcal{F}'\times T_e\tilde{\mathcal{E}}\rightarrow\mathcal{F}'$, and on $(\pi^{\mathcal{F}})^{-1}(0)=\{0\}\times\tilde{\mathcal{E}}$ and $\{0\}\times T_e\tilde{\mathcal{E}}$ agrees with the identification between $\tilde{E}$ and a subset of $T_e\tilde{\mathcal{E}}$ chosen in the proof of Theorem 6.10. Define

$$\begin{aligned} Y^{\mathcal{F}}&=\mathbb{R}^d\times\mathcal{I}_{X'},\quad Z=\mathcal{O}_{X'}\times T_e\tilde{\mathcal{E}}\times\mathcal{K}_{X'}\times L^p_{k,\boldsymbol{\mu}}(X'),\\ T&=\bigl\{u\in L^p_{k-2,\boldsymbol{\mu}-2}(X'):\textstyle\int_{X'}u\,\mathrm{d}V_g=0\bigr\}\quad\text{and}\\ W^{\mathcal{F}}&=\bigl\{(s,\beta,\gamma,\hat{e},v,f)\in Y^{\mathcal{F}}\times Z:(s,\hat{e})\in\tilde{\mathcal{E}}^{\mathcal{F}'}\subset\mathbb{R}^d\times T_e\tilde{\mathcal{E}},\ (\beta+\mathrm{d}v,f)\in\mathcal{D}_{X'}\bigr\}.\end{aligned}$$

Then $0\in Z$ is $(0,e,0,0)$. Choose any norms on the finite-dimensional spaces $\mathbb{R}^d,\mathcal{I}_{X'},\mathcal{O}_{X'},T_e\tilde{\mathcal{E}},\mathcal{K}_{X'}$, and use the usual norms on $L^p_{k,\boldsymbol{\mu}}(X')$ and T. Then $Y^{\mathcal{F}},Z,T$ are Banach spaces, and $W^{\mathcal{F}}$ is an open neighbourhood of $(0,0)$ in $Y^{\mathcal{F}}\times Z$, as in Theorem 6.9.

Define a map $G^{\mathcal{F}}:W^{\mathcal{F}}\rightarrow T$ by $G(s,\beta,\gamma,\hat{e},v,f)=\gamma+F^{\mathcal{F}}(s,\hat{e},\beta+\mathrm{d}v,f)$. This is a smooth map of Banach manifolds, by Proposition 7.8, and $G^{\mathcal{F}}(0,0)=G^{\mathcal{F}}(0,0,0,e,0,0)=0$ as $F^{\mathcal{F}}(0,e,0,0)=0$. The map $\mathrm{d}G^{\mathcal{F}}_{(0,0)}|_Z$ is given by

$$\mathrm{d}G^{\mathcal{F}}_{(0,0)}|_Z:(\gamma,y,v,f)\mapsto\gamma+\mathrm{d}F^{\mathcal{F}}_{(0,e,0,0)}(0,y,\mathrm{d}v,f)=\gamma+\mathrm{d}F_{(e,0,0)}(y,\mathrm{d}v,f),\quad(62)$$

since $F^{\mathcal{F}}|_{s=0}\equiv F$, as in Definition 7.6.

Comparing (62) with (55) we see that $\mathrm{d}G^{\mathcal{F}}_{(0,0)}|_Z:Z\rightarrow T$ agrees with $\mathrm{d}G_{(0,0)}|_Z:Z\rightarrow T$ in the proof of Theorem 6.10. Therefore $\mathrm{d}G^{\mathcal{F}}_{(0,0)}|_Z$ is an isomorphism of topological vector spaces as in the proof of Theorem 6.10, and we can apply Theorem 6.9 to $Y^{\mathcal{F}},Z,T,W^{\mathcal{F}}$ and $G^{\mathcal{F}}$. The rest of the proof is a straightforward modification of that of Theorem 6.10. □

Here is the analogue of Corollary 6.11. Note the similarity to Theorem 2.13.

Corollary 7.10 *Let (M,J,ω,Ω) be an almost Calabi–Yau m-fold, X a compact SL m-fold in M with stable conical singularities, let $\bigl\{(M,J^s,\omega^s,\Omega^s):s\in\mathcal{F}\bigr\}$ be a smooth family of deformations of (M,J,ω,Ω) for $\mathcal{F}\subset\mathbb{R}^d$ with $\iota_*(\gamma)\cdot[\omega^s]=0$ and $[X]\cdot[\operatorname{Im}\Omega^s]=0$ for all $\gamma\in H_2(X,\mathbb{R})$ and $s\in\mathcal{F}$, and let $\mathcal{M}_X,\mathcal{M}_X^{\mathcal{F}},\pi^{\mathcal{F}}$, and $\mathcal{I}_{X'}$ be as in Theorem 7.9.*

Then $\mathcal{M}_X^{\mathcal{F}}$ is a smooth manifold of dimension $d+\dim\mathcal{I}_{X'}$ and $\pi^{\mathcal{F}}:\mathcal{M}_X^{\mathcal{F}}\rightarrow\mathcal{F}$ a smooth submersion. For all $s\in\mathcal{F}$ sufficiently close to 0 the fibre $(\pi^{\mathcal{F}})^{-1}(s)$ is a nonempty smooth manifold of dimension $\dim\mathcal{I}_{X'}$, with $(\pi^{\mathcal{F}})^{-1}(0)=\mathcal{M}_X$.

Here $\pi^{\mathcal{F}}:\mathcal{M}_X^{\mathcal{F}}\rightarrow\mathcal{F}$ is a *submersion* means that $\pi^{\mathcal{F}}:T_{(s,\hat{X})}\mathcal{M}_X^{\mathcal{F}}\rightarrow T_s\mathcal{F}=\mathbb{R}^d$ is surjective for all $(s,\hat{X})\in\mathcal{M}_X^{\mathcal{F}}$. This follows near $(0,X)\in\mathcal{M}_X^{\mathcal{F}}$ as $\Xi^{\mathcal{F}}$ is a diffeomorphism from $U^{\mathcal{F}} \subset \mathcal{F} \times U$ to a neighbourhood of $(0, X) \in \mathcal{M}_X^{\mathcal{F}}$ which identifies the projections $\pi^{\mathcal{F}} : \mathcal{M}_X^{\mathcal{F}} \to \mathcal{F}$ and $\pi^{\mathcal{F}} : \mathcal{F} \times U \to \mathcal{F}$. Thus it holds near every $(s, \hat{X}) \in \mathcal{M}_X^{\mathcal{F}}$, by applying Theorem 7.9 with (M, J, ω, Ω) replaced by $(M, J^s, \omega^s, \Omega^s)$ and X by $\hat{X}$.

Corollary 7.10 implies the analogue of Theorem 2.13 for compact SL m-folds X in M with stable conical singularities. That is, it shows that there are no local obstructions to deforming X to nearby almost Calabi–Yau structures $(J^s, \omega^s, \Omega^s)$ on M, except the obvious cohomological ones.

Here are the analogues of Definition 6.12 and Corollary 6.13.

Definition 7.11 Let (M, J, ω, Ω) be an almost Calabi–Yau m-fold, X a compact SL m-fold in M with conical singularities, $\{(M, J^s, \omega^s, \Omega^s) : s \in \mathcal{F}\}$ a smooth family of deformations of (M, J, ω, Ω) for $\mathcal{F} \subset \mathbb{R}^d$ with $\iota_*(\gamma) \cdot [\omega^s] = 0$ and $[X] \cdot [\operatorname{Im} \Omega^s] = 0$ for all $\gamma \in H_2(X, \mathbb{R})$ and $s \in \mathcal{F}$, and let $\mathcal{I}_{X'}, \mathcal{O}_{X'}, U^{\mathcal{F}}, \Phi$ and $\Phi^{\mathcal{F}}$ be as in Theorem 7.9.

We call X *transverse in* $\mathcal{F}$ if the linear map $\mathrm{d}\Phi^{\mathcal{F}}|_{(0,0)} : \mathbb{R}^d \times \mathcal{I}_{X'} \to \mathcal{O}_{X'}$ is surjective. This definition is independent of the choices made in defining $\mathcal{I}_{X'}, \mathcal{O}_{X'}, U^{\mathcal{F}}$ and $\Phi^{\mathcal{F}}$. Since the restriction of $\mathrm{d}\Phi^{\mathcal{F}}|_{(0,0)}$ to $\mathcal{I}_{X'} \subset \mathbb{R}^d \times \mathcal{I}_{X'}$ is $\mathrm{d}\Phi|_0$, we see that if X is *transverse* in the sense of Definition 6.12 then it is also transverse in $\mathcal{F}$, for any family $\mathcal{F}$.

Corollary 7.12 *Let (M, J, ω, Ω) be an almost Calabi–Yau m-fold, X a compact SL m-fold in M with conical singularities, let $\{(M, J^s, \omega^s, \Omega^s) : s \in \mathcal{F}\}$ be a smooth family of deformations of (M, J, ω, Ω) for $\mathcal{F} \subset \mathbb{R}^d$ with $\iota_*(\gamma) \cdot [\omega^s] = 0$ and $[X] \cdot [\operatorname{Im} \Omega^s] = 0$ for all $\gamma \in H_2(X, \mathbb{R})$ and $s \in \mathcal{F}$, and let $\mathcal{M}_X^{\mathcal{F}}, \mathcal{I}_{X'}$ and $\mathcal{O}_{X'}$ be as in Theorem 7.9. Suppose X is transverse in $\mathcal{F}$. Then $\mathcal{M}_X^{\mathcal{F}}$ is near $(0, X)$ a smooth manifold of dimension $d + \dim \mathcal{I}_{X'} - \dim \mathcal{O}_{X'}$, and $\pi^{\mathcal{F}} : \mathcal{M}_X^{\mathcal{F}} \to \mathcal{F}$ is a smooth map near $(0, X)$.*

Here Theorem 7.9 implies that near $(0, X)$ we can identify $\mathcal{M}_X^{\mathcal{F}}$ with a submanifold of $\mathcal{F} \times U$, and $\pi^{\mathcal{F}}$ then coincides with the projection $\pi^{\mathcal{F}} : \mathcal{F} \times U \to \mathcal{F}$, so $\pi^{\mathcal{F}}$ is smooth near $(0, X)$. Corollary 7.12 will be important in §9, as we will show that for any compact SL m-fold X in (M, J, ω, Ω) with conical singularities, there exists a family of deformations $\{(M, J^s, \omega^s, \Omega^s) : s \in \mathcal{F}\}$ of (M, J, ω, Ω) such that X is transverse in $\mathcal{F}$.

8 Other extensions of Theorems 6.10 and 7.9

Section 7 discussed the extension of the deformation theory of §5–§6 to *families* of almost Calabi–Yau m-folds. We now briefly consider other possible extensions of the theory, first to *immersed* rather than *embedded* submanifolds, and secondly to ways in which we can allow the SL cones $C_1, \ldots, C_n$ to vary over the moduli spaces $\mathcal{M}_X, \mathcal{M}_X^{\mathcal{F}}$, rather than being the same at every point. Allowing the C_i to vary reduces the dimension of the obstruction space $\mathcal{O}_{X'}$, and so increases the (expected) dimension of $\mathcal{M}_X, \mathcal{M}_X^{\mathcal{F}}$.

8.1 Immersions

So far, for simplicity, we have worked throughout with *embedded* submanifolds. In fact, nearly everything we have done can be generalized to *immersed* submanifolds in an obvious way, with only trivial, notational changes. Here are a few of the details involved in doing this.

Instead of regarding compact SL m-folds X in (M, J, ω, Ω) with conical singularities as subsets of M, we instead regard X as a *Riemannian manifold with conical singularities*, in the sense of [12, §2], together with an *isometric immersion* $\iota : X \to M$, which is locally but not necessarily globally injective. The singular points $x_1, \ldots, x_n \in X$ are distinct, but their images $\iota(x_1), \ldots, \iota(x_n) \in M$ may not be.

The Σ_i become compact Riemannian manifolds with isometric immersions $\Sigma_i \to \mathcal{S}^{2m-1}$, and the cones C_i on Σ_i become *Riemannian cones* in the sense of [12, §2.1], with isometric immersions $C_i \to \mathbb{C}^m$ which need not be locally injective near 0. The Υ_i can still be embeddings, but their images may overlap. The $\phi_i, \iota_i, \Phi_{C_i}, \Phi_{X'}$, etc., should be taken to be immersions.

The only point the author is aware of where there is a significant problem in changing from embeddings to immersions is in the Geometric Measure Theory of [12, §6], in particular Theorem 4.11 above, where the tangent cone C must have $C \setminus \{0\}$ a genuine embedded submanifold. However, we do not use Theorem 4.11 in this paper, so this does not affect the results of §5–§7.

Suppose C is an embedded SL cone in $\mathbb{C}^m$ with an isolated singularity at 0, so that $\Sigma = C \cap \mathcal{S}^{2m-1}$ is a compact $(m-1)$-manifold. If Σ is not simply-connected we may be able to take a *finite cover* $\pi : \tilde{\Sigma} \to \Sigma$. Then $\tilde{\Sigma}$ is an *immersed* minimal Legendrian $(m-1)$-fold in $\mathcal{S}^{2m-1}$, with a corresponding immersed SL cone $\tilde{C}$ in $\mathbb{C}^m$.

This construction considerably increases the supply of possible SL cones available as model singularities in the immersed case. It is particularly effective when $m = 3$, as then Σ is an oriented Riemann surface of genus $g \geqslant 1$, and so admits many finite covers. A similar phenomenon is described in [9, Th. 11.6], which constructs a large family of immersed SL 3-folds in $\mathbb{C}^3$ diffeomorphic to $\mathcal{S}^1 \times \mathbb{R}^2$, which are asymptotic at infinity to the double cover of an embedded SL T^2-cone in $\mathbb{C}^3$.

8.2 Cones C_i with multiple ends

The moduli spaces $\mathcal{M}_X$ and $\mathcal{M}_X^{\mathcal{F}}$ defined in §5 and §7.1 have the *same* set of SL cones $C_1, \ldots, C_n$ (up to $\mathrm{SU}(m)$ equivalence) for every $\hat{X} \in \mathcal{M}_X$ or $(s, \hat{X}) \in \mathcal{M}_X^{\mathcal{F}}$. There are various ways of relaxing this, and enlarging the moduli spaces $\mathcal{M}_X, \mathcal{M}_X^{\mathcal{F}}$ by allowing the SL cones C_i to vary. Consider the case in which $\Sigma_1, \ldots, \Sigma_n$ are not all connected, so that $b^0(\Sigma_i) > 1$ for at least one i. We shall explain two ways to generalize $\mathcal{M}_X$ and $\mathcal{M}_X^{\mathcal{F}}$.

The first way is to regard X as an *immersed* SL m-fold in M with conical singularities, as in §8.1. That is, instead of X having n singular points $x_1, \ldots, x_n$, we regard it as having $\check{n} = \sum_{i=1}^n b^0(\Sigma_i)$ distinct singular points $y_1, \ldots, y_{\check{n}}$, where

$\check{n} > n$, which happen to be mapped to n points in M in groups of $b^0(\Sigma_i)$ for $i = 1, \ldots, n$ by the immersion $\iota : X \to M$.

Essentially, we replace X by $\check{X} = X' \cup \{y_1, \ldots, y_{\check{n}}\}$, where each y_i compactifies one of the $\check{n}$ noncompact ends of X'. Then we deform $\check{X}$ to get a moduli space $\check{\mathcal{M}}_X$ or $\check{\mathcal{M}}_X^{\mathcal{F}}$ of immersed SL m-folds with $\check{n}$ singular points. Note that for general elements of $\check{\mathcal{M}}_X$ or $\check{\mathcal{M}}_X^{\mathcal{F}}$, there will be up to $\check{n}$ distinct singular points in M, rather than just n.

The second way is to retain the number n of singular points, but to allow the $b^0(\Sigma_i)$ components of C_i' to move around separately under SU(m) rotations. Let Σ_i^j be the connected components of Σ_i for $j = 1, \ldots, b^0(\Sigma_i)$, and let C_i^j be the cone on Σ_i^j in $\mathbb{C}^m$, so that $C_i = \bigcup_{j=1}^{b^0(\Sigma_i)} C_i^j$.

Then in defining $\mathcal{M}_X, \mathcal{M}_X^{\mathcal{F}}$ we allow the SL m-folds $\hat{X}$ with conical singularities at $\hat{x}_1, \ldots, \hat{x}_n$ to have cones $\hat{C}_i = \bigcup_{j=1}^{b^0(\Sigma_i)} B_i^j C_i^j$ for $B_i^j \in \mathrm{SU}(m)$ with $B_i^1 = 1$. This enlarges the family of SL cones allowed in $\mathcal{M}_X, \mathcal{M}_X^{\mathcal{F}}$, and so enlarges $\mathcal{M}_X, \mathcal{M}_X^{\mathcal{F}}$.

In §5–§7 we have to enlarge $\mathcal{E}$, etc., by including possible values of B_i^j near 1 for $j > 1$. The main effect that this has on the final results is that it *reduces* the dimension of the obstruction space $\mathcal{O}_{X'}$, and thus *increases* the (expected) dimension of $\mathcal{M}_X, \mathcal{M}_X^{\mathcal{F}}$. The old formula (52) for $\dim \mathcal{O}_{X'}$ should be replaced by

$$\dim \mathcal{O}_{X'} = \sum_{i=1}^{n} \Bigl(-2m + \sum_{j=1}^{b^0(\Sigma_i)} \bigl(\text{s-ind}(C_i^j) + 2m \bigr) \Bigr). \tag{63}$$

If $b^0(\Sigma_i) > 1$ one can show that this does strictly reduce $\dim \mathcal{O}_{X'}$.

The new obstruction space $\mathcal{O}_{X'}$ is a quotient of the old by a vector subspace, which is the extra obstructions we can overcome by moving the C_i^j around separately under SU(m). The new infinitesimal deformation space $\mathcal{I}_{X'}$ is the same as the old one.

There is one special case to be considered above. In Definition 3.6 and throughout we have assumed that the SL cone C_i has an isolated singularity at 0. It could be that if $b^0(\Sigma_i) > 1$ then some of the C_i^j above are SL planes $\mathbb{R}^m$ in $\mathbb{C}^m$, and thus are nonsingular at 0, and so are not covered by Definition 3.6.

In this case (8) fails for $\Sigma_i^j = \mathcal{S}^{m-1}$, as $m_{\Sigma_i^j}(1) = m$. To compensate for this, the appropriate value of s-ind(C_i^j) in (63) is s-ind$(C_i^j) = -m$. This is because the term s-ind$(C_i^j) + 2m$ in (63) contains a contribution $2m$ on the assumption that $m_{\Sigma_i^j}(1) = 2m$, and this has to be reduced from $2m$ to m.

8.3 Families of special Lagrangian cones

Let X be a compact SL m-fold in (M, J, ω, Ω) with conical singularities at $x_1, \ldots, x_n$ with cones $C_1, \ldots, C_n$. Here is a more general way of relaxing the condition that the SL m-folds $\hat{X}$ in $\mathcal{M}_X$, $\mathcal{M}_X^{\mathcal{F}}$ must all have the *same* SL cones $C_1, \ldots, C_n$ at their singular points.

Suppose $\mathcal{C}_i$ is a *smooth, connected family* of *distinct* SL cones in $\mathbb{C}^m$ with $C_i \in \mathcal{C}_i$ for $i = 1, \ldots, n$. Since we can always move cones through SU(m)

rotations by changing the identifications υ_i, suppose for simplicity that $\mathcal{C}_i$ is closed under the action of $\mathrm{SU}(m)$. Then in defining $\mathcal{M}_X, \mathcal{M}_X^{\mathcal{F}}$ we allow the SL m-folds $\hat{X}$ with conical singularities at $\hat{x}_1, \ldots, \hat{x}_n$ to have cones $\hat{C}_i \in \mathcal{C}_i$ for $i = 1, \ldots, n$.

If $\mathcal{C}_i$ is the $\mathrm{SU}(m)$-orbit of C_i, then this yields exactly the same moduli spaces $\mathcal{M}_X, \mathcal{M}_X^{\mathcal{F}}$ as in §5–§7. In the situation of §8.2, if $C_i = \bigcup_{j=1}^{b^0(\Sigma_i)} C_i^j$ and we take $\mathcal{C}_i$ to be an open subset of the product of the $\mathrm{SU}(m)$-orbits of C_i^j for $j = 1, \ldots, b^0(\Sigma_i)$, so that $\mathcal{C}_i$ consists of cones $\hat{C}_i$ got by moving the C_i^j about independently with $\mathrm{SU}(m)$ rotations, then this recovers the 'second way' of §8.2.

But if $\mathcal{C}_i$ contains *nontrivial* deformations of C_i not obtained by $\mathrm{SU}(m)$ rotations of the components of C_i', then this is a true generalization of the problem, which will enlarge $\mathcal{M}_X, \mathcal{M}_X^{\mathcal{F}}$ and their (expected) dimension. Intuitively one might expect that special Lagrangian cones are pretty rigid things and will not admit nontrivial deformations in this way, so that there do not exist any interesting families $\mathcal{C}_i$ to use in this construction.

However, at least when $m = 3$, this is not true. There exists a complicated theory which describes all *special Lagrangian* T^2*-cones* in $\mathbb{C}^3$ using *integrable systems*, which is described in McIntosh [20] and the author [10]. It establishes a 1-1 correspondence between SL T^2-cones in $\mathbb{C}^3$ up to isometry and collections of *spectral data*, including a Riemann surface Y with even genus called the *spectral curve*, and a holomorphic line bundle $L \to Y$.

As [20, §4.2] and [10, §4.3], it turns out that an SL T^2-cone with spectral curve Y of genus $2d \geqslant 4$ is part of a smooth $(d-2)$-dimensional family of SL T^2-cones up to isometries of $\mathbb{C}^3$, which have the same spectral curve Y but varying line bundles $L \to Y$. Ian McIntosh (personal communication) and Emma Carberry have recently announced a proof of the existence of SL T^2-cones with spectral curves of every even genus. Thus there exist smooth families $\mathcal{C}_i$ of SL T^2-cones in $\mathbb{C}^3$ with arbitrarily high dimension, to which we can apply this deformation theory.

The main changes to the final results are that we replace the definition of $\text{s-ind}(C_i)$ in (9) by $\text{s-ind}_{\mathcal{C}_i}(C_i) = N_{\Sigma_i}(2) - b^0(\Sigma_i) - 2m - \dim \mathcal{C}_i$, the *stability index of* C_i *in* $\mathcal{C}_i$, and then the old formula (52) for $\dim \mathcal{O}_{X'}$ should be replaced by $\dim \mathcal{O}_{X'} = \sum_{i=1}^n \text{s-ind}_{\mathcal{C}_i}(C_i)$. The new infinitesimal deformation space $\mathcal{I}_{X'}$ is the same as the old one.

9 Transversality and genericity results

Finally we discuss the question: if (M, J, ω, Ω) is a *generic* almost Calabi–Yau m-fold, are moduli spaces $\mathcal{M}_X$ of compact SL m-folds X in M with conical singularities necessarily smooth?

Consider what we mean by *generic* here. The conditions $\iota_*(\gamma) \cdot [\omega] = 0$ for $\gamma \in H_2(X, \mathbb{R})$ and $[X] \cdot [\mathrm{Im}\,\Omega] = 0$ mean that when $[\omega], [\mathrm{Im}\,\Omega]$ are generic there *will not exist* any such SL m-folds X in (M, J, ω, Ω). Thus, choosing (M, J, ω, Ω) generically in the family of all almost Calabi–Yau m-folds is too strong. Instead, we shall require only that ω *is generic in its Kähler class.*

That is, given an almost Calabi–Yau m-fold (M,J,ω,Ω) containing a compact SL m-fold X with conical singularities, we consider generic perturbations $(M,J,\check\omega,\Omega)$ with $\check\omega=\omega+\mathrm{d}(J\,\mathrm{d}f)$ for some *Kähler potential* $f\in C^\infty(M)$, so that $[\check\omega]=[\omega]\in H^2(M,\mathbb{R})$. Then there are no cohomological obstructions to the existence of SL m-folds $\check X$ with conical singularities in $(M,J,\check\omega,\Omega)$ isotopic to X, and we wish to know whether the moduli space $\check{\mathcal{M}}_X$ of such $\check X$ is smooth.

We begin by showing that for any compact SL m-fold X with conical singularities, there exists a family of deformations $\mathcal{F}$ with X transverse in $\mathcal{F}$.

Theorem 9.1 *Let (M,J,ω,Ω) be an almost Calabi–Yau m-fold and X a compact SL m-fold in M with conical singularities. Let $\mathcal{I}_{X'},\mathcal{O}_{X'}$ be as in §6. Then there exists a smooth family of deformations $\bigl\{(M,J,\omega^s,\Omega):s\in\mathcal{F}\bigr\}$ of (M,J,ω,Ω) with $[\omega^s]=[\omega]\in H^2(M,\mathbb{R})$ for all $s\in\mathcal{F}$, such that X is transverse in $\mathcal{F}$, in the sense of Definition 7.11, and* $\dim\mathcal{F}=\dim\mathcal{O}_{X'}$. *Hence the moduli space $\mathcal{M}_X^{\mathcal{F}}$ of §7 is a manifold near $(0,X)$.*

Proof. Use the notation of §6–§7. Recall from Definition 6.8 that $\mathcal{O}_{X'}$ consists of smooth, compactly-supported functions v on X' with $\int_{X'}v\,\mathrm{d}V_g=0$. Since $H^m_{\mathrm{cs}}(X',\mathbb{R})=0$, we see that each such v may be written as $\mathrm{d}^*(\psi^m\alpha)$ for α a smooth, compactly-supported 1-form on X'. Let $d=\dim\mathcal{O}_{X'}$, and choose smooth, compactly-supported 1-forms $\alpha_1,\ldots,\alpha_d$ on X' with

$$\mathcal{O}_{X'}=\bigl\langle\mathrm{d}^*(\psi^m\alpha_1),\ldots,\mathrm{d}^*(\psi^m\alpha_d)\bigr\rangle. \tag{64}$$

Suppose $f\in C^\infty(M)$ with $f|_{X'}\equiv 0$. Then $\mathrm{d}f|_{X'}\in C^\infty(\nu^*)$, where $\nu\rightarrow X'$ is the normal bundle to X' in M. But the complex structure J induces an isomorphism $\nu\cong TX'$, so we can regard $\mathrm{d}f|_{X'}$ as an element of $C^\infty(T^*X')$, that is, a 1-form on X'.

Choose smooth functions $f_1,\ldots,f_d\in C^\infty(M)$ such that $f_j|_{X'}\equiv 0$, and f_j is supported on a small open neighbourhood U_j in M of the support of α_j in X' with $x_i\notin U_j$ for $i=1,\ldots,n$, and $\mathrm{d}f_j|_{X'}$ is identified with α_j under the isomorphism $C^\infty(\nu^*)\cong C^\infty(T^*X')$ above, for $j=1,\ldots,d$. It is easy to show that this is possible.

For $s=(s_1,\ldots,s_d)\in\mathbb{R}^d$, define a closed real (1,1)-form ω^s on M by

$$\omega^s=\omega+\textstyle\sum_{j=1}^d s_j\,\mathrm{d}\bigl(J(\mathrm{d}f_j)\bigr). \tag{65}$$

Choose an open neighbourhood $\mathcal{F}$ of 0 in $\mathbb{R}^d$ such that ω^s is the Kähler form of a Kähler metric g^s on (M,J) for all $s\in\mathcal{F}$. This is true for small $s\in\mathbb{R}^d$. Then $\bigl\{(M,J,\omega^s,\Omega):s\in\mathcal{F}\bigr\}$ is a smooth family of deformations of (M,J,ω,Ω), in the sense of Definition 2.11.

The definition of f_j implies that $(J\,\mathrm{d}f_j)|_{X'}=\alpha_j$. Thus (65) gives

$$\omega^s|_{X'}=\textstyle\sum_{j=1}^d s_j\,\mathrm{d}\alpha_j. \tag{66}$$

Applying Theorem 4.9 gives $0\in\mathcal{F}'\subseteq\mathcal{F}$ and family of maps $\Phi^s_{X'}:U_{X'}\rightarrow M$ for $s\in\mathcal{F}'$ with $(\Phi^s_{X'})^*(\omega^s)=\hat\omega$. Identifying X' with the zero section in $U_{X'}$, we see from (65) and (66) that

$$(\Phi^s_{X'})^*(\omega)|_{X'} = -\sum_{j=1}^d s_j \,\mathrm{d}\alpha_j + O\bigl(|s|^2\bigr) \quad \text{for small } s\in\mathcal{F}'. \tag{67}$$

As the restriction of $\hat\omega$ on $U_{X'}$ to the graph $\Gamma(\alpha)$ of a 1-form α is $-\mathrm{d}\alpha$, examining the proof of Theorem 4.9 in [12] we find that we can choose $\Phi^s_{X'}$ such that

$$\Phi^s_{X'}(x) = \Phi_{X'}\bigl(\textstyle\sum_{j=1}^d s_j\,\alpha_j\bigr) + O\bigl(|s|^2\bigr) \quad \text{for } x\in X' \text{ and small } s\in\mathcal{F}'. \tag{68}$$

That is, the image of the zero section under $\Phi^s_{X'}$ approximates the image of the graph of $\sum_{j=1}^d s_j\,\alpha_j$ under $\Phi_{X'}$.

The proof of Proposition 2.10 now shows that

$$(\Phi^s_{X'})^*(\operatorname{Im}\Omega)|_{X'} = -\sum_{j=1}^d s_j\,\mathrm{d}^*(\psi^m\alpha_j)\mathrm{d}V_g + O\bigl(|s|^2\bigr) \quad \text{for small } s\in\mathcal{F}'. \tag{69}$$

But $\Phi^{s,e^s}_{X'} = \Phi^s_{X'}$ in Theorem 7.3 and (60) in Definition 7.6 imply that

$$(\Phi^s_{X'})^*(\operatorname{Im}\Omega)|_{X'} = F^{\mathcal{F}}(s,e^s,0,0)\mathrm{d}V_g. \tag{70}$$

Combining equations (64), (69) and (70) shows that the projection to $\mathcal{O}_{X'}$ of the derivative $\mathrm{d}\mathcal{F}^{\mathcal{F}}|_{(0,e,0,0)}$ is surjective. It easily follows that in Theorem 7.9, the map $\mathrm{d}\Phi^{\mathcal{F}}|_{(0,0)} : \mathbb{R}^d\times\mathcal{I}_{X'}\to\mathcal{O}_{X'}$ is surjective. Hence X is transverse in $\mathcal{F}$ by Definition 7.11. The last part follows from Corollary 7.12. □

Let $F : P \to Q$ be a smooth map between finite-dimensional manifolds. Recall that $q\in Q$ is called a *critical value* of F if $q = F(p)$ for some $p\in P$ for which $\mathrm{d}F|_p : T_pP\to T_qQ$ is not surjective. Points $q\in Q$ which are not critical values are called *regular values*. Then *Sard's Theorem* (see Bredon [2, §II.6 & App. C] for a proof) says that the set of critical values of F is of measure zero in Q. Thus, almost all points in Q are regular values.

This is important because if $q\in Q$ is a regular value then $F^{-1}(q)$ is a *submanifold* of Q, of dimension $\dim P - \dim Q$. Now in Theorem 9.1 we know that $\mathcal{M}^{\mathcal{F}}_X$ is a manifold and $\pi^{\mathcal{F}} : \mathcal{M}^{\mathcal{F}}_X \to \mathcal{F}$ a smooth map near $(0,X)$. Thus Sard's Theorem shows that $(\pi^{\mathcal{F}})^{-1}(s)$ is a manifold near $(0,X)$ for small generic $s\in\mathcal{F}$. So we prove:

Corollary 9.2 *In the situation of Theorem 9.1, for small generic $s\in\mathcal{F}$ the moduli space $\mathcal{M}^s_X = (\pi^{\mathcal{F}})^{-1}(s)\subset\mathcal{M}^{\mathcal{F}}_X$ of deformations of X in (M,J,ω^s,Ω) is near $(0,X)$ a manifold of dimension $\dim\mathcal{I}_{X'} - \dim\mathcal{O}_{X'}$.*

If $\dim\mathcal{I}_{X'} - \dim\mathcal{O}_{X'} < 0$ then $\mathcal{M}^s_X$ is empty near $(0,X)$ for small generic s. We can generalize Theorem 9.1 in the following way. As transversality is an open condition, $\hat X$ is transverse to $\mathcal{F}$ for $\hat X$ in an open neighbourhood of X in $\mathcal{M}_X$. In the same way, for each $\tilde X\in\mathcal{M}_X$ we can construct a family of deformations $\mathcal{F}_{\tilde{X}}$ of (M,J,ω,Ω) and an open neighbourhood of $\tilde{X}$ in $\mathcal{M}_X$ in which all $\hat{X}$ are transverse to $\mathcal{F}_{\tilde{X}}$.

Let $W \subseteq \mathcal{M}_X$ be compact. Taking a finite subcover of W from this collection of open neighbourhoods in $\mathcal{M}_X$, we get families of deformations $\mathcal{F}_1,\ldots,\mathcal{F}_l$ of (M,J,ω,Ω) such that every $\hat{X}\in W$ is transverse in $\mathcal{F}_j$ for some $j=1,\ldots,l$. Choose a family $\mathcal{F}$ of deformations of (M,J,ω,Ω) containing open neighbourhoods of 0 in $\mathcal{F}_1,\ldots,\mathcal{F}_l$. This is easily done, as the $\mathcal{F}_j$ are open neighbourhoods of ω in affine subspaces $\mathcal{A}_1,\ldots,\mathcal{A}_l$ of the Kähler class of ω, and we can take $\mathcal{F}$ to be an open neighbourhood of ω in the affine subspace spanned by $\mathcal{A}_1,\ldots,\mathcal{A}_l$. Then all $\hat{X}\in W$ are transverse in $\mathcal{F}$, giving:

Theorem 9.3 *Let (M,J,ω,Ω) be an almost Calabi–Yau m-fold and X a compact SL m-fold in M with conical singularities. Let $\mathcal{M}_X,\mathcal{I}_{X'},\mathcal{O}_{X'}$ be as in §5–§6, and suppose $W\subseteq\mathcal{M}_X$ is a compact subset. Then there exists a smooth family of deformations $\{(M,J,\omega^s,\Omega): s\in\mathcal{F}\}$ of (M,J,ω,Ω) with $[\omega^s]=[\omega]\in H^2(M,\mathbb{R})$ for all $s\in\mathcal{F}$, such that $\hat{X}$ is transverse in $\mathcal{F}$ for all $\hat{X}\in W$. Hence the moduli space $\mathcal{M}_X^{\mathcal{F}}$ of §7 is a manifold near $\{0\}\times W$.*

The analogue of Corollary 9.2 is:

Corollary 9.4 *In the situation of Theorem 9.3, for small generic $s\in\mathcal{F}$ the moduli space $\mathcal{M}_X^s=(\pi^{\mathcal{F}})^{-1}(s)\subset\mathcal{M}_X^{\mathcal{F}}$ of deformations of X in (M,J,ω^s,Ω) is near $\{0\}\times W$ a manifold of dimension* $\dim\mathcal{I}_{X'}-\dim\mathcal{O}_{X'}$.

Roughly speaking, Corollaries 9.2 and 9.4 imply that for a small generic perturbation $(M,J,\breve{\omega},\Omega)$ of (M,J,ω,Ω) in the same Kähler class, the perturbed moduli space $\breve{\mathcal{M}}_X$ is a manifold near X, or more generally near a compact subset W of $\mathcal{M}_X$. Of course, X and W do not lie in $\breve{\mathcal{M}}_X$, but the idea does make sense. We conjecture that if $\breve{\omega}$ is sufficiently generic then $\breve{\mathcal{M}}_X$ is a manifold everywhere.

Conjecture 9.5 *Let (M,J,ω,Ω) be an almost Calabi–Yau m-fold and X a compact SL m-fold in M with conical singularities, and let $\mathcal{I}_{X'},\mathcal{O}_{X'}$ be as in §6. Then for a second category subset of Kähler forms $\breve{\omega}$ in the Kähler class of ω, the moduli space $\breve{\mathcal{M}}_X$ of compact SL m-folds $\hat{X}$ with conical singularities in $(M,J,\breve{\omega},\Omega)$ isotopic to X is a manifold of dimension* $\dim\mathcal{I}_{X'}-\dim\mathcal{O}_{X'}$.

Recall that a subset of a topological space is of *second category* if it can be written as the intersection of a countable number of open dense sets. Using the *Baire category theorem* one can show that second category subsets of the Kähler class of ω are dense. Thus, the conjecture implies that $\breve{\mathcal{M}}_X$ is smooth for generic $\breve{\omega}$.

As a countable intersection of second category subsets is second category, the conjecture also implies that by choosing $\breve{\omega}$ generically we can make a countable number of moduli spaces $\breve{\mathcal{M}}_{X_1},\breve{\mathcal{M}}_{X_2},\ldots$ simultaneously smooth. However, we have not extended Conjecture 9.5 to the tempting, much simpler statement that for generic $\breve{\omega}$, *all* the moduli spaces $\breve{\mathcal{M}}_X$ are smooth.

This is because, as in §8.3, there can exist smooth, positive-dimensional families of SL cones in $\mathbb{C}^m$ which are distinct under $\mathrm{SU}(m)$ transformations. Now with the definitions of §5, every $\hat{X} \in \mathcal{M}_X$ has the same cones $C_1, \ldots, C_n$. If these cones C_i are allowed to vary in positive-dimensional families, we would get corresponding uncountable families of moduli spaces $\check{\mathcal{M}}_X^t$, and it is too much to expect all of these to be simultaneously smooth.

Results similar to Conjecture 9.5 are proved by Donaldson and Kronheimer [3, §4.3] for moduli spaces of instantons on 4-manifolds w.r.t. a generic C^l metric, and by McDuff and Salamon [18, §3] for smoothness of moduli spaces of pseudo-holomorphic curves on a symplectic manifold w.r.t. a generic C^l or smooth almost complex structure.

Following these proofs, the author has a sketch proof of a version of Conjecture 9.5 using C^l Kähler forms $\check{\omega}$ rather than smooth Kähler forms, for large $l \geqslant 3$. It involves messy issues in infinite-dimensional analysis, so we will not give it. The reason for using C^l Kähler forms is to be able to apply the Sard–Smale Theorem, a version of Sard's Theorem for Banach manifolds. The author cannot yet see how to extend this to smooth Kähler forms.

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