DPhil Thesis

Nikolaos Fountoulakis
St. Hugh's College
Oxford

March 12, 2004

Contents

Chapter 1

Introduction

In this chapter, following a necessary section with definitions and notation, we present an overview of the thesis and, in particular, we discuss threshold phenomena.

1.1 Definitions and notation

A *graph* $G = (V, E)$ is an ordered pair of disjoint sets, where $V = V(G)$ is the set of *vertices* (or nodes) and $E = E(G)$ is a multiset of unordered pairs of vertices. The elements of E are called *edges*. In the present thesis, V and E are assumed to be finite. If E contains at least two copies of the same edge or an edge where a specific vertex appears twice, then G is said to be a *multigraph*, otherwise the graph is said to be *simple*. In a multigraph, an edge consisting of two copies of some vertex is called a *loop*, whereas if an edge has more than one copy in E, then it is called a *multiple edge*. Any two vertices that form an edge are called *adjacent* and these are said to be the *endvertices* of the edge. We set $v_G = |V(G)|$ (the so-called *order* of the graph) and $e_G = |E(G)|$ (called the *size* of the graph). For a vertex $v \in V(G)$, we define its *degree* to be the number of occurrences of v as a member of an element of $E(G)$. A subset of vertices of a graph is called *stable* (or *independent*) if no two elements of it are adjacent. Two graphs $G = (V, E)$ and $G' = (V', E')$ are *isomorphic* if there is a bijection from V onto V', which preserves adjacencies and multiplicities. An *automorphism* of G is a bijection from V onto itself that preserves adjacencies and multiplicities.

A graph with no edges and at least one vertex is called an *empty* graph and a graph with no vertices is called the *null* graph. Note that if $v_G = n$ and G is a simple graph, then $e_G \le \binom{n}{2}$. If $e_G = \binom{n}{2}$, then the simple graph is said to be *complete* and it is denoted by K_n. For a set

V, let $V^{(2)}$ denote the set of unordered pairs of elements of V (including (v, v) for every $v \in V$) and whenever A is multiset then $A \cap V^{(2)}$ will denote the multiset of those elements of $V^{(2)}$ that belong to A with their multiplicities preserved. A graph $G = (V, E)$ where $V = V_1 \cup V_2$ and $E \cap (V_1^{(2)} \cup V_2^{(2)}) = \emptyset$ is called *bipartite*. A bipartite simple graph $G = (V_1 \cup V_2, E)$ where E is the set of all possible edges between V_1 and V_2 is a *complete bipartite graph*. The complete bipartite graph, where the two sets of the bipartition are of cardinality m and n respectively, is denoted by $K_{m,n}$. For some positive integer r, we say that a graph G is *r-regular* if each vertex of it has degree r. Also, a graph G is *regular* if it is r-regular, for some positive integer r.

For a graph $G = (V, E)$ any graph $G' = (V', E')$ such that $V' \subseteq V$ and $E' \subseteq E$ is called a *subgraph* of G. If $G = (V, E)$ is a graph and $V' \subseteq V(G)$, then $G[V']$ denotes the restriction of G to V' and this is the graph with vertex set V' and edge set $E(G) \cap V'^{(2)}$. This is called an *induced subgraph* of G. If the subgraph $G' = (V', E')$ is such that $V' = V$, then G' is said to be a *spanning subgraph* of G. A *perfect matching* on V, where $|V|$ is even, is a simple graph $G = (V, E)$ such that for every $v \in V$ there exists $e \in E$ such that $v \in e$ and for every $e, e' \in E$ we have $e \cap e' = \emptyset$.

A *path* is a graph $P = (V, E)$, with $V = \{v_1, \dots, v_l\}$ and $E = \{(v_1, v_2), (v_2, v_3), \dots, (v_{l-1}, v_l)\}$. It is denoted by $v_1 v_2 \cdots v_l$. The vertices v_1, v_l are the *endvertices* of P. We say that P is a path *from v_1 to v_l* (or vice versa). A graph $G = (V, E)$ is *connected* if for every pair $\{x, y\}$ of distinct vertices there is a path (as a subgraph of G) from x to y. A *maximal connected subgraph* is called a *component* of the graph. If such a subgraph has at least one edge, then it is said to be a *non-trivial* component. A simple graph G is a *tree* if it is connected and between any two of its vertices there is a unique path.

The set $\mathcal{A} = \{\cup_{n \in \mathbb{Z}^+} \mathcal{A}(n)\}$, where $\mathcal{A}(n)$ is a subset of graphs defined on the set $V_n = \{1, \dots, n\}$ and it is closed under automorphisms, is called a *graph property $\mathcal{A}$*. Now, a *monotonically increasing* (or just *increasing*) graph property $\mathcal{A}$ is a (graph) property such that for any $n \in \mathbb{Z}^+$, whenever $G \in \mathcal{A}(n)$ and G is a subgraph of $H = H(V_n, E)$, then $H \in \mathcal{A}(n)$. Similarly, we define the *monotonically decreasing* properties. A graph property which is either increasing or decreasing is said to be *monotone*.

Let $G = (V, E)$ be a graph, where $V \neq \emptyset$. In general, for any positive integer k, a *k-colouring* of G is a mapping $S : V \to \{1, \dots, k\}$. If there exists a k-colouring such that no two adjacent vertices have the same image (i.e. colour) under S, then the graph is said to be *k-colourable* and the corresponding colouring is said to be *proper*. The *chromatic number* $\chi(G)$ of a graph G is the least positive integer k for which the graph is k-colourable. Note that the property of k-colourability is monotonically decreasing.

A *random graph* (rather crudely) is a simple graph which is created randomly, i.e. it is the outcome of a random experiment. The random experiment consists in the random choice of the set E on a fixed set of vertices V. For the sake of convenience, we consider graphs with n vertices on $V_n = \{1, \ldots, n\}$, where n is a positive integer. More specifically, there are several models for the random construction of graphs, but here we will describe the basic ones. The most frequently used models are the $\mathcal{G}_{n,m}$ (the so-called *uniform* model) and the $\mathcal{G}_{n,p}$ (known as the *binomial* model), where n is a positive integer.

The first consists of all simple graphs with vertex set $V_n = \{1, \ldots, n\}$ having m edges, where $0 \leq m \leq \binom{n}{2}$, in which the graphs have the same probability. Thus $\mathcal{G}_{n,m}$ has $\binom{\binom{n}{2}}{m}$ equiprobable elements. That is, every element occurs with probability $\binom{\binom{n}{2}}{m}^{-1}$. In this model the random experiment consists in choosing E uniformly among the m-subsets of the set of all possible edges.

The model $\mathcal{G}_{n,p}$, where $0 \leq p \leq 1$, consists of all simple graphs with vertex set $V_n = \{1, \ldots, n\}$ in which each edge appears independently and with probability p. In other words, if G is a graph on V_n having $m \leq \binom{n}{2}$ edges, then its probability to occur is equal to $p^m(1-p)^{\binom{n}{2}-m}$. Note that the number of edges follows the binomial distribution $\mathrm{Bin}\left(\binom{n}{2}, p\right)$.

Whenever reasons of simplicity force it, we will be working in the so called $\mathcal{G}^*_{n,m}$ model. This is slightly different from the $\mathcal{G}_{n,m}$ model. Here, we have available the set of all $\binom{n}{2}$ possible edges between the vertices $\{1, \ldots, n\}$. We construct our random graph selecting m times uniformly and independently and with replacement the edges which are going to participate in the graph. The resulting graph can be considered as an ordered word, consisting of m symbols and each of them corresponds to an edge. All these words are equiprobable. Typically, we regard any multiple edges copies of one edge. In what follows, we will be using the symbols $\mathcal{G}^*_{n,m}$, $\mathcal{G}_{n,m}$ and $\mathcal{G}_{n,p}$ to denote the random outcome of the corresponding model. We say that a property $\mathcal{A}$ is satisfied *asymptotically almost surely* (a.a.s.) if $\lim_{n\to\infty} \mathbb{P}[\mathcal{G}_{n,m} \in \mathcal{A}] = 1$ (we give the same definition for the $\mathcal{G}_{n,p}$ and $\mathcal{G}^*_{n,m}$ models of random graphs).

Finally, we give some definitions from probability theory. Let $\{(\Omega_n, \mathcal{F}_n, \mathbb{P}_n)\}_{n\in\mathbb{Z}^+}$ be a sequence of probability triples and let $(X_n)_{n\in\mathbb{Z}^+}$ be a sequence of real-valued random variables defined on them. If $(a_n)_{n\in\mathbb{Z}^+}$ is a sequence of positive real numbers, we write $X_n = o_p(a_n)$, whenever $|X_n|/a_n$ *converges in probability* to 0, i.e. for every $\varepsilon > 0$ we have $\lim_{n\to\infty} \mathbb{P}_n[|X_n|/a_n > \varepsilon] = 0$ (we write $|X_n|/a_n \xrightarrow{p} 0$). Also, we write $X_n = O_p(a_n)$, if for every $\delta > 0$ there exists $C > 0$ such that $\mathbb{P}_n[|X_n| > Ca_n] < \delta$ for n sufficiently large, and $X_n = O_C(a_n)$ if there exists real $C > 0$ such that $\lim_{n\to\infty} \mathbb{P}_n[|X_n| \leq Ca_n] = 1$. Moreover, $X_n = \Theta_p(a_n)$, if for every $\delta > 0$ there exist constants $c_\delta, C_\delta > 0$ such that $\mathbb{P}[c_\delta a_n \leq X_n \leq C_\delta a_n] > 1 - \delta$ for n sufficiently large. If X is a real-valued random variable on some probability space, we say that X_n *converges in distribution* to X if for every $x \in \mathbb{R}$, where x is a point of continuity of the distribution function of X, we have $\lim_{n\to\infty} \mathbb{P}_n[X_n \leq x] = \mathbb{P}[X \leq x]$ (we write $X_n \xrightarrow{d} X$). Also, for some real $z > 0$ the symbol $\mathrm{Po}(z)$ denotes the Poisson distribution of mean z. For a positive integer k, we shall denote by $[k]$ the set $\{1, \ldots, k\}$ and $[V]^k$ will denote the set of all k-subsets of a set V. Finally, for an even positive integer $n = 2k$, we set $(n-1)!! = \prod_{i=0}^{k-1}(2i+1)$.

For an introduction (and much more than this) to the theory of random graphs, see the classical monograph of B. Bollobás [6] or the recently appeared book by S. Janson, T. Łuczak and A. Ruciński [23].

1.1.1 Equivalence between random graph models

As we mentioned previously, whenever it is simpler, we will be working in the $\mathcal{G}^*_{n,m}$ model. It can be easily seen that for any increasing graph property $\mathcal{A}$ the following holds

$$\mathbb{P}[\mathcal{G}_{n,m} \in \mathcal{A}] \geq \mathbb{P}[\mathcal{G}^*_{n,m} \in \mathcal{A}].$$

Therefore, we have proved the following:

Proposition 1.1.1 *For any increasing graph property $\mathcal{A}$, if $\lim_{n\to\infty} \mathbb{P}[\mathcal{G}^*_{n,m} \in \mathcal{A}] = 1$, then $\lim_{n\to\infty} \mathbb{P}[\mathcal{G}_{n,m} \in \mathcal{A}] = 1$ as well.*

Note that many central graph properties, such as non-k-colourability (see the next chapter for the definition), are increasing.

The following proposition links the binomial with the uniform model (see [23] for a proof):

Proposition 1.1.2 *Let $\mathcal{A}$ be a monotonically increasing property of graphs and let $m = m(n) : \mathbb{Z}^+ \to \mathbb{N}$ be such that $m \leq N$, where $N = \binom{n}{2}$. Then if $p = p(n) = m/N$ we have*

$$(1-o(1))\mathbb{P}\left[\mathcal{G}_{n,\left(\left\lfloor m-\sqrt{\omega(n)m\left(1-\frac{m}{N}\right)}\right\rfloor\right)^+} \in \mathcal{A}\right] \leq \mathbb{P}[\mathcal{G}_{n,p} \in \mathcal{A}] \leq \mathbb{P}\left[\mathcal{G}_{n,\left\lceil m+\sqrt{\omega(n)m\left(1-\frac{m}{N}\right)}\right\rceil \wedge N} \in \mathcal{A}\right] + o(1),$$

where $\omega(n)$ is a positive real-valued function on $\mathbb{N}$ tending to infinity as $n \to \infty$.

1.2 Aims and overview

Is this section, we present the main themes of this thesis. We work on the approximation of the 3-colourability/non-3-colourability phase transition threshold of sparse random graphs, as well as on the investigation of the typical structure of a random graph near the alleged critical point.

In the last decade, there has been an extensive interest in the understanding of phase transitions of combinatorial problems such as the k-colourability of random graphs or the satisfiability of random k-SAT formulae. More specifically, in the case of random graphs for each $k \geq 3$ it is believed that there exists a constant θ_k, such that whenever the average degree of a random graph exceeds it, then the random graph is asymptotically almost surely not k-colourable, whereas if the average degree is less than this bound, then the random graph is k-colourable asymptotically almost surely. Similarly, for each $k \geq 3$ it is believed that there exists a constant c_k so that if the ratio $\#clauses/\#variables$ of a random k-SAT boolean formula is bigger than c_k, then the random formula is asymptotically almost surely unsatisfiable, whereas if this ratio is less than c_k, then the random formula is asymptotically almost surely satisfiable. The target in both cases is to find the values of θ_k and c_k, for each $k \geq 3$ (assuming this exists).

The reason for trying to tackle these problems is that, apart from being interesting for their own sake as both colourability and satisfiability are important properties, the understanding of the phase transitions will probably lead to a better understanding of the nature of colourability (and similarly of satisfiability in the case of boolean formulae). More specifically, we are aimining towards a better understanding of the effect that the increase of the average degree has on the structure of the graph, that actually forces it to be non-k-colourable.

Here, we focus on the property of 3-colourability of random graphs. The aims of the present thesis are, on the one hand, to give better approximations to the alleged constant θ_3 and, on the other hand, to improve our understanding of the picture of a random graph near the (assumed) critical point.

The main obstacle to the exact calculation of the 3-colourability threshold is that the number of proper 3-colourings of a $\mathcal{G}_{n,m}$ random graph, where $m = \lceil \theta n/2 \rceil$ and θ is a small constant is not concentrated around its mean (unless θ is quite large). If this were the case, then an application of a first and a (probably hard) second moment method would suffice. More specifically, we could determine a critical value, such that if the average degree of the random graph is above this then the expected number of proper 3-colourings tends to 0 as n tends to infinity, whereas if the average degree is below this value then the expected number of proper 3-colourings tends to infinity as n tends to infinity. Then Markov's inequality would suffice to show almost sure non-3-colourability above this critical value and Chebyschev's inequality would show that below the critical value the number of proper 3-colourings is actually concentrated around its expected value, thus yielding almost sure 3-colourability. But in fact the behaviour of this random variable is more intriguing. As we shall see, there are values of the average degree for which though the expected number of proper 3-colourings tends to infinity as the order of the random graph tends to infinity, the random graph has no proper 3-colourings asymptotically almost surely. The reason of this behaviour is the occurrence of the so-called jackpot phenomena: the existence of a huge number of proper 3-colourings with very small probability. We try to identify a source of this phenomenon and make an attempt to eliminate (at least partially) its impact. More generally, we shall try to eliminate it by resorting to a sub-family of proper 3-colourings, which are called rigid 3-colourings. This way of attacking the problem has been inspired by an analogous attempt to approximate the satisfiability threshold of random 3-SAT boolean formulae. (At this stage, we would like to point out the similarity between the natures of the two phase transitions - see [34].) More specifically, we have proved that if the average degree of a $\mathcal{G}_{n,m}$ random graph is at least 5, then the random graph is asymptotically almost surely not 3-colourable.

Also, we apply (unsuccessfully) the idea of rigid 3-colourings in an attempt to show that a random 5-regular graph is not 3-colourable asymptotically almost surely. However, before this we establish the non-k-colourability of r-regular graphs for any $k \geq 3$ and for suitable small r's.

In view of the above story, we try to improve our understanding of the typical structure of a sparse $\mathcal{G}_{n,m}$ random graph and more specifically of the structure of a subgraph where the transition from 3-colourability to non-3-colourability occurs. This is the (induced) subgraph where each vertex has degree at least 3 and is called the 3-core (see Chapter 4 for a precise definition - the notion of the k-core for $k \geq 2$ is defined analogously). Note that a simple graph is k-colourable if and only if its k-core is k-colourable, for any $k \geq 2$. So it is plausible that the better our understanding of the structure of the 3-core is, the more we are likely to know about the transition from 3-colourability to non-3-colourability. In general, for any $k \geq 3$ the average degree at which the emergence of the k-core takes place as well as its asymptotic order has been discovered in [37] by B. Pittel, J. Spencer and N. Wormald. Our contribution is the estimation of the proportion of vertices in the k-core for any $k \geq 2$ having various fixed degrees. That is we have determined the degree sequence of the k-core. To achieve this, we have used and extended the techniques that have been developed in [37].

For $k = 2$, we focus our analysis on sparse $\mathcal{G}_{n,m}$ random graphs with average degree greater than 1, that is after the emergence of the giant component. It is known that the 2-core at that stage consists of a giant component along with a few isolated small cycles. Moreover, the giant component consists of a 2-vertex-connected component (see Chapter 5 for the definition) containing almost all of the vertices of the giant component of the 2-core as well as a few small cycles attached to it by a path with 0,1 or more internal vertices. Using the information about the degree sequence of the 2-core and the fact that this is uniformly distributed among those graphs having this degree sequence, we determine the asymptotic distributions of the number of cycles that are isolated in the 2-core as well as of those that are not isolated there. Furthermore, we find the asymptotic distributions of the number of cycles that are attached to the giant 2-vertex-connected component.

We close this chapter with a study of threshold functions of random graphs. We distinguish between coarse and sharp thresholds and we focus on some properties of the latter.

In Chapter 2, we begin analysing the notion of the so-called jackpot phenomena and we try to identify a source of them. Meanwhile, we give a quite precise asymptotic expression for the expected number of proper 3-colourings of sparse $\mathcal{G}_{n,m}$ random graphs and we obtain two upper bounds on the non-3-colourability threshold. After that, we continue with the main result of that chapter, which is that when the average degree of a random graph is above 5, then the graph is asymptotically almost surely not 3-colourable. We achieve this bound by analysing the expected number of a specific kind of proper 3-colourings the so-called rigid 3-colourings. We end this chapter with some minor improvements on this bound.

In Chapter 3, we focus on the chromatic number of r-regular graphs where r is a small positive constant. Applying a simple first moment method, for each $k \geq 3$ we determine a bound $f(k)$ so that almost all r-regular graphs with $r > f(k)$ are not k-colourable. Then we try to prove that almost all 5-regular graphs are not 3-colourable using the idea of the rigid 3-colourings that was introduced in Chapter 2. In order to do this we analyse the asymptotic behaviour of a variation of the classical coupon collector problem.

In Chapter 4, we investigate the structure of the k-core of a sparse $\mathcal{G}_{n,m}$ random graph after its appearance. More specifically, by extending previous techniques, we obtain asymptotic results concerning the degree sequence of the k-core for any $k \geq 2$.

Finally, in Chapter 5, we apply the results of Chapter 4 in order to analyse the structure of the 2-core (or simply the core) of a sparse $\mathcal{G}_{n,m}$ random graph after the appearance of the giant component. We obtain the asymptotic distributions of the numbers of isolated cycles and non-isolated cycles of various fixed lengths in the core as well as the asymptotic distributions of the numbers of 2-edge-connected and 2-vertex-connected components of the giant component of the core (for the definitions see the introductory section of that chapter).

1.3 Thresholds

To close this introductory chapter, we present the notion of a threshold of a graph property. The results we present here concern mainly the uniform model. Similar results can be deduced for the binomial model. One of the most surprising facts about properties of random graphs is the "sudden" appearance of an increasing graph property when the number of edges increases.

Let us be more specific starting with the uniform model. Let $\widehat{M}(n)$ be a function such that $\widehat{M} : \mathbb{Z}^+ \to \mathbb{R} \setminus \mathbb{R}^-$ (in what follows, we assume that every function is of this form). For an increasing graph property $\mathcal{A}$, we say that $\widehat{M} = \widehat{M}(n)$ is a *threshold* if

$$\lim_{n\to\infty} \mathbb{P}\left[\mathcal{G}_{n,m} \in \mathcal{A}\right] = \begin{cases} 1 & \text{if } m \gg \widehat{M} \\ 0 & \text{if } m \ll \widehat{M} \end{cases}, \tag{1.1}$$

where $m = m(n) : \mathbb{Z}^+ \to \mathbb{N}$ such that $m \leq \binom{n}{2}$ (whenever we use a function as the edge parameter of a random graph model, we assume that it has this form). If $m(n)$ and $m'(n)$ are two functions defined on $\mathbb{Z}^+$ taking values in $\mathbb{R} \setminus \mathbb{R}^-$, we write $m(n) \gg m'(n)$ if $m(n)/m'(n) \to \infty$, as $n \to \infty$. To avoid trivial complications we assume that $\widehat{M} \geq 1$, or at least $\inf_{n\in\mathbb{Z}^+} \widehat{M}(n) > 0$.

Similarly, in the $\mathcal{G}_{n,p}$ model, a sequence $\widehat{p} = \widehat{p}(n)$ is called a *threshold* if

$$\lim_{n\to\infty} \mathbb{P}\left[\mathcal{G}_{n,p} \in \mathcal{A}\right] = \begin{cases} 1 & \text{if } p \gg \widehat{p} \\ 0 & \text{if } p \ll \widehat{p} \end{cases}. \tag{1.2}$$

We assume throughout this section (unless otherwise stated) that $\mathcal{A}$ is a monotonically increasing graph property which is non-trivial, i.e. it is neither always true nor always false. Note that similar results can be deduced for monotonically decreasing graph properties, since they are complements of increasing ones. Hence, thresholds for decreasing graph properties are defined as thresholds of their complements.

One could also see thresholds from the viewpoint of the *hitting time* of property $\mathcal{A}$. We define the random graph process on a set $V_n = \{1, \ldots, n\}$ (the set of vertices) $\{\mathcal{G}_{n,m}\}_{m=0}^N$, where $N = \binom{n}{2}$, to be the following Markov process, with time running through the set $\{0, 1, \ldots, N\}$. It begins with the empty graph at time 0 and adds new edges, one at a time; each new edge is selected at random, uniformly among all edges not already present in the graph. Note that at the m-th stage the graph that has been formed so far can be considered as an outcome of the $\mathcal{G}_{n,m}$ model. For a monotonically increasing property $\mathcal{A}$, define a random variable $\widetilde{M} = \widetilde{M}_n$ to be the number of edges selected when the property appears for the first time. This is called the *hitting* *time* of the property $\mathcal{A}$. Thus,

$$\mathbb{P}\left[\widetilde{M}_n \leq m\right] = \mathbb{P}\left[\mathcal{G}_{n,m} \in \mathcal{A}\right].$$

One can see that $\widehat{M} = \widehat{M}(n)$ is a threshold function if and only if for every $\delta > 0$ there exist constants $c_\delta, C_\delta > 0$ and n_0 such that $\mathbb{P}[c_\delta \widehat{M}(n) \leq \widetilde{M}_n \leq C_\delta \widehat{M}(n)] \geq 1 - \delta$, for every $n \geq n_0$.

Let us investigate threshold functions more closely. For any $a \in [0, 1]$, we define

$$M(a) = M(a; n) = \min\{m : \mathbb{P}\left[\mathcal{G}_{n,m} \in \mathcal{A}\right] \geq a\}.$$

The definition yields the following:

$$\mathbb{P}\left[\mathcal{G}_{n,M(a)-1} \in \mathcal{A}\right] < a \leq \mathbb{P}\left[\mathcal{G}_{n,M(a)} \in \mathcal{A}\right]. \quad (1.3)$$

The following result appears in [23].

Proposition 1.3.1 *Let $\mathcal{A}$ be an increasing graph property. Then a function $\widehat{M}(n)$ is a threshold for this property if and only if $M(a; n) = \Theta(\widehat{M}(n))$, for every $0 < a < 1$.*

Proof. Suppose first that $\widehat{M}(n)$ is a threshold. If $0 < a < 1$ but $M(a) \neq \Theta(\widehat{M})$, then there exists a strictly increasing sequence of integers $(n_i)_{i \in \mathbb{N}}$, such that $M(a; n_i)/\widehat{M}(n_i) \to 0$ or $M(a; n_i)/\widehat{M}(n_i) \to \infty$, as $i \to \infty$. In the first case, by the definition of the threshold we have $\mathbb{P}\left[\mathcal{G}_{n_i, M(a;n_i)} \in \mathcal{A}\right] \to 0$ as $i \to \infty$, and this contradicts (1.3). For the other case, we have $M(a; n_i) - 1 \gg \widehat{M}(n_i)$, for $i \in \mathbb{N}$, and, by the definition of the threshold, we have $\mathbb{P}\left[\mathcal{G}_{n_i, M(a;n_i)-1} \in \mathcal{A}\right] \to 1$ as $i \to \infty$, which is again a contradiction. Therefore, for every $a \in (0, 1)$, $M(a; n) = \Theta(\widehat{M}(n))$.

Conversely, suppose that $\widehat{M}(n)$ is not a threshold. Then there exists a function $m = m(n)$ such that $\lim_{n\to\infty} m(n)/\widehat{M}(n) = 0$ and $\liminf_{n\to\infty} \mathbb{P}\left[\mathcal{G}_{n,m} \in \mathcal{A}\right] > 0$, or $\lim_{n\to\infty} m(n)/\widehat{M}(n) = \infty$ and $\limsup_{n\to\infty} \mathbb{P}\left[\mathcal{G}_{n,m} \in \mathcal{A}\right] < 1$. Hence, in the first case there exists a strictly increasing sequence of integers $(n_i)_{i\in\mathbb{N}}$ along which $\mathbb{P}\left[\mathcal{G}_{n_i, m(n_i)} \in \mathcal{A}\right] \geq a$, for some $a > 0$, and thus $M(a; n_i) \leq m(n_i) \ll \widehat{M}(n_i)$. Similarly, for the second case, there exists $0 < a < 1$ and a strictly increasing sequence of integers $(n_i)_{i\in\mathbb{N}}$ along which $\mathbb{P}[\mathcal{G}_{n_i, m(n_i)} \in \mathcal{A}] < a$ and therefore $M(a; n_i) > m(n_i) \gg \widehat{M}(n_i)$. In both cases, we do not have $M(a; n_i) = \Theta(\widehat{M}(n_i))$ for every $0 < a < 1$. ∎

The previous proposition implies that if a property $\mathcal{A}$ has a threshold, then $M(1/2; n)$ is a threshold function for $\mathcal{A}$. A similar result can be proved in a similar way for the $\mathcal{G}_{n,p}$ model.

We can use this result to prove the following which appeared in [7]:

Theorem 1.3.1 *Every monotone property has a threshold.*

Proof. Assume that property $\mathcal{A}$ is increasing. As we have seen, the same result can be stated for monotonically decreasing properties, since these are complements of increasing ones. Let a be such that $0 < a < 1$ and let m be an integer such that $(1-a)^m \leq a$. Consider m independent copies of $\mathcal{G}_{n,M(a)}$. Since $\mathcal{A}$ is increasing we have

$$\mathbb{P}\left[\mathcal{G}^{(1)}_{n,M(a)} \cup \cdots \cup \mathcal{G}^{(m)}_{n,M(a)} \in \mathcal{A}\right] \leq \mathbb{P}\left[\mathcal{G}_{n,mM(a)} \in \mathcal{A}\right].$$

Here the union of the m independent copies of $\mathcal{G}_{n,M(a)}$ is the union of the sets of edges that have been chosen in each of them. Also, we have

$$\begin{aligned}\mathbb{P}\left[\mathcal{G}^{(1)}_{n,M(a)} \cup \cdots \cup \mathcal{G}^{(m)}_{n,M(a)} \notin \mathcal{A}\right] &\leq \mathbb{P}\left[\mathcal{G}^{(i)}_{n,M(a)} \notin \mathcal{A}, \text{ for } i = 1, \ldots, m\right] \\ &= \left(1 - \mathbb{P}\left[\mathcal{G}_{n,M(a)} \in \mathcal{A}\right]\right)^m \leq (1-a)^m \leq a.\end{aligned}$$

Hence,

$$\mathbb{P}\left[\mathcal{G}_{n,mM(a)} \in \mathcal{A}\right] \geq \mathbb{P}\left[\mathcal{G}^{(1)}_{n,M(a)} \cup \cdots \cup \mathcal{G}^{(m)}_{n,M(a)} \in \mathcal{A}\right] \geq 1 - a$$

and therefore $mM(a) \geq M(1-a)$. Thus, for $0 < a < 1/2$,

$$\frac{1}{m} M(1-a) \leq M(a) \leq M(1/2) \leq M(1-a) \leq mM(a).$$

Note that m depends solely on a. This means that $M(a) = \Theta(M(1/2))$, for any $0 < a < 1$. The previous proposition implies that $M(1/2; n)$ is a threshold function for property $\mathcal{A}$. ∎

1.3.1 Sharp thresholds

Paul Erdös and Alfred Rényi in their seminal paper [14] pointed out that a number of interesting increasing graph properties exhibit a sharp threshold behaviour. Namely, for each such property $\mathcal{A}$, there exists a critical number of edges $m_{\mathcal{A}}(n)$ such that for m around $m_{\mathcal{A}}(n)$ the probability a $\mathcal{G}_{n,m}$ random graph having $\mathcal{A}$ changes very rapidly from near 0 to near 1. In this subsection we present two classes of thresholds.

We begin by defining the *threshold interval* for the uniform model. This is $\delta_M(\varepsilon) = \delta_M(\varepsilon; n) = M(1-\varepsilon; n) - M(\varepsilon; n)$, for $0 < \varepsilon < 1/2$. Note that this measures the concentration of the random variable $\widetilde{M}_n$.

The proof of the previous theorem implies that $\delta_M(\varepsilon) = O(\widehat{M})$. More precisely, it implies that for $0 < \varepsilon < 1/2$,

$$1 \leq \frac{M(1-\varepsilon)}{M(\varepsilon)} \leq m = m(\varepsilon),$$

for some $m = m(\varepsilon) > 1$. This means that $M(\varepsilon)/M(1/2)$ is bounded from above and below by universal constants for every fixed $\varepsilon \in (0,1)$.

However, this transition is sharper for some properties. Let $\widehat{M} : \mathbb{Z}^+ \to \mathbb{R} \setminus \mathbb{R}^-$ be a function. We say that for an increasing property $\mathcal{A}$ the function $\widehat{M} = \widehat{M}(n)$ is a *sharp threshold* if, for $m = m(n)$ and for every $\eta > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left[\mathcal{G}_{n,m} \in \mathcal{A}\right] = \begin{cases} 0 & \text{if } m \leq (1-\eta)\widehat{M} \\ 1 & \text{if } m \geq (1+\eta)\widehat{M} \end{cases}.$$

Similarly, in the $\mathcal{G}_{n,p}$ model $\widehat{p} = \widehat{p}(n)$ is a *sharp threshold* if for every $\eta > 0$

$$\lim_{n\to\infty} \mathbb{P}\left[\mathcal{G}_{n,p} \in \mathcal{A}\right] = \begin{cases} 0 & \text{if } p \leq (1-\eta)\widehat{p} \\ 1 & \text{if } p \geq (1+\eta)\widehat{p} \end{cases}.$$

The second class of thresholds is the class of the so-called *coarse thresholds*. We say that a threshold $\widehat{M}(n)$ is *coarse* if there exists an $\varepsilon > 0$ such that $\delta_M(\varepsilon) = \Theta(\widehat{M})$. In what follows, we focus on sharp thresholds, giving various characterisations of such boundaries.

We shall be concerned with the uniform model. We assume that $\widehat{M}(n) \to \infty$. The following propositions are well-known facts about sharp thresholds, but since their proofs do not seem to appear elsewhere we prove them here for completeness. First, we shall show the following:

Proposition 1.3.2 *A function $\widehat{M} = \widehat{M}(n)$ is a sharp threshold for an increasing property $\mathcal{A}$ if and only if for every $0 < \varepsilon < 1$ we have $\widehat{M}(n) = M(\varepsilon; n)(1 + o(1))$.*

Proof. Assume that $\widehat{M}$ is a sharp threshold. Suppose that there exists an $0 < \varepsilon < 1$, an increasing sequence of positive integers $(n_i)_{i\in\mathbb{N}}$ and some $\eta > 0$ such that

$$\left| \frac{M(\varepsilon; n_i)}{\widehat{M}(n_i)} - 1 \right| > \eta,$$

for every $i \in \mathbb{N}$. This means that there exists an infinite subsequence $(n_{i_k})_{k\in\mathbb{N}}$ along which

$$\text{either } \frac{M(\varepsilon; n_{i_k})}{\widehat{M}(n_{i_k})} > 1 + \eta \text{ or } \frac{M(\varepsilon; n_{i_k})}{\widehat{M}(n_{i_k})} < 1 - \eta,$$

that is

$$\text{either } M(\varepsilon; n_{i_k}) > \widehat{M}(n_{i_k})(1+\eta) \text{ or } M(\varepsilon; n_{i_k}) < \widehat{M}(n_{i_k})(1-\eta),$$

for every $k \in \mathbb{N}$, and $(\widehat{M}(n_{i_k}))_{k\in\mathbb{N}}$ is increasing. If the latter happens the sharpness of the threshold and the definition of $M(\varepsilon; n)$ imply that there is a $K \in \mathbb{N}$, such that for $k \geq K$, we have $\mathbb{P}[\mathcal{G}_{n_{i_k},M(\varepsilon;n_{i_k})} \in \mathcal{A}] < \varepsilon$ which is a contradiction. Now, suppose that the former holds. This implies that

$$M(\varepsilon; n_{i_k}) - 1 > \widehat{M}(n_{i_k}) \left(1 + \eta - \frac{1}{\widehat{M}(n_{i_k})}\right),$$

for every $k \in \mathbb{N}$. Since $\widehat{M}(n_{i_k}) \to \infty$, as $k \to \infty$, and is increasing there is an N such that for $k \geq N$ we have $\eta - 1/\widehat{M}(n_{i_k}) > 0$. We choose an N such that $\eta - 1/\widehat{M}(n_{i_N}) = \eta/2$. So,

$$M(\varepsilon; n_{i_k}) - 1 > \widehat{M}(n_{i_k})(1 + \eta/2),$$

for $k \geq N$. But this is a contradiction, since $\mathbb{P}\left[\mathcal{G}_{n_{i_k},M(\varepsilon;n_{i_k})-1}\right] < \varepsilon$ and $\widehat{M}(n_{i_k})$ is a sharp threshold.

Conversely, for a function $\widehat{M}(n)$ assume that for any $\varepsilon > 0$, we have $\widehat{M}(n) = M(\varepsilon; n)(1+o(1))$ but $\widehat{M}(n)$ is not a sharp threshold. The definition implies that there exists an $\eta_0 > 0$ such that for some integer valued function $M(n)$ and some strictly increasing sequence of integers $(n_i)_{i\in\mathbb{N}}$, for any $i \in \mathbb{N}$, we have either $M(n_i) \leq (1 - \eta_0)\widehat{M}(n_i)$ and $\mathbb{P}\left[\mathcal{G}_{n_i,M(n_i)} \in \mathcal{A}\right] \geq \tau$, for some $\tau > 0$, or $M(n_i) \geq (1+\eta_0)\widehat{M}_{n_i}$ and $\mathbb{P}\left[\mathcal{G}_{n_i,M(n_i)} \in \mathcal{A}\right] < 1-\tau$, for some $\tau > 0$. Suppose that the first is true. Then, along this sequence we have $M(n_i) \geq M(\tau; n_i)$ and therefore $M(\tau; n_i) \leq (1 - \eta_0)\widehat{M}(n_i)$. This means that for every $i \in \mathbb{N}$, we have

$$\frac{\widehat{M}(n_i)}{M(\tau; n_i)} \geq \frac{1}{1 - \eta_0} > 1,$$

and this contradicts our assumption (in fact the contradiction comes from the first inequality).

On the other hand, if the second case is true, then for every $i \in \mathbb{N}$, $M(1 - \tau; n_i) > M(n_i) \geq (1 + \eta_0)\widehat{M}(n_i)$. This implies that for every $i \in \mathbb{N}$,

$$\frac{M(1 - \tau; n_i)}{\widehat{M}(n_i)} \geq 1 + \eta_0 > 1,$$

which again contradicts our assumption. ∎

Thus, the following is immediate:

Corollary 1.3.3 *An increasing graph property $\mathcal{A}$ has a sharp threshold if and only if $M(1/2; n)$ is a sharp threshold.*

Now, we give one more characterisation of the existence of sharp threshold of an increasing property $\mathcal{A}$:

Proposition 1.3.4 *An increasing property $\mathcal{A}$ has a sharp threshold $\widehat{M} = \widehat{M}(n)$ if and only if $\forall \varepsilon > 0$, we have*

$$\lim_{n\to\infty} \mathbb{P}\left[\left|\frac{\widetilde{M}_n}{M(1/2;n)} - 1\right| > \varepsilon\right] = 0, \tag{1.4}$$

where the probability is taken over the uniform space of the $N!$ ordered sequences of the $N = \binom{n}{2}$ possible edges.

Proof. Assume first that $\widehat{M}$ is a sharp threshold of property $\mathcal{A}$. Fix an $\varepsilon > 0$. The sharpness of the threshold implies that for any $\varepsilon_0 > 0$ and $\varepsilon' > 0$ we can find an $I = I(\varepsilon_0, \varepsilon') \in \mathbb{N}$ such that

$$1 - \mathbb{P}\left[(1-\varepsilon_0)\widehat{M}(n) \leq \widetilde{M}_n \leq (1+\varepsilon_0)\widehat{M}(n)\right] < \varepsilon',$$

for any $n > I$. On the other hand, the previous proposition implies that for any $\varepsilon_1 > 0$ there exists an $N = N(\varepsilon_1)$ such that

$$\left|\frac{\widehat{M}(n)}{M\,(1/2;n)} - 1\right| < \varepsilon_1,$$

that is

$$M\,(1/2;n)\,(1-\varepsilon_1) < \widehat{M}(n) < M\,(1/2;n)\,(1+\varepsilon_1),$$

for any $n \geq N$. Therefore, for any $n \geq \max\{I, N\}$, we have

$$1 - \mathbb{P}\left[M\,(1/2;n)\,(1-\varepsilon_0)(1-\varepsilon_1) < \widetilde{M}_n < M\,(1/2;n)\,(1+\varepsilon_0)(1+\varepsilon_1)\right] < \varepsilon'$$

or

$$1 - \mathbb{P}\left[M\,(1/2;n)\,(1-\varepsilon_0-\varepsilon_1+\varepsilon_0\varepsilon_1) < \widetilde{M}_n < M\,(1/2;n)\,(1+\varepsilon_0+\varepsilon_1+\varepsilon_0\varepsilon_1)\right] < \varepsilon'.$$

This implies that

$$1 - \mathbb{P}\left[M\,(1/2;n)\,(1-\varepsilon_0-\varepsilon_1-\varepsilon_0\varepsilon_1) < \widetilde{M}_n < M\,(1/2;n)\,(1+\varepsilon_0+\varepsilon_1+\varepsilon_0\varepsilon_1)\right] < \varepsilon',$$

for $n \geq \max\{I, N\}$. Now, choosing $\varepsilon_0, \varepsilon_1$ such that $\varepsilon = \varepsilon_0 + \varepsilon_1 + \varepsilon_0\varepsilon_1$, we obtain the desired result.

Conversely, assume that for any $\varepsilon > 0$, we have

$$\lim_{n\to\infty} \mathbb{P}\left[\left|\frac{\widetilde{M}_n}{M(1/2;n)} - 1\right| > \varepsilon\right] = 0.$$

We shall prove that the function $M(1/2;n)$ is a sharp threshold. If not, then two things may happen. To see the first, for some $\eta > 0$ there is a strictly increasing sequence of integers $(n_i)_{i\in\mathbb{N}}$

such that for some integer-valued function $M(n)$ we have $M(n_i) < (1-\eta)M(1/2; n_i)$ and there is an $\varepsilon > 0$ such that $\mathbb{P}\left[\mathcal{G}_{n_i,M(n_i)} \in \mathcal{A}\right] > \varepsilon$, for any $i \in \mathbb{N}$. On the other hand, our first assumption immediately implies that for any $\tau > 0$ there exists $N = N(\tau)$ such that for $n > N$ we have:

$$\mathbb{P}\left[(1-\eta)M(1/2;n) \leq \widetilde{M}_n \leq (1+\eta)M(1/2;n)\right] \geq 1-\tau.$$

That is

$$\mathbb{P}\left[(1-\eta)M(1/2;n) > \widetilde{M}_n \text{ or } \widetilde{M}_n > (1+\eta)M(1/2;n)\right] \leq \tau.$$

But along the sequence $(n_i)_{i\in\mathbb{N}}$, we have

$$\mathbb{P}\left[\widetilde{M}_{n_i} < (1-\eta)M(1/2;n_i)\right] \geq \mathbb{P}\left[\widetilde{M}_{n_i} \leq M(n_i)\right] = \mathbb{P}\left[\mathcal{G}_{n_i,M(n_i)} \in \mathcal{A}\right] > \varepsilon$$

and, therefore,

$$\mathbb{P}\left[(1-\eta)M(1/2;n_i) > \widetilde{M}_{n_i} \text{ or } \widetilde{M}_{n_i} > (1+\eta)M(1/2;n_i)\right] > \varepsilon,$$

for any $i \in \mathbb{N}$. Choosing $\tau = \varepsilon$, we obtain

$$\varepsilon < \mathbb{P}\left[(1-\eta)M(1/2;n_i) > \widetilde{M}_{n_i} \text{ or } \widetilde{M}_{n_i} > (1+\eta)M(1/2;n_i)\right] \leq \varepsilon,$$

for all $i \in \mathbb{N}$ such that $n_i > N(\varepsilon)$, and this is a contradiction.

For the second case, for some $\eta > 0$ there is a strictly increasing sequence of integers $(n_i)_{i\in\mathbb{N}}$ such that for some integer-valued function $M(n)$ we have $M(n_i) > (1+\eta)M(1/2;n_i)$ and there is an $0 < \varepsilon < 1$ such that $\mathbb{P}\left[\mathcal{G}_{n_i,M(n_i)} \in \mathcal{A}\right] < 1-\varepsilon$, for any $i \in \mathbb{N}$. This implies that along this sequence

$$\mathbb{P}\left[\widetilde{M}_{n_i} > (1+\eta)M(1/2;n_i)\right] \geq \mathbb{P}\left[\widetilde{M}_{n_i} > M(n_i)\right] = \mathbb{P}\left[\mathcal{G}_{n_i,M(n_i)} \notin \mathcal{A}\right] > \varepsilon.$$

Hence,

$$\mathbb{P}\left[(1-\eta)M(1/2;n_i) > \widetilde{M}_{n_i} \text{ or } \widetilde{M}_{n_i} > (1+\eta)M(1/2;n_i)\right] > \varepsilon.$$

Choosing, as above, $\tau = \varepsilon$, we find once again

$$\varepsilon < \mathbb{P}\left[(1-\eta)M(1/2;n_i) > \widetilde{M}_{n_i} \text{ or } \widetilde{M}_{n_i} > (1+\eta)M(1/2;n_i)\right] \leq \varepsilon,$$

for $i \in \mathbb{N}$ such that $n_i > N(\varepsilon)$, and this is a contradiction. Therefore, $M(1/2;n)$ is a sharp threshold for property $\mathcal{A}$. ∎

Another observation about sharp thresholds is the following:

Proposition 1.3.5 *An increasing property $\mathcal{A}$ has a sharp threshold $\widehat{M} = \widehat{M}(n)$ if and only if for any $0 < \varepsilon < 1/2$, we have $\delta_M(\varepsilon) = o(M(1/2; n))$.*

Proof. Assume that $\widehat{M}(n)$ is a sharp threshold for the property $\mathcal{A}$ but for some $0 < \varepsilon < 1/2$ there is a strictly increasing sequence of integers $(n_i)_{i \in \mathbb{N}}$ such that for some $\tau > 0$,

$$\frac{M(1-\varepsilon; n_i) - M(\varepsilon; n_i)}{M(1/2; n_i)} > \tau,$$

for any $i \in \mathbb{N}$. Hence, we obtain

$$\Delta \equiv M(1-\varepsilon; n_i) - M(\varepsilon; n_i) > \tau M(1/2; n_i),$$

for any $i \in \mathbb{N}$. On the other hand, the sharpness of the threshold and Proposition 1.3.2 imply that for any $\eta > 0$ there exists some I such that for $i > I$, we have

$$(1-\eta) M(1/2; n_i) \leq M(\varepsilon; n_i)$$

and

$$M(1-\varepsilon; n_i) \leq (1+\eta) M(1/2; n_i).$$

Therefore, for $i > I$ we have

$$\Delta \leq M(1/2; n_i)(1 + \eta - 1 + \eta) = 2\eta M(1/2; n_i).$$

Choosing $\eta = \tau/2$, we eventually find

$$\tau M(1/2, n_i) > \tau M(1/2, n_i),$$

which gives a contradiction.

Conversely, assume that for any $0 < \varepsilon < 1/2$ and for every $\tau > 0$ there exists N such that for $n > N = N(\tau, \varepsilon)$, we have

$$\frac{M(1-\varepsilon; n) - M(\varepsilon; n)}{M(1/2; n)} < \tau.$$

Consider a fixed τ, which will be specified later. We shall prove that $M(1/2; n)$ is a sharp threshold. Recall that $\mathcal{A}$ is assumed to be monotonically increasing. Suppose that $M(1/2; n)$ is not a sharp threshold. Thus, two things may happen. First, assume that for some $0 < \eta < 1$ there is a strictly increasing sequence of integers $(n_i)_{i \in \mathbb{N}}$ such that for some integer-valued function $M(n)$ we have $M(n_i) \leq (1-\eta) M(1/2; n_i)$ and there is an $0 < \varepsilon < 1/2$ such that $\mathbb{P}\left[\mathcal{G}_{n_i, M(n_i)} \in \mathcal{A}\right] > \varepsilon$, for any $i \in \mathbb{N}$. Thus, $M(n_i) > M(\varepsilon; n_i)$, for any $i \in \mathbb{N}$. Therefore, we have

$$M(1-\varepsilon; n_i) < (1 + \tau - \eta) M(1/2; n_i),$$

for any $i \in \mathbb{N}$ such that $n_i \geq N(\tau, \varepsilon)$. In this case, choosing $\tau = \eta/2$, we obtain

$$M(1-\varepsilon; n_i) < (1-\eta/2)M(1/2; n_i) < M(1/2; n_i),$$

for any $i \in \mathbb{N}$ such that $n_i \geq N(\eta/2, \varepsilon)$. But this is a contradiction since $\varepsilon < 1/2$ and $\mathcal{A}$ is increasing.

To see the second case, assume that for some $\eta > 0$ there is a strictly increasing sequence of integers $(n_i)_{i\in\mathbb{N}}$ such that for some integer-valued function $M(n)$ we have $M(n_i) \geq (1 + \eta)M(1/2; n_i)$ and there is an $0 < \varepsilon < 1/2$ such that $\mathbb{P}\left[\mathcal{G}_{n_i, M(n_i)} \in \mathcal{A}\right] < 1 - \varepsilon$, for any $i \in \mathbb{N}$. The definition of the function $M(1-\varepsilon; n)$ implies that $M(1-\varepsilon; n_i) \geq M(n_i)$, for any $i \in \mathbb{N}$. In this case, we obtain

$$M(1/2; n_i) < (1 + \eta - \tau)M(1/2; n_i) < M(\varepsilon; n_i),$$

choosing some $0 < \tau < \eta$, and for $i \in \mathbb{N}$ such that $n_i \geq N(\tau, \varepsilon)$, which is again a contradiction. ∎

Chapter 2

Upper bounds on the non-3-colourability threshold

2.1 Introduction

In this chapter, we are concerned with the property of 3-colourability of graphs. More specifically, we investigate the property of 3-colourability of sparse random graphs. After giving some definitions, we describe previous related work. We then discuss "jackpot phenomena" and their elimination, and in particular we prove Theorem 2.2.1. This leads to the main result of this chapter, which is presented in Section 2.3 and proved in the succeeding sections.

Let $\mathcal{C}(G)$ denote the set of all proper 3-colourings of G and $C(G)$ its cardinality. If G is the null graph, then we set $\mathcal{C}(G) = \emptyset$ and $C(G) = 0$.

Now, let $\mathcal{G}_{n,m}$ be a random graph in the uniform model, having $m = \lceil \theta n/2 \rceil$ edges, where $\theta > 0$ is fixed. Note that the average degree of this random graph is $\theta + o(1)$. Since a graph is 2-colourable (i.e. bipartite) if and only if it contains no odd cycles, the probability of non-2-colourability is bounded away from 0, for any $\theta > 0$, and keeps increasing with θ, tending to 1 as $n \to \infty$ during the emergence of the giant component at $\theta = 1$. However, for $k > 2$, our understanding of k-colourability is not thorough. Erdös in [14, 5] asked whether for each $k > 2$ there exists a constant θ_k such that for any $\epsilon > 0$, whenever $m = \lceil (\theta_k/2 - \epsilon)n \rceil$ the random graph is k-colourable with probability tending to 1 as $n \to \infty$, and if $m = \lceil (\theta_k/2 + \epsilon)n \rceil$ the random graph is not k-colourable with probability tending to 1 as $n \to \infty$. Note that θ_k would specify a critical average degree of the random graph. In the case of 3-colourability, the experiments in [20] suggest $\theta_3 \simeq 4.6$. We say that θ is an *upper bound* on the non-3-colourability threshold, if for all

$\rho > \theta$ we have

$$\lim_{n\to\infty} \mathbb{P}\left[\chi\left(\mathcal{G}_{n,\lceil \rho n/2\rceil}\right) \le 3\right] = 0.$$

Recently, E. Friedgut in [17] made great progress in our understanding of threshold phenomena in random graphs by giving necessary conditions for a property to have a coarse threshold. Using the main theorem of [17], D. Achlioptas and E. Friedgut proved in [1] that for each fixed $k > 2$, there exists a function $d_k(n)$ such that for any $\epsilon > 0$, if $m \le (1-\epsilon)\left(\frac{d_k(n)}{2}n\right)$ the random graph is k-colourable with probability tending to 1 as $n \to \infty$ and if $m \ge (1+\epsilon)\left(\frac{d_k(n)}{2}n\right)$ the random graph is not k-colourable with probability tending to 1 as $n \to \infty$. A simple application of the first moment method as well as results concerning the lower bound on the non-3-colourability threshold suggest the form of the threshold function and show that the function $d_3(n)$ is asymptotically bounded below and above by non-zero constants. In general, while it is widely believed that $\lim_{n\to\infty} d_k(n)$ exists, confirming this conjecture and determining the limit θ_k seems challenging. Observe that the function $d_k(n)$ is unique in that every other function with the above property is asymptotically equal to $d_k(n)$.

But let us see how this function behaves. We define the sets C_k^- and C_k^+ as follows: $C_k^+ \equiv \{\theta > 0 : \lim_{n\to\infty} \mathbb{P}[\chi(\mathcal{G}_{n,\lceil \frac{\theta n}{2}\rceil}) > 3] = 1\}$ and $C_k^- \equiv \{\theta > 0 : \lim_{n\to\infty} \mathbb{P}[\chi(\mathcal{G}_{n,\lceil \frac{\theta n}{2}\rceil}) > 3] = 0\}$. Clearly, these sets are intervals. Also, let $\theta_k^+ = \inf C_k^+$ and $\theta_k^- = \sup C_k^-$. One can prove that $\limsup_{n\to\infty} d_k(n) = \theta_k^+$ and $\liminf_{n\to\infty} d_k(n) = \theta_k^-$. To see the first, assume that there is some $\theta > 0$ such that $\limsup_{n\to\infty} d_k(n) < \theta < \theta_k^+$. Then there exists some $\eta > 0$ and $N \in \mathbb{N}$ such that $d_k(n)(1+\eta) < \theta$, for $n \ge N$. The definition of the sharp threshold implies that $\lim_{n\to\infty} \mathbb{P}[\chi(\mathcal{G}_{n,\lceil \frac{\theta n}{2}\rceil}) > 3] = 1$ and therefore $\theta \in C_k^+$. Since the choice of the real number θ was arbitrary, we have $\limsup_{n\to\infty} d_k(n) \ge \theta_k^+$. It is easy to see that equality holds. Similarly, assume that there exists some $\theta > 0$ such that $\liminf_{n\to\infty} d_k(n) > \theta > \theta_k^-$. This means that there exists some $\eta > 0$ and $N \in \mathbb{N}$ such that $\theta < d_k(n)(1-\eta)$, for $n \ge N$. Again, the definition of the sharp threshold yields $\lim_{n\to\infty} \mathbb{P}[\chi(\mathcal{G}_{n,\lceil \frac{\theta n}{2}\rceil}) > 3] = 0$ and therefore $\theta \in C_k^-$. Since the choice of the real number θ was arbitrary, we have $\liminf_{n\to\infty} d_k(n) \le \theta_k^-$. In fact, equality holds as well. Note that the limit of $d_k(n)$ exists if and only if $\theta_k^+ = \theta_k^- = \lim_{n\to\infty} d_k(n)$.

The main method to establish k-colourability for small values of $k > 2$ has been based on the elementary fact that if a graph has no induced subgraph with minimum degree at least k, then it is k-colourable. First Łuczak in [28] proved that with high probability when $m = \lceil \theta n/2\rceil$, for $\theta < 1.0001$, a random graph remains 3-colourable after the emergence of the giant component by showing that asymptotically almost surely the random graph has no subgraph of minimum degree at least 3. Shortly afterwards, V. Chvátal in [10] improved this greatly by showing that asymptotically almost surely a random graph with $m = \lceil \theta n/2 \rceil$ edges has no subgraph with minimum degree at least 3 for $\theta \leq 2.88$, thus implying that $\theta_3^- \geq 2.88$. B. Reed and M. Molloy in [35] improved the bound even further to $\theta_3^- \geq 3.34$. Finally, B. Pittel, J.H. Spencer and N.C. Wormald proved in [37] that, in fact, for all $k > 2$ there exists γ_k such that for $\theta < \gamma_k$, a random graph with $m = \lceil \theta n/2 \rceil$ edges has no subgraph with minimum degree at least k asymptotically almost surely, while for $\theta > \gamma_k$ there exists such a subgraph asymptotically almost surely (see Chapter 4 later). Moreover, they determined γ_k for all such k. For instance, $\gamma_3 = 3.350\ldots$. M. Molloy has proved in [33] that $\theta_k^- > \gamma_k$ for all $k \geq 4$ and conjectured $\theta_3^- \neq \gamma_3$ as well. This conjecture was verified recently by D. Achlioptas and M. Molloy in [2] after analysing the performance of a greedy "list-colouring" heuristic on random graphs. This argument gave $\theta_3^- \geq 3.846$. Most recently, D. Achlioptas and C. Moore in [4] proved that $\theta_3^- \geq 4.03$. For a review on the previous results as well as the following ones see [34].

The upper bounds on θ_3^+ are obtained using various applications of the first moment method. There are 3^n partitions of n vertices into 3 sets and for any such partition at least 1/3 of the pairs of vertices are contained in the same set. This immediately implies that $\mathbb{E}[C(\mathcal{G}_{n,m})] \leq 3^n\,(2/3)^m$, which yields $\theta_3^+ \leq 5.41$ via Markov's inequality. As we shall see in the next section, this upper bound on $\mathbb{E}[C(\mathcal{G}_{n,m})]$ is within a constant factor of the exact value for $m = O(n)$. However, the resulting bound on θ_3^+ can be improved significantly. In doing so, researchers have adopted the following paradigm. For a graph G let $\mathcal{C}'(G) \subseteq \mathcal{C}(G)$ satisfy $\mathcal{C}'(G) \neq \emptyset$ whenever $\mathcal{C}(G) \neq \emptyset$. We call such a family of colourings *adequate*. Let us denote $|\mathcal{C}'(G)|$ by $C'(G)$. By Markov's inequality,

$$\mathbb{P}(\chi(\mathcal{G}_{n,m}) \leq 3) \leq \mathbb{E}[C'(\mathcal{G}_{n,m})]. \tag{2.1}$$

Of course equality holds here if $C'(G) \leq 1$ always. The aim is to find small adequate families which are simple enough to handle. We may also use an extension of the Markov inequality - see (2.3) below.

2.2 Jackpot phenomena and their elimination

The term *"jackpot phenomena"* appears in M. Molloy's survey paper (see [34]) and is quite illuminating, in so far as our attempts and the methodology are concerned. Quoting M. Molloy, let us give an idea of what these phenomena really are: "Consider a lottery with a jackpot of 4^n pounds. The probability of winning the jackpot is 2^{-n}, and there are no other prizes. Your expected winnings are 2^n, but with very high probability, your winnings will be 0". Now, let us see how this applies in our setting. It may be the case that the expected number of proper 3-colourings is exponentially high, but almost surely there are no proper 3-colourings. And the reason for this behaviour: if there is at least one proper 3-colouring, then there will probably be exponentially many proper 3-colourings.

We shall try to identify a source of jackpot phenomena that occur in the 3-colourability of sparse random graphs. In particular, if a $\mathcal{G}_{n,m}$ random graph with $m = \lceil \theta n/2 \rceil$ and $\theta > 1$ is 3-colourable, then typically an exponential number of proper 3-colourings are induced from its tree-components as well as from the forest that is attached to its core (see below for the precise definitions), since a fixed proportion of vertices belong to them asymptotically. Taking these into account we prove the following:

Theorem 2.2.1 *For any $\theta \geq 5.299$ and for $m = \lceil \theta n/2 \rceil$, there exists a $\delta > 0$ such that for n sufficiently large*

$$\mathbb{P}\left[\chi\left(\mathcal{G}_{n,m}\right) \leq 3\right] \leq 2^{-\delta n}.$$

Thus, $\theta_3^+ \leq 5.299$. Note that this improves on the upper bound that is obtained by a straight application of the first moment method which gives $\theta_3^+ \leq 5.41$, as we discussed above. Using a similar but stronger argument M. Molloy and B. Reed proved that $\theta_3^+ \leq 5.142$ (see [32]) - they proved that for $\theta > 5.142$ the core of a $\mathcal{G}_{n,m}$ random graph is not 3-colourable a.a.s..

We proceed with the proof of Theorem 2.2.1. For a graph $G = (V_n, E)$, let $T = T(G)$ be the number of tree components, $T_k = T_k(G)$ be the number of tree components of order $k \geq 1$ and finally $X = X(G)$ and $X_k = X_k(G)$ be the number of vertices that belong to tree-components and to tree-components of order k, respectively. Also, let $complex(G)$ be the subgraph of G that is the union of those components with at least one circuit. Note that for any graph G this is unique. We set $\mathrm{cr}(G)$ to be the maximal subgraph of minimum degree at least 2 - this is called the *core* of G. Observe that $\mathrm{cr}(G)$ is a subgraph of $complex(G)$ and, if it exists, it is unique. If the graph G has no such subgraph, then $\mathrm{cr}(G)$ and $complex(G)$ are isomorphic to the null graph. Now, a tree component of order k has exactly $3 \cdot 2^{k-1}$ proper 3-colourings. Thus, we obtain the following:

Lemma 2.2.2 *For any natural numbers s, t, x, n and m, where $0 < m \leq \binom{n}{2}$, we have*

$$\begin{aligned}\mathbb{P}[\chi(\mathcal{G}_{n,m}) \leq 3] \quad &\leq \quad 2^{-s}\left(\frac{2}{3}\right)^t 2^{-x}\mathbb{E}[C(\mathcal{G}_{n,m})] + \mathbb{P}[T(\mathcal{G}_{n,m}) < t] + \mathbb{P}[X(\mathcal{G}_{n,m}) < x] + \\ &\mathbb{P}[||V(complex(\mathcal{G}_{n,m}))| - |V(\mathrm{cr}(\mathcal{G}_{n,m}))|| < s].\end{aligned}$$

Proof. Since for any graph G the set of vertices $V(complex(G)) \setminus V(\mathrm{cr}(G))$ induces a forest and every tree of this forest is "attached" by exactly one edge to $\mathrm{cr}(G)$, we deduce that every proper 3-colouring of $\mathrm{cr}(G)$ induces $2^{|V(complex(G))|-|V(\mathrm{cr}(G))|}$ proper 3-colourings of this forest. This observation along with the number of proper 3-colourings of the tree-components and the fact that whenever $\mathrm{cr}(G)$ is non-empty then $C(\mathrm{cr}(G)) \neq 0$ if and only if $C(G) \neq 0$ yield the following:

$$\begin{aligned} C(G) &\geq C(\mathrm{cr}(G))\, 2^{|V(complex(G))|-|V(\mathrm{cr}(G))|} \prod_{k=1}^{n} \left(3 \cdot 2^{k-1}\right)^{T_k} \\ &= C(\mathrm{cr}(G))\, 2^{|V(complex(G))|-|V(\mathrm{cr}(G))|} \left(\frac{3}{2}\right)^{T} 2^{\sum_{k=1}^{n} kT_k} \\ &= C(\mathrm{cr}(G))\, 2^{|V(complex(G))|-|V(\mathrm{cr}(G))|} \left(\frac{3}{2}\right)^{T} 2^{X}. \end{aligned}$$

In particular, if $T(G) \geq t$ and $X(G) \geq x$, then

$$C(\mathrm{cr}(G)) \leq C(G)\, 2^{-(|V(complex(G))|-|V(\mathrm{cr}(G))|)} \left(\frac{2}{3}\right)^{t} 2^{-x}. \tag{2.2}$$

Now, we use a form of Markov's inequality. Let V be a non-negative random variable, and let A be an event on the probability space where V is defined . Then

$$\mathbf{1}_{\{V \geq 1\}} \leq V \mathbf{1}_A + \mathbf{1}_{\overline{A}}$$

and so

$$\mathbb{P}[V \geq 1] \leq \mathbb{E}[V; A] + \mathbb{P}[\overline{A}]. \tag{2.3}$$

Let A be the event "$\{T(\mathcal{G}_{n,m}) \geq t\} \cap \{X(\mathcal{G}_{n,m}) \geq x\} \cap \{||V(complex(\mathcal{G}_{n,m}))| - |V(\mathrm{cr}(\mathcal{G}_{n,m}))|| \geq s\}$".

Then, by (2.2) and (2.3) we obtain:

$$\begin{aligned} \mathbb{P}[\chi(\mathcal{G}_{n,m}) \leq 3] &= \mathbb{E}[\mathbf{1}_{\{\chi(\mathrm{cr}(\mathcal{G}_{n,m})) \leq 3\}}] = \mathbb{E}[\mathbf{1}_{\{C(\mathrm{cr}(\mathcal{G}_{n,m})) \geq 1\}}] \\ &\leq \mathbb{E}[C(\mathrm{cr}(\mathcal{G}_{n,m}))\mathbf{1}_A] + \mathbb{P}[\overline{A}] \\ &\leq 2^{-s} \left(\frac{2}{3}\right)^{t} 2^{-x}\mathbb{E}[C(\mathcal{G}_{n,m})] + \mathbb{P}[\overline{A}] \\ &\leq 2^{-s} \left(\frac{2}{3}\right)^{t} 2^{-x}\mathbb{E}[C(\mathcal{G}_{n,m})] + \mathbb{P}[T(\mathcal{G}_{n,m}) < t] + \mathbb{P}[X(\mathcal{G}_{n,m}) < x] \\ &\quad + \mathbb{P}[||V(complex(\mathcal{G}_{n,m}))| - |V(\mathrm{cr}(\mathcal{G}_{n,m}))|| < s], \end{aligned}$$

and this conclude the proof of the lemma. ∎

Before giving an asymptotic expression for $\mathbb{E}[C(\mathcal{G}_{n,m})]$, i.e. the expected number of proper 3-colourings of the random graph $\mathcal{G}_{n,m}$, we present some definitions, which will be used in this section as well as in the following ones.

If S is a 3-colouring of a graph $G = (V_n, E)$, then we let S_1, S_2 and S_3 be the induced colour classes and s_1, s_2, s_3 be their cardinalities. We define $\alpha = \alpha(n)$, $\beta = \beta(n)$ and $\gamma = \gamma(n)$ to be such that $s_1 = \alpha n$, $s_2 = \beta n$ and $s_3 = \gamma n$ respectively, where α, β, $\gamma \geq 0$ and $\alpha + \beta + \gamma = 1$. If we set $k = k(n) = \alpha(n) + \beta(n)$, then $e(S)$, the number of possible edges between the 3 colour classes, can be expressed as follows:

$$\begin{aligned} e(S) &= n^2 (\alpha\beta + \beta\gamma + \gamma\alpha) \\ &= n^2 \left(k(1-\alpha) - (k-\alpha)^2\right) = n^2 \phi(k,\alpha). \end{aligned} \tag{2.4}$$

where we define $\phi(k, \alpha) = k(1 - \alpha) - (k - \alpha)^2$.

We set

$$\mathrm{D} = \{(k,\alpha) :\ 0 \leq k \leq 1,\ 0 \leq \alpha \leq k\},$$

and let D^o be its interior and $\mathrm{P} = \{(0,0),(1,0),(1,1)\}$. For each positive integer n let

$$\mathrm{D}^{(n)} = \left\{\mathrm{D} \cap \frac{1}{n}\mathbb{Z}^2\right\} \setminus \mathrm{P}.$$

Thus, for each positive integer n and each $(k, \alpha) \in \mathrm{D}^{(n)}$, let $\mathcal{C}(k, \alpha, n)$ denote the set of all partitions of V_n into three sets S_1, S_2, S_3, where $s_1 = |S_1| = \alpha n$, $s_2 = |S_2| = \beta n$, $s_3 = |S_3| = \gamma n$, $\beta = k - \alpha$ and $\gamma = 1 - k$.

Similarly, we define

$$\mathrm{D}_1(\theta) = \{(k,\alpha) :\ 0 < C_1(\theta) \leq \alpha \leq k \leq C_2(\theta) < 1\},$$

for some $C_1(\theta)$ and $C_2(\theta)$ which will be specified later and for each positive integer n

$$\mathrm{D}_1^{(n)}(\theta) = \mathrm{D}_1(\theta) \cap \frac{1}{n}\mathbb{Z}^2.$$

For any fixed $\theta > 0$ and $(k, \alpha) \in \mathrm{D} \setminus \mathrm{P}$ let

$$h(k,\alpha,\theta) = H(k) + kH\left(\frac{\alpha}{k}\right) + \frac{\theta}{2}\ \log_2(2\phi(k,\alpha)),$$

where $H(\cdot)$ is the entropy function, and let $\psi(\theta) = \sup_{(k,\alpha)\in \mathrm{D}\setminus \mathrm{P}} h(k, \alpha, \theta)$. We state without proof the following lemmas will be used in the proof of Lemma 2.2.5 as well as in the next sections. This can be proved by elementary methods:

Lemma 2.2.3 *The function $\phi(k,\alpha)$ is continuous and strictly concave on D and for each $(k,\alpha) \in \mathrm{D}$ we have $0 \leq \phi(k,\alpha) \leq 1/3$, with the maximum attained at $(k,\alpha) = (2/3, 1/3)$. Moreover, $\phi(k,\alpha) = 0$ if and only if $(k,\alpha) \in \mathrm{P}$.*

Also, the following standard lemma on approximating binomial coefficients may be proved using Stirling's formula (see [41]):

Lemma 2.2.4 *Let $0 < c < C$. Then uniformly over $cn \leq r \leq u \leq Cn$ we have*

$$\binom{u}{r} = \Theta(n^{-1/2}) \left(\frac{u}{r}\right)^r \left(\frac{u}{u-r}\right)^{u-r} = \Theta(n^{-1/2})\ 2^{uH(\mu)},$$

where $\mu = r/u$ and $H(x) = -x\log_2 x - (1-x)\log_2(1-x)$ for $0 \leq x \leq 1$ (when $x = 0$ or $x = 1$, then $H(x) = 0$, identically) is the entropy function. Moreover, $\binom{u}{r} \leq 2^{uH(\mu)}$ always.

We observe that for any fixed $\theta > 0$, the function $h(k,\alpha,\theta)$ is concave in $\mathrm{D}\backslash\mathrm{P}$, since the entropy functions involved there are concave over D and $\log_2$ is an increasing and concave function. The supremum of $h(k,\alpha,\theta)$ is attained at $k = 2/3$ and $\alpha = 1/3$. Thus, choosing $C_1(\theta) < 1/3$ and $C_2(\theta) > 2/3$, we have $\sup_{(k,\alpha)\in\mathrm{D}\backslash\mathrm{P}} h(k,\alpha,\theta) = \max_{(k,\alpha)\in\mathrm{D}_1(\theta)} h(k,\alpha,\theta)$. Temporarily, we will be working in the $\mathcal{G}^*_{n,m}$ model of random graphs and we show the following:

Lemma 2.2.5 *For any fixed $\theta > 0$, we have*

$$\mathbb{E}[C(\mathcal{G}^*_{n,m})] = 2^{\psi(\theta)n + O(\log_2 n)},$$

where $m = \lceil \theta n/2 \rceil$.

Proof. Note that for any fixed $\theta > 0$ we have

$$\begin{aligned}
\mathbb{E}[C(\mathcal{G}^*_{n,m})] &= \sum_{(k,\alpha)\in\mathrm{D}^{(n)}} \binom{n}{nk}\binom{nk}{\alpha n}\left(\frac{e(S)}{\binom{n}{2}}\right)^m \\
&= \sum_{(k,\alpha)\in\mathrm{D}_1^{(n)}(\theta)} \binom{n}{nk}\binom{nk}{\alpha n}\left(\frac{e(S)}{\binom{n}{2}}\right)^m + \sum_{(k,\alpha)\in\mathrm{D}^{(n)}\backslash\mathrm{D}_1^{(n)}(\theta)} \binom{n}{nk}\binom{nk}{\alpha n}\left(\frac{e(S)}{\binom{n}{2}}\right)^m \\
&= \Theta(n^{-1}) \sum_{(k,\alpha)\in\mathrm{D}_1^{(n)}(\theta)} 2^{nH(k)+nkH(\frac{\alpha}{k})}(2\phi(k,\alpha))^m\left(1-\frac{1}{n}\right)^{-m} \\
&\quad +O(n^2)\ 2^{n\sup_{(k,\alpha)\in\mathrm{D}^{(n)}\backslash\mathrm{D}_1^{(n)}(\theta)} h(k,\alpha,\theta)} \\
&= \Theta(n^{-1}) \sum_{(k,\alpha)\in\mathrm{D}^{(n)}(\theta)} \left(2^{H(k)+kH(\frac{\alpha}{k})}(2\phi(k,\alpha))^{\frac{\theta}{2}}\right)^n + O(1)\ 2^{\delta n},
\end{aligned}$$

where $\sup_{(k,\alpha)\in D^{(n)}\setminus D_1^{(n)}(\theta)} h(k,\alpha,\theta) < \delta < \psi(\theta)$ and note that such a constant exists if we choose $C_1(\theta)$, $C_2(\theta)$ sufficiently close to 0 and 1 respectively, (and definitely $C_1(\theta) < 1/3$ and $C_2(\theta) > 2/3$), by Lemma 2.2.3. Thus,

$$\mathbb{E}[C(\mathcal{G}^*_{n,m})] = O(n^2) \max_{(k,\alpha)\in D_1^{(n)}(\theta)} \left\{2^{nh(k,\alpha,\theta)}\right\} + O(2^{\delta n}) = O(n^2)\ 2^{n\psi(\theta,n)} + O(2^{\delta n}),$$

where $\psi(\theta,n) = \max_{(k,\alpha)\in D_1^{(n)}(\theta)} h(k,\alpha,\theta)$ and the multiplicative factor is $\Omega(n^{-1})$ as well.

As we have already mentioned, by the choice of $C_1(\theta)$ and $C_2(\theta)$, the function $h(k,\alpha,\theta)$ attains its maximum at an internal point of $D_1(\theta)$, namely at $\mathbf{x}^* = (2/3, 1/3)$. This is a stationary point and one can also see that h is differentiable on D^o and its derivatives are continuous. The latter implies that for any $\varepsilon > 0$ there exists an open ball U containing $\mathbf{x}^*$ where $\|\nabla h\| < \varepsilon$. For each $n \geq 1$, let $\mathbf{x}_n \in D_1^{(n)}(\theta)$ be such that $\|\mathbf{x}_n - \mathbf{x}^*\| = \min\{\|\mathbf{x} - \mathbf{x}^*\| \ :\ \mathbf{x} \in D_1^{(n)}(\theta)\}$. For n sufficiently large, we have $\mathbf{x}_n \in D_1^{(n)}(\theta) \cap U$. Moreover,

$$\|\mathbf{x}_n - \mathbf{x}^*\| \leq \left(2\ \frac{1}{n^2}\right)^{1/2} = \frac{\sqrt{2}}{n}.$$

By the Mean Value Theorem, we have

$$\psi(\theta) - h(\mathbf{x}_n,\theta) \leq \sqrt{2}\varepsilon n^{-1}.$$

Hence,

$$n(\psi(\theta) - \psi(\theta,n)) = O(1).$$

Therefore,

$$\mathbb{E}[C(\mathcal{G}^*_{n,m})] = 2^{\psi(\theta)n+O(\log_2 n)} + O(2^{\delta n}) = 2^{\psi(\theta)n+O(\log_2 n)},$$

which was to be proved. ∎

Now, we give the following lemma which actually shows that the expected numbers of proper 3-colourings in the $\mathcal{G}^*_{n,m}$ and the $\mathcal{G}_{n,m}$ models, respectively, are essentially the same.

Lemma 2.2.6 *For any $\theta > 0$, we have*

$$\mathbb{E}[C(\mathcal{G}_{n,m})] = \Theta(1)\ \mathbb{E}[C(\mathcal{G}^*_{n,m})], \tag{2.5}$$

where $m = \lceil \theta n/2 \rceil$.

Proof. To see the one direction note that

$$\mathbb{E}[C(\mathcal{G}^*_{n,m})] \geq \mathbb{E}[C(\mathcal{G}_{n,m})],$$

by the monotonicity of the property of 3-colourability.

We have $e(S) = n^2\phi(k,\alpha) \geq \tau n^2$, where $\tau = \inf_{(k,\alpha)\in \mathrm{D}_1(\theta)} \phi(k,\alpha)$, for any colouring $S \in \mathcal{C}(k,\alpha,n)$, such that $(k,\alpha) \in \mathrm{D}_1^{(n)}(\theta)$, whence we obtain $e(S)/m^2 \geq \eta > 0$, for some η and for n sufficiently large. Thus,

$$\begin{aligned} \left(1 - \frac{m}{e(S)}\right)^m &= \exp\left(m \ln\left(1 - \frac{m}{e(S)}\right)\right) \\ &\geq \exp(-2m^2/e(S)) \geq e^{-2/\eta}. \end{aligned} \tag{2.6}$$

Hence, for any such S, we have

$$\begin{aligned} \mathbb{P}[S \in \mathcal{C}(\mathcal{G}_{n,m})] &= \frac{e(S)}{\binom{n}{2}} \frac{e(S)-1}{\binom{n}{2}-1} \cdots \frac{e(S)-m+1}{\binom{n}{2}-m+1} \\ &\geq \left(\left(\frac{e(S)}{\binom{n}{2}}\right)\left(1 - \frac{m}{e(S)}\right)\right)^m \\ &\geq e^{-2/\eta}\, \mathbb{P}[S \in \mathcal{C}(\mathcal{G}^*_{n,m})]. \end{aligned}$$

Therefore,

$$\mathbb{P}[S \in \mathcal{C}(\mathcal{G}^*_{n,m})] \leq e^{2/\eta}\, \mathbb{P}[S \in \mathcal{C}(\mathcal{G}_{n,m})],$$

for any colouring $S \in \mathcal{C}(k,\alpha,n)$, such that $(k,\alpha) \in \mathrm{D}_1^{(n)}(\theta)$. Finally, as we have seen in the proof of the previous lemma, we can choose $C_1(\theta)$, $C_2(\theta)$ sufficiently close to 0 and 1 respectively, so that there exists $\delta > 0$ such that $\sup_{(k,\alpha)\in \mathrm{D}^{(n)} \setminus \mathrm{D}_1^{(n)}(\theta)} h(k,\alpha,\theta) < \delta < \psi(\theta)$. Thus, the above inequality yields:

$$\begin{aligned} \mathbb{E}[C(\mathcal{G}^*_{n,m})] &= (1+o(1)) \sum_{(k,\alpha)\in \mathrm{D}_1^{(n)}(\theta)} \sum_{S\in \mathcal{C}(k,\alpha,n)} \mathbb{P}[S \in \mathcal{C}(\mathcal{G}^*_{n,m})] \\ &\leq (1+o(1))e^{2/\eta} \sum_{(k,\alpha)\in \mathrm{D}_1^{(n)}(\theta)} \sum_{S\in \mathcal{C}(k,\alpha,n)} \mathbb{P}[S \in \mathcal{C}(\mathcal{G}_{n,m})] \\ &\leq C\, \mathbb{E}[C(\mathcal{G}_{n,m})], \end{aligned}$$

which concludes the proof of the lemma. ∎

Combining the two previous lemmas and recalling that for a graph G on a non-empty set of vertices the symbol $C(G)$ denotes the total number of proper 3-colourings, we obtain:

Theorem 2.2.7 *For any $\theta > 0$, we have*

$$\mathbb{E}[C(\mathcal{G}_{n,m})] = 2^{\psi(\theta)n + O(\log_2 n)},$$

where $m = \lceil \theta n/2 \rceil$ and $\psi(\theta)$ is the supremum of the function $h(k,\alpha,\theta)$ over the domain $\mathrm{D} \setminus \mathrm{P}$.

It is easy to see that $\psi(\theta) < 0$, for any $\theta > 5.41$ (actually $\psi(\theta)$ decreases as θ increases), and the following is immediate:

Theorem 2.2.8 *For any $\theta \geq 5.41$ and for $m = \lceil \theta n/2 \rceil$, there exists a $\delta > 0$ such that for n sufficiently large*

$$\mathbb{P}\left[\chi\left(\mathcal{G}_{n,m}\right) \leq 3\right] \leq 2^{-\delta n}.$$

This implies that $\theta_3^+ \leq 5.41$ and note that this is weaker than Theorem 2.2.1.

We shall give estimates for the random variables T and X. Let us begin with the expectation of $T = T\left(\mathcal{G}_{n,m}\right)$.

Lemma 2.2.9 *For any fixed $1 < \theta < \infty$ and $m = \lceil \theta n/2 \rceil$, we have*

$$\mathbb{E}\left[\sum_{k=\lfloor n^{1/3} \rfloor+1} T_k(\mathcal{G}_{n,m})\right] = o(n^{-5/2}),$$

which implies that in a $\mathcal{G}_{n,m}$ random graph asymptotically almost surely there are no tree components of order greater that $n^{1/3}$.

Proof. Using Pittel's inequality (see [23] page 17 or [6]) which relates probabilities of an event in the models $\mathcal{G}_{n,m}$ and $\mathcal{G}_{n,p}$, where $p = m/\binom{n}{2}$, it can be shown that

$$\sum_{k=\lfloor n^{1/3} \rfloor+1}^{n} \mathbb{E}\left[T_k\left(\mathcal{G}_{n,m}\right)\right] \leq 3\sqrt{m} \sum_{k=\lfloor n^{1/3} \rfloor+1}^{n} \mathbb{E}\left[T_k\left(\mathcal{G}_{n,p}\right)\right] = o(n^{-5/2}),$$

where $p = m/\binom{n}{2}$ (since the above expectation in the $\mathcal{G}_{n,p}$ model is $o(n^{-3})$ - see [6] equation (5.14) page 105). ∎

We shall use the above lemma to deduce the following:

Proposition 2.2.10 *For any fixed $1 < \theta < \infty$ and for $m = \lceil \theta n/2 \rceil$, we have*

$$\mathbb{E}\left[T\left(\mathcal{G}_{n,m}\right)\right] = nc_1(\theta) + O(1),$$

where $c_1(\theta) = \frac{1}{\theta} \sum_{k=1}^{\infty} \frac{k^{k-2}}{k!} \left(\theta e^{-\theta}\right)^k$.

Proof. First observe that the previous lemma implies that

$$\begin{aligned} \mathbb{E}\left[T\left(\mathcal{G}_{n,m}\right)\right] &= \sum_{k=1}^{n} \mathbb{E}\left[T_k\left(\mathcal{G}_{n,m}\right)\right] = \sum_{k=1}^{\lfloor n^{1/3} \rfloor} \mathbb{E}\left[T_k\left(\mathcal{G}_{n,m}\right)\right] + \sum_{k=\lfloor n^{1/3} \rfloor+1}^{n} \mathbb{E}\left[T_k\left(\mathcal{G}_{n,m}\right)\right] \\ &= \sum_{k=1}^{\lfloor n^{1/3} \rfloor} \mathbb{E}\left[T_k\left(\mathcal{G}_{n,m}\right)\right] + o(n^{-5/2}). \end{aligned}$$

Now, using standard estimates, we can see that there exists a constant $C > 0$ such that for any $1 \le k \le \lfloor n^{1/3} \rfloor$, we have

$$\begin{aligned} \mathbb{E}\left[T_k\left(\mathcal{G}_{n,m}\right)\right] &= \binom{n}{k} k^{k-2} \frac{\binom{\binom{n-k}{2}}{m-k+1}}{\binom{\binom{n}{2}}{m}} \\ &= n^k \frac{k^{k-2}}{k!} \left(\frac{\theta}{n}\right)^{k-1} \exp(-k\theta) \left(1 \pm C\frac{k^2}{n}\right) \end{aligned}$$

Hence,

$$\begin{aligned} &\left|\mathbb{E}[T(\mathcal{G}_{n,m})] - n\sum_{k=1}^{\infty} \frac{k^{k-2}}{k!}\theta^{k-1}e^{-k\theta}\right| \le \\ &\quad \left|\sum_{k=1}^{\lfloor n^{1/3}\rfloor} \mathbb{E}[T_k(\mathcal{G}_{n,m})] - n\sum_{k=1}^{\infty}\frac{k^{k-2}}{k!}\theta^{k-1}e^{-k\theta}\right| + o(n^{-5/2}) \\ &\le \left| n\sum_{k=1}^{\lfloor n^{1/3}\rfloor} \frac{k^{k-2}}{k!}\ \theta^{k-1}\ \exp(-k\theta) - n\sum_{k=1}^{\infty}\frac{k^{k-2}}{k!}\theta^{k-1}e^{-k\theta}\right| + C\sum_{k=1}^{\lfloor n^{1/3}\rfloor}\frac{k^k}{k!}\ \theta^{k-1}e^{-k\theta} + o(n^{-5/2}) \\ &\le n\sum_{\lfloor n^{1/3}\rfloor+1}^{\infty} \frac{k^{k-2}}{k!}\theta^{k-1}e^{-k\theta} + C\sum_{k=1}^{\lfloor n^{1/3}\rfloor} (\theta e^{-(\theta-1)})^k + o(n^{-5/2}) \\ &\le n\sum_{\lfloor n^{1/3}\rfloor+1}^{\infty} (\theta e^{-(\theta-1)})^k + C\sum_{k=1}^{\infty}(\theta e^{-(\theta-1)})^k + o(n^{-5/2}) = O(1). \end{aligned}$$

∎

Now, we state without proof a lemma which will be used in the proof of Lemma 2.2.12 as well as in other lemmas. This is a special case of Theorem 7.4 of [30], see also Example 7.3 there:

Lemma 2.2.11 *Let f be a function on simple graphs on V_n such that, if G' is obtained from G by adding an edge and deleting an edge, then $|f(G) - f(G')| \le c$. Let $\mu = \mathbb{E}[f(\mathcal{G}_{n,m})]$. Then for any $t \ge 0$*

$$\mathbb{P}[f(\mathcal{G}_{n,m}) - \mu \ge t] \le \exp\left(-2t^2/mc^2\right)$$

and

$$\mathbb{P}[f(\mathcal{G}_{n,m}) - \mu \le -t] \le \exp\left(-2t^2/mc^2\right).$$

Now, we can prove the following:

Lemma 2.2.12 *For any fixed $\theta > 1$ and $m = \lceil \theta n/2 \rceil$ we have*

$$\mathbb{P}\left[T\left(\mathcal{G}_{n,m}\right) \le \mathbb{E}\left[T\left(\mathcal{G}_{n,m}\right)\right] - n^{\frac{1}{2}} \ln n\right] = o(1).$$

Proof. Let G and G' be two simple graphs on V_n such that G' is obtained from G by adding an edge and deleting an edge. Then

$$\left|T(G) - T(G')\right| \leq 1,$$

because in this case the difference in the number of tree-components is at most one. Hence,

$$\begin{aligned}\mathbb{P}\left[T\left(\mathcal{G}_{n,m}\right) \leq \mathbb{E}\left[T\left(\mathcal{G}_{n,m}\right)\right] - n^{\frac{1}{2}}\ln n\right] &\leq \exp\left(-\frac{2\left(n^{\frac{1}{2}}\ln n\right)^2}{m}\right) \\ &= \exp\left(-\Theta\left(\ln^2 n\right)\right) = o(1),\end{aligned}$$

by Lemma 2.2.11. ∎

Now, we investigate the random variable $X = X(\mathcal{G}_{n,m})$, that is the number of vertices that belong to tree-components. Let $\mathcal{Q}(\mathcal{G}_{n,m})$ be the event "there exists a tree-component of order greater than $n^{1/3}$". The following is now immediate from Lemma 2.2.9:

Corollary 2.2.13 *For any fixed $\theta > 1$ and $m = \lceil \theta n/2 \rceil$, we have*

$$\mathbb{P}[\mathcal{Q}(\mathcal{G}_{n,m})] = o(1).$$

Let $Y = Y(\mathcal{G}_{n,m})$ be the number of vertices that belong to trees of order at most $n^{1/3}$. Lemma 2.2.9 also yields

Corollary 2.2.14 *For any fixed $\theta > 1$ and $m = \lceil \theta n/2 \rceil$, we have*

$$\mathbb{E}[X(\mathcal{G}_{n,m})] = \mathbb{E}[Y(\mathcal{G}_{n,m})] + o(1).$$

As we shall use it, we prove the following proposition:

Proposition 2.2.15 *For any fixed $\theta > 1$ and for $m = \lceil \theta n/2 \rceil$, we have*

$$\mathbb{E}\left[Y\left(\mathcal{G}_{n,m}\right)\right] = nc_2(\theta) + O(1),$$

where $c_2(\theta) = \frac{1}{\theta}\sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} \left(\theta e^{-\theta}\right)^k$.

Proof. Again, using standard estimates, we can see that there exists a constant $C > 0$ such that for any $k \in \mathbb{N}$ with $1 \leq k \leq n^{1/3}$, we have

$$\begin{aligned}\mathbb{E}\left[kT_k\left(\mathcal{G}_{n,m}\right)\right] &= \binom{n}{k} k^{k-1} \frac{\binom{\binom{n-k}{2}}{m-k+1}}{\binom{\binom{n}{2}}{m}} \\ &= n^k \frac{k^{k-1}}{k!} \left(\frac{\theta}{n}\right)^{k-1} \exp(-k\theta) \left(1 \pm C\frac{k^2}{n}\right) \\ &= n\frac{k^{k-1}}{k!} \theta^{k-1} \exp(-k\theta) \left(1 \pm C\frac{k^2}{n}\right).\end{aligned}$$

Therefore, we obtain

$$\left|\mathbb{E}[\sum_{k=1}^{\lfloor n^{1/3}\rfloor} kT_k(\mathcal{G}_{n,m})] - n\sum_{k=1}^{\lfloor n^{1/3}\rfloor} \frac{k^{k-1}}{k!}\ \theta^{k-1}\ \exp(-k\theta)\right| \leq C\sum_{k=1}^{\lfloor n^{1/3}\rfloor} \frac{k^{k+1}}{k!}\ \theta^{k-1}\ \exp(-k\theta) = O(1).$$

∎

Now, we apply the bounded differences lemma to obtain the following:

Lemma 2.2.16 *For any fixed $\theta > 1$ and $m = \lceil \theta n/2 \rceil$, we have*

$$\mathbb{P}\left[Y\left(\mathcal{G}_{n,m}\right) \leq \mathbb{E}\left[Y\left(\mathcal{G}_{n,m}\right)\right] - n^{\frac{5}{6}}\ln n\right] = o(1).$$

Proof. Once again, we are using the standard machine. If G and G' are two simple graphs on V_n such that G' is obtained from G by adding an edge and deleting an edge, then

$$\left|Y(G) - Y(G')\right| \leq 2n^{\frac{1}{3}},$$

because in this case the difference in the number of tree-components of order at most $n^{1/3}$ is at most two. Hence, for n sufficiently large

$$\begin{aligned}\mathbb{P}\left[Y\left(\mathcal{G}_{n,m}\right) \leq \mathbb{E}\left[Y\left(\mathcal{G}_{n,m}\right)\right] - n^{\frac{5}{6}}\ln n\right] &\leq \exp\left(-\frac{2\left(n^{\frac{5}{6}}\ln n\right)^2}{4m\left(n^{2/3}\right)}\right)\\ &= \exp\left(-\Theta\left(\ln^2 n\right)\right) = o(1),\end{aligned}$$

by Lemma 2.2.11. ∎

Now, we can see how the random variables X and Y are related. For any function $\omega(n)$ which tends to infinity sufficiently slowly, we have:

$$\begin{aligned}&\mathbb{P}\left[X\left(\mathcal{G}_{n,m}\right) \leq \mathbb{E}\left[X\left(\mathcal{G}_{n,m}\right)\right] - \omega(n)\right]\\ &\quad< \mathbb{P}\left[X\left(\mathcal{G}_{n,m}\right) \leq \mathbb{E}\left[X\left(\mathcal{G}_{n,m}\right)\right] - \omega(n) \mid \overline{\mathcal{Q}(\mathcal{G}_{n,m})}\right] + \mathbb{P}\left[\mathcal{Q}(\mathcal{G}_{n,m})\right]\\ &\quad= \mathbb{P}\left[X\left(\mathcal{G}_{n,m}\right) \leq \mathbb{E}\left[X\left(\mathcal{G}_{n,m}\right)\right] - \omega(n) \mid \overline{\mathcal{Q}(\mathcal{G}_{n,m})}\right] + o(1).\end{aligned}$$

Let E_X be the event "$X\left(\mathcal{G}_{n,m}\right) \leq \mathbb{E}\left[X\left(\mathcal{G}_{n,m}\right)\right] - n^{5/6}\ln n$" and E_Y be the event "$Y\left(\mathcal{G}_{n,m}\right) \leq \mathbb{E}\left[X\left(\mathcal{G}_{n,m}\right)\right] - n^{5/6}\ln n$". Then

$$\begin{aligned}\mathbb{E}[\mathbf{1}_{E_X}\mathbf{1}_{\overline{\mathcal{Q}}}] &= \mathbb{E}[\mathbf{1}_{E_Y}\mathbf{1}_{\overline{\mathcal{Q}}}]\\ &\leq \mathbb{E}[\mathbf{1}_{E_Y}]^{\frac{1}{2}}\mathbb{E}[\mathbf{1}_{\overline{\mathcal{Q}}}]^{\frac{1}{2}} < \mathbb{E}[\mathbf{1}_{E_Y}]^{\frac{1}{2}} = \left(\mathbb{P}\left[Y\left(\mathcal{G}_{n,m}\right) \leq \mathbb{E}\left[X\left(\mathcal{G}_{n,m}\right)\right] - n^{5/6}\ln n\right]\right)^{\frac{1}{2}}\\ &= \left(\mathbb{P}\left[Y\left(\mathcal{G}_{n,m}\right) \leq \mathbb{E}\left[Y\left(\mathcal{G}_{n,m}\right)\right] - n^{5/6}\ln n + o(1)\right]\right)^{\frac{1}{2}} = o(1),\end{aligned}$$

by the Cauchy-Schwarz inequality, Corollary 2.2.14 and Lemma 2.2.16. Since

$$\mathbb{P}\left[X\left(\mathcal{G}_{n,m}\right) \leq \mathbb{E}\left[X\left(\mathcal{G}_{n,m}\right)\right] - n^{5/6}\ln n \mid \overline{\mathcal{Q}}\right] = \frac{\mathbb{E}[\mathbf{1}_{E_X}\mathbf{1}_{\overline{\mathcal{Q}}}]}{\mathbb{P}[\overline{\mathcal{Q}}]}$$

and $\mathbb{P}[\overline{\mathcal{Q}}] = 1 - o(1)$, we obtain

$$\mathbb{P}\left[X\left(\mathcal{G}_{n,m}\right) \leq \mathbb{E}\left[X\left(\mathcal{G}_{n,m}\right)\right] - n^{5/6}\ln n\right] = o(1). \tag{2.7}$$

It is well known that for any fixed $\theta > 1$ and $m = \lceil \theta n/2 \rceil$ $complex(\mathcal{G}_{n,m})$ consists of a "giant" multicyclic component of linear order along with unicyclic components of total order $O_p(1)$ (see [6]). Moreover, the order of the "giant" multicyclic component, which is essentially the order of $complex(\mathcal{G}_{n,m})$, is $\rho(\theta)n + O_C(\ln n\sqrt{n})$, where ρ is the solution of $\rho + e^{-\rho\theta} = 1$ (see [23]). On the other hand, the order of $\mathrm{cr}(\mathcal{G}_{n,m})$ is $(1-T)\rho(\theta)n + O_C(\ln^2 n\sqrt{n})$, where $T = T(\theta)$ is the solution of the equation $xe^{-x} = \theta e^{-\theta}$, with $x < 1$ (see [36]). Therefore, for any fixed $\theta > 1$, taking $t = \mathbb{E}\left[T\left(\mathcal{G}_{n,m}\right)\right] - n^{\frac{1}{2}}\ln n$, $x = \mathbb{E}\left[X\left(\mathcal{G}_{n,m}\right)\right] - n^{\frac{5}{6}}\ln n$, and $s = T\rho(\theta)n - \varepsilon n$ for some $\varepsilon > 0$, which will be chosen arbitrarily small Lemma 2.2.2 implies the following:

$$\begin{aligned}
&\mathbb{P}[\chi(\mathcal{G}_{n,m}) \leq 3] \leq \\
&\qquad 2^{o(n)}\ 2^{-T\rho(\theta)n+\varepsilon n}\ \left(\frac{2}{3}\right)^{\mathbb{E}[T(\mathcal{G}_{n,m})]}\ 2^{-\mathbb{E}[Y(\mathcal{G}_{n,m})]}\ \mathbb{E}[C(\mathcal{G}_{n,m})] + o(1) \\
&\leq\ 2^{o(n)}\left(2^{\psi(\theta)}\ 2^{-T\rho(\theta)+\varepsilon}\ \left(\frac{2}{3}\right)^{\frac{1}{\theta}\sum_{k=1}^{\infty}\frac{k^{k-2}}{k!}\left(\theta e^{-\theta}\right)^k}\ 2^{-\frac{1}{\theta}\sum_{k=1}^{\infty}\frac{k^{k-1}}{k!}\left(\theta e^{-\theta}\right)^k}\right)^n + o(1),
\end{aligned}$$

by Propositions 2.2.10, 2.2.15, Lemma 2.2.12, Corollary 2.2.14 and Equation (2.7).

A calculation with Maple reveals that the expression

$$2^{-T\rho(\theta)+\varepsilon}\left(\frac{2}{3}\right)^{\frac{1}{\theta}\sum_{k=1}^{\infty}\frac{k^{k-2}}{k!}\left(\theta e^{-\theta}\right)^k}\ 2^{-\frac{1}{\theta}\sum_{k=1}^{\infty}\frac{k^{k-1}}{k!}\left(\theta e^{-\theta}\right)^k}\ 2^{\psi(\theta)}$$

is less than 1 for $\theta = \theta_0 = 5.299$, for ε sufficiently small. Hence, by the monotonicity of the property of 3-colourability, we deduce Theorem 2.2.1. This result transfers to the $\mathcal{G}_{n,p}$ model when $p = m/\binom{n}{2}$, by Proposition 1.1.2.

2.3 The adequate family of rigid 3-colourings

As we have seen, when we consider the adequate family of all possible 3-colourings, that is we take $\mathcal{C}'(G) = \mathcal{C}(G)$ for each G, we find that $\mathbb{E}[C'(\mathcal{G}_{n,m})] \to 0$ as $n \to \infty$ if $\theta = 5.41$, and so $\theta_3^+ \leq 5.41$.

This bound was improved by Dunne and Zito [13] to $\theta_3^+ \leq 5.2057$ by considering a certain adequate family; and then further improved by Achlioptas and Molloy [3] to $\theta_3^+ \leq 5.044$ by considering a smaller adequate family, namely the 'rigid' 3-colourings. The idea of considering such colourings came from the success of the method of 'locally maximum' satisfying truth assignments in [25] for investigating the unsatisfiability threshold for random 3-SAT problems, see the survey [34].

Definition 2.3.1 *A proper 3-colouring with stable sets S_1, S_2, S_3 is called* rigid *if each vertex in $S_2 \cup S_3$ is adjacent to some vertex in S_1, and each vertex in S_3 is adjacent to some vertex in S_2.*

The above bound from [3] was obtained independently by the author and C.J.H. McDiarmid in [16]. In recent work, Kaporis *et al.* [24] give a tighter estimate of the expected number of rigid 3-colourings, and obtain $\theta_3^+ \leq 4.99$. We (the author and C.J.H. McDiarmid) obtained this result independently and concurrently (see [16]), and present here a more complete analysis of the expected number of rigid 3-colourings, which shows in particular that the last bound cannot be improved, in the sense that with average degree 4.989 the expected number of rigid 3-colourings tends to ∞. Most recently, O. Dubois and J. Mandler in [12] announced an improvement on this result obtaining $\theta_3^+ \leq 4.854$.

Now, let us introduce our central theorem. Let $m = \lceil \theta n/2 \rceil$. For a graph G, let $\mathcal{R}(G)$ denote the set of rigid 3-colourings of G, and let $R(G)$ denote the cardinality of this set. For θ in a certain interval and $\mathbf{x}$ in a domain $\mathcal{D} = \mathcal{D}(\theta) \subseteq [0,1]^3$, we introduce a function $h(\mathbf{x}, \theta)$ (defined in (2.27) below), and let $\mu(\theta) = \sup_{\mathbf{x} \in \mathcal{D}} h(\mathbf{x}, \theta)$.

Theorem 2.3.1 *There exist positive real numbers $\theta_0 < \theta_1$ such that:*

1. *The function $\mu(\theta)$ is continuous on $[\theta_0, \theta_1]$, and for every $\theta \in [\theta_0, \theta_1]$*

$$\mathbb{E}\left[R\left(\mathcal{G}_{n,m}\right)\right] = 2^{\mu(\theta)n + O(\log_2 n)}, \tag{2.8}$$

   where $m = \lceil \theta n/2 \rceil$. Moreover, $\mu(\theta_0) > 0$ and $\mu(\theta_1) < 0$.

2. *We have*

$$4.9893 < \theta_0 < \theta_1 < 4.9895. \tag{2.9}$$

In order to prove this result, one half of the battle is to show (2.8), and the other half is to show (2.9). The main step in proving (2.9) is to show that for $\theta = 4.9895$, we have $\mu(\theta) < 0$. This involves considerable computation: we are more explicit about this side of matters than has been the custom in previous papers in this area. From the above theorem and the Markov inequality (2.1) we obtain

Corollary 2.3.2 *Let* $\theta = 4.9895$ *and let* $m = \lceil \theta n/2 \rceil$. *Then there exists* $\delta > 0$ *such that* $\mathbb{P}\left[\chi(\mathcal{G}_{n,m}) \leq 3\right] \leq 2^{-\delta n}$*, for* n *sufficiently large.*

Therefore, $\theta_3^+ \leq 4.9895$. Following that, we obtain a minor improvement, based on the earlier work, but now considering the expected number of rigid 3-colourings of the giant component.

Theorem 2.3.3 *Let* $\theta = 4.98887$ *and let* $m = \lceil \theta n/2 \rceil$. *Then there exists* $\delta > 0$ *such that* $\mathbb{P}\left[\chi(\mathcal{G}_{n,m}) \leq 3\right] \leq 2^{-\delta n}$*, for* n *sufficiently large.*

By the equivalence between the probabilistic models and the fact that non-3-colourability is an increasing property, similar results can be deduced for the $\mathcal{G}_{n,p}$ model using Proposition 1.1.2. We find that if $\theta > 4.98887$ and $p = \theta/n$, then $\mathbb{P}\left[\chi\left(\mathcal{G}_{n,p}\right) > 3\right] \to 1$ as $n \to \infty$.

2.4 Overview of proofs

Now, let us give a few more technical details. The plan of the proof of Theorem 2.3.1 is as follows. We first give an exact formula for $\mathbb{E}[R(\mathcal{G}^*_{n,m})]$; see Lemma 2.5.1 below. We show that we can discard the "tails" of the sum that appears there, and give good approximations to the central terms. These terms include probabilities $p(k, \alpha, n, \theta)$, which are investigated in Section 3. These probabilities can be written as a sum over a single quantity λ, where the summands involve binomial coefficients and certain probabilities that concern the random throwing of balls into bins. Again we show that we can discard the tails in the sums; and give good approximations to the central terms, now involving Stirling numbers of the second kind. We use known asymptotic expressions of these numbers thus rephrasing the summands as $2^{h(\mathbf{x},\theta)\mathbf{n}}$. This yields an approximation for $\mathbb{E}[R(\mathcal{G}^*_{n,m})]$ as $2^{\mu(\theta)n+O(\log_2 n)}$ - see (2.31) - which in fact is an approximation for $\mathbb{E}[R(\mathcal{G}_{n,m})]$.

Half of the battle is to show (2.8); the other half is to show (2.9), by giving rigorous estimates for $\mu(\theta)$. In fact, we show that for $\theta = 4.9895$, we have $\mu(\theta) < 0$. The main step in proving this is to show that $h(\mathbf{x})$ is concave over $\mathcal{D}$. We find numerically a first approximation for a point which gives the maximum value of h inside this area. Then we define a suitably fine grid on a box around this point and we seek the maximum value of h over it. Let $\hat{\mathbf{x}}$ be the point where this maximum occurs. We determine an upper bound for h on the surface of the box (by computing values of h and its partial derivatives on a fine grid and using concavity). We find that this bound is less than the value of h at $\hat{\mathbf{x}}$, and so we know that the box contains the maximum over $\mathcal{D}$. Further computations handle the region inside the box.

The above computations and the monotonicity of the property of non-3-colourability establish Theorem 2.3.1.

To establish Theorem 2.3.3, we define a function $\tau(\theta) > 0$ such that if $\mu(\theta) < \tau(\theta)$, then a random graph is asymptotically almost surely not 3-colourable. As we mentioned earlier, the argument is based on the observation that if G is 3-colourable and has at least k non-trivial components, then G has at least 2^k rigid 3-colourings. The function $\tau(\theta)$, as it has been defined here, is determined by the expected number of non-trivial tree components in a random graph $\mathcal{G}_{n,m}$, where $m = \lceil \theta n/2 \rceil$ for some specific $\theta > 0$. After computations much as described above, we find that $\mu(\theta) < \tau(\theta)$, for $\theta = 4.98887$, and we obtain Theorem 2.3.3.

2.5 Starting the proofs

For the sake of simplicity, we carry out the probability calculations in the $\mathcal{G}^*_{n,m}$ model. In this model, we form the random graph by choosing at random m times, each time independently, uniformly and with replacement, an edge out of the $\binom{n}{2}$ possible 2-subsets of $V_n = \{1, \ldots, n\}$. We ignore any repetitions of an edge, so the random graph may have less than m edges. Our results transfer to the $\mathcal{G}_{n,m}$ model - see Lemma 2.7.3 below. Every probability, unless otherwise stated, is meant to be taken over the $\mathcal{G}^*_{n,m}$ model.

As before, let

$$\mathrm{D} = \{(k, \alpha) : \ 0 \leq k \leq 1, \ 0 \leq \alpha \leq k\},$$

D^o be its interior, $\mathrm{P} = \{(0,0), (1,0), (1,1)\}$ and for each positive integer n let

$$\mathrm{D}^{(n)} = \left\{ \mathrm{D} \cap \frac{1}{n}\mathbb{Z}^2 \right\} \setminus \mathrm{P}.$$

Recall that for each $(k, \alpha) \in \mathrm{D}$ we set $\phi(k, \alpha) = k(1 - \alpha) - (k - \alpha)^2$.

Let $\mathcal{C}(n)$ be the set of all 3-colourings of V_n, i.e. the set of all possible mappings from V_n to $\{1, 2, 3\}$, or equivalently the set of all partitions of V_n into three sets S_1, S_2 and S_3 (some of them possibly empty). Recall that for each positive integer n and each $(k, \alpha) \in \mathrm{D}^{(n)}$, $\mathcal{C}(k, \alpha, n)$ denotes the set of all partitions of V_n into three sets S_1, S_2, S_3, where $s_1 = |S_1| = \alpha n$, $s_2 = |S_2| = \beta n$ and $s_3 = |S_3| = \gamma n$, and where $k = \alpha + \beta$. Let

$$p(k, \alpha, n, m) = \mathbb{P}[S \text{ is rigid} | \ S \text{ is proper}], \tag{2.10}$$

where $S \in \mathcal{C}(k, \alpha, n)$. Recall that $R(G)$ denotes the number of rigid 3-colourings of G.

Lemma 2.5.1 *For all positive integers n and $0 < m \le \binom{n}{2}$,*

$$\mathbb{E}[R(\mathcal{G}^*_{n,m})] = (1-\frac{1}{n})^{-m} \sum_{(k,\alpha)\in \mathrm{D}^{(n)}} \binom{n}{kn}\binom{kn}{\alpha n} (2\phi(k,\alpha))^m \ p(k,\alpha,n,m).$$

Proof. By the linearity of the expected value, we have

$$\begin{aligned}\mathbb{E}[R(\mathcal{G}^*_{n,m})] &= \sum_{S\in\mathcal{C}(n)} \mathbb{P}[S \text{ is rigid}] \\ &= \sum_{S\in\mathcal{C}(n)} \mathbb{P}[S \text{ is rigid}|S \text{ is proper}]\cdot \mathbb{P}[S \text{ is proper}]. \quad (2.11)\end{aligned}$$

Let us take a fixed colouring S with stable sets S_1, S_2, S_3 and as in Section 2.2 we let

$$e(S) = \binom{n}{2} - \left(\binom{s_1}{2} + \binom{s_2}{2} + \binom{s_3}{2}\right).$$

Then,

$$\mathbb{P}[S \text{ is proper}] = \left(\frac{e(S)}{\binom{n}{2}}\right)^m = (2\phi(k,\alpha))^m \left(1-\frac{1}{n}\right)^{-m}.$$

Now notice that the family $\mathcal{C}(k,\alpha,n)$ consists of exactly $\binom{n}{kn}\cdot\binom{kn}{\alpha n}$ colourings.

So, rephrasing the sum in (2.11) in terms of k and α we obtain the following:

$$\begin{aligned}\mathbb{E}[R(\mathcal{G}^*_{n,m})] &= \sum_{S\in\mathcal{C}(n)} \mathbb{P}[S \text{ is rigid}] \\ &= \left(1-\frac{1}{n}\right)^{-m} \sum_{(k,\alpha)\in \mathrm{D}^{(n)}} \binom{n}{kn}\binom{kn}{\alpha n} (2\phi(k,\alpha))^m \ p(k,\alpha,n,m),\end{aligned}$$

since $p(k,\alpha,n,m)$ depends solely on the sizes of the independent sets induced by S. ∎

We next check that we may ignore the extreme values of α and k. We will break the above sum into two pieces. Let

$$\mathrm{D}_1 = \{(k,\alpha):\ \alpha \ge 0.2,\ k \le 0.8,\ k-\alpha \ge 0.2\} \quad (2.12)$$

(i.e. $\alpha,\beta,\gamma \ge 0.2$); and let $\mathrm{D}_1^{(n)} = \mathrm{D}_1 \cap \frac{1}{n}\mathbb{Z}^2$. Doing some elementary calculations, we obtain the following:

Lemma 2.5.2 *For the function $\phi(k,\alpha)$*

$$\sup_{(k,\alpha)\in \mathrm{D}\setminus \mathrm{D}_1} \phi(k,\alpha) = 0.32,$$

$$\min_{(k,\alpha)\in \mathrm{D}_1} \phi(k,\alpha) = 0.28.$$

Let

$$\mathcal{S}_1 = (1 - \frac{1}{n})^{-m} \sum_{(k,\alpha)\in \mathrm{D}_1^{(n)}} \binom{n}{kn} \binom{kn}{\alpha n} \; (2\phi(k,\alpha))^m \; p(k,\alpha,n,m), \tag{2.13}$$

and let

$$\mathcal{S}_2 = (1 - \frac{1}{n})^{-m} \sum_{(k,\alpha)\in \mathrm{D}^{(n)}\setminus \mathrm{D}_1^{(n)}} \binom{n}{kn} \binom{kn}{\alpha n} \; (2\phi(k,\alpha))^m \; p(k,\alpha,n,m),$$

so that by Lemma 2.5.1

$$\mathbb{E}[R(\mathcal{G}^*_{n,m})] = \mathcal{S}_1 + \mathcal{S}_2. \tag{2.14}$$

We will see that the second 'error' term here is negligible for a specific choice of m; and then we may focus on the first term.

Lemma 2.5.3 *There exists $\delta > 0$ and $n_0 \in \mathbb{N}$ such that the following holds. If $m = \lceil \theta n/2 \rceil$, for $\theta \in [a,b]$, where $4.98 \leq a < b < \infty$, and $n \geq n_0$, we have:*

$$\mathcal{S}_2 \leq 2^{-\delta n}.$$

Proof. By Lemma 2.5.2, we have

$$\sigma = \sup_{(k,\alpha)\in \mathrm{D}\setminus \mathrm{D}_1} \phi(k,\alpha) = 0.32.$$

Thus,

$$\begin{aligned} \mathcal{S}_2 &\leq \sum_{(k,\alpha)\in \mathrm{D}^{(n)}\setminus \mathrm{D}_1^{(n)}} \binom{n}{kn} \binom{kn}{\alpha n} \left(\frac{n}{n-1} \; 2\phi(k,\alpha) \right)^m \\ &\leq 3^n \left(\frac{n}{n-1} \; 2\sigma \right)^{\frac{\theta n}{2}} \\ &= \left(\frac{n}{n-1} \right)^{\theta n/2} \left(3 \; (2\sigma)^{\frac{\theta}{2}} \right)^n = O\left(\left(3 \; (2\sigma)^{\theta/2} \right)^n \right). \end{aligned}$$

But $3 \; (0.64)^{\frac{4.98}{2}} < 1$, and we obtain $\mathcal{S}_2 = O(2^{-\delta' n})$, where $\delta' = -\log_2 \left(3 \; (0.64)^{\frac{4.98}{2}} \right)$. Choosing any $\delta \in (0, \delta')$, the lemma follows. ∎

For $m = \lceil \theta n/2 \rceil$, we set

$$p(k,\alpha,n,\theta) = p(k,\alpha,n,m) \tag{2.15}$$

(the use of the same letter p should not cause confusion). Lemmas 2.5.1, 2.2.4 imply the following:

Lemma 2.5.4 *For $m = \lceil \theta n/2 \rceil$, uniformly over θ in any interval $[a,b]$ with $0 < a < b < \infty$,*

$$\mathcal{S}_1 = \Theta(n^{-1}) \sum_{(k,\alpha)\in D_1^{(n)}} \left(2^{H(k)+kH(\alpha/k)} (2\phi(k,\alpha))^{\theta/2} \right)^n \; p(k,\alpha,n,\theta).$$

In the next section we consider the term $p(k,\alpha,n,\theta)$.

2.6 Calculations for $p(k,\alpha,n,\theta)$

In this section, we derive an asymptotic formula for $p(k,\alpha,n,\theta)$, which was defined in (2.15) as $\mathbb{P}[S \text{ is rigid}|S \text{ is proper}]$, where $S \in \mathcal{C}(k,\alpha,n)$, $m = \lceil \theta n/2 \rceil$ and we are working on the $\mathcal{G}^*_{n,m}$ model. See Lemma 2.6.2 below.

For positive integers $t \geq r$, we let $p(t,r)$ denote the probability that, when we throw t balls uniformly at random into r bins, each bin ends up non-empty. Consider $(k,\alpha) \in \mathrm{D}^{(n)}$. Let $L^{(n)} = L^{(n)}(k,\alpha) = \left\{\lambda : \frac{n(1-\alpha)}{m} \leq \lambda \leq 1 - \frac{n(1-k)}{m}\right\} \cap (\frac{1}{m}\mathbb{Z})$. Then

$$p(k,\alpha,n,\theta) \;=\; \sum_{\lambda \in L^{(n)}} b(\lambda m; m, p)\; p(\lambda m, n(1-\alpha))\; p((1-\lambda)m, n(1-k)), \qquad (2.16)$$

where, $b(\lambda m; m, p)$ is the probability that a random variable distributed according to $\mathrm{Bin}(m,p)$, where

$$p = 1 - ((1-k)(k-\alpha))/\phi(k,\alpha), \qquad (2.17)$$

is equal to λm. We are proving (2.16).

Let us fix a 3-colouring $S \in \mathcal{C}(k,\alpha,n)$ and assume that it is proper. This means that our "pool" of edges E' contains now $e(S)$ elements and we choose at random m times, uniformly, independently and with replacement, an edge in E'. Let $v \in V$ be a vertex such that either $v \in S_2$ or $v \in S_3$. We set $B^c_v = \{(v,u) : \ u \in S_c\}$, i.e. the subset of E' that contains those edges that "start" from v and "end" on a vertex in S_c, where $c \in \{1,2,3\}$. If $v \in S_c$, then $B^c_v = \emptyset$. It is clear that

$$E' = \left(\cup_{v\in S_3} B^1_v\right) \bigcup \left(\cup_{v\in S_3} B^2_v\right) \bigcup \left(\cup_{v\in S_2} B^1_v\right).$$

These sets are disjoint. The probability of choosing an edge that belongs to a specific set B^c_v among these is $\frac{s_c}{e(S)}$. Thus, this probability for the sets B^1_v where either $v \in S_3$ or $v \in S_2$ is $\frac{s_1}{e(S)}$ and for the sets B^2_v where $v \in S_3$ is $\frac{s_2}{e(S)}$.

Let us try to rephrase our random experiment. We consider a 1-to-1 correspondence between the collection of sets B^c_v, for $v \in S_2$ or $v \in S_3$, that belong to any of the aforementioned elements of the partition of E', and a collection of bins. Hence, we have a collection of $2s_3 + s_2$ bins. Thus, a first step in the formation of the random graph is equivalent to the random placement of m distinct balls into $2s_3 + s_2$ distinct bins. The problem is that these bins are not equilikely to be occupied. So, we divide this collection of bins into two parts, each of them containing equiprobable bins. Namely, the first part contains the bins that correspond to the sets B^1_v, where either $v \in S_3$ or $v \in S_2$, and the second one consists of those bins that correspond to the sets B^2_v, where $v \in S_3$. Thus, our random experiment has two stages. In the first stage, for each ball separately we decide which of these two parts the ball will be thrown into. During the second stage, we make two independent random experiments, one for each part of bins, throwing the balls that are to be placed in each part randomly to the corresponding bins. In each of these two parts, the bins are equally likely to be occupied by a ball. The probability $p(k, \alpha, n, \theta)$ as it was defined in (2.15) is exactly the probability that each bin contains at least one ball.

Now, we observe that the sample space of all random graphs properly colourable by S is partitioned into disjoint sets. Each of these is the set of sample points for which i balls are to be thrown in the first part of the bins, for $i = 0, \ldots, m$, i.e. there are i edges between S_1 and $S_2 \cup S_3$. Thus, the sample space is partitioned into $m + 1$ disjoint sets. The probability of each of these sets (events) follows the binomial distribution. Namely, this is $\mathrm{Bin}(m, p)$, where $p = \frac{s_1(s_2+s_3)}{e(S)}$. This is the case, because p is the probability of choosing one of the edges that belong to a set B_v^1 where either $v \in S_3$ or $v \in S_2$, i.e. to a bin of the first part. Let ε_i denote the i-th such event. So,

$$\mathbb{P}[\text{All bins non-empty}] = \sum_{i=0}^{m} \mathbb{P}[\text{All bins non-empty}|\varepsilon_i] \cdot \mathbb{P}[\varepsilon_i]. \tag{2.18}$$

Now, let us fix such an event ε_i and assume that it has been realised. This means that i balls are to be placed into the bins of the first part and $m - i$ in the second part. The set ε_i of sample points can be divided into $\binom{m}{i}$ equiprobable sets. Each of these corresponds to the event that a specific set of i balls is to be placed into the bins of the first part. Let ϵ_j denote this event. Therefore,

$$\mathbb{P}[\text{All bins non-empty}|\varepsilon_i] = \sum_{j=1}^{\binom{m}{i}} \mathbb{P}[\text{All bins non-empty}|\varepsilon_i, \epsilon_j] \cdot \mathbb{P}[\epsilon_j|\varepsilon_i].$$

The second probability in the above sum is equal to $1/\binom{m}{i}$. For the first one, we must take into consideration that we are in the second stage of our random experiment and the placement of the balls in each of the two parts of bins is being done independently. In each part, all the bins are equally likely to receive a ball. So, the probability of at least one ball occupying each bin is equal to the number of the ways of placing the balls into the bins without leaving any bin empty over the total number of ways of placing the balls into the bins. Hence, for both parts this will be

$$\mathbb{P}[\text{All bins non-empty}|\varepsilon_i, \epsilon_j] = p(i, s_2 + s_3) \cdot p(m - i, s_3),$$

and Equation (2.16) follows.

2.6.1 Discarding the tails

We next check that we may discard the extreme values of λ. This is a technical exercise for which we need one preliminary lemma.

Lemma 2.6.1 *For positive integers $t > r$*

$$\frac{t}{2(t-r)}p(t-1,r) \le p(t,r) \le \frac{t}{t-r}p(t-1,r).$$

Proof. Let $W(t,r)$ be the set of all arrangements of t balls into r bins leaving no empty bins and $w(t,r)$ be its cardinality. What we want to prove will follow from the following inequality:

$$\frac{rt}{2(t-r)}w(t-1,r) \le w(t,r) \le \frac{rt}{t-r}w(t-1,r). \tag{2.19}$$

To see this, consider ordered pairs of balls and bins, i.e. if T is the set of balls and R the set of bins, take the Cartesian product of them $P = T \times R$. Each such pair (b, B), where $b \in T$ and $B \in R$, corresponds to the fact that the ball b is in bin B. For each such pair arrange the remaining $t-1$ balls into the r bins leaving no empty bins. Thus, we form the set $\mathcal{W} = \{(p,w) : \ p \in P, \ w \in W(t-1,r)\}$. Note that we have a surjective mapping from the set $\mathcal{W}$ onto the set $W(t,r)$. Clearly, $\mathcal{W}$ is of cardinality $rtw(t-1,r)$. The mapping induces a natural partition on this set and each of the parts, which is the set of pairs that are mapped to a specific arrangement of t balls into r bins without leaving any empty bins, is of cardinality equal to the number of balls which are not the only ball in their bins, which is at least $t-r$ and at most $2(t-r)$. Thus, (2.19) has been established. Therefore,

$$\frac{rt}{2(t-r)}\frac{w(t-1,r)}{r^t} \le \frac{w(t,r)}{r^t} \le \frac{rt}{t-r}\frac{w(t-1,r)}{r^t},$$

and the lemma follows. ∎

For $(k,\alpha) \in \mathrm{D}_1$, let

$$L_1 = L_1(k,\alpha,\theta) = \left\{\lambda : \ \frac{2(1-\alpha)}{\theta}(1+0.24) \le \lambda \le 1 - \frac{2(1-k)}{\theta}(1+0.065)\right\}, \tag{2.20}$$

and

$$L_1^{(n)} = L_1^{(n)}(k,\alpha,\theta) = L_1 \cap (\frac{1}{m}\mathbb{Z}).$$

(The extra terms 0.24 and 0.065 here are to exclude extreme values which, as we shall see shortly, are negligible, but which would cause awkwardness later.) It is convenient to restrict θ to a range $[\theta_l, \theta_u]$, as we need to obtain approximations uniformly over θ. We let

$$\theta_l = 4.98 \qquad \theta_u = 4.99. \tag{2.21}$$

Lemma 2.6.2 *Uniformly over $\theta \in [\theta_l, \theta_u]$ and $(k, \alpha) \in \mathrm{D}_1^{(n)}$*

$$p(k, \alpha, n, \theta) \quad = \quad \Theta(1) \sum_{\lambda \in L_1^{(n)}} b(\lambda m; m, p)\ p(\lambda m, n(1-\alpha))\ p((1-\lambda)m, n(1-k)). \tag{2.22}$$

Proof. Let $f(\lambda)$ be the general term in the sum of equation (2.16). We will compare the term $f(\lambda)$, for some λ which will be specified later, with the adjacent term $f(\lambda - 1/m)$. Note that $f(\lambda) = p(t, r)p(m - t, r')b(\lambda m; m, p)$, where $r = n(1 - \alpha)$ and $r' = n(1 - k)$, and $t = \lambda m$. We assume that $t = n(1 - \alpha) + \lfloor \eta n(1 - \alpha) \rfloor$, for some $\eta > 0$. By Lemma 2.6.1 we have

$$p(t-1, r)p(m-t+1, r') \leq 2 \left(\frac{t-r}{t} \right) \left(\frac{m-t+1}{m-t+1-r'} \right) p(t, r)p(m-t, r').$$

Let us investigate the coefficients on the right hand side of the above inequality:

$$\begin{aligned}
& 2 \cdot \frac{t-r}{t} \cdot \left(\frac{m-t+1}{m-t+1-r'} \right) \\
\leq \quad & 2 \cdot \frac{n(1-\alpha) + \eta n(1-\alpha) - n(1-\alpha)}{n(1-\alpha) + \eta n(1-\alpha)} \cdot \left(\frac{1}{\frac{\theta}{2(1-k)} - \frac{1-\alpha}{1-k}(1+\eta) - 1} + 1 \right) \cdot (1 + o(1)) \\
= \quad & 2 \cdot \frac{\eta}{1+\eta} \cdot \left(\frac{1}{\frac{\theta}{2(1-k)} - \frac{1-\alpha}{1-k}(1+\eta) - 1} + 1 \right) \cdot (1 + o(1)).
\end{aligned}$$

Also, note that $m - t + 1 \geq (1 + \varepsilon)r' > r'$, for $\theta \geq \theta_l$ and for a suitable η, where

$$\begin{aligned}
\varepsilon \quad \geq \quad & \frac{\lceil \frac{\theta n}{2} \rceil - n(1-\alpha)(1+\eta) - n(1-k)}{n(1-k)} + o(1) \geq \frac{\frac{\theta n}{2} - n(1-\alpha)(1+\eta) - n(1-k)}{n(1-k)} + o(1) \\
= \quad & \frac{\theta/2 - (1-\alpha)(1+\eta) - (1-k)}{(1-k)} + o(1) = \frac{\theta}{2(1-k)} - \frac{1-\alpha}{1-k}(1+\eta) - 1 + o(1).
\end{aligned}$$

On the other hand, note that the term $b(\lambda m; m, p) = b(t; m, p)$ is of the form $\binom{m}{\lambda m} p^{\lambda m}(1 - p)^{m(1-\lambda)}$, where $p = 1 - ((1 - k)(k - \alpha))/\phi(k, \alpha)$. It can be easily seen that

$$b(t-1; m, p) = \frac{\lambda m}{m(1-\lambda)+1} \frac{1-p}{p} b(t; m, p). \tag{2.23}$$

Thus, we obtain

$$\begin{aligned}
& b(t-1; m, p)p(t-1, r)p(m-t+1, r') \leq \\
\leq \quad & \frac{\frac{2(1-\alpha)}{\theta}(1+\eta)}{1 - \frac{2(1-\alpha)}{\theta}(1+\eta)}(1+o(1)) \cdot \frac{(1-k)(k-\alpha)}{\alpha(1-\alpha)} \cdot 2 \cdot \frac{\eta}{1+\eta} \times \\
& \left(\frac{1}{\frac{\theta}{2(1-k)} - \frac{1-\alpha}{1-k}(1+\eta) - 1} + 1 \right) b(t; m, p)p(t, r)p(m-t, r') \\
\leq \quad & \frac{\frac{2(1-0.2)}{\theta}(1+\eta)}{1 - \frac{2(1-0.2)}{\theta}(1+\eta)}(1+o(1)) \cdot \frac{(1-0.6)(0.6-0.2)}{0.2(1-0.2)} \cdot 2 \cdot \frac{\eta}{1+\eta} \times \\
& \left(\frac{1}{\frac{\theta}{2(1-0.4)} - \frac{1-0.2}{1-0.4}(1+\eta) - 1} + 1 \right) b(t; m, p)p(t, r)p(m-t, r'),
\end{aligned}$$

using the fact that $(k, \alpha) \in \mathrm{D}_1$. Straightforward verification shows that for $\eta = 0.25$, the factor on the right hand side is less than $0.449 < 1/2$, for any $\theta \in [\theta_l, \theta_u]$ and for n sufficiently large. This is the case because the above expression is increasing with respect to η. Therefore, the sum of the terms for t from $n(1-\alpha)$ up to $n(1-\alpha) + \lfloor 0.25(1-\alpha)n \rfloor - 1$ can be bounded as follows:

$$\sum_{t=n(1-\alpha)}^{n(1-\alpha)+\lfloor 0.25(1-\alpha)n\rfloor -1} f\left(\frac{t}{m}\right) \leq f\left(\frac{n(1-\alpha)+\lfloor 0.25(1-\alpha)n\rfloor}{m}\right),$$

by the geometric sum formula.

Following the same treatment, we can bound the other tail of the sum. Here, assume that $m - t = n(1-k) + \lfloor \eta(1-k)n \rfloor$. Thus, $t = m - n(1-k) - \lfloor \eta(1-k)n \rfloor$. So, $t \geq (1+\varepsilon)r > r$, for $\theta \geq \theta_l$ and for some $\eta > 0$, where

$$\begin{aligned}
\varepsilon &\geq \frac{\lceil \frac{\theta n}{2} \rceil - n(1-k)(1+\eta) - n(1-\alpha)}{n(1-\alpha)} + o(1) \geq \frac{\frac{\theta n}{2} - n(1-k)(1+\eta) - n(1-\alpha)}{n(1-\alpha)} + o(1) \\
&= \frac{\theta/2 - (1-k)(1+\eta) - (1-\alpha)}{(1-\alpha)} + o(1) = \frac{\theta}{2(1-\alpha)} - \frac{1-k}{1-\alpha}(1+\eta) - 1 + o(1).
\end{aligned}$$

From Lemma 2.6.1 and (2.23), we obtain

$$\begin{aligned}
&b(t;m,p)p(t,r)p(m-t,r') \leq \\
&\leq \frac{t}{t-r}\,\frac{2(m-t+1-r')}{m-t+1}\,\frac{(1-\lambda)m+1}{\lambda m}\,\frac{p}{1-p}\,b(t-1;m,p)p(t-1,r)p(m-t+1,r').
\end{aligned}$$

Therefore, this becomes

$$\begin{aligned}
&b(t;m,p)p(t,r)p(m-t,r') \leq \\
&\leq \frac{\frac{2(1-k)}{\theta}(1+\eta)}{1-\frac{2(1-k)}{\theta}(1+\eta)}(1+o(1)) \cdot \frac{\alpha(1-\alpha)}{(1-k)(k-\alpha)} \cdot 2 \cdot \frac{\eta}{1+\eta} \times \\
&\quad \left(\frac{1}{\frac{\theta}{2(1-\alpha)} - \frac{1-k}{1-\alpha}(1+\eta) - 1} + 1\right) b(t-1;m,p)p(t-1,r)p(m-t+1,r') \\
&\leq \frac{\frac{2(1-0.4)}{\theta}(1+\eta)}{1-\frac{2(1-0.4)}{\theta}(1+\eta)}(1+o(1)) \cdot \frac{0.6(1-0.6)}{(1-0.8)(0.8-0.6)} \cdot 2 \cdot \frac{\eta}{1+\eta} \times \\
&\quad \left(\frac{1}{\frac{\theta}{2(1-0.2)} - \frac{1-0.4}{1-0.2}(1+\eta) - 1} + 1\right) b(t-1;m,p)p(t-1,r)p(m-t+1,r'),
\end{aligned}$$

since $(k, \alpha) \in \mathrm{D}_1$. One can see that for $\eta = 0.07$, the factor on the right hand side is less than 0.4809, for any $\theta \in [\theta_l, \theta_u]$ and for n sufficiently large. As in the previous case, this expression is increasing with respect to η. Therefore, the sum of the terms for t from $m - n(1-k) - \lfloor 0.07(1-$

$k)n\rfloor + 1$ up to to $m - n(1-k)$ can be bounded as follows:

$$\sum_{t=m-n(1-k)-\lfloor 0.07(1-k)n\rfloor+1}^{m-n(1-k)} f\left(\frac{t}{m}\right) \leq f\left(\frac{m-n(1-k)-\lfloor 0.07(1-k)n\rfloor}{m}\right),$$

by the geometric sum formula.

Now, the lemma follows from the above observations along with the fact that each term is non-negative, which means that removing a few terms from the sum gives a lower bound on it. ■

2.6.2 Introducing Stirling numbers of the second kind

For positive integers $t \geq r$ the Stirling number of the second kind $S(t,r)$ is defined to be $1/r!$ times the number of surjective functions from a set of cardinality t to a set of cardinality r. Thus

$$p(t,r) = \frac{r!S(t,r)}{r^t}.$$

Hence, we may rewrite Lemma 2.6.2 as follows:

Lemma 2.6.3 *Uniformly over $\theta \in [\theta_l, \theta_u]$ and $(k,\alpha) \in \mathrm{D}_1^{(n)}$, we have:*

$$p(k,\alpha,n,\theta) = \\ = \Theta(1) \sum_{\lambda \in L_1^{(n)}} \binom{\lceil \theta n/2\rceil}{\lambda\lceil \theta n/2\rceil} \left(\frac{(1-k)(k-\alpha)}{\phi(k,\alpha)}\right)^{\lceil\frac{\theta n}{2}\rceil(1-\lambda)} \left(1-\frac{(1-k)(k-\alpha)}{\phi(k,\alpha)}\right)^{\lambda\lceil\frac{\theta n}{2}\rceil} \times \\ \frac{(n(1-\alpha))!S(\lambda\lceil\theta n/2\rceil, n(1-\alpha))}{(n(1-\alpha))^{\lambda\lceil\frac{\theta n}{2}\rceil}} \frac{(n(1-k))!S((1-\lambda)\lceil\theta n/2\rceil, n(1-k))}{(n(1-k))^{\lceil\frac{\theta n}{2}\rceil(1-\lambda)}}. \quad (2.24)$$

2.6.3 Asymptotics for Stirling numbers of the second kind

An essential part of our probability calculations involves asymptotic expressions for the Stirling numbers of the second kind. We need some preliminary definitions and results.

For $0 < u \leq 1$ let

$$E_u(x) = 1 - e^{-x} - ux = 0.$$

For $0 < u < 1$ let $x_0 = x_0(u)$ be the unique positive root of $E_u(x)$. See Figure 2.1. Note that $E_1(x)$ has the unique root $x = 0$: we let $x_0(1) = 0$. Then $x_0(u)$ is a continuous function on $(0,1]$, and

$$x_0(u) + 1 - 1/u > 0$$

for each $0 < u < 1$.

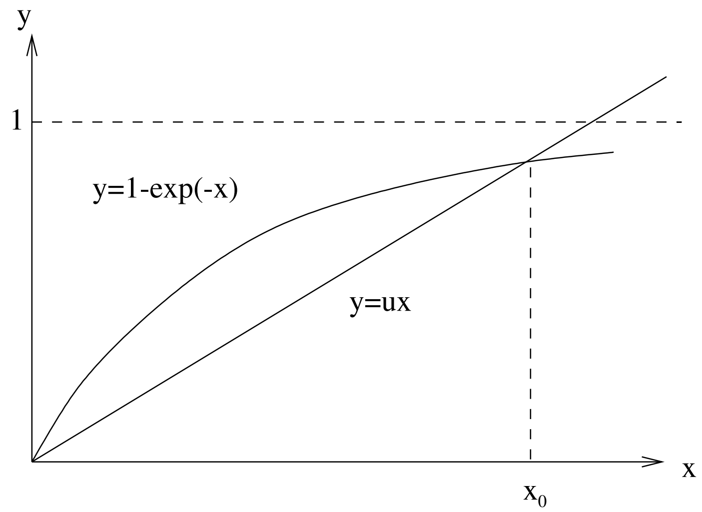

Figure 2.1: The function $x_0(u)$

For $0 < u < 1$ let

$$f(u) = \left(\frac{1-u}{x_0(u)+1-1/u}\right)^{1/2},$$

and let $f(1) = 0$. Then f is continuous function on $(0,1]$, and $0 \le f(u) \le 1$ (see [11]). In Appendix A, we give a positive lower bound on $f(u)$.

It is shown in [39] that, for positive integers $t > r$,

$$S(t,r) = (e^{x_0}-1)^r (x_0(u))^{-t} e^{-(t-r)} (t-r)^{(t-r)} \binom{t}{r} f(u)(1+\varepsilon(t,r)), \tag{2.25}$$

where $u = r/t$ and $\max_{0<r<t} \varepsilon(t,r) \to 0$ as $t \to \infty$.

In Appendix A, we present some useful properties of x_0 as function of r and t.

2.6.4 The estimate

Recall that D_1 and L_1 are defined in (2.12) and (2.20) above. Let

$$\mathcal{D} = \mathcal{D}(\theta) = \{(k,\alpha,\lambda) : (k,\alpha) \in \mathrm{D}_1, \lambda \in L_1(k,\alpha,\theta)\}, \tag{2.26}$$

and let

$$\mathcal{D}^{(n)} = \mathcal{D}^{(n)}(\theta) = \mathcal{D} \cap \left((\frac{1}{n}\mathbb{Z}^2) \times (\frac{1}{m}\mathbb{Z}) \right) = \{(k,\alpha,\lambda) : (k,\alpha) \in \mathrm{D}_1^{(n)}, \lambda \in L_1^{(n)}(k,\alpha)\}.$$

For $(k,\alpha,\lambda) \in \mathcal{D}$, let

$$P(k,\alpha,\lambda,\theta) =$$

$$(1-k)^{-\frac{\theta}{2}}\ 2^{\frac{\theta}{2}H(\lambda)}\ \left(\frac{1-k}{1-\alpha}\right)^{\frac{\lambda\theta}{2}} \times$$
$$(e^{x_1}-1)^{(1-\alpha)}\ (e^{x_2}-1)^{(1-k)}\ (x_1)^{-\frac{\lambda\theta}{2}}\ (x_2)^{-\frac{\theta}{2}(1-\lambda)}\ e^{-\frac{\theta}{2}} \times$$
$$\left(\frac{\lambda\theta}{2}\right)^{\frac{\theta\lambda}{2}}\ \left(\frac{\theta(1-\lambda)}{2}\right)^{\frac{\theta(1-\lambda)}{2}}\ \left(\frac{(1-k)(k-\alpha)}{\phi(k,\alpha)}\right)^{\frac{\theta}{2}(1-\lambda)}\ \left(1-\frac{(1-k)(k-\alpha)}{\phi(k,\alpha)}\right)^{\frac{\lambda\theta}{2}},$$

where $x_1 = x_0\left(\frac{2(1-\alpha)}{\theta\lambda}\right)$ and $x_2 = x_0\left(\frac{2(1-k)}{\theta(1-\lambda)}\right)$. We shall prove the following lemma:

Lemma 2.6.4 *Uniformly over $\theta \in [\theta_l, \theta_u]$ and $(k,\alpha) \in \mathrm{D}_1$,*

$$p(k,\alpha,n,\theta) = \Theta(n^{-1/2}) \sum_{\lambda \in L_1^{(n)}} \{P(k,\alpha,\lambda,\theta)\}^n .$$

Proof. In what follows, the "error" term $\lceil \theta n/2 \rceil - \theta n/2$ yields a $\Theta(1)$ factor for each term of the sum in Lemma 2.6.3, since $(k,\alpha) \in \mathrm{D}_1^{(n)}$. We use Lemma 2.2.4 to take asymptotic expressions for the binomial coefficients. Note that since $(k,\alpha) \in \mathrm{D}_1$ the assumptions of Lemma 2.2.4 are satisfied. This is also true for the coefficient that involves θ, since θ is assumed to be in a closed interval and is bounded away from 0. Thus, using (2.25) and Stirling's approximation for the factorials, Lemma 2.6.3 implies the following:

$$p(k,\alpha,n,\theta) = \Theta(n^{-1/2}) \sum_{\lambda \in L_1^{(n)}} \left\{P'(k,\alpha,\lambda,\theta)\right\}^n ,$$

where, for $(k,\alpha,\lambda) \in \mathcal{D}$,

$$\begin{aligned}
P'(k,\alpha,\lambda,\theta) &= \\
&= \frac{n^{(2-\alpha-k)}(1-\alpha)^{(1-\alpha)}(1-k)^{(1-k)}e^{-(2-\alpha-k)}}{n^{\frac{\theta}{2}}(1-k)^{\frac{\theta}{2}}}\ 2^{\frac{\theta}{2}H(\lambda)}\ \left(\frac{1-k}{1-\alpha}\right)^{\frac{\lambda\theta}{2}} \times \\
&\quad n^{\left(\frac{\theta\lambda}{2}+\alpha-1\right)}\ t(1-\alpha,\lambda\theta/2)\ n^{\left(\frac{\theta}{2}-\frac{\theta\lambda}{2}-1+k\right)}\ t(1-k,(1-\lambda)\theta/2) \times \\
&\quad f\left(\frac{2(1-\alpha)}{\theta\lambda}\right)\ f\left(\frac{2(1-k)}{\theta(1-\lambda)}\right)\ \left(\frac{(1-k)(k-\alpha)}{\phi(k,\alpha)}\right)^{\frac{\theta}{2}(1-\lambda)}\ \left(1-\frac{(1-k)(k-\alpha)}{\phi(k,\alpha)}\right)^{\frac{\lambda\theta}{2}} \\
&= \frac{(1-\alpha)^{(1-\alpha)}(1-k)^{(1-k)}e^{-(2-\alpha-k)}}{(1-k)^{\frac{\theta}{2}}}\ 2^{\frac{\theta}{2}H(\lambda)}\ \left(\frac{1-k}{1-\alpha}\right)^{\frac{\lambda\theta}{2}}\ f\left(\frac{2(1-\alpha)}{\theta\lambda}\right)\ f\left(\frac{2(1-k)}{\theta(1-\lambda)}\right) \times \\
&\quad t(1-\alpha,\lambda\theta/2)\ t(1-k,(1-\lambda)\theta/2)\ \left(\frac{(1-k)(k-\alpha)}{\phi(k,\alpha)}\right)^{\frac{\theta}{2}(1-\lambda)}\ \left(1-\frac{(1-k)(k-\alpha)}{\phi(k,\alpha)}\right)^{\frac{\lambda\theta}{2}},
\end{aligned}$$

and

$$t(x,y) = (e^{x_0}-1)^x\ (x_0)^{-y}\ e^{-(y-x)}\ (y-x)^{(y-x)}\ \left(\frac{y}{x}\right)^x\ \left(\frac{y}{y-x}\right)^{(y-x)},$$

where $x_0 = x_0(x/y)$. Doing some calculations, we obtain:

$$
\begin{aligned}
&P'(k,\alpha,\lambda,\theta) = \\
&= (1-k)^{-\frac{\theta}{2}}\ 2^{\frac{\theta}{2}H(\lambda)}\ \left(\frac{1-k}{1-\alpha}\right)^{\frac{\lambda\theta}{2}} \times \\
&\quad (e^{x_1}-1)^{(1-\alpha)}\ (e^{x_2}-1)^{(1-k)}\ (x_1)^{-\frac{\lambda\theta}{2}}\ (x_2)^{-\frac{\theta}{2}(1-\lambda)}\ e^{-\frac{\theta}{2}} \times \\
&\quad f\left(\frac{2(1-\alpha)}{\theta\lambda}\right)\ f\left(\frac{2(1-k)}{\theta(1-\lambda)}\right) \times \\
&\quad \left(\frac{\lambda\theta}{2}\right)^{\frac{\theta\lambda}{2}}\ \left(\frac{\theta(1-\lambda)}{2}\right)^{\frac{\theta(1-\lambda)}{2}}\ \left(\frac{(1-k)(k-\alpha)}{\phi(k,\alpha)}\right)^{\frac{\theta}{2}(1-\lambda)}\ \left(1-\frac{(1-k)(k-\alpha)}{\phi(k,\alpha)}\right)^{\frac{\lambda\theta}{2}} \\
&= P(k,\alpha,\lambda,\theta)\ f\left(\frac{2(1-\alpha)}{\theta\lambda}\right)\ f\left(\frac{2(1-k)}{\theta(1-\lambda)}\right),
\end{aligned}
$$

where $x_1 = x_0\left(\frac{2(1-\alpha)}{\theta\lambda}\right)$ and $x_2 = x_0\left(\frac{2(1-k)}{\theta(1-\lambda)}\right)$. For the elementary but tedious calculations see Appendix B.

Also, note that by (A.4) in Appendix A, the monotonicity of the lower bound for $f(u)$, where $u = \frac{2(1-\alpha)}{\theta\lambda}$ or $u = \frac{2(1-k)}{\theta(1-\lambda)}$, and the fact that $(k,\alpha) \in \mathrm{D}_1$, it follows that in both cases the function f is at least

$$\sqrt{\frac{1-\frac{2}{5\theta}}{x_0\left(\frac{2}{5\theta}\right)}},$$

and so the two factors including f in the expression above yield a $\Theta(1)$ term. This concludes the proof of the lemma. ∎

2.7 Proof of Theorem 2.3.1 and Corollary 2.3.2

For $\theta \in [\theta_l, \theta_u]$ and $(k,\alpha,\lambda) \in \mathcal{D}$, let

$$
\begin{aligned}
&h(k,\alpha,\lambda,\theta) = \\
&\quad H(k) + kH\left(\frac{\alpha}{k}\right) + \frac{\theta}{2} - \frac{\theta}{2}\log_2(e) + \frac{\theta}{2}\log_2\left(\frac{\theta}{2}\right) \\
&\quad + \frac{\theta}{2}\Big(-\lambda\log_2\lambda - (1-\lambda)\log_2(1-\lambda) + (1-\lambda)\log_2(1-k) \\
&\quad + (1-\lambda)\log_2(k-\alpha) + \lambda\log_2\alpha + \lambda\log_2(1-\alpha)\Big) \\
&\quad + (1-k)\log_2(e^{x_2}-1) - \frac{\theta}{2}(1-\lambda)\log_2(x_2) - \frac{\theta(1-\lambda)}{2}\log_2(1-k) + \frac{\theta(1-\lambda)}{2}\log_2(1-\lambda) \\
&\quad + (1-\alpha)\log_2(e^{x_1}-1) - \frac{\lambda\theta}{2}\log_2(x_1) - \frac{\lambda\theta}{2}\log_2(1-\alpha) + \frac{\lambda\theta}{2}\log_2\lambda, \qquad (2.27)
\end{aligned}
$$

(see (2.26) for the definition of $\mathcal{D}$, and (B.1), (B.2) for the definitions of x_1 and x_2). We will show the following lemma:

Lemma 2.7.1 *Uniformly over $\theta \in [\theta_l, \theta_u]$,*

$$\mathcal{S}_1 = \Theta(n^{-3/2}) \sum_{(k,\alpha,\lambda)\in\mathcal{D}^{(n)}} 2^{h(k,\alpha,\lambda,\theta)n}.$$

Proof. Lemmas 2.5.4 and 2.6.4 imply that uniformly over θ in any closed interval bounded away from 0:

$$\begin{aligned}
\mathcal{S}_1 &= \Theta(n^{-3/2}) \sum_{(k,\alpha,\lambda)\in\mathcal{D}^{(n)}} \left(2^{H(k)}\ 2^{kH(\frac{\alpha}{k})}\ (2\phi(k,\alpha))^{\frac{\theta}{2}}\ P(k,\alpha,\lambda,\theta)\right)^n \\
&= \Theta(n^{-3/2}) \sum_{(k,\alpha,\lambda)\in\mathcal{D}^{(n)}} \left(2^{h(k,\alpha,\lambda,\theta)}\right)^n,
\end{aligned}$$

since for $(k,\alpha,\lambda)\in\mathcal{D}$,

$$\begin{aligned}
&2^{H(k)}\ 2^{kH(\frac{\alpha}{k})}\ (2\phi(k,\alpha))^{\frac{\theta}{2}}\ P(k,\alpha,\lambda,\theta) = \\
&= 2^{H(k)}\ 2^{kH(\frac{\alpha}{k})}\ (2\phi(k,\alpha))^{\frac{\theta}{2}}\ 2^{\frac{\theta}{2}H(\lambda)}\ (1-k)^{-\frac{\theta}{2}}\ \left(\frac{1-k}{1-\alpha}\right)^{\frac{\lambda\theta}{2}} \times \\
&\quad (e^{x_1}-1)^{(1-\alpha)}\ (e^{x_2}-1)^{(1-k)}\ (x_1)^{-\frac{\lambda\theta}{2}}\ (x_2)^{-\frac{\theta}{2}(1-\lambda)}\ e^{-\frac{\theta}{2}} \times \\
&\quad \left(\frac{\lambda\theta}{2}\right)^{\frac{\theta\lambda}{2}}\ \left(\frac{\theta(1-\lambda)}{2}\right)^{\frac{\theta(1-\lambda)}{2}}\ \left(\frac{(1-k)(k-\alpha)}{\phi(k,\alpha)}\right)^{\frac{\theta}{2}(1-\lambda)}\ \left(1-\frac{(1-k)(k-\alpha)}{\phi(k,\alpha)}\right)^{\frac{\lambda\theta}{2}} \\
&= 2^{H(k)}\ 2^{kH(\frac{\alpha}{k})}\ (2\phi(k,\alpha))^{\frac{\theta}{2}}\ 2^{\frac{\theta}{2}H(\lambda)}\ (1-k)^{-\frac{(1-\lambda)\theta}{2}}\ (1-\alpha)^{-\frac{\lambda\theta}{2}} \times \\
&\quad (e^{x_1}-1)^{(1-\alpha)}\ (e^{x_2}-1)^{(1-k)}\ (x_1)^{-\frac{\lambda\theta}{2}}\ (x_2)^{-\frac{\theta}{2}(1-\lambda)}\ e^{-\frac{\theta}{2}}\ \left(\frac{\theta}{2}\right)^{\frac{\theta}{2}} \times \\
&\quad 2^{\frac{\theta}{2}(\lambda\log_2\lambda+(1-\lambda)\log_2(1-\lambda))}\ \left(\frac{(1-k)(k-\alpha)}{\phi(k,\alpha)}\right)^{\frac{\theta}{2}(1-\lambda)}\ \left(\frac{\phi(k,\alpha)-(1-k)(k-\alpha)}{\phi(k,\alpha)}\right)^{\frac{\lambda\theta}{2}} \\
&= 2^{H(k)}\ 2^{kH(\frac{\alpha}{k})}\ (2\phi(k,\alpha))^{\frac{\theta}{2}}\ 2^{\frac{\theta}{2}H(\lambda)}(1-k)^{-\frac{(1-\lambda)\theta}{2}}\ (1-\alpha)^{-\frac{\lambda\theta}{2}} \times \\
&\quad (e^{x_1}-1)^{(1-\alpha)}\ (e^{x_2}-1)^{(1-k)}\ (x_1)^{-\frac{\lambda\theta}{2}}\ (x_2)^{-\frac{\theta}{2}(1-\lambda)}\ e^{-\frac{\theta}{2}} \times \\
&\quad \left(\frac{\theta}{2}\right)^{\frac{\theta}{2}}\ 2^{-\frac{\theta}{2}H(\lambda)}\ ((1-k)(k-\alpha))^{\frac{\theta}{2}(1-\lambda)}\ (\alpha(1-\alpha))^{\frac{\lambda\theta}{2}}\ \phi(k,\alpha)^{-\frac{\theta}{2}} \\
&= 2^{H(k)}\ 2^{kH(\frac{\alpha}{k})}\ 2^{\frac{\theta}{2}}\ \left(\frac{\theta}{2}\right)^{\frac{\theta}{2}}\ e^{-\frac{\theta}{2}} \times \\
&\quad 2^{\frac{\theta}{2}H(\lambda)}((1-k)(k-\alpha))^{\frac{\theta}{2}(1-\lambda)}\ (\alpha(1-\alpha))^{\frac{\lambda\theta}{2}} \times \\
&\quad (1-k)^{-\frac{(1-\lambda)\theta}{2}}\ (1-\alpha)^{-\frac{\lambda\theta}{2}}(e^{x_1}-1)^{(1-\alpha)}\ (e^{x_2}-1)^{(1-k)}\ (x_1)^{-\frac{\lambda\theta}{2}}\ (x_2)^{-\frac{\theta}{2}(1-\lambda)} \times \\
&\quad 2^{\frac{\theta}{2}(\lambda\log_2\lambda+(1-\lambda)\log_2(1-\lambda))} \\
&= 2^{H(k)}\ 2^{kH(\frac{\alpha}{k})}\ 2^{\frac{\theta}{2}}\ \left(\frac{\theta}{2}\right)^{\frac{\theta}{2}}\ e^{-\frac{\theta}{2}} \times
\end{aligned}$$

$$
\begin{aligned}
& 2^{\frac{\theta}{2}H(\lambda)}((1-k)(k-\alpha))^{\frac{\theta}{2}(1-\lambda)} \ (\alpha(1-\alpha))^{\frac{\lambda\theta}{2}} \times \\
& (1-\alpha)^{-\frac{\lambda\theta}{2}}(e^{x_1}-1)^{(1-\alpha)} \ (x_1)^{-\frac{\lambda\theta}{2}} 2^{\frac{\theta}{2}(\lambda \log_2 \lambda)} \times \\
& (1-k)^{-\frac{(1-\lambda)\theta}{2}} \ (e^{x_2}-1)^{(1-k)} \ (x_2)^{-\frac{\theta}{2}(1-\lambda)} \ 2^{\frac{\theta}{2}((1-\lambda)\log_2(1-\lambda))} \\
= \ & 2^{h(k,\alpha,\lambda,\theta)}.
\end{aligned}
$$

Hence, the lemma has been established. ∎

Therefore, (2.13) can be expressed as follows:

$$
\begin{aligned}
\mathcal{S}_1 &= c(n,\theta) \left(\max_{(k,\alpha,\lambda)\in\mathcal{D}^{(n)}} \left\{ 2^{h(k,\alpha,\lambda,\theta)} \right\} \right)^n \\
&= c(n,\theta) \ 2^{\mu(\theta,n)n}, \qquad (2.28)
\end{aligned}
$$

where $c(n,\theta) = \Omega(n^{-3/2})$, $c(n,\theta) = O(n^{3/2})$ and

$$
\mu(\theta,n) = \max_{(k,\alpha,\lambda)\in\mathcal{D}^{(n)}} \{h(k,\alpha,\lambda,\theta)\}.
$$

Let $\mathcal{D}^{(\infty)} = \limsup \mathcal{D}^{(n)}$ and note that it is the set of rationals contained in $\mathcal{D}$ and, moreover, it is dense inside it. Thus, we have

$$
\lim_{n\to\infty} \mu(\theta,n) = \mu(\theta) = \sup_{(k,\alpha,\lambda)\in\mathcal{D}^{(\infty)}} h(k,\alpha,\lambda,\theta) = \max_{(k,\alpha,\lambda)\in\mathcal{D}} h(k,\alpha,\lambda,\theta),
$$

since $h(k,\alpha,\lambda,\theta)$ is continuous on $\mathcal{D}$, which is a compact subset of $\mathbb{R}^3$. In fact, we can say a bit more than this. We know that $h(k,\alpha,\lambda,\theta)$ attains its maximum at an internal point of $\mathcal{D}$, say $\mathbf{x}^*$. Note that this is a stationary point and one can also see that h is differentiable on $\mathcal{D}$ and its derivatives are continuous. The latter implies that for any $\varepsilon > 0$ there exists an open ball U containing $\mathbf{x}^*$ where $\|\nabla h\| < \varepsilon$. For n sufficiently large, there is a point $\mathbf{x}_n \in \mathcal{D}^{(n)} \cap U$ with

$$
\|\mathbf{x}_n - \mathbf{x}^*\|^2 \leq \left(2\left(\frac{1}{2n}\right)^2 + \left(\frac{1}{m}\right)^2 \right) \leq \left(\frac{1}{2} + \frac{4}{\theta_l^2}\right) n^{-2},
$$

and then

$$
\mu(\theta,n) \geq \mu(\theta) - \varepsilon \left(\frac{1}{2} + \frac{4}{\theta_l^2}\right)^{1/2} n^{-1},
$$

by the Mean Value Theorem. Hence,

$$
\sup_{\theta\in[\theta_l,\theta_u]} n(\mu(\theta) - \mu(\theta,n)) = O(1).
$$

This fact along with (2.28) imply the following:

Lemma 2.7.2 *Uniformly over $\theta \in [\theta_l, \theta_u]$,*

$$\mathcal{S}_1 = 2^{\mu(\theta)n+O(\log_2 n)}.$$

Since $h(k, \alpha, \lambda, \theta)$ is continuous on its domain $\mathcal{D}$, and h as a function of θ is also continuous, the function $\mu(\theta)$ is continuous as well. As we shall see later, for $\theta_0 = 4.9893$, we have

$$\mu(\theta_0) > 0. \tag{2.29}$$

The numerical investigation in the next section shows that

$$\mu(\theta_2) < 0, \tag{2.30}$$

for $\theta_2 = 4.9895$. By Lemma 2.5.3, there exists $\tilde{\delta} > 0$ such that $\mathcal{S}_2 \leq 2^{-\tilde{\delta}n}$ for n sufficiently large, uniformly over $\theta \geq 4.98$. We set $\delta' = \min\{\tilde{\delta}, -\mu(\theta_2)\}$. Let

$$\theta_1 = \inf\{\theta \geq \theta_0 : \ \mu(\theta) \leq -\delta'/2\}.$$

Note that (2.29) implies that $\theta_0 \leq \theta_1 < \theta_2$. Thus, we have $\mu(\theta_1) = -\delta'/2$ and $\mu(\theta) \geq -\delta'/2$ for each $\theta \in [\theta_0, \theta_1]$. Therefore, Lemma 2.7.2 implies that for any $\theta \in [\theta_0, \theta_1]$, we have

$$\mathbb{E}[R(\mathcal{G}^*_{n,m})] \ = \ 2^{\mu(\theta)n+O(\log_2 n)} + \mathcal{S}_2 \leq 2^{\mu(\theta)n+O(\log_2 n)} + 2^{-\delta' n} = 2^{\mu(\theta)n+O(\log_2 n)}$$

and

$$\mathbb{E}[R(\mathcal{G}^*_{n,m})] \geq 2^{\mu(\theta)n+O(\log_2 n)}.$$

The latter follows by simply deleting the term $\mathcal{S}_2$, since every term is non-negative. Therefore for $\theta \in [\theta_0, \theta_1]$, we have

$$\mathbb{E}[R(\mathcal{G}^*_{n,m})] = 2^{\mu(\theta)n+O(\log_2 n)}. \tag{2.31}$$

Now, to establish Theorem 2.3.1, i.e. to show that this result in fact is also true in the $\mathcal{G}_{n,m}$ model, we have to do some more work. We prove the following:

Lemma 2.7.3 *Uniformly over $\theta \in [\theta_l, b]$ for any $b \geq \theta_l$ we have*

$$\mathbb{E}[R(\mathcal{G}_{n,m})] = \Theta(1)\ \mathbb{E}[R(\mathcal{G}^*_{n,m})], \tag{2.32}$$

where $m = \lceil \theta n/2 \rceil$.

Proof. To see the one direction note that

$$\mathbb{E}[R(\mathcal{G}^*_{n,m})] \geq \mathbb{E}[R(\mathcal{G}^*_{n,m}) || E(\mathcal{G}^*_{n,m})| = m]\ \mathbb{P}[|E(\mathcal{G}^*_{n,m})| = m],$$

where the probability on the right-hand side is bounded below by a constant and

$$\mathbb{E}[R(\mathcal{G}^*_{n,m}) || E(\mathcal{G}^*_{n,m})| = m] = \mathbb{E}[R(\mathcal{G}_{n,m})].$$

Recall that $\ln(1-x) \geq -\frac{x}{1-x} \geq -2x$, for $0 \leq x \leq 1/2$. Hence, if $2m \leq \binom{n}{2}$,

$$\begin{aligned}
\mathbb{P}[|E(\mathcal{G}^*_{n,m})| = m] &= \prod_{i=0}^{m-1} \left(1 - \frac{i}{\binom{n}{2}}\right) \\
&= \exp\left(\sum_{i=0}^{m-1} \ln\left(1 - \frac{i}{\binom{n}{2}}\right)\right) \\
&\geq \exp\left(-2\sum_{i=0}^{m-1} \frac{i}{\binom{n}{2}}\right) = \exp\left(-2\frac{\binom{m}{2}}{\binom{n}{2}}\right).
\end{aligned}$$

This expression is bounded away from 0 uniformly for θ in the closed interval $[\theta_l, b]$.

The other direction is a bit more tricky. To see this note that

$$\mathbb{P}[S \in \mathcal{R}(\mathcal{G}_{n,m})] = \mathbb{P}[S \text{ is proper for } \mathcal{G}_{n,m}]\ \mathbb{P}[S \in \mathcal{R}(\mathcal{G}_{n,m}) | S \text{ is proper for } \mathcal{G}_{n,m}].$$

By Lemma 2.5.2, we have $e(S) = n^2\phi(k,\alpha) \geq 0.28n^2$, for a colouring $S \in \mathcal{C}(k,\alpha,n)$, where $(k,\alpha) \in \mathrm{D}_1$, whence we obtain $e(S)/m^2 \geq \eta > 0$, for some η (depending only on b). Thus,

$$\begin{aligned}
\left(1 - \frac{m}{e(S)}\right)^m &= \exp\left(m\ \ln\left(1 - \frac{m}{e(S)}\right)\right) \\
&\geq \exp(-2m^2/e(S)) \geq e^{-2/\eta}. \qquad (2.33)
\end{aligned}$$

Hence, for such an S, we have

$$\begin{aligned}
\mathbb{P}[S \text{ is proper for } \mathcal{G}_{n,m}] &= \frac{e(S)}{\binom{n}{2}}\ \frac{e(S)-1}{\binom{n}{2}-1} \cdots \frac{e(S)-m+1}{\binom{n}{2}-m+1} \\
&\geq \left(\left(\frac{e(S)}{\binom{n}{2}}\right)\left(1 - \frac{m}{e(S)}\right)\right)^m \\
&\geq e^{-2/\eta}\ \mathbb{P}[S \text{ is proper for } \mathcal{G}^*_{n,m}].
\end{aligned}$$

On the other hand,

$$\mathbb{P}[S \in \mathcal{R}(\mathcal{G}_{n,m}) | S \text{ is proper for } \mathcal{G}_{n,m}] \geq \mathbb{P}[S \in \mathcal{R}(\mathcal{G}^*_{n,m}) | S \text{ is proper for } \mathcal{G}^*_{n,m}],$$

since adding edges to a proper 3-colouring increases the probability that this is rigid. Therefore,

$$\mathbb{P}[S \in \mathcal{R}(\mathcal{G}^*_{n,m})] \quad \leq \quad e^{2/\eta}\, \mathbb{P}[S \in \mathcal{R}(\mathcal{G}_{n,m})].$$

Finally, by Lemma 2.5.3, since $\theta \in [\theta_l, b]$,

$$\begin{aligned}
\mathbb{E}[R(\mathcal{G}^*_{n,m})] &= (1+o(1)) \sum_{(k,\alpha)\in \mathrm{D}_1^{(n)}} \sum_{S\in\mathcal{C}(k,\alpha,n)} \mathbb{P}[S \in \mathcal{R}(\mathcal{G}^*_{n,m})] \\
&\leq (1+o(1))\, e^{2/\eta} \sum_{(k,\alpha)\in \mathrm{D}_1^{(n)}} \sum_{S\in\mathcal{C}(k,\alpha,n)} \mathbb{P}[S \in \mathcal{R}(\mathcal{G}_{n,m})] \\
&\leq (1+o(1))\, e^{2/\eta}\, \mathbb{E}[R(\mathcal{G}_{n,m})].
\end{aligned}$$

∎

Thus, Lemma 2.7.3 along with Lemma 2.7.2 and inequalities (2.29) and (2.30) conclude the proof of Theorem 2.3.1. Corollary 2.3.2 follows immediately.

2.8 Numerical computations

Let us quickly dispose of the easy result (2.29). As above, let $\theta_0 = 4.9893$. Also, we set $k = 0.698$, $\alpha = 0.362$ and $\lambda = 0.691$. We find using Maple with 10 digits precision that $h(k, \alpha, \lambda, \theta_0) > 0$. In fact, to one digit it equals 5×10^{-6}. These values for k, α, λ were found using the Complex method [9]: see [16] for an implementation of this method in Maple.

To complete the proof of Theorem 2.3.1, and thus of Corollary 2.3.2, it remains to establish (2.30). Let $h = h(k, \alpha, \lambda) = h(k, \alpha, \lambda, \theta_2)$, where as above $\theta_2 = 4.9895$. Then h is continuous over $\mathcal{D}$, and we shall see in Appendix C that it is strictly concave over the interior of $\mathcal{D}$. Thus, if $\mathcal{C} \subseteq \mathcal{D}$ and $\mathbf{y} \in \mathcal{C}^o$ (the interior of $\mathcal{C}$) are such that $h(\mathbf{y}) > h(\mathbf{x})$ for every $\mathbf{x}$ in the boundary of $\mathcal{C}$, then h has a unique maximum point $\mathbf{x}^*$ over $\mathcal{D}$ and $\mathbf{x}^* \in \mathcal{C}^o$. Using concavity, we can estimate numerically where the maximum of h is located, and give an upper bound on h over this domain, as follows.

We obtain a first approximation of the maximum of $h(k, \alpha, \lambda)$ numerically, using a method for constrained optimisation called the Complex Method. This algorithm has been developed by M. J. Box in [8]. We have implemented the method as it appears in [9]. In our implementation, we have chosen α, which is the reflection parameter, to be 1.3. The termination conditions are $\sigma^2 < 10^{-8}$ and $d_{\max} < 10^{-4}$, the standard deviation of the set of points and the maximum distance between any two of them, respectively. Strengthening the convergence conditions, we can obtain arbitrarily good approximation for the maximum. The maximum value of $h(k,\alpha,\lambda)$ in $\mathcal{D}$ is almost equal to -0.00003938856489 for $k = 0.6980552884$, $\alpha = 0.3622352192$ and $\lambda = 0.6910563713$. The starting point is for $k = 0.66$, $\alpha = 0.33$ and $\lambda = 0.6$ (we can easily check that this satisfies the constraints). Thus, our attention is directed to the cube $\mathcal{C} \subseteq \mathcal{D}$, where $0.6980 \leq k \leq 0.6981$, $0.3622 \leq \alpha \leq 0.3623$ and $0.6910 \leq \lambda \leq 0.6911$. Divide the surface of the cube $\mathcal{C}$ into squares of side $s = 5 \times 10^{-6}$. For a square centred at $\mathbf{a}$, by the concavity of h we obtain

$$h(\mathbf{b}) \leq h(\mathbf{a}) + (s/\sqrt{2})\|\nabla h(\mathbf{a})\|$$

for each point $\mathbf{b}$ in the square. By checking each square, we find that $h(\mathbf{x}) \leq -3.937721 \times 10^{-5}$ for each point $\mathbf{x}$ on the surface of the cube $\mathcal{C}$. But there is a point $\mathbf{y}$ inside $\mathcal{C}$ with $h(\mathbf{y})$ strictly greater than this bound. More specifically, we may define a cubic grid inside $\mathcal{C}$ each cube having side equal to 5×10^{-6}. The maximum value we find by searching this grid is equal to -3.937414×10^{-5} and it is strictly greater than the upper bound on h on the surface of $\mathcal{C}$. Since h is concave on $\mathcal{D}$, it follows as noted above that h attains its maximum over $\mathcal{D}$ inside $\mathcal{C}$.

Now we obtain an upper bound on h inside $\mathcal{C}$, using the aforementioned grid. By concavity, for each point $\mathbf{b}$ in the subcube with its centre located at $\mathbf{a}$ and edge of length equal to s, we have

$$h(\mathbf{b}) \leq h(\mathbf{a}) + \frac{\sqrt{3}s}{2}\|\nabla h(\mathbf{a})\|.$$

By checking through each subcube, we find that $h(\mathbf{x})$ is less than -3.9×10^{-5}, for each $\mathbf{x} \in \mathcal{C}$, and thus for each $\mathbf{x} \in \mathcal{D}$.

2.9 Proof of Theorem 2.3.3

For a graph $G = (V,E)$, let $t(G)$ denote the number of components of G that are non-trivial trees (that is, trees with at least one edge).

Lemma 2.9.1 *For any $t \geq 0$ and any positive integers n and m with $0 < m \leq \binom{n}{2}$,*

$$\mathbb{P}[\chi(\mathcal{G}_{n,m}) \leq 3] \leq 2^{-t}\ \mathbb{E}[R(\mathcal{G}_{n,m})] + \mathbb{P}[t(\mathcal{G}_{n,m}) < t].$$

Proof. For a graph $G = (V_n, E)$, let $giant(G)$ denote the lexicographically first component of maximum order. For each unlabelled tree T, let $X_T(G)$ be the number of components of G which are isomorphic to T. Then

$$R(G) \geq R(giant(G)) \prod_T R(T)^{X_T(G)}, \tag{2.34}$$

where the product is over all unlabelled trees T, except that we do not count $giant(G)$, if it is a tree. Since $R(T) \geq 2$ for each non-trivial tree, we obtain:

$$R(G) \geq R(giant(G))\ 2^{t(G)-1}. \tag{2.35}$$

We proceed using inequality (2.35) (see next section for a further treatment based on (2.34)). In particular, if $t(G) \geq t+1$, then

$$R(giant(G)) \leq 2^{-t}\ R(G). \tag{2.36}$$

Recall that for a non-negative random variable X, and an event A on the same probability space, we have

$$\mathbb{P}[X \geq 1] \leq \mathbb{E}[X; A] + \mathbb{P}[\overline{A}]. \tag{2.37}$$

Take A to be the event that "$t(\mathcal{G}_{n,m}) \geq t+1$". Then, by (2.36) and (2.37) we obtain:

$$\begin{aligned}
\mathbb{P}[\chi(\mathcal{G}_{n,m}) \leq 3] &\leq \mathbb{E}[\mathbf{1}_{\{\chi(giant(\mathcal{G}_{n,m}))\leq 3\}}] = \mathbb{E}[\mathbf{1}_{\{R(giant(\mathcal{G}_{n,m}))\geq 1\}}] \\
&\leq \mathbb{E}[R(giant(\mathcal{G}_{n,m}))\mathbf{1}_A] + \mathbb{P}[\overline{A}] \\
&\leq 2^{-t}\mathbb{E}[R(\mathcal{G}_{n,m})] + \mathbb{P}[\overline{A}] \leq 2^{-t}\mathbb{E}[R(\mathcal{G}_{n,m})] + \mathbb{P}[t(\mathcal{G}_{n,m}) < t+1]
\end{aligned}$$

and this concludes the proof of the lemma. ∎

For $\theta > 0$ let

$$\tau(\theta) = \sum_{t=2}^{\infty} \frac{\theta^{t-1} e^{-t\theta} t^{t-2}}{t!},$$

and $m = \lceil \theta n/2 \rceil$. We shall use standard methods to prove:

Lemma 2.9.2 *Let $\theta > 0$. For any $\varepsilon > 0$ there exists $\delta_1 > 0$ such that*

$$\mathbb{P}[t(\mathcal{G}_{n,m}) \leq \tau(\theta)n - \varepsilon n] = O(e^{-\delta_1 n}).$$

Proof. Following the proof of Proposition 2.2.10, we can see that

$$\mathbb{E}[t(\mathcal{G}_{n,m})] = n \left(\sum_{t=2}^{\infty} \theta^{t-1} e^{-\theta t} t^{t-2}/t! \right) + O(1) = n \cdot \tau(\theta) + O(1).$$

Lemma 2.2.11 will complete the proof, since if G' is obtained from G by adding an edge and deleting an edge then $|t(G) - t(G')| \leq 2$. ∎

Therefore, setting $t = \tau(\theta)n - \varepsilon n$ in Lemma 2.9.1, and using also Lemmas 2.9.2, 2.7.2 and 2.7.3 along with equation (2.14) we obtain the following:

$$\begin{aligned}
\mathbb{P}\left[\chi(\mathcal{G}_{n,m}) \leq 3\right] &\leq 2^{-n\tau(\theta)+\varepsilon n}\mathbb{E}\left[R(\mathcal{G}_{n,m})\right] + C\ e^{-\delta_1 n} \\
&= 2^{\varepsilon n}\ 2^{-n\tau(\theta)}\ (2^{n\mu(\theta)+O(\log_2 n)} + \mathcal{S}_2) + C\ e^{-\delta_1 n},
\end{aligned}$$

for a positive constant C and for n sufficiently large. We shall see that for $\theta = 4.98887$

$$\mu(\theta) < \tau(\theta), \tag{2.38}$$

and Theorem 2.3.3 will then follow easily.

From the proof of Lemma 2.5.3 we deduce that for $\theta = 4.98887$, $\mathcal{S}_2$ is at most $2^{-\delta_2 n}$ for some $\delta_2 > 0$, for n sufficiently large. We keep the definition of the region $\mathcal{D}$ unchanged. Inside $\mathcal{D}$, we may perform numerical investigations similar to those in the case of the rigid 3-colourings. Let us prove (2.38). Letting $h(k, \alpha, \lambda) = h(k, \alpha, \lambda, \theta)$ for $\theta = 4.98887$, we must show that for $(k, \alpha, \lambda) \in \mathcal{D}$ we have $\tilde{h}(k, \alpha, \lambda) = h(k, \alpha, \lambda) - \tau(\theta) < 0$, for this specific θ. Observe that $\tilde{h}(k, \alpha, \lambda)$ is also continuous on $\mathcal{D}$. In fact, we have to show that $\tilde{\mu}(\theta) = \max_{(k,\alpha,\lambda)\in\mathcal{D}} \tilde{h}(k, \alpha, \lambda) = \mu(\theta) - \tau(\theta) < 0$. Note that all the partial derivatives of $\tilde{h}(k, \alpha, \lambda)$ are exactly those of $h(k, \alpha, \lambda)$, since we have added just a constant.

Arguing as in subsection 2.8, we may bound $\tilde{h}$ inside $\mathcal{D}$. We observed that the maximum value of $\tilde{h}$ inside $\mathcal{D}$ is attained inside the cube $\mathcal{C} \subseteq \mathcal{D}$, where $0.6980 \leq k \leq 0.6981$, $0.3622 \leq \alpha \leq 0.3623$ and $0.6910 \leq \lambda \leq 0.6911$. More specifically, we have applied the Complex Method with the same terminating parameters as before in $\mathcal{D}$, since concavity has been established there. For $\theta = 4.98887$, the maximum value of $\tilde{h}(k, \alpha, \lambda)$ is almost equal to -0.00001213643454 for $k = 0.6980500153$, $\alpha = 0.3622539825$ and $\lambda = 0.6910760813$. The starting point is for $k = 0.67$, $\alpha = 0.35$ and $\lambda = 0.69$ (we can easily check that this satisfies the constraints). We have followed the same analysis as in the case of the rigid colourings. We have checked the value of the function for more than 9000 points in a very fine area around the values of k, α and λ that mentioned above. In particular, we have checked $\tilde{h}(k, \alpha, \lambda)$ in the cuboid $0.6980 \leq k \leq 0.6981$, $0.3622 \leq \alpha \leq 0.3623$ and $0.6910 \leq \lambda \leq 0.6911$. Let (k, α, λ) be a point on its boundary. We have defined a grid on the boundary with step size equal to 0.000005. Again, concavity yields that $\tilde{h}(k, \alpha, \lambda) + 0.000005 \cdot |\nabla\tilde{h}(k, \alpha, \lambda)| \cdot \frac{\sqrt{2}}{2}$ is an upper bound on the value of the function at points whose distance from any point of this grid is not bigger than $0.000005 \cdot \frac{\sqrt{2}}{2}$. Thus, we obtain an upper bound on the value of the function at any point on the boundary. Having checked the value of this expression for the points of this grid, the value of the objective function on that is now no more than -0.00001214075434. Furthermore, we have defined a fine grid inside the cuboid with step size equal to 0.000005. The maximum value we have found by searching this grid is equal to -0.00001213744444 and it is greater than the upper bound of the function for the boundary points. Since the objective function is concave in a region that contains this specific one, we deduce that its real maximum must be attained inside the cuboid. The value of $h(k, \alpha, \lambda)$ at points whose distance from any point of the grid (k, α, λ), over the region where the function is concave, is no bigger than $0.000005 \cdot \frac{\sqrt{3}}{2}$ cannot be more than $\tilde{h}(k, \alpha, \lambda) + 0.000005 \cdot |\nabla \tilde{h}(k, \alpha, \lambda)| \cdot \frac{\sqrt{3}}{2}$. Checking through the grid points, for any point (k, α, λ), where $0.6980 \leq k \leq 0.6981$, $0.3622 \leq \alpha \leq 0.3623$ and $0.6910 \leq \lambda \leq 0.6911$, $\tilde{\mu}(\theta)$ is less than -0.00001213582426 for $\theta = 4.98887$. Hence, $\tilde{\mu}(\theta)$ is less than -1.2×10^{-5} inside $\mathcal{D}$, for $\theta = 4.98887$. Note that for $k = 0.698$, $\alpha = 0.362$, $\lambda = 0.691$ and $\theta = 4.9888$, $\tilde{h}(k, \alpha, \lambda) > 0$.

Therefore, choosing $\varepsilon = \min\{-\tilde{\mu}(\theta)/2, \delta_2/2\}$, where $\theta = 4.98887$, we deduce that for $n \geq n_0$, for some n_0 sufficiently large, there exists a constant $\delta > 0$ such that

$$\mathbb{P}\left[\chi\left(\mathcal{G}_{n,m}\right) \leq 3\right] \leq 2^{-\delta n},$$

for $m = \lceil \theta n/2 \rceil$.

2.10 A full treatment

In fact, the above treatment can be viewed as a special case of a more general approach. We begin with some definitions. Let $K = \{1, \ldots, k\}$ and $\mathcal{T}_k$ be the family of all possible labelled trees on the vertex set K, where $k \geq 2$ is a fixed integer. There are k^{k-2} of them. Now, let $f_k : \mathcal{T}_k \to [k^{k-2}]$ be an arbitrary bijective map, which can be considered as an enumeration of these trees. Let $T_l^{(k)} = f_k^{-1}(l)$, for $l \in [k^{k-2}]$. For $A \in [V_n]^k$ (the set of k-subsets of V_n) define $\phi_A : K \to A$ to be the bijective map for which for any $i, j \in K$, where $i \neq j$, we have $i < j$ if and only if $\phi_A(i) < \phi_A(j)$. Now, let $\Phi_A : \mathcal{T}_k \to G[A]$ be the map from the set of trees to the set of all simple graphs on A, such that for any $i, j \in K$, we have $\{i, j\} \in E(T)$, where $T \in \mathcal{T}_k$, if and only if $\{\phi_A(i), \phi_A(j)\} \in E(\Phi_A(T))$. Now, for $T \in \mathcal{T}_k$, a graph $G = (V_n, E)$ and some $A \in [V_n]^k$, we define the indicator function $\varepsilon_A^T(G)$ which equals 1 if and only if $\Phi_A(T)$ is a component of G (otherwise it is 0). Note that $\prod_{l=1}^{k^{k-2}} R(T_l^{(k)})^{\sum_{A \in [V_n]^k} \varepsilon_A^{T_l^{(k)}}(G)}$ is a lower bound on $R(G)$. For any fixed positive integer $k \geq 2$ and $l \in [k^{k-2}]$, we set

$$Y_l^{(k)} = Y_l^{(k)}(\mathcal{G}_{n,m}) = \sum_{A \in [V_n]^k} \varepsilon_A^{T_l^{(k)}}\left(\mathcal{G}_{n,m}\right).$$

Therefore, Lemma 2.9.1 can be extended as follows:

Lemma 2.10.1 *Let $K_0 \geq 2$ be a fixed positive integer and $\{\{t_l^{(k)}\}_{l=1,\ldots,k^{k-2}}\}_{k=2,\ldots,K_0}$ be a family of sets of positive integers. Then for any positive integers n and m such that $0 < m \leq \binom{n}{2}$, we* *have*

$$\mathbb{P}\left[\chi(\mathcal{G}_{n,m}) \leq 3\right] \leq \prod_{k=2}^{K_0}\prod_{l=1}^{k^{k-2}} R(T_l^{(k)})^{-t_l^{(k)}}\mathbb{E}\left[R(\mathcal{G}_{n,m})\right] + \sum_{k=2}^{K_0}\sum_{l=1}^{k^{k-2}} \mathbb{P}\left[\sum_{A\in[V_n]^k} \varepsilon_A^{T_l^{(k)}}(\mathcal{G}_{n,m}) < t_l^{(k)}\right].$$

Proof. The proof is very similar to the proof of Lemma 2.9.1 For a graph $G = (V_n, E)$, let $giant(G)$ denote the lexicographically first component of maximum order. Let us consider a fixed $K_0 \geq 2$. Then

$$R(G) \geq R(giant(G)) \prod_{k=2}^{K_0}\prod_{l=1}^{k^{k-2}} R(T_l^{(k)})^{\sum_{A\in[V_n]^k} \varepsilon_A^{T_l^{(k)}}(G)},$$

and we obtain

$$R(giant(G)) \leq R(G) \prod_{k=2}^{K_0}\prod_{l=1}^{k^{k-2}} R(T_l^{(k)})^{-\sum_{A\in[V_n]^k} \varepsilon_A^{T_l^{(k)}}(G)}. \tag{2.39}$$

Let us recall inequality (2.37)

$$\mathbb{P}[X \geq 1] \leq \mathbb{E}[X; A] + \mathbb{P}[\overline{A}].$$

Let A be the event

"for each $2 \leq k \leq K_0$ and $l \in [k^{k-2}]$, we have $\sum_{A\in[V_n]^k} \varepsilon_A^{T_l^{(k)}}(\mathcal{G}_{n,m}) \geq t_l^{(k)}$".

Then, by (2.39) and the previous inequality we obtain:

$$\begin{aligned}
\mathbb{P}[\chi(\mathcal{G}_{n,m}) \leq 3] &\leq \mathbb{E}[\mathbf{1}_{\{\chi(giant(\mathcal{G}_{n,m}))\leq 3\}}] = \mathbb{E}[\mathbf{1}_{\{R(giant(\mathcal{G}_{n,m}))\geq 1\}}] \\
&\leq \mathbb{E}[R(giant(\mathcal{G}_{n,m}))\mathbf{1}_A] + \mathbb{P}[\overline{A}] \\
&\leq \prod_{k=2}^{K_0}\prod_{l=1}^{k^{k-2}} R(T_l^{(k)})^{-t_l^{(k)}}\ \mathbb{E}[R(\mathcal{G}_{n,m})] + \mathbb{P}[\overline{A}] \\
&\leq \prod_{k=2}^{K_0}\prod_{l=1}^{k^{k-2}} R(T_l^{(k)})^{-t_l^{(k)}}\mathbb{E}[R(\mathcal{G}_{n,m})] \\
&\quad + \sum_{k=2}^{K_0}\sum_{l=1}^{k^{k-2}} \mathbb{P}\left[\sum_{A\in[V_n]^k} \varepsilon_A^{T_l^{(k)}}(\mathcal{G}_{n,m}) < t_l^{(k)}\right],
\end{aligned}$$

and this concludes the proof of the lemma. ∎

In the following lemma, we determine the expectation of $Y_l^{(k)}$.

Lemma 2.10.2 *For any fixed $k \geq 2$ and $l \in [k^{k-2}]$, where $0 < \theta < \infty$ is fixed and $m = \lceil \theta n/2 \rceil$, we have*

$$\mathbb{E}[Y_l^{(k)}(\mathcal{G}_{n,m})] = n\ \frac{\theta^{k-1}}{k!}\ \exp(-k\theta) + O(1).$$

Proof. Using standard estimates as in the proof of Proposition 2.2.10, for any fixed $k \geq 2$ and $l \in [k^{k-2}]$, we have

$$\begin{aligned}\mathbb{E}\left[\sum_{A\in[V_n]^k} \varepsilon_A^{T_l^{(k)}}(\mathcal{G}_{n,m})\right] &= \binom{n}{k}\frac{\binom{\binom{n-k}{2}}{m-k+1}}{\binom{\binom{n}{2}}{m}} = n\,\frac{\theta^{k-1}}{k!}\,\exp(-k\theta)\,\left(1\pm C\frac{1}{n}\right)\\ &= n\,\frac{\theta^{k-1}}{k!}\,\exp(-k\theta)+O(1),\end{aligned}$$

for some $C > 0$. ∎

Applying the concentration bound for $Y_l^{(k)}$, we obtain the following lemma:

Lemma 2.10.3 *Let $k \geq 2$ be fixed and $l \in [k^{k-2}]$. Then, uniformly for any $\theta \in [a, b]$, where $0 < a < b < \infty$, we have*

$$\mathbb{P}\left[Y_l^{(k)}(\mathcal{G}_{n,m}) \leq \mathbb{E}\left[Y_l^{(k)}(\mathcal{G}_{n,m})\right] - n^{1/2}\ln n\right] = o(1).$$

Proof. Let G and G' be two simple graphs on V_n such that G' is obtained from G by adding an edge and deleting an edge. Then

$$\left|Y_l^{(k)}(G) - Y_l^{(k)}(G')\right| \leq 3, \tag{2.40}$$

because if two graphs differ in only one edge the difference in the number of tree-components on k vertices of "type" l is at most three. Hence,

$$\begin{aligned}\mathbb{P}\left[Y_l^{(k)}(\mathcal{G}_{n,m}) \leq \mathbb{E}\left[Y_l^{(k)}(\mathcal{G}_{n,m})\right] - n^{\frac{1}{2}}\ln n\right] &\leq \exp\left(-\frac{2\left(n^{\frac{1}{2}}\ln n\right)^2}{9m}\right)\\ &= \exp\left(-\Theta\left(\ln^2 n\right)\right) = o(1),\end{aligned}$$

by Lemma 2.2.11, uniformly over $\theta \in [a, b]$. ∎

Now, setting $t_l^{(k)} = \mathbb{E}\left[Y_l^{(k)}(\mathcal{G}_{n,m})\right] - n^{1/2}\ln n$, the sum in Lemma 2.10.1 is $o(1)$. By Lemmas 2.5.3, 2.5.4 and 2.10.2, we obtain:

$$\begin{aligned}\mathbb{P}\left[\chi(\mathcal{G}_{n,m}) \leq 3\right] &\leq C\prod_{k=2}^{K_0}\prod_{l=1}^{k^{k-2}} R(T_l^{(k)})^{-n\,\frac{\theta^{k-1}}{k!}\,\exp(-k\theta)}\mathbb{E}\left[R(\mathcal{G}_{n,m})\right] + o(1)\\ &= 2^{o(n)}\prod_{k=2}^{K_0}\left(\prod_{l=1}^{k^{k-2}} R(T_l^{(k)})\right)^{-n\,\frac{\theta^{k-1}}{k!}\,\exp(-k\theta)} 2^{n\mu(\theta)} + o(1),\end{aligned}$$

for some positive constant C, uniformly for θ in any interval $[a, b]$, where $4.98 < a < b$. Setting $K_0 = 5$, we have computed explicitly each term of the product appearing in the above inequality.

This computation has been carried through using Maple and representing each tree by its Prüfer code. That is, for each tree of k vertices, where $2 \leq k \leq 5$, we have found the number of its rigid 3-colourings (see Appendix D) and then we multiplied these numbers together in order to obtain the basis of the power that comprises each term. Since some of these numbers are huge (e.g. about 80 digits long) we are giving the final values of the terms to 10 digits precision. Namely,

$$\begin{aligned}
\left(\prod_{l=1}^{2^{2-2}} R(T_l^{(2)})\right)^{-\frac{\theta^{2-1}}{2!}\exp(-2\theta)} &= 0.9999197393 \\
\left(\prod_{l=1}^{3^{3-2}} R(T_l^{(3)})\right)^{-\frac{\theta^{3-1}}{3!}\exp(-3\theta)} &= 0.9999972718 \\
\left(\prod_{l=1}^{4^{4-2}} R(T_l^{(4)})\right)^{-\frac{\theta^{4-1}}{4!}\exp(-4\theta)} &= 0.9999997836 \\
\left(\prod_{l=1}^{5^{5-2}} R(T_l^{(5)})\right)^{-\frac{\theta^{5-1}}{5!}\exp(-5\theta)} &= 0.9999999853,
\end{aligned}$$

for $\theta = 4.98887$. Their product is equal to 0.9999167802. We observe that this is not very different from the first term, which corresponds to tree components that are isolated edges, and also not very different from the corresponding term that we used in the previous section (that was equal to 0.9999168807). This reveals that the dominating term in the jackpot phenomenon that occurs due to the exponential number of rigid 3-colourings of the tree-components comes from the isolated edges.

2.11 Concluding remarks

After analysing the expected number of proper 3-colourings of a $\mathcal{G}_{n,m}$ random graph and identifying a source of jackpot phenomena, we considered the adequate family consisting of the rigid 3-colourings of a graph, and investigated carefully the expected number of such colourings in the random graph $\mathcal{G}_{n,m}$. We thus obtained an upper bound on the non-3-colourability threshold θ_3^+, which appears independently in [24]. We then improved this upper bound slightly, by taking into account the number of non-trivial tree components. When we considered the number of non-trivial tree components in Section 2.9, in the proof of Lemma 2.9.1 we used the fact that $R(T) \geq 2$ for any non-trivial tree T. We can be more precise; for example $R(T) \geq 3$ unless T is a star. By computing the value of $R(T)$ for each 'small' non-trivial tree (those having at most 5 vertices), and then following the general approach in Section 2.9, we showed in Section 2.10 that essentially the isolated edges give the main contribution to the jackpot phenomena that occur due to the tree-components of the $\mathcal{G}_{n,m}$ random graph.

We may also consider an adequate subfamily of the rigid 3-colourings of a graph G, namely the *leftmost* 3-colourings. These are the proper 3-colourings S_1, S_2, S_3 where $|S_3|$ is minimal and, subject to this, $|S_2|$ is minimal. Note that any such 3-colouring must be rigid and, further, this family is adequate. Unfortunately it seems to be hard to study leftmost 3-colourings, but we can work with related families such as those defined in terms of "Ψ-gadgets".

Given a 3-colouring S_1, S_2, S_3 of a graph G, a Ψ_{12}-gadget is a component of the subgraph induced on $S_1 \cup S_2$, which is a star with centre in S_1 and at least 2 leaves (which must belong to S_2). We may define Ψ_{13}- and Ψ_{23}-gadgets similarly. Call a rigid 3-colouring Ψ-*gadget free* if there are no Ψ_{12} or Ψ_{13} or Ψ_{23} gadgets. Note that these 3-colourings form an adequate family, since each leftmost 3-colouring is Ψ-gadget free. By analysing such families of 3-colourings we may reduce the upper bound on θ_3^+ slightly.

Chapter 3

On the chromatic number of random regular graphs

3.1 Introduction

In this chapter, we investigate the k-colourability of random r-regular graphs, for any integer $k \geq 3$. As we have already seen D. Achlioptas and E. Friedgut established in [1], for each fixed $k > 2$, the existence of a function $d_k(n)$ such that for any $\epsilon > 0$ and any $m = m(n)$, if $m \leq (1-\epsilon)\left(\frac{d_k(n)}{2}n\right)$ a $\mathcal{G}_{n,m}$ random graph is k-colourable a.a.s. and if $m \geq (1+\epsilon)\left(\frac{d_k(n)}{2}n\right)$ a $\mathcal{G}_{n,m}$ random graph is not k-colourable a.a.s.. In particular, if $k = 3$, the best known results so far concern the asymptotic upper and lower bounds on the function $d_3(n)$. As we have already seen, it has been shown that

$$4.03 \leq \liminf_{n\to\infty} d_3(n) \leq \limsup_{n\to\infty} d_3(n) < 4.99$$

see [4] and [24] or the previous chapter. The monotonicity of 3-colourability implies in particular that asymptotically almost surely a uniform random graph of average degree $r \geq 5$, is not 3-colourable. Application of the first moment method can show non-k-colourability a.a.s. for any integer $k > 3$, for a specific average degree onwards. Although a random graph of average degree r may be regular, this does not imply that a random r-regular graph is also asymptotically almost surely not k-colourable. On the other hand, the chromatic and the independence numbers of random r-regular graphs for r a large constant have been studied in [18] by A. Frieze and T. Łuczak. In particular, they have proved that for $r_0 \leq r = o(n^\theta)$ for some fixed $\theta < 1$, the chromatic number of a random r-regular graph on V_n is with probability tending to 1 as $n \to \infty$

equal to $r/2 \ln r(1 + o(1))$, where the asymptotics here is with respect to r. Apart from the very recent result by D. Achlioptas and C. Moore (see [4]) which shows that with positive probability a random 4-regular graph is 3-colourable, there are no other results concerning the k-colourability of r-regular random graphs when r is a fixed "small" positive integer. Let $\mathbb{G}(n, r)$ be a uniformly random simple r-regular graph on V_n. The main result of this chapter is the following:

Theorem 3.1.1 *For any integer $k \geq 3$, there exists a $\delta = \delta(k) > 0$ such that for n sufficiently large*

$$\mathbb{P}[\mathbb{G}(n, r) \text{ is } k\text{-colourable}] \leq e^{-\delta n},$$

for any $r > 2 \ln k / \ln(k/(k-1))$; thus for r as it is specified above asymptotically almost surely a random simple r-regular graph is not k-colourable.

To show this, we resort to the configuration model of r-regular multigraphs, which will be described in the next section. Using a simple first moment method and the fact that an r-regular multigraph which is an outcome of the configuration model is simple with probability bounded away from zero, we derive Theorem 3.1.1. Note that the value given in the above theorem is for large k equal to $2k \ln k(1 + o(1))$ (here we take asymptotics in terms of k). Moreover, the bound obtained by by A. Frieze and T. Łuczak in [18] implies that the chromatic number of an r-regular graph is greater than k, for $r \geq 2(k + 1) \ln(k + 1)(1 + o(1))$. Observe that the ratio of these asymptotic estimates tends to 1 as k grows.

In the last two sections of this chapter, we make an (unsuccessful) attempt to prove that almost all 5-regular graphs are not 3-colourable. Note that the bound obtained by the previous theorem implies that almost all simple 6-regular graphs are not 3-colourable. In order to do this, we resort to the analysis of the adequate family of rigid 3-colourings of random 5-regular graphs. Before this, in Section 3.3 we obtain asymptotic results for a variation of the coupon collector problem (that was essentially used to handle the family of rigid 3-colourings of $\mathcal{G}_{n,m}$ random graphs) and we use them in order to obtain in Section 3.4 a tight estimate of the expected number of rigid 3-colourings of random 5-regular graphs. This is joint work with C.J.H. McDiarmid.

3.2 Probability calculations

We work with r-regular graphs on $V_n = \{1, \ldots, n\}$ (where rn is even) and we carry out our probability calculations in the *configuration model* on the set $W = V_n \times [r]$ - we call the elements of W *points*. A *configuration* is a partition of W into $rn/2$ pairs; these are called the *edges* of the configuration. Thus a configuration is a perfect matching on W. The natural projection of the set W onto V_n maps each configuration C to a multigraph $\pi(C)$ on V_n, that is the projection may contain loops and multiple edges. However, if $\pi(C)$ has no loops and multiple edges it is a simple r-regular graph. It can be easily seen that there are $(rn-1)!! \equiv (rn-1)(rn-3)\cdots 3\cdot 1 = (rn)!/\left(2^{rn/2}(rn/2)!\right)$ different configurations on W.

Thus, we define the *random r-regular multigraph* $\mathbb{G}^*(n,r)$ to be the multigraph $\pi(C)$ obtained from a uniformly random configuration C. The following can be proved (see [23]):

Proposition 3.2.1 *The following holds for any $r \geq 1$*

$$\lim_{n\to\infty} \mathbb{P}[\mathbb{G}^*(n,r) \textit{is simple}] = \exp(-(r^2-1)/4).$$

Note that the random r-regular multigraph that is obtained is not uniformly distributed among the r-regular multigraphs on V_n, since the number of configurations whose projection gives a certain multigraph depends on the number of loops and multiple edges. However, if we condition on the event that $\mathbb{G}^*(n,r)$ is simple, the r-regular random graph that is obtained by a random configuration is uniformly distributed (in this case there are $r!^n$ configurations whose projection is a certain simple graph on V_n). Thus, the following is immediate (see also [23]):

Proposition 3.2.2 *Any graph property holding a.a.s. for $\mathbb{G}^*(n,r)$, also holds a.a.s. for $\mathbb{G}(n,r)$.*

Assume that $k \geq 3$ is an integer. For $c = 1,\dots,k$, $\lambda_c = \lambda_c(n) > 0$ and $\sum_{c=1}^{k} \lambda_c = 1$, let $\mathcal{C}(\lambda_1,\dots,\lambda_k,n)$ be the family of k-colourings of V_n, where for each $S = S(V_n) \in \mathcal{C}(\lambda_1,\dots,\lambda_k,n)$ the colour classes $S_1,\dots,S_k$ are such that $|S_c| = \lambda_c n$, for $c = 1,\dots,k$. We define

$$\begin{aligned} \mathrm{D} &= \{(\lambda_1,\dots,\lambda_k) \in \mathbb{R}^k : \ \forall c \in \{1,\dots,k\}\ \lambda_c > 0, \sum_{c=1}^{k} \lambda_c = 1\}, \\ \mathrm{D}^{(n)} &= \mathrm{D} \cap \frac{1}{n}\mathbb{Z}^k. \end{aligned}$$

For an r-regular multigraph G, let $\mathcal{C}_k(G)$ be the set of proper k-colourings of G and $C_k = C_k(G)$ its cardinality. By Markov's inequality the following is immediate:

$$\begin{aligned} \mathbb{P}[\mathcal{C}_k(\mathbb{G}^*(n,r)) \neq \emptyset] &\leq \mathbb{E}[C_k(\mathbb{G}^*(n,r))] \\ &= \sum_{(\lambda_1,\dots,\lambda_k)\in \mathrm{D}^{(n)}} \ \sum_{S\in \mathcal{C}_k(\lambda_1,\dots,\lambda_k,n)} \mathbb{P}[S \in \mathcal{C}_k(\mathbb{G}^*(n,r))]. \end{aligned} \tag{3.1}$$

Assuming that $(\lambda_1,\dots,\lambda_k) \in \mathrm{D}^{(n)}$, we fix a colouring $S \in \mathcal{C}(\lambda_1,\dots,\lambda_k,n)$, with colour classes $S_1,\dots,S_k$. For $c = 1,\dots,k$, let s_c be $S_c \times [r]$, , i.e. the induced colour classes in the configuration model each having $r\lambda_c n$ points, for $c = 1, \ldots, k$. Let $f(|S_1|, \ldots, |S_k|)$ be the number of perfect matchings between points such that for each $c = 1, \ldots, k$, the set s_c is stable. Then

$$\mathbb{P}[S \in \mathcal{C}_k(\mathbb{G}^*(n,r))] = \frac{f(|S_1|, \ldots, |S_k|)}{(rn-1)!!}. \tag{3.2}$$

Note that for any permutation σ on the set $\{1, \ldots, k\}$, we have $f(|S_1|, \ldots, |S_k|) = f(|S_{\sigma(1)}|, \ldots, |S_{\sigma(k)}|)$. The following holds:

Lemma 3.2.3 *If $i \equiv n \mod k$, with $i \in \{0, \ldots, k-1\}$, then we have*

$$\max_{(\lambda_1,\ldots,\lambda_k)\in \mathrm{D}^{(n)}} \left\{ \sum_{S \in \mathcal{C}_k(\lambda_1,\ldots,\lambda_k,n)} \mathbb{P}[S \in \mathcal{C}_k(\mathbb{G}^*(n,r))] \right\} =$$

$$\binom{n}{\overbrace{\lceil\frac{n}{k}\rceil \cdots \lceil\frac{n}{k}\rceil}^{i} \lfloor\frac{n}{k}\rfloor \cdots \lfloor\frac{n}{k}\rfloor} \frac{f\left(\overbrace{\lceil\frac{n}{k}\rceil, \ldots, \lceil\frac{n}{k}\rceil}^{i}, \lfloor\frac{n}{k}\rfloor, \ldots, \lfloor\frac{n}{k}\rfloor\right)}{(rn-1)!!}.$$

Proof. Given non-negative integers $a_1, \ldots, a_k$, let $f'(a_1, \ldots, a_k)$ be the number of perfect matchings of points in W for which a specific ordered partition of W into k sets with the i-th set having a_i points, for $i = 1, \ldots, k$, is such that each of the sets is stable. For convenience define $f'(a_1, \ldots, a_k) = 0$, if $a_i < 0$ for some $i \in [k]$.

Claim 3.2.4 *If $a_1 \leq a_2 - 2$, then $f'(a_1 + 1, a_2 - 1, \ldots, a_k) \geq f'(a_1, a_2, \ldots, a_k)$.*

Proof of Claim 3.2.4.

Consider an ordered partition of W into k non-empty sets, say $s_1, \ldots, s_k$, such that for each $c = 1, \ldots, k$, we have $|s_c| = a_c$ and $a_1 \leq a_2 - 2$, which is a minimal counterexample to the claim. If we pick a point $p \in s_2$, then the number of perfect matchings on W leaving all the sets stable can be expressed, according to the set the neighbour of p belongs to, as follows:

$$f'(a_1, a_2, \ldots, a_k) = \sum_{i=3}^{k} a_i f'(a_1, a_2 - 1, \ldots, a_i - 1, \ldots, a_k) + a_1 f'(a_1 - 1, a_2 - 1, \ldots, a_k).$$

Therefore,

$$f'(a_1 + 1, a_2 - 1, \ldots, a_k) = \sum_{i=3}^{k} a_i f'(a_1 + 1, a_2 - 2, \ldots, a_i - 1, \ldots, a_k) + (a_1 + 1) f'(a_1, a_2 - 2, \ldots, a_k).$$

Thus, for each $i = 3, \ldots, k$, we have

$$f'(a_1 + 1, a_2 - 2, \ldots, a_i - 1, \ldots, a_k) \geq f'(a_1, a_2 - 1, \ldots, a_i - 1, \ldots, a_k)$$

and

$$f'(a_1, a_2 - 2, \ldots, a_k) \geq f'(a_1 - 1, a_2 - 1, \ldots, a_k),$$

by the minimality of this (counter)example and this gives a contradiction. ∎

Now consider an ordered partition of V_n that maximises f, say $S_1^{max}, \ldots, S_k^{max}$. If $S_1, \ldots, S_k$ is an ordered partition of V_n into k sets such that $f(|S_1|, \ldots, |S_k|)$ is maximised subject to the condition that $\sum_{c=1}^{k} \left||S_c| - \frac{n}{k}\right|$ is minimised, then $f(|S_1^{max}|, \ldots, |S_k^{max}|) = f(|S_1|, \ldots, |S_k|)$. Here is the proof. If $S_1^{max}, \ldots, S_k^{max}$ is not such that $\sum_{c=1}^{k} \left||S_c^{max}| - \frac{n}{k}\right|$ is minimum, then there exist two sets, say S_1^{max} and S_2^{max}, without loss of generality since f is invariant under any permutation of its arguments, such that $S_1^{max} < \lfloor n/k \rfloor$ and $S_2^{max} \geq \lceil n/k \rceil$. (The case where $S_2^{max} > \lceil n/k \rceil$ and $S_1^{max} \leq \lfloor n/k \rfloor$ is similar.) Removing any vertex from S_2 and adding it to S_1, thus creating a partition $S_1^{max'}, \ldots, S_k^{max'}$, would yield $\sum_{c=1}^{k} \left||S_c^{max'}| - \frac{n}{k}\right| < \sum_{c=1}^{k} \left||S_c^{max}| - \frac{n}{k}\right|$, but $f(|S_1^{max}|, \ldots, |S_k^{max}|) = f'(r|S_1^{max}|, \ldots, r|S_k^{max}|) \leq f'(r|S_1^{max}| + r, r|S_2^{max}| - r, \ldots, r|S_k^{max}|) = f(|S_1^{max'}|, |S_2^{max'}|, \ldots, |S_k^{max'}|)$, by a repeated application of the last claim. Our claim then follows, if we repeat this until the constraint function reaches its minimum.

To conclude the proof of the lemma, note that the multinomial coefficient $\binom{n}{|S_1| \cdots |S_k|}$ is maximised when $\lfloor n/k \rfloor \leq |S_c| \leq \lceil n/k \rceil$, for $c = 1, \ldots, k$. ∎

Therefore, the above lemma yields:

$$\mathbb{E}[C_k(\mathbb{G}^*(n, r))] = O(n^{k-1}) \binom{n}{\overbrace{\lceil \frac{n}{k} \rceil \cdots \lceil \frac{n}{k} \rceil}^{i} \lfloor \frac{n}{k} \rfloor \cdots \lfloor \frac{n}{k} \rfloor} \frac{f\left(\overbrace{\lceil \frac{n}{k} \rceil, \ldots, \lceil \frac{n}{k} \rceil}^{i}, \lfloor \frac{n}{k} \rfloor, \ldots, \lfloor \frac{n}{k} \rfloor\right)}{(rn-1)!!},$$

where the first factor is also bounded away from zero. Consider an arbitrary ordered partition of V_n into k sets, namely $S_1, \ldots, S_k$, where $|S_c| = \lambda_c n$, for each $c \in \{1, \ldots, k\}$. Given a perfect matching on W leaving each set $S_c \times [r]$ stable, for every such c, we let $\lambda_{c'}^{(c)}$ be the proportion of points in $S_c \times [r]$ that are matched to a point in $S_{c'} \times [r]$, for any $c' \in \{1, \ldots, k\} \setminus \{c\}$. Thus[1],

$f(|S_1|, \ldots, |S_k|) =$

[1]Here for notational convenience we set $\binom{a}{a_1 \cdots a_k} = \frac{a!}{a_1! \cdots a_k!}$.

$$\sum_{\{\lambda_{c'}^{(c)}, c\neq c'\}\in \mathbf{d}(\lambda_1,\ldots,\lambda_k,r,n)} \frac{\prod_{c=1}^{k} \binom{r\lambda_c n}{\lambda_1^{(c)} r\lambda_c n \cdots \lambda_{c-1}^{(c)} r\lambda_c n \ \lambda_{c+1}^{(c)} r\lambda_c n \cdots \lambda_k^{(c)} r\lambda_c n} \prod_{c,c'\in\{1,\ldots,k\},\ c<c'} \left(\lambda_{c'}^{(c)} r\lambda_c n\right)!}{(rn-1)!!}$$

$$= \sum_{\{\lambda_{c'}^{(c)}, c\neq c'\}\in \mathbf{d}(\lambda_1,\ldots,\lambda_k,r,n)} \frac{\prod_{c=1}^{k}(r\lambda_c n)!}{\prod_{c,c'\in\{1,\ldots,k\},\ c<c'} \left(\lambda_{c'}^{(c)} r\lambda_c n\right)!} \frac{1}{(rn-1)!!}, \qquad (3.3)$$

where $\mathbf{d}(\lambda_1, \ldots, \lambda_k, r, n)$ is the set that contains the non-negative solutions of the following system:

$$\begin{aligned} \lambda_c \lambda_{c'}^{(c)} &= \lambda_{c'} \lambda_c^{(c')},\ r\lambda_c\lambda_{c'}^{(c)} n \in \mathbb{N}\ \forall c, c' \in \{1,\ldots,k\},\ c\neq c'. \\ \sum_{c'\neq c} \lambda_{c'}^{(c)} &= 1,\ \forall c \in \{1,\ldots,k\}. \end{aligned} \qquad (3.4)$$

If we ignore these constraints, the term in the sum of (3.3) is maximised when the terms in its denominator differ by at most one pairwise. We are going to show that in fact there is a solution to (3.4) that asymptotically looks like that.

Proposition 3.2.5 *Assume that $k \geq 3$ and let $t = \lfloor n/k \rfloor$. If $i \equiv n \bmod k$ and, whenever $i \geq 1$, for $j = 1, \ldots, i$, we have $\lambda_j = (t+1)/n$ and for $j = i+1, \ldots, k$, we have $\lambda_j = t/n$, then the above system has at least one non-negative solution, such that $\lambda_{c'}^{(c)} = 1/(k-1) + o(1)$ for every $c, c' \in \{1, \ldots, k\}$ with $c \neq c'$.*

Proof. Assuming that rn is even, we let S be a partition on V_n such that, if $i \geq 1$, then $|S_1|, \ldots, |S_i| = \lceil n/k \rceil = t+1$, and $|S_{i+1}|, \ldots, |S_k| = \lfloor n/k \rfloor = t$. Then each of the first i colour classes has $\lfloor n/k \rfloor$ vertices along with an additional vertex, whereas each of the remaining colour classes has precisely $\lfloor n/k \rfloor$ vertices. For each of the k sets divide the $\lfloor n/k \rfloor$ vertices into $k-1$ subsets, thus leaving ρ vertices out of this partition, for some $0 \leq \rho \leq k-2$. Now notice that each of the i first colour classes consists of $k-1$ subsets of vertices (we call them *main*) each having $\lfloor \lfloor n/k \rfloor/(k-1) \rfloor$ elements along with $\rho+1$ *extra* vertices. The remaining colour classes have the same structure, that is $k-1$ main subsets of cardinality $\lfloor \lfloor n/k \rfloor/(k-1) \rfloor$ but the number of extra vertices is now equal to ρ. Then, this induces a partition on $W = V_n \times [r]$, so that for each induced colour class $s_c = S_c \times [r]$, where $c = 1, \ldots, k$, there are $r(\rho+1)$ extra points for the i first colour classes and $r\rho$ for the remaining ones. The total number of extra points is $rk\rho + ri$ and notice that it is even, for

$$rn = rk(k-1)\left\lfloor \frac{\lfloor n/k \rfloor}{k-1} \right\rfloor + rk\rho + ri$$

and the quantity on the left-hand side as well as the first quantity on the right-hand side are even. Now, we are able to match the points whose projections on V_n belong to a main set of vertices:

repeatedly for each pair of colour classes find an unmatched main set in each of them and establish a matching between the corresponding points. We will show that there exists a perfect matching on the set of the extra points leaving the subsets of those points belonging to the same colour class stable. Before we proceed with the proof, we need the following claim, which we now prove.

Claim 3.2.6 *Let s_1, s_2, s_3 be three sets of points such that $|s_1| + |s_2| + |s_3| = m$ is even and assume that $|s_1|, |s_2|, |s_3| \leq m/2$. Then there is a perfect matching of these points leaving each set stable.*

Proof of Claim 3.2.6. We are trying to solve the linear system (3.4). Let $\alpha = |s_1|/m$, $\beta = |s_2|/m$ and $\gamma = |s_3|/m$. Assume that there are $\alpha'\alpha m$ edges between s_1 and s_2, $\beta'\alpha m$ edges between s_1 and s_3 and $\gamma'\beta m$ edges between s_2 and s_3. Note that

$$\alpha m = \alpha'\alpha m + \beta'\beta m,$$

that is

$$\alpha' + \beta' = 1. \tag{3.5}$$

Also, since we do not allow edges in each set, we obtain

$$\begin{aligned} \beta m - \alpha'\alpha m &= \gamma m - \beta'\alpha m \\ \beta - \alpha'\alpha &= \gamma - (1-\alpha')\alpha \\ \beta - \alpha'\alpha &= \gamma - \alpha + \alpha'\alpha \\ \alpha + \beta - \gamma &= 2\alpha'\alpha \\ \alpha' &= \frac{\alpha+\beta-\gamma}{2\alpha}, \end{aligned}$$

using (3.5). Thus,

$$\beta' = 1 - \alpha' = 1 - \frac{\alpha+\beta-\gamma}{2\alpha} = \frac{\alpha+\gamma-\beta}{2\alpha}.$$

Similarly, we have

$$\begin{aligned} \gamma m &= \beta'\alpha m + \gamma'\beta m \\ \gamma &= \beta'\alpha + \gamma'\beta. \end{aligned}$$

Hence,

$$\gamma' = \frac{1}{\beta}\left(\gamma - \beta'\alpha\right) = \frac{1}{\beta}\left(\gamma - \frac{\alpha+\gamma-\beta}{2}\right) = \frac{\beta+\gamma-\alpha}{2\beta}.$$

Therefore,

$$\begin{aligned}\alpha' &= \frac{\alpha+\beta-\gamma}{2\alpha},\\ \beta' &= \frac{\alpha+\gamma-\beta}{2\alpha},\\ \gamma' &= \frac{\beta+\gamma-\alpha}{2\beta},\end{aligned}$$

which establishes the existence of the required perfect matching. ∎

Assume that $\rho > 0$. As we mentioned above for each of the first i colour classes, there is a set of $r(\rho+1)$ extra points, whereas for each of the rest of them there is a set of $r\rho$ extra points. Thus, the partition is naturally divided into two parts. We are dealing with the above sets of extra vertices. We have to consider four cases.

1. i, $k-i$ are both even. In this case there exists an obvious such perfect matching. Namely, we divide the sets of each part in pairs and for each pair we create a matching taking a bijection between the points of the two sets.

2. i is odd and $k-i$ is even (or vice versa). We pick one set from the odd part and two sets from the even part. The remaining sets can be matched as in the first case. Note that the total number of points in the three selected sets is even. Claim 3.2.6 implies the existence of a perfect matching between these three sets leaving them stable (it is trivial to check that the assumption of the claim is satisfied: $r\rho, r\rho + r \leq (3r\rho + r)/2$ since $\rho \geq 1$). If $i = 0$, then we pick three sets of extra points and we create a perfect matching leaving each of them stable using Claim 3.2.6. Note that each of them has $r\rho$ points and that this is an even number - so the assumption of Claim 3.2.6 is satisfied. Now, an even number of sets of extra points remains and we match them in pairs as in the first case.

3. i, $k-i$ are both odd. In this case, we have $k \geq 4$. To create the suitable perfect matching, we choose any three sets from the largest part and one set from the other part (the total number of points contained in these four sets is even). Then pick two of them, so that, if possible, both belong to the second part, and treat them as one. A perfect matching leaving each of the (now) three sets stable is guaranteed by Claim 3.2.6, since its assumption is obviously satisfied. Now, an even number of sets remain in each part. For these sets, we complete the matching as in the first case.

If $\rho = 0$ and $i \neq 1$, then if i is odd, we match an arbitrary triple of sets, using Claim 3.2.6 and then the rest of them in pairs, whereas if i is even we simply match them in pairs. However, if

$i = 1$, then first note that r must be an even number. We are going to redirect the matching we have already created between those points that correspond to main vertices. All of the parts apart from one have t vertices and let A be the odd part. To create the required perfect matching, let B, C be two parts different from A. Among those points that correspond to main vertices (and which are already matched), choose $r/2$ points in B that are adjacent to points in A and $r/2$ points in B that are adjacent to points in C. We match half of the r extra points in A to the points of the first $r/2$-tuple and the second half of them to the neighbours of the second $r/2$-tuple in C. Then match the $r/2$ neighbours of the first $r/2$-tuple in A to the points of the second $r/2$-tuple in B. In this case, $\lambda_B^{(C)} = 1/(k-1) - r/2/r\lambda_C n$, $\lambda_C^{(B)} = 1/(k-1) - r/2/r\lambda_B n$.

Hence, we have found a solution to the system (3.4), with the λ_c's specified as in the proposition. Note that for each pair $c, c' \in \{1, \ldots, k\}$ with $c \neq c'$, we have

$$\frac{1}{k-1}\left(1 - \frac{(k-1)r}{r\lambda_c n}\right) \leq \lambda_{c'}^{(c)} \leq \frac{1}{k-1}\left(1 + \frac{(k-1)(r + r\rho)}{r\lambda_c n}\right),$$

and this concludes the proof of the proposition. ∎

Now, if for any $c \in \{1, \ldots, k\}$, either $\lambda_c n = \lceil n/k \rceil$ or $\lambda_c n = \lfloor n/k \rfloor$, then $\lambda_c = 1/k \pm O(n^{-1})$. The above proposition implies that the leading term in the sum (3.3) corresponds to $\lambda_{c'}^{(c)} = 1/(k-1)(1 \pm O(n^{-1}))$ for any $c, c' \in \{1, \ldots, k\}$ with $c \neq c'$. We are going to express the terms of sum (3.3) in their first form. For any $\{\lambda_{c'}^{(c)}, c \neq c'\}$ as in the previous proposition we have

$$\binom{r\lambda_c n}{\lambda_1^{(c)} r\lambda_c n \cdots \lambda_{c-1}^{(c)} r\lambda_c n \ \lambda_{c+1}^{(c)} r\lambda_c n \cdots \lambda_k^{(c)} r\lambda_c n} = \Theta\left(n^{-\frac{k-2}{2}}\right) \exp\left(\frac{rn\ln(k-1)}{k}\right),$$

and

$$\begin{aligned}
\left(\lambda_{c'}^{(c)} r\lambda_c n\right)! &= O(n^{1/2}) \left(\lambda_{c'}^{(c)} \lambda_c\right)^{\lambda_{c'}^{(c)} r\lambda_c n} (rn)^{\lambda_{c'}^{(c)} r\lambda_c n} \exp\left(-\lambda_{c'}^{(c)} r\lambda_c n\right) \\
&= O(n^{1/2}) \left(\frac{1}{k(k-1)}\right)^{\frac{rn}{k(k-1)}} (rn)^{\lambda_{c'}^{(c)} r\lambda_c n} \exp\left(-\lambda_{c'}^{(c)} r\lambda_c n\right) \\
&= O(n^{1/2}) \exp\left(-\frac{rn\ln(k(k-1))}{k(k-1)}\right) (rn)^{\lambda_{c'}^{(c)} r\lambda_c n} \exp\left(-\lambda_{c'}^{(c)} r\lambda_c n\right).
\end{aligned}$$

Moreover,

$$(rn-1)!! = \Theta(1)(rn)^{rn/2} \ \exp\left(-rn/2\right).$$

Since $\mathbf{d}(\lambda_1, \ldots, \lambda_k, r, n)$ has $O(n^{k^2})$ elements for any $(\lambda_1, \ldots, \lambda_k) \in \mathrm{D}^{(n)}$, the last three previous equations yield

$$f\left(\overbrace{\lceil\frac{n}{k}\rceil, \ldots, \lceil\frac{n}{k}\rceil}^{i}, \lfloor\frac{n}{k}\rfloor, \ldots, \lfloor\frac{n}{k}\rfloor\right) =$$

$$
\begin{aligned}
& O(n^{3k^2/2}) \frac{\prod_{c=1}^{k} \exp\left(\frac{rn\ln(k-1)}{k}\right) \prod_{c,c' \in \{1,\dots,k\},\ c<c'} \exp\left(-\frac{rn\ln(k(k-1))}{k(k-1)}\right) (rn)^{\lambda_{c'}^{(c)} r\lambda_c n} \exp\left(-\lambda_{c'}^{(c)} r\lambda_c n\right)}{(rn-1)!!} \\
= \ & O(n^{3k^2/2}) \frac{\exp\left(rn\ln(k-1)\right)\ \exp\left(-\frac{rn\ln(k(k-1))}{2}\right)\ (rn)^{rn/2}\ \exp\left(-rn/2\right)}{(rn)^{rn/2}\ \exp\left(-rn/2\right)} \\
= \ & O(n^{3k^2/2}) \exp\left(rn\left(\ln(k-1) - \frac{\ln(k(k-1))}{2}\right)\right).
\end{aligned}
$$

On the other hand,

$$
\binom{n}{\overbrace{\lceil \frac{n}{k} \rceil \cdots \lceil \frac{n}{k} \rceil}^{i}\ \lfloor \frac{n}{k} \rfloor \cdots \lfloor \frac{n}{k} \rfloor} = \Theta\left(n^{-\frac{k-1}{2}}\right)\ \exp\left(n\ln k\right).
$$

Therefore, we obtain:

$$
\begin{aligned}
\mathbb{E}[C_k(\mathbb{G}^*(n,r))] &= O(n^{3k^2/2})\ \exp\left(n\left(\ln k + r\left(\ln(k-1) - \frac{\ln(k(k-1))}{2}\right)\right)\right) \\
&= \exp\left(n\left(\ln k + r\left(\ln(k-1) - \frac{\ln(k(k-1))}{2}\right)\right) + o(n)\right),
\end{aligned}
$$

where the multiplicative coefficient is $\Omega\left(n^{-k^2/2}\right)$ as well. The following has been proved,

Theorem 3.2.7 *For any fixed integer $r \geq 1$, we have*

$$
\mathbb{E}[C_k(\mathbb{G}^*(n,r))] = \exp\left(\mu_k(r)n + o(n)\right),
$$

where $\mu_k(r) = \ln k + r\left(\ln(k-1) - \frac{1}{2}\ln(k(k-1))\right)$.

Note that for any integer $r > 2\ln k/\ln(k/(k-1))$, we have $\mu_k(r) < 0$, and, moreover, $\mu_k(r)$ is decreasing with respect to r, since its coefficient is negative for any $k \geq 3$. By (3.1), Proposition 3.2.2 and Theorem 3.2.7, we deduce Theorem 3.1.1.

3.3 The restricted coupon collector problem

In this section we introduce the restricted coupon collector problem and give an asymptotic solution to it. The motivation for studying this problem comes from the application of the rigid 3-colourings technique in an attempt to prove that almost all 5-regular graphs are not 3-colourable. As we shall see in the next section, the combinatorial numbers whose asymptotic behaviour we are going to analyse here play the role of the Stirling numbers of the second kind, which were used in the analysis of the expected number of rigid 3-colourings in $\mathcal{G}_{n,m}$ random graphs in the previous chapter. But now let us focus on the restricted coupon collector problem. Assume that there are k types of coupons each of them having d *distinct* copies, where $d \geq 2$. Also, the coupon collector has a list of permissible numbers of coupons of any type, namely $l = (l_1, \ldots, l_h) \in \mathbb{N}^h$ with $1 \leq l_1 < \cdots < l_h \leq d$ for $h > 1$. That is, for any type of coupon, the number of copies she is allowed to collect must be one of the elements of the above list. Given any natural number t, we let $C_{d,l}(t,k)$ be the number of ways of collecting a total of t copies of coupons, so that the constraints induced by the list are satisfied. Note that $C_{d,l}(t,k)$ is the coefficient of z^t in the following generating function:

$$G(z) = R(z)^k = \left(\binom{d}{l_1} z^{l_1} + \cdots + \binom{d}{l_h} z^{l_h}\right)^k.$$

Cauchy's integral formula gives

$$C_{d,l}(t,k) = \frac{1}{2\pi i} \int_C \frac{R(z)^k}{z^{t+1}} dz,$$

where the integration is taking place over a closed contour containing the origin.

Theorem 3.3.1 *For integers $d, h \geq 2$ and $l = (l_1, \ldots, l_h) \in \mathbb{N}^h$, where $1 \leq l_1 < \cdots < l_h \leq d$, assume that $1 \leq \min_{1 \leq j \leq h}\{l_j\} < t/k < \max_{1 \leq j \leq h}\{l_j\} \leq d$ and $k \to \infty$. Let r_0 be the unique positive solution to the equation*

$$\frac{rR'(r)}{R(r)} = \frac{t}{k} \tag{3.6}$$

and we set

$$s = r_0 \frac{d}{dx} \frac{xR'(x)}{R(x)}\bigg|_{x=r_0}.$$

Then

$$C_{d,l}(t,k) = \frac{1}{\sqrt{2\pi k s}} \frac{R(r_0)^k}{r_0^t}(1 + o(1)), \tag{3.7}$$

uniformly for $\min_{1 \leq j \leq h}\{l_j\} < t/k < \max_{1 \leq j \leq h}\{l_j\}$.

Proof. The proof was inspired by [10]. Letting C be a circle of radius r, where r will be specified later, containing the origin and using polar coordinates, we obtain

$$\begin{aligned} C_{d,l}(t,k) = \frac{1}{2\pi i} \int_C \frac{R(re^{i\phi})^k}{r^{t+1} e^{it\phi} e^{i\phi}} d(re^{i\phi}) &= \frac{1}{2\pi i} \int_C \frac{R(re^{i\phi})^k}{r^{t+1} e^{it\phi} e^{i\phi}} ire^{i\phi} d\phi \\ &= \frac{1}{2\pi r^t} \int_{-\pi}^{\pi} \frac{R(re^{i\phi})^k}{e^{it\phi}} d\phi. \end{aligned}$$

Once again letting $z = re^{i\phi}$, we obtain

$$\left|\binom{d}{l_1} z^{l_1} + \cdots + \binom{d}{l_h} z^{l_h}\right| =$$

$$
\begin{aligned}
& \left(\left(\sum_{j=1}^{h} r^{l_j}\binom{d}{l_j}\cos l_j\phi\right)^2 + \left(\sum_{j=1}^{h} r^{l_j}\binom{d}{l_j}\sin l_j\phi\right)^2\right)^{1/2} \\
= & \left(\sum_{j_1,j_2=1}^{h} r^{l_{j_1}+l_{j_2}}\binom{d}{l_{j_1}}\binom{d}{l_{j_2}}\left(\cos l_{j_1}\phi\cos l_{j_2}\phi+\sin l_{j_1}\phi\sin l_{j_2}\phi\right)\right)^{1/2} \\
= & \left(\sum_{j_1,j_2=1}^{h} r^{l_{j_1}+l_{j_2}}\binom{d}{l_{j_1}}\binom{d}{l_{j_2}}\cos\left((l_{j_1}-l_{j_2})\phi\right)\right)^{1/2} \\
= & \left(\sum_{j=1}^{h} r^{2l_j}\binom{d}{l_j}^2 + \sum_{j_1,j_2=1,j_1\neq j_2}^{h} r^{l_{j_1}+l_{j_2}}\binom{d}{l_{j_1}}\binom{d}{l_{j_2}}\cos\left((l_{j_1}-l_{j_2})\phi\right)\right)^{1/2} \\
= & \left(\sum_{j=1}^{h} r^{2l_j}\binom{d}{l_j}^2 + \sum_{j_1,j_2=1,j_1\neq j_2}^{h} r^{l_{j_1}+l_{j_2}}\binom{d}{l_{j_1}}\binom{d}{l_{j_2}}\right. \\
& \left. - \sum_{j_1,j_2=1,j_1\neq j_2}^{h} r^{l_{j_1}+l_{j_2}}\binom{d}{l_{j_1}}\binom{d}{l_{j_2}}\left(1-\cos\left((l_{j_1}-l_{j_2})\phi\right)\right)\right)^{1/2} \\
= & \left(\left(\sum_{j=1}^{h} r^{l_j}\binom{d}{l_j}\right)^2 - \sum_{j_1,j_2=1,j_1\neq j_2}^{h} r^{l_{j_1}+l_{j_2}}\binom{d}{l_{j_1}}\binom{d}{l_{j_2}}\left(1-\cos\left((l_{j_1}-l_{j_2})\phi\right)\right)\right)^{1/2} \\
\leq & \; R(r)\left(1-c\left(1-\cos\phi\right)\right)^{1/2},
\end{aligned}
$$

for some positive constant $c=c(r)$.Therefore,

$$
\begin{aligned}
\left|\int_{-\pi}^{-\delta}\frac{R(re^{it\phi})^k}{e^{it\phi}}d\phi\right| + \left|\int_{\delta}^{\pi}\frac{R(re^{it\phi})^k}{e^{it\phi}}d\phi\right| & \leq \int_{-\pi}^{-\delta}\left|R(re^{it\phi})\right|^k d\phi + \int_{\delta}^{\pi}\left|R(re^{it\phi})\right|^k d\phi \\
& \leq 2\pi R(r)^k\left(1-c(1-\cos\delta)\right)^{k/2} \\
& \leq 2\pi R(r)^k \exp\left(-\frac{kc(1-\cos\delta)}{2}\right).
\end{aligned}
$$

Since $(1-\cos\delta)/2\delta^2 \to 1/4$ as $\delta \to 0$, choosing $\delta=\delta(k)$ such that $k\delta^2/\ln k \to \infty$ and $\delta \to 0$ as $k \to \infty$, we obtain

$$
\left|\int_{-\pi}^{-\delta}\frac{R(re^{it\phi})^k}{e^{it\phi}}d\phi\right| + \left|\int_{\delta}^{\pi}\frac{R(re^{it\phi})^k}{e^{it\phi}}d\phi\right| \leq 2\pi R(r)^k k^{-\omega(k)}, \tag{3.8}
$$

where $\omega(k) \to \infty$ as $k \to \infty$.

Note that

$$
\lim_{r\to 0}\frac{rR'(r)}{R(r)} = \min_{1\leq j\leq h}\{l_j\}, \quad \lim_{r\to\infty}\frac{rR'(r)}{R(r)} = \max_{1\leq j\leq h}\{l_j\}.
$$

Also, this function is increasing with respect to r for $r>0$. To see this note that

$$
\left(\frac{rR'(r)}{R(r)}\right)' = \frac{(R'(r)+rR''(r))R(r)-r(R'(r))^2}{(R(r))^2}.
$$

The numerator is

$$
\begin{aligned}
&(R'(r)+rR''(r))R(r)-r(R'(r))^2 = \\
&\quad \sum_{i=1}^{h} r^{l_i}\binom{d}{l_i}\sum_{j=1}^{h} r^{l_j-1}\binom{d}{l_j}l_j + r\sum_{i=1}^{h} r^{l_i}\binom{d}{l_i}\sum_{j=1}^{h} r^{l_j-2}\binom{d}{l_j}l_j(l_j-1) \\
&\quad -r\sum_{i=1}^{h} r^{l_i-1}\binom{d}{l_i}l_i\sum_{j=1}^{h} r^{l_j-1}\binom{d}{l_j}l_j \\
= &\quad \sum_{i,j=1}^{h} r^{l_i}r^{l_j-1}\binom{d}{l_i}\binom{d}{l_j}l_j + \sum_{i,j=1}^{h} r^{l_i}r^{l_j-1}\binom{d}{l_i}\binom{d}{l_j}l_j(l_j-1) - \sum_{i,j=1}^{h} r^{l_i}r^{l_j-1}\binom{d}{l_i}\binom{d}{l_j}l_il_j \\
= &\quad \sum_{i,j=1}^{h}(l_j^2-l_il_j)r^{l_i+l_j-1}\binom{d}{l_i}\binom{d}{l_j} = \sum_{i<j}(l_j^2-l_il_j+l_i^2-l_il_j)r^{l_i+l_j-1}\binom{d}{l_i}\binom{d}{l_j} \\
= &\quad \sum_{i<j}(l_i-l_j)^2r^{l_i+l_j-1}\binom{d}{l_i}\binom{d}{l_j} \geq 0,
\end{aligned}
$$

for any $r \geq 0$. This shows that Equation (3.6) has a unique positive solution.

We let $r = r_0$ and set

$$
\begin{aligned}
f(r,\phi) &= R(re^{i\phi})\exp\left(-i\frac{t}{k}\phi\right) \\
&= \exp\left(-i\frac{t}{k}\phi\right)\left(\sum_{j=1}^{h} r^{l_j}\binom{d}{l_j}(\cos l_j\phi + i\sin l_j\phi)\right).
\end{aligned}
$$

Thus,

$$
\frac{\partial f(r,\phi)}{\partial\phi} = \exp\left(-i\frac{t}{k}\phi\right)\left(\sum_{j=1}^{h} r^{l_j}\binom{d}{l_j}\left(\frac{t}{k}-l_j\right)(\sin l_j\phi - i\cos l_j\phi)\right).
$$

Note that

$$
\begin{aligned}
\left.\frac{\partial f(r,\phi)}{\partial\phi}\right|_{r=r_0,\phi=0} &= -i\left(\frac{t}{k}\sum_{j=1}^{h} r_0^{l_j}\binom{d}{l_j} - \sum_{j=1}^{h} r_0^{l_j}\binom{d}{l_j}l_j\right) \\
&= -i\left(\frac{t}{k}R(r_0) - r_0R'(r_0)\right) = 0.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\frac{\partial^2 f(r,\phi)}{\partial\phi^2} = &-i\frac{t}{k}\exp\left(-i\frac{t}{k}\phi\right)\left(\sum_{j=1}^{h} r^{l_j}\binom{d}{l_j}\left(\frac{t}{k}-l_j\right)(\sin l_j\phi - i\cos l_j\phi)\right) \\
&+\exp\left(-i\frac{t}{k}\phi\right)\left(\sum_{j=1}^{h} r^{l_j}\binom{d}{l_j}\left(\frac{t}{k}-l_j\right)l_j(\cos l_j\phi + i\sin l_j\phi)\right).
\end{aligned}
$$

On the other hand,

$$\begin{aligned}\sum_{j=1}^{h} r^{l_j}\binom{d}{l_j} l_j^2 &= \sum_{j=1}^{h} r^{l_j}\binom{d}{l_j} l_j(l_j-1) + \sum_{j=1}^{h} r^{l_j}\binom{d}{l_j} l_j \\ &= r^2 R''(r) + rR'(r).\end{aligned}$$

Therefore,

$$\begin{aligned}\left.\frac{\partial^2 f(r,\phi)}{\partial \phi^2}\right|_{r=r_0,\phi=0} &= -\frac{t}{k}\sum_{j=1}^{h} r_0^{l_j}\binom{d}{l_j}\left(\frac{t}{k}-l_j\right) + \sum_{j=1}^{h} r_0^{l_j}\binom{d}{l_j}\left(\frac{t}{k}-l_j\right) l_j \\ &= -\left(\frac{t}{k}\right)^2 R(r_0) + 2\frac{t}{k} r_0 R'(r_0) - r_0^2 R''(r_0) - r_0 R'(r_0) \\ &= -\frac{(r_0R'(r_0))^2}{R(r_0)^2} R(r_0) + 2\frac{(r_0R'(r_0))^2}{R(r_0)} - r_0^2 R''(r_0) - r_0 R'(r_0) \\ &= \frac{(r_0R'(r_0))^2}{R(r_0)} - r_0^2 R''(r_0) - r_0 R'(r_0) \\ &= \frac{r_0^2 R'(r_0)^2 R(r_0) - r_0^2 R''(r_0) R(r_0)^2 - r_0 R'(r_0) R(r_0)^2}{R(r_0)^2} \\ &= -R(r_0) r_0 \frac{-r_0 R'(r_0)^2 + r_0 R''(r_0) R(r_0) + R'(r_0) R(r_0)}{R(r_0)^2} \\ &= -R(r_0) r_0 \frac{(r_0 R''(r_0) + R'(r_0)) R(r_0) - r_0 R'(r_0)^2}{R(r_0)^2} \\ &= -R(r_0) r_0 \left.\frac{d}{dx}\,\frac{xR'(x)}{R(x)}\right|_{x=r_0} \in \mathbb{R}^-. \qquad (3.9)\end{aligned}$$

Since t/k is bounded below away from zero and above as well, there are two positive constants a, b such that $a \le r_0 \le b$, for k sufficiently large. The continuity of f on the compact set $a \le r \le b$ and $-\pi \le \phi \le \pi$ implies that there is a positive constant δ_0 such that whenever $|\phi| \le \delta_0$ we have $Ref(r,\phi) > 0$. Let c_1 be the maximum modulus of the third partial derivative of $Imf(r,\phi)/Ref(r,\phi)$ with respect to ϕ over the compact set $a \le r \le b$ and $|\phi| \le \delta_0$. As long as the first two partial derivatives of $Imf(r,\phi)$ with respect to ϕ vanish when $r = r_0$ and $\phi = 0$ and $f(r_0, 0) = R(r_0) \ne 0$, Taylor's theorem implies that

$$\left|\frac{Imf(r_0,\phi)}{Ref(r_0,\phi)}\right| \le \frac{c_1\phi^3}{6},$$

for any ϕ with $|\phi| \le \delta_0$. On the other hand, we have (see [10] for the details)

$$\left|\frac{Re(z^k)}{Re(z)^k} - 1\right| \le \epsilon\left(k, \left|\frac{Imz}{Rez}\right|\right),$$

with

$$\epsilon(k,x) = (1+x)^k - 1 - kx.$$

Since $\epsilon(k,x)$ increases for $x \geq 0$, we have

$$1-\epsilon\left(k, \frac{c_1\delta^3}{6}\right) \leq \frac{Re(f(r_0,\phi)^k)}{Re(f(r_0,\phi))^k} \leq 1+\epsilon\left(k, \frac{c_1\delta^3}{6}\right),$$

whenever $|\phi| \leq \delta \leq \delta_0$.

Moreover, equation (3.9) implies that $\partial^2 \ln Re(f(r,\phi))/\partial\phi^2|_{r=r_0,\phi=0} = -s$. Thus, by Taylor's theorem

$$\left|\ln Re(f(r_0,\phi)) - \left(\ln R(r_0) - \frac{s\phi^2}{2}\right)\right| \leq \frac{c_2\phi^3}{6},$$

where c_2 is the maximum modulus of the 3rd partial derivative of $\ln Re(f(r,\phi))$ with respect to ϕ over the compact set $a \leq r \leq b$ and $|\phi| \leq \delta_0$. It follows that

$$\exp\left(-\frac{c_2 k\delta^3}{6}\right) \leq \frac{Re(f(r_0,\phi))^k}{R(r_0)^k \exp(-sk\phi^2/2)} \leq \exp\left(\frac{c_2 k\delta^3}{6}\right).$$

If we choose δ such that $k\delta^3 \to 0$ as $k \to \infty$, we have

$$\epsilon\left(k, \frac{c_1\delta^3}{6}\right) \leq \exp\left(\frac{c_1 k\delta^3}{6}\right) - 1 = o(1), \text{ and } \exp\left(\frac{c_2 k\delta^3}{6}\right) = 1 + o(1),$$

proving that

$$Re(f(r_0,\delta)^k) = R(r_0)^k \exp(-sk\phi^2/2)(1+o(1)),$$

for any δ as it was chosen above. Thus, Equation (3.8) along with the above one imply that

$$\begin{aligned} \int_{-\pi}^{\pi} \frac{R(r_0e^{i\phi})^k}{e^{it\phi}} d\phi &= \int_{-\delta}^{\delta} \frac{R(r_0e^{i\phi})^k}{e^{it\phi}} d\phi + \int_{\delta}^{2\pi-\delta} \frac{R(r_0e^{i\phi})^k}{e^{it\phi}} d\phi \\ &= \int_{-\delta}^{\delta} f(r_0,\phi)^k d\phi + \int_{\delta}^{2\pi-\delta} \frac{R(r_0e^{i\phi})^k}{e^{it\phi}} d\phi \\ &= R(r_0)^k \left(\int_{-\delta}^{\delta} \exp(-sk\phi^2/2) d\phi + o(1)\right). \end{aligned} \tag{3.10}$$

Now, observe that

$$\int_{-\delta}^{\delta} \exp\left(-\frac{sk\phi^2}{2}\right) d\phi = \frac{1}{\sqrt{k}} \int_{-\delta\sqrt{k}}^{\delta\sqrt{k}} \exp\left(-\frac{sy^2}{2}\right) dy.$$

Choosing δ such that $\delta\sqrt{k} \to \infty$ as $k \to \infty$, we obtain

$$\int_{-\delta\sqrt{k}}^{\delta\sqrt{k}} \exp\left(-\frac{sy^2}{2}\right) dy = \sqrt{\frac{2\pi}{s}} - o(1).$$

Thus, using again Equation (3.8)

$$\int_{-\pi}^{\pi} \frac{R(r_0e^{i\phi})^k}{e^{it\phi}} d\phi = \sqrt{\frac{2\pi}{ks}} R(r_0)^k (1+o(1)).$$

and, therefore, Equation (3.7) follows. ∎

We present a few examples of the validity of the asymptotic formula that we give in the above theorem, presenting the relative error between the actual value of $C_{d,l}(t,k)$ and its approximation. Here, we have set $d = 3$ and $l = \{1, 2\}$. The following data have been collected using Maple.

t, k	relative error
$t = 12$, $k = 8$	0.03068
$t = 35$, $k = 20$	0.01787
$t = 112$, $k = 100$	0.00703
$t = 290$, $k = 150$	0.00833
$t = 500$, $k = 300$	0.00097

3.4 On the 3-colourability of random 5-regular graphs

The purpose of studying these numbers in the previous section comes from an attempt to apply the approach of rigid 3-colourings in order to investigate the 3-colourability of random 5-regular graphs. Here, these numbers play the role of the Stirling numbers of the second kind in the case of ordinary random graphs (see the previous chapter). Recall that for a graph G a proper 3-colouring S of its vertices with colour classes S_1, S_2, S_3 is rigid if every vertex in S_2 and S_3 is adjacent to a vertex in S_1 and every vertex in S_3 is also adjacent to a vertex in S_2. We are working on r-regular graphs on $V_n = \{1, \ldots, n\}$, where $r \geq 1$. Thus, if S is a 3-colouring of V_n, let s be the induced colouring on the set of points $V_n \times [r]$, with colour classes s_1, s_2, s_3. If S is a proper 3-colouring of an r-regular graph, then the assumption of Claim 3.2.6 is satisfied and, therefore, the number of edges between the colour classes is determined by the orders of the colour classes by means of the formulae that are given at the end of the proof of that claim. Now, each of the colour classes s_2, s_3 corresponds to a collection of coupons each having r distinct copies: each coupon corresponds to a vertex and the r distinct copies of the coupon correspond to the r distinct points in the configuration model that correspond to this vertex. The number of ways to choose those points in s_2 that will be adjacent to points in s_1 so that each vertex in S_2 will be adjacent to a vertex in S_1 is equal to $C_{r,(1,\ldots,r)}(e_{S_1,S_2}, |S_2|)$, where e_{S_1,S_2} is the number of edges between S_1 and S_2. On the other hand, the number of ways to choose those points in s_3 that will be adjacent to points in s_1 so that each vertex in S_3 will be adjacent to a vertex in S_1 *as well as* to a vertex in S_2 is equal to $C_{r,(1,\ldots,r-1)}(e_{S_1,S_3}, |S_3|)$, where e_{S_1,S_3} is the number of edges between S_1 and S_3. This is the case because if we leave at least one point for each vertex in S_3 unmatched when we match the points in s_3 with those in s_1, then this will be necessarily matched to a point in s_2. Note that if $|s_1| = r\alpha n$, $|s_2| = r\beta n$ and $|s_3| = r(1-\alpha-\beta)n$, then by the proof of Claim 3.2.6, we have $e_{S_1,S_2} = r\alpha'\alpha n$ and $e_{S_1,S_3} = r\beta'\alpha n$, where $\alpha' = (2\alpha+2\beta-1)/2\alpha$ and $\beta' = (1-2\beta)/2\alpha$. Moreover, the number of edges between S_2 and S_3 will be equal to $r\gamma'\beta n$, where $\gamma' = (1-2\alpha)/2\beta$. Using Stirling's formula for the asymptotic expansion of the factorials, Lemma 2.2.4 (or its analogue for multinomial coefficients) and Theorem 3.3.1, we have for real $0 < \alpha, \beta, (1-\alpha-\beta) < 1/2$ such that $1 < r\alpha'\alpha/\beta < r$ and $1 < r\beta'\alpha/(1-\alpha-\beta) < r-1$,

$$\begin{aligned}
\sum_{S \in \mathcal{C}_3(\alpha,\beta,1-\alpha-\beta,n)} \mathbb{P}[S \text{ is rigid}] &= \binom{n}{\alpha n \;\; \beta n \;\; (1-\alpha-\beta)n} \times \\
&\frac{\binom{r\alpha n}{r\alpha'\alpha n}\, C_{r,(1,\dots,r)}(r\alpha'\alpha n, \beta n)\; C_{r,(1,\dots,r-1)}(r\beta'\alpha n, (1-\alpha-\beta)n)\, (r\alpha'\alpha n)!\, (r\beta'\alpha n)!\, (r\gamma'\beta n)!}{(rn-1)!!} \\
= \;\; & O(n^{\lambda})\; \exp\Big(n\Big(H(\alpha,\beta) + r\alpha H(\alpha') + \beta\ln\left((r_1+1)^r - 1\right) - r\alpha'\alpha\ln r_1 \\
& +(1-\alpha-\beta)\ln\left((r_2+1)^r - 1 - r_2^r\right) - r\beta'\alpha\ln r_2 + r\alpha'\alpha\ln\left(\alpha'\alpha\right) + r\beta'\alpha\ln\left(\beta'\alpha\right) \\
& + r\gamma'\beta\ln\left(\gamma'\beta\right)\Big)\Big),
\end{aligned}$$

for some $\lambda \in \mathbb{R}$, where $H(\cdot)$ denotes the entropy function and r_1 and r_2 are the unique positive roots of the equations

$$\frac{rx(1+x)^{r-1}}{(1+x)^r - 1} = \frac{r\alpha'\alpha}{\beta}$$

and

$$\frac{x(r(1+x)^{r-1} - rx^{r-1})}{(1+x)^r - 1 - x^r} = \frac{r\beta'\alpha}{1-\alpha-\beta},$$

respectively. Also, note that the multiplicative factor is $\Omega(n^{\lambda'})$ for some $\lambda' \in \mathbb{R}$. Unfortunately, for $r = 5$ if we set $\alpha = 0.36$ and $\beta = 0.32$ (it can be easily checked that these satisfy the conditions), then the coefficient of n in the above exponential becomes equal (up to 3 decimal places) to 0.044, which implies that the expected number of rigid 3-colourings of a random 5-regular graph tends to infinity, as $n \to \infty$. Therefore, in order to determine the 3-colourability of random 5-regular graphs we need to resort to more sophisticated techniques. However, we would like to point out the fact that although $\mathbb{E}[R(\mathcal{G}_{n,\lceil 5n/2 \rceil})] \to 0$, as $n \to \infty$, the expected number of rigid 3-colourings of a random 5-regular multigraph goes to infinity.

Chapter 4

The degree sequence of the k-core of a random graph

4.1 Introduction

In this chapter, we are concerned with the degree sequence of the k-core of $\mathcal{G}_{n,m}$ random graphs. Recall that a $\mathcal{G}_{n,m}$ random graph is a uniformly drawn element of the set of simple graphs on $V_n = \{1, \ldots, n\}$ with $m = m(n)$ edges, where $0 \leq m \leq \binom{n}{2}$. We will be working on sparse $\mathcal{G}_{n,m}$ random graphs, i.e. $m = \lceil \theta n/2 \rceil$, where $\theta = \theta(n) = \Theta(1)$. For a graph $G = (V, E)$ and a natural number $k \geq 2$ (k will be a natural number throughout the chapter), the k-core of G is the maximal subgraph of minimum degree at least k (and it is empty if there is no such graph). For a graph G, we denote its k-core by k–cr(G). Also, we set cr$(G) = 2$–cr(G).

In [37], B. Pittel, J. Spencer and N. Wormald showed that for $k \geq 3$ the k-core appears suddenly in the random graph process and, more specifically, at the stage where the number of edges is proportional to the number of vertices. Before this, it was proved by T. Łuczak in [28], that for $k \geq 3$ and any m, such that $0 \leq m \leq \binom{n}{2}$, the k-core of a $\mathcal{G}_{n,m}$ random graph a.a.s. either is empty or has linear order. Moreover, he proved that if the k-core is non-empty, then it is a.a.s. k-connected. For $k = 2$, B. Pittel proved in [36] that above the threshold for the appearance of the giant component (i.e. for any fixed $\theta > 1$), in a $\mathcal{G}_{n,m}$ random graph there is a.a.s. a giant 2-core of order $n(1 - T)(1 - T/\theta)$, where T is the smaller solution of the equation $Te^{-T} = \theta e^{-\theta}$. For $\theta \leq 1$, that is before the appearance of the giant component, the 2-core is with asymptotically positive probability non-empty and consists of very few "small" components (which are isolated cycles for any fixed $\theta < 1$).

However, a full and precise analysis of the appearance and the order of the k-core was given in [37]. Let $p_k(z)$ be the probability of a Poisson random variable of mean z being at least k and note that $p_k(z) = e^{-z}\sum_{j\geq k} z^j/j!$. For any $k \geq 2$, let

$$\gamma_k = \inf\left\{\frac{\lambda}{p_{k-1}(\lambda)} \ : \ \lambda > 0\right\}.$$

Now, for any $k \geq 2$ and $\theta > \gamma_k$ let $\lambda_k(\theta)$ be the larger root of the equation $\theta = \lambda/p_{k-1}(\lambda)$, if $k \geq 3$, or the unique root, if $k = 2$. Also, The main result in [37] is the following: for m as above, $k \geq 3$ and $\delta \in (0, 1/2)$, if $\theta = \theta(n) \geq \gamma_k + n^{-\delta}$, then a.a.s. the k-core of the random graph $\mathcal{G}_{n,m}$ is non-empty and has order $np_k(\lambda_k(\theta)) + O_C(n^\gamma)$, for some constant $\gamma \in (0, 1)$, and if $\theta \leq \gamma_k - n^{-\delta}$, then a.a.s. the k-core is empty.

For a simple graph G on V_n, the sequence $(v_0(G), \ldots, v_{n-1}(G))$, where for $j = 0, \ldots, n-1$ $v_j(G)$ is the number of vertices of G of degree j, is said to be the *degree sequence* of G. In [27], T. Łuczak investigates the structure of the "giant" 2-core above the critical point, though quite close to it. Namely, for each constant natural number $d \geq 2$, he gave an asymptotic expression for $v_d(\mathrm{cr}(\mathcal{G}_{n,m}))$, when $1 + n^{-1/3}\omega(n) \leq \theta = \theta(n) \leq 1 + 1/\omega(n)$ and $\omega(n)$ is a function going to infinity with n (where for the first inequality we need $\omega(n) = o(n^{1/3})$). However, this analysis does not apply when θ is well above this stage of the random graph process, e.g. for any constant $\theta > 1$. Furthermore, for $k \geq 3$ the structure of the k-core after its appearance was not studied in [37] (apart from its order, which is given above). In this chapter, we provide an asymptotic estimation for the number of vertices of every constant degree in the k-core from immediately after its appearance, for any $k \geq 2$. These results will be used in the next chapter, in a study of the structure of the core of a $\mathcal{G}_{n,m}$ sparse random graph with $\theta > 1$. Let $\rho_j(z)$ be the probability that a Poisson distributed random variable of mean z is equal to j. We prove the following:

Theorem 4.1.1 *For $k \geq 3$ ($k = 2$, respectively), let $\theta \geq \gamma_k + n^{-\delta}$, where $\delta \in (0, 1/2)$ (let $\theta > \gamma_2$ be fixed). For every fixed $l \geq k$, there exist constants $c > 0$ and $\gamma, \tau \in (0, 1)$, such that $v_l(k\text{–}\mathrm{cr}(\mathcal{G}_{n,m})) = n\rho_l(\lambda_k(\theta)) + O_C(n^\gamma)$, with probability at least $1 - O\left(e^{-cn^\tau}\right)$. Moreover, with the same bound on the probability the number of edges in the k-core is $n\frac{\lambda_k^2(\theta)}{2\theta} + O_C(n^\gamma)$. In fact, for every $k \geq 2$ and every $\theta \in (0, 1/2)$, if $k \geq 3$, or $\theta \in (0, 1/3)$, if $k = 2$, whenever $\theta \geq \gamma_k + n^{-\delta}$, then for every natural number $l \geq k$ we have $v_l(k\text{–}\mathrm{cr}(\mathcal{G}_{n,m})) = n\rho_l(\lambda_k(\theta))(1 + o_p(1))$.*

As we shall see later (see the remark at the end of Section 5), we re-express Łuczak's results that appear in [27] concerning the degree sequence of the 2-core of a $\mathcal{G}_{n,m}$ random graph in the supercritical phase. The proof of this theorem is based on the techniques that have been developed in [37]. Namely, we provide an analysis of a vertex deletion process applied to a $\mathcal{G}_{n,m}$ random graph that delivers the k-core if this exists, or a graph with each vertex having degree less than k, if the k-core is empty. We show that this process induces a Markov chain on a suitably defined state space, where the states are extensions of the states that were considered in [37]. Namely, the state space consists of integer non-negative $(l+1)$-vectors for a fixed $l > k$, with their elements corresponding to the number of edges and to the number of vertices of each degree less than l. Then, we give an asymptotic expression for the transition probabilities of the Markov chain, thus obtaining a Markov chain with its trajectory being close to that of the initial one. We approximate this trajectory by the solution of a system of differential equations. The asymptotic (with respect to n) solution to a sub-system of it provides the number of vertices of specific degrees (greater than k) that are encoded in the state space by the end of the deletion process.

Before discussing the proof of this theorem, we show how it can be applied to bound the probability of a $\mathcal{G}_{n,m}$ random graph being 3-colourable. We note that if the 3-core of $\mathcal{G}_{n,m}$ random graph has n' vertices and m' edges then, provided that the maximum degree is $o(\ln^2 n')$, the expected number of 3-colourings of the 3-core is at most $3^{n'}(2/3)^{m'}$ (we omit the proof which is similar to but slightly more complicated than the corresponding proof for the expected number of proper 3-colourings of a $\mathcal{G}_{n,m}$ random graph). The previous theorem as well as the work in [37] imply that $m'/n' = \lambda_3^2(\theta)/(p_3(\lambda_3(\theta))2\theta)(1 + o_p(1))$, for any $\theta > \gamma_3$, and $\text{max.deg.}(3\text{–cr}(\mathcal{G}_{n,m})) \leq \text{max.deg.}(\mathcal{G}_{n,m}) = O_p(\ln n/\ln\ln n)$ (see [6]). Moreover, for $\theta = 5.15$ we have $\lambda_3^2(\theta)/(p_3(\lambda_3(\theta))2\theta) > 2.705$. Let A_n be the event "$|n' - np_3(\lambda_3(\theta))| \leq Cn^\gamma$ and $|m' - n2\theta/\lambda_3^2(\theta)| \leq C'n^{\gamma'}$ and $\text{max.deg.}(3\text{–cr}(\mathcal{G}_{n,m})) \leq \ln n$", where $C, C' > 0$, $\gamma, \gamma' \in (0,1)$ are constants which are implicit in the above theorem as well as in [37]. Then for any $\theta > \gamma_3$ we have

$$\begin{aligned}\mathbb{P}[\chi(\mathcal{G}_{n,m}) \geq 3] &\leq \mathbb{E}[C(3\text{–cr}(\mathcal{G}_{n,m}))\mathbf{1}_{A_n}] + \mathbb{P}[\overline{A_n}] \\ &\leq 3^{np_3(\lambda_3(\theta))}(2/3)^{n\lambda_3^2(\theta)/2\theta}(1+o(1)) + o(1) = o(1),\end{aligned}$$

where the last equality holds for $\theta \geq 5.15$.

4.2 Definitions and preliminaries

We analyse the deletion process that was introduced and studied in [37], in order to determine the asymptotic behaviour of the degree sequence of the k-core of a $\mathcal{G}_{n,m}$ random graph. For a graph $G = (V, E)$ and a fixed positive integer $k \geq 2$, let the *light* vertices be the vertices of degree less than k, and let the *heavy* vertices be those vertices of degree at least k. At each step, we choose a vertex uniformly at random amongst the non-isolated light vertices and delete all the edges incident to it, thus making it isolated. This step is repeated so long as there are edges to be deleted and the current set of heavy vertices, say H, is non-empty. At the end, either $H \neq \emptyset$ and so H is the vertex set of the k-core in the initial graph, or $H = \emptyset$ and so the k-core is empty.

The graph on which the above procedure applies is a $\mathcal{G}_{n,m}$ random graph on V_n having $m = \lceil \theta n/2 \rceil$ edges and $\theta = \theta(n) = \Theta(1)$. We shall see that the deletion process induces a Markov chain on a suitably defined set of states. Namely, as in [37] but in a slightly different way, the set of states S_l, with $l > k$, consists of $l+1$-tuples of nonnegative integers $\mathbf{w} = (\mathbf{v}, \mu)$, such that $\mathbf{v} = (v_0, v_1, \dots, v_{l-1})$, where as we shall see shortly v_j corresponds to the number of vertices of degree j and μ to the total number of edges of the underlying graph in the course of the deletion process. For a state $\mathbf{w} = (v_0, \dots, v_{l-1}, \mu)$, we define

$$\bar{v} = \bar{v}(\mathbf{w}) = n - v_0 - v, \quad \text{where } v = v(\mathbf{w}) = \sum_{j=1}^{k-1} v_j,$$

and

$$\bar{\bar{v}} = \bar{\bar{v}}(\mathbf{w}) = n - v_0 - v - \sum_{j=k}^{l-1} v_j.$$

We set $s = s(\mathbf{w}) = \sum_{j=1}^{k-1} j v_j$ and $t = t(\mathbf{w}) = 2\mu - s$. Moreover, let $\bar{\bar{t}} = \bar{\bar{t}}(\mathbf{w}) = 2\mu - \sum_{j=1}^{l-1} j v_j$.

Also, for an $l+1$-tuple $\mathbf{w} = (v_0, \dots, v_{l-1}, \mu)$, we set $v_j(\mathbf{w}) = v_j$, for $j = 0, \dots, l-1$, and $\mu(\mathbf{w}) = \mu$. The aim is to describe the deletion process by a sequence $\{\mathbf{w}(\tau)\}$, where $\mathbf{w}(\tau)$ is the state at step τ. Here we consider l to be fixed (i.e. not depending on n).

Naturally, the deletion process induces a sequence of graphs $\{G(\tau)\}$, where $G(0) = \mathcal{G}_{n,m}$, i.e. the first element of the sequence is distributed uniformly on the set of all simple graphs on V_n with m edges. For a given graph G, we introduce $\mathbf{w}(G) = \mathbf{w}_l(G) = (\mathbf{v}(G), \mu(G))$ with $\mathbf{v}(G) = \{v_0(G), \dots, v_{l-1}(G)\}$, where $v_j(G)$ is the number of vertices of degree j in G, and $\mu(G)$ is the number of edges of G. Similarly, we set $v(G) = \sum_{j=1}^{k-1} v_j(G)$ (the number of non-isolated light vertices), $\bar{v}(G) = n - v_0(G) - v(G)$ (the number of heavy vertices), and $\bar{\bar{v}}(G) = n - v_0(G) - v(G) - \sum_{j=k}^{l-1} v_j(G)$, the number of vertices of degree greater than $l-1$. We call these vertices *very heavy*. Also, we set $s(G) = \sum_{j=1}^{k-1} j v_j(G)$, $t(G) = 2\mu(G) - \sum_{j=1}^{k-1} j v_j(G)$ and $\bar{\bar{t}}(G) = 2\mu(G) - \sum_{j=1}^{l-1} j v_j(G)$. This is the total degree of the vertices that are light, heavy and very heavy, respectively. For $\mathbf{w} \in S_l$, we define $\mathcal{H}(\mathbf{w}) = \{G : \ G \text{ is simple}, \ \mathbf{w}(G) = \mathbf{w}\}$ and let $h(\mathbf{w}) = |\mathcal{H}(\mathbf{w})|$. As we mentioned before, we choose the initial graph G uniformly at random amongst the simple graphs on V_n with m edges and we start the deletion process. We obtain a random graph sequence

$\{G(\tau)\}_{0\le\tau\le T}$, where T is the total number of deletion steps and $G(\tau)$ is the graph at the τ-th step of the deletion process, for each $0 \le \tau \le T$. According to the description of the deletion process, either there are no heavy vertices at $G(T)$, or besides the isolated vertices there are only heavy vertices left (i.e. there are no light non-isolated vertices). Now let $\{\mathbf{w}(\tau)\} = \{\mathbf{w}(G(\tau))\}_{0\le\tau\le T}$. Note that if $\mathbf{w}(T) = \mathbf{w}$, the above observation translates into

$$\bar{v}(\mathbf{w}) = 0 \ \text{ or } \ \bar{v}(\mathbf{w}) > 0 \text{ but } v(\mathbf{w}) = 0.$$

Such a $\mathbf{w}$ is called *terminal.* We can extend both sequences setting $G(\tau) = G(T)$ and $\mathbf{w}(\tau) = \mathbf{w}(T)$, for $\tau > T$.

4.3 The deletion process and its asymptotics

In this section, we present a more detailed analysis of the deletion process. We prove that it induces a Markov chain on S_l and we give asymptotic estimates for the transition probabilities. We make the convention that the dashed quantities refer to $\mathbf{w}'$ (e.g. $v' = v(\mathbf{w}')$). We begin with the following:

Proposition 4.3.1 *(a)The sequence $\{\mathbf{w}(\tau)\}$ is a Markov chain: for every non-terminal $\mathbf{w}$ such that $\mathbb{P}[\mathbf{w}(\tau) = \mathbf{w}] > 0$*

$$\begin{aligned} p[\mathbf{w}' \mid \mathbf{w}] &= \mathbb{P}[\mathbf{w}(\tau+1) = \mathbf{w}' \mid \mathbf{w}(\tau) = \mathbf{w}] \\ &= \frac{1}{v}\,\frac{h(\mathbf{w}')}{h(\mathbf{w})}\,v'_0 \prod_{j=0}^{l} \binom{v'_j - \delta_{j0}}{u_{j+1}}, \end{aligned} \tag{4.1}$$

where $v'_l = \bar{\bar{v}}'$ and $\mathbf{u} = \{u_j\}_{1\le j\le l+1}$ is the solution of the system

$$\begin{aligned} v_j &= v'_j - u_{j+1} + u_j + \delta_{ij}, \quad 0 \le j \le l-1, \\ \bar{\bar{v}} &= \bar{\bar{v}}' + u_l, \\ \sum_{j=1}^{l+1} u_j &= i = \mu - \mu', \end{aligned} \tag{4.2}$$

setting $u_0 = -1$ and δ being the Kronecker symbol, provided that $\mathbf{u} \ge \mathbf{0}$. If $\mathbf{u} \not\ge \mathbf{0}$, then $p[\mathbf{w}' \mid \mathbf{w}] = 0$. (Notice that in a transition $G \to G'$, the parameters u_j $(1 \le j \le l)$ and u_{l+1} express the number of edges in G connecting the chosen light vertex with the vertices of degree j and of degree $> l$, respectively.)

(b) For every τ, conditioned on $\{\mathbf{w}(\nu)\}_{0\le\nu\le\tau}$, the random graph $G(\tau)$ is distributed uniformly, i.e. for every $\{\mathbf{w}^0(\nu)\}_{0\le\nu\le\tau}$ such that $\mathbb{P}[\mathbf{w}(\nu) = \mathbf{w}^0(\nu); 0 \le \nu \le \tau] > 0$,

$$\mathbb{P}[G(\tau) = G \mid \mathbf{w}(\nu) = \mathbf{w}^0(\nu); 0 \le \nu \le \tau] = \frac{1}{h(\mathbf{w}^0(\tau))}, \; \forall G \in \mathcal{H}(\mathbf{w}^0(\tau)). \tag{4.3}$$

Consequently, if a stopping time $\mathcal{T}$ adapted to $\{\mathbf{w}(\nu)\}$ and $\mathbf{w}$ are such that $\mathbb{P}[\mathbf{w}(\mathcal{T}) = \mathbf{w}] > 0$, then

$$\mathbb{P}[G(\mathcal{T}) = G \mid \mathbf{w}(\mathcal{T}) = \mathbf{w}] = \frac{1}{h(\mathbf{w})}, \; \forall G \in \mathcal{H}(\mathbf{w}). \tag{4.4}$$

Proof. We will use induction. Assume that for some $\tau \ge 0$ the sequence $\{\mathbf{w}(\nu)\}_{0\le\nu\le\tau}$ is Markov with one step probabilities defined by (4.1) and the relation (4.3) holds. (This is true when $\tau = 0$, which is the base case.)

Now, for every sequence of nonterminal states $\mathbf{w}^0(\nu) = (\mathbf{v}^0(\nu), \mu^0(\nu))$, where $0 \le \nu \le \tau + 1$, such that $\mathbb{P}[\mathbf{w}(\nu) = \mathbf{w}^0(\nu); 0 \le \nu \le \tau] > 0$ and every $G' \in \mathcal{H}(\mathbf{w}^0(\tau + 1))$ we have

$$\begin{aligned}
&\mathbb{P}[G(\tau+1) = G' \mid \mathbf{w}(\nu) = \mathbf{w}^0(\nu); 0 \le \nu \le \tau] = \\
&= \sum_{G \in \mathcal{H}(\mathbf{w}^0(\tau))} \mathbb{P}[G(\tau+1) = G', G(\tau) = G \mid \mathbf{w}(\nu) = \mathbf{w}^0(\nu); 0 \le \nu \le \tau] \\
&= \sum_{G \in \mathcal{H}(\mathbf{w}^0(\tau))} \mathbb{P}[G(\tau+1) = G' \mid G(\tau) = G] \; \mathbb{P}[G(\tau) = G \mid \mathbf{w}(\nu) = \mathbf{w}^0(\nu); 0 \le \nu \le \tau] \\
&= \frac{1}{h(\mathbf{w}^0(\tau))} \sum_{G \in \mathcal{H}(\mathbf{w}^0(\tau))} \mathbb{P}[G(\tau+1) = G' \mid G(\tau) = G] \\
&= \frac{1}{h(\mathbf{w}^0(\tau)) \; v^0(\tau)} \; N(G', \mathbf{w}^0(\tau)),
\end{aligned} \tag{4.5}$$

by the induction hypothesis, where $N(G', \mathbf{w}^0(\tau))$ is the total number of graphs G from $\mathcal{H}(\mathbf{w}^0(\tau))$ such that G' can be obtained from G by deleting all edges of a non-isolated light vertex.

The function $N(G', \mathbf{w}^0(\tau))$ depends only on $\mathbf{w}^0(\tau+1) = \mathbf{w}(G')$ and $\mathbf{w}^0(\tau)$. To see this, note we can choose the vertex that has been deleted in v'_0 ways. Before the deletion, the "deleted" vertex has u_j vertices of degree j $(0 \le j \le l)$ and u_{l+1} vertices of degree at least $l+1$ adjacent to it. The vector $\mathbf{u}$ is determined by $\mathbf{w}^0(\tau+1)$ and $\mathbf{w}^0(\tau)$, as it is the solution of the linear system (4.2). Therefore, for each j, where $0 \le j \le l-1$, the vertices of degree $j+1$ that were adjacent to the deleted vertex can be chosen in $\binom{v'_j - \delta_{j0}}{u_{j+1}}$ ways. Also, the vertices of degree at least $l+1$ that were adjacent to the "deleted" vertex can be chosen in $\binom{\bar{\bar{v}}'}{u_{l+1}}$ ways among the $\bar{\bar{v}}'$ very heavy vertices. Thus,

$$N(G', w^0(\tau)) = t(\mathbf{w}^0(\tau), \mathbf{w}^0(\tau+1)) = v'_0 \prod_{j=0}^{l} \binom{v'_j - \delta_{j0}}{u_{j+1}}. \tag{4.6}$$

(Recall that $v'_l = \bar{\bar{v}}'$.) Hence, equation (4.5) along with (4.6) imply that

$$\begin{aligned}\mathbb{P}[\mathbf{w}(\tau+1) = \mathbf{w}' \mid \mathbf{w}(\nu) = \mathbf{w}^0(\nu),\ 0 \le \nu \le \tau] &= \mathbb{P}[\mathbf{w}(\tau+1) = \mathbf{w}' \mid \mathbf{w}(\tau) = \mathbf{w}] \\ &= \frac{1}{v}\,\frac{h(\mathbf{w}')}{h(\mathbf{w})}\ t(\mathbf{w}, \mathbf{w}').\end{aligned}$$

So, the induction hypothesis implies that $\{\mathbf{w}(\nu)\}_{0\le\nu\le\tau+1}$ is Markov, with one-step transition probabilities $p[\mathbf{w}' \mid \mathbf{w}]$, where $\mathbf{w}$ is non-terminal, given by (4.1).

To see (4.3), notice that for every $G' \in \mathcal{H}(\mathbf{w}')$ we have

$$\begin{aligned}&\mathbb{P}[G(\tau+1) = G' \mid \mathbf{w}(\nu) = \mathbf{w}^0(\nu); 0 \le \nu \le \tau+1] = \\ &= \frac{\mathbb{P}[G(\tau+1) = G' \mid \mathbf{w}(\nu) = \mathbf{w}^0(\nu); 0 \le \nu \le \tau]}{\mathbb{P}[\mathbf{w}(\tau+1) = \mathbf{w}' \mid \mathbf{w}(\nu) = \mathbf{w}^0(\nu); 0 \le \nu \le \tau]} \\ &= \frac{h(\mathbf{w}')^{-1} p[\mathbf{w}' \mid \mathbf{w}]}{p[\mathbf{w}' \mid \mathbf{w}]} \\ &= \frac{1}{h(\mathbf{w}')}.\end{aligned}$$

The induction is now complete. ∎

Let us now give a slightly different expression for $t(\mathbf{w}, \mathbf{w}')$. For $j \neq 0$, we have $\binom{v'_j - \delta_{j0}}{u_{j+1}} = \frac{v'_j!}{(v'_j - u_{j+1})! u_{j+1}!}$ and

$$\begin{aligned}v'_0 &= v_0 + u_1 + 1, \\ v'_j &= v_j - u_j + u_{j+1} - \delta_{ij}, \quad 1 \le j \le l-1, \\ \bar{\bar{v}}' &= \bar{\bar{v}} - u_l.\end{aligned}$$

So, for $1 \le j \le l-1$, we obtain

$$\binom{v'_j}{u_{j+1}} = \frac{v'_j!}{(v_j - u_j - \delta_{ij})! u_{j+1}!} = ((v_i - 1)\delta_{ij} + 1)\ \frac{v'_j!}{v_j!}\ \frac{(v_j - \delta_{ij})!}{(v_j - u_j - \delta_{ij})! u_{j+1}!},$$

and

$$\binom{v'_l}{u_{l+1}} = \frac{v'_l!}{(v'_l - u_{l+1})! u_{l+1}!} = \frac{\bar{\bar{v}}'!}{(\bar{\bar{v}} - u_l - u_{l+1})! u_{l+1}!}.$$

Therefore,

$$\begin{aligned}t(\mathbf{w}, \mathbf{w}') &= v'_0 \prod_{j=0}^{l} \binom{v'_j - \delta_{j0}}{u_{j+1}} \\ &= v_i \prod_{j=1}^{l-1} \frac{v'_j!}{v_j!} \prod_{j=1}^{l-1} \binom{v_j - \delta_{ij}}{u_j}\ \frac{v'_0 (v'_0 - 1)!}{v_0!}\ \frac{\bar{\bar{v}}'}{\bar{\bar{v}}}\ \frac{\bar{\bar{v}}!}{u_l! u_{l+1}! (\bar{\bar{v}} - u_l - u_{l+1})!},\end{aligned} \tag{4.7}$$

where $i = \mu - \mu'$.

In what follows, we will try to give asymptotic expressions for $h(\mathbf{w})$ and $p[\mathbf{w}' \mid \mathbf{w}]$. Let $\mathbf{d} = \{d_1, \ldots, d_n\}$ be a non-decreasing sequence of nonnegative integers with even sum and $\hat{h}(\mathbf{d})$ be the total number of simple graphs on V_n such that the i-th vertex has degree d_i for each $i \in V_n$. Then,

$$h(\mathbf{w}) = \frac{n!}{\prod_{j=0}^{l} v_j!} \sum_{\mathbf{d} \in \mathbf{D}} \hat{h}(\mathbf{d}), \tag{4.8}$$

for $\mathbf{w} = (\mathbf{v}, \mu)$, where $v_l = \bar{\bar{v}} = n - \sum_{j=0}^{l-1} v_j$ and $\mathbf{D} = \mathbf{D}(\mathbf{w})$ is the set of all nonnegative n-tuples $\mathbf{d}$ such that

$$\begin{aligned}
&d_1 = \cdots = d_{v_0} = 0,\\
&d_{v_0+1} = \cdots = d_{v_0+v_1} = 1,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
&d_{\sum_{j=0}^{l-2} v_j + 1} = \cdots = d_{\sum_{j=0}^{l-1} v_j} = l - 1,\\
&d_{\sum_{j=0}^{l-1} v_j + 1}, \cdots, d_n \geq l,\\
&\sum_{j=1}^{n} d_j = 2\mu.
\end{aligned}$$

For $\mathbf{D}$ to be nonempty, it is necessary that

$$\bar{\bar{t}} = 2\mu - \sum_{j=1}^{l-1} j v_j \geq l \bar{\bar{v}}.$$

We will use an asymptotic formula for $\hat{h}(\mathbf{d})$ which is due to McKay and Wormald [31].

For any natural number $r \geq 1$, set $M_r = \sum_{1 \leq j \leq n} (d_j)_r$, where $(x)_r = x(x-1)\cdots(x-r+1)$, (for instance $M_1 = 2\mu = \sum_j d_j$), and let $d_{\max} = \max_{1 \leq j \leq n} d_j$. If $M_1 \to \infty$ and $d_{\max} = o(M_1^{1/3})$ as $n \to \infty$, then

$$\begin{aligned}
\hat{h}(\mathbf{d}) \quad = \quad & \frac{(M_1 - 1)!!}{\prod_{j=1}^{n} d_j!} \times \\
& \exp\left(-\frac{M_2}{2M_1} - \frac{M_2^2}{4M_1^2} - \frac{M_2^2 M_3}{2M_1^4} + \frac{M_2^4}{4M_1^5} + \frac{M_3^2}{6M_1^3} + O\left(\frac{d_{\max}^3}{M_1} \right) \right),
\end{aligned} \tag{4.9}$$

where $(M_1 - 1)!! = 1 \cdot 3 \cdots (M_1 - 1)$. We will show that the likely $\mathbf{w}$'s are those for which

$$\hat{h}(\mathbf{d}) = (1 + o(1)) \, \frac{(M_1 - 1)!!}{\prod_{j=1}^{n} d_j!} \exp\left(-\frac{M_2}{2M_1} - \frac{M_2^2}{4M_1^2} \right).$$

Given $\mathbf{w}$ such that $h(\mathbf{w}) > 0$, let $G(\mathbf{w})$ be the random graph chosen uniformly at random from $\mathcal{H}(\mathbf{w})$. For a fixed $b \in (0, 1/3)$ define

$$\mathcal{E}_n = \mathcal{E}_n(b) = \left\{ G = (V_n, E) \ : \ \text{max.deg.}(G) \ > n^b \text{ or } \sum_{j \geq l} j^4 v_j(G) > 2n\mathbb{E}[Z^4(\theta)] \right\},$$

where $Z(\theta)$ is a Poisson random variable of mean θ, which is the average degree of the random graph. Also, let $g(\mathbf{w}) = g(\mathbf{w}, b) = \mathbb{P}[G(\mathbf{w}) \in \mathcal{E}_n]$. Assume that $\mathbf{w}'$ is such that $p[\mathbf{w}' \mid \mathbf{w}] > 0$. Then $h(\mathbf{w}') > 0$, too. If we delete the edges incident to a randomly chosen light vertex of the random graph $G(\mathbf{w})$, then we produce a random graph G'. Note that $\mathbb{P}[\mathbf{w}(G') = \mathbf{w}'] = p[\mathbf{w}' \mid \mathbf{w}] > 0$, and if we condition on this event then G' is the random graph $G(\mathbf{w}')$. So there is a probability space that accommodates both $G(\mathbf{w})$ and $G(\mathbf{w}')$ in such a way that $G(\mathbf{w}') \subset G(\mathbf{w})$. Since the event $\mathcal{E}_n$ is non-decreasing, we obtain:

$$g(\mathbf{w}) \geq g(\mathbf{w}'), \qquad \text{if } p[\mathbf{w}' \mid \mathbf{w}] > 0.$$

This implies that the random sequence $\{g(\mathbf{w}(\tau))\}$ is non-increasing. We will show the following:

Lemma 4.3.2 *For any fixed $b \in (0, 1/3)$, we have*

$$\mathbb{P}\left[g(\mathbf{w}(0)) \geq \frac{1}{n}\right] = O(e^{-n^b}).$$

Proof. Clearly, the expected value of $g(\mathbf{w}(0))$ is equal to $\mathbb{P}[\mathcal{G}_{n,m} \in \mathcal{E}_n]$. So by Markov's inequality, we have

$$\begin{aligned} \mathbb{P}\left[g(\mathbf{w}(0)) \geq \frac{1}{n}\right] &\leq n\mathbb{P}[\mathcal{G}_{n,m} \in \mathcal{E}_n] \\ &\leq n(P_1 + P_2), \end{aligned} \tag{4.10}$$

where

$$\begin{aligned} P_1 &= \mathbb{P}[\text{max.deg.}(\mathcal{G}_{n,m}) > n^b] \\ P_2 &= \mathbb{P}\left[\sum_{j \geq l} j^4 v_j(\mathcal{G}_{n,m}) > 2n\mathbb{E}[Z^4(\theta)]\right]. \end{aligned}$$

We will show that these two probabilities are tiny, using a double conditioning technique.

Note that the graph process $\mathcal{G}_{n,m}$ can be viewed as the multigraph process $MG(n, m)$ conditioned on the event $A_n = \{MG(n, m) \text{ has no loops and no multiple edges}\}$. (Recall that at each stage of the multigraph process we form an edge by picking uniformly at random, independently and with replacement two vertices, i and j, ignoring the previous choices; if $i = j$ then the multigraph gets a loop at i.) Using $\mathbb{P}[U \mid V] \le \mathbb{P}[U]/\mathbb{P}[V]$ (this is the first conditioning), and denoting by P_i' the analogous probabilities in $MG(n, m)$, for $i = 1, 2$, we have

$$P_i \le \frac{P_i'}{A_n} = O(P_i'), \tag{4.11}$$

since

$$\lim_{n\to\infty} \mathbb{P}[A_n] = \lim_{n\to\infty} \frac{\binom{\binom{n}{2}}{m} m! 2^m}{n^{2m}} = \exp(-\theta/2 - \theta^2/4) > 0.$$

The second conditioning comes from the fact that the degree sequence of the random multigraph produced by the multigraph process can be viewed as the random sequence of occupancy numbers when we throw $2m$ distinguishable balls into n bins. These are jointly distributed as n independent copies, say $\{Z_j(\theta)\}_{j=1,\dots,n}$, of $Z(\theta)$ (a Poisson random variable with mean θ) conditioned on $S_n = \sum_{j=1}^{n} Z_j(\theta) = 2m$. Since S_n is $Z(2m)$, we have

$$\mathbb{P}[S_n = 2m] = \mathbb{P}[Z(2m) = 2m] = e^{-2m}\frac{(2m)^{2m}}{(2m)!} \ge \text{const.} \times m^{-1/2}.$$

Therefore,

$$P_i' = O(n^{1/2} P_i''), \tag{4.12}$$

where

$$\begin{aligned} P_1'' &= \mathbb{P}\left[\max_{1\le j\le n} Z_j(\theta) > n^b\right], \\ P_2'' &= \mathbb{P}\left[\sum_{j=1}^{n} Z_j^4(\theta) > 2n\mathbb{E}[Z^4(\theta)]\right]. \end{aligned}$$

By Chernoff's inequality,

$$P_2'' = O(e^{-\alpha(\theta)n}), \qquad \alpha(\theta) > 0, \tag{4.13}$$

and

$$P_1'' \le n\mathbb{P}[Z(\theta) > n^b] = n \sum_{r > n^b} e^{-\theta}\frac{\theta^r}{r!} = \exp\left(-\Omega\left(n^b \ln n\right)\right). \tag{4.14}$$

The estimates (4.10)-(4.14) imply that

$$\mathbb{P}\left[g(\mathbf{w}(0)) \ge \frac{1}{n}\right] = O(e^{-n^b}).$$

∎

Thus, with probability at least $1 - O(e^{-n^b})$, the $\mathbf{w}$'s we find in the deletion process are such that

$$g(\mathbf{w}) \le 1/n. \tag{4.15}$$

This yields

$$h(\mathbf{w}) = \left(1 + O\left(\frac{1}{n}\right)\right) h_1(\mathbf{w}),$$

where

$$h_1(\mathbf{w}) = \left| \left\{ G \in \mathcal{H}(\mathbf{w}) \ : \ \text{max.deg.}(G) \le n^b, \ \sum_{j \ge l} j^4 v_j(G) \le dn \right\} \right|.$$

Let the set $\mathbf{D}_1 \subset \mathbf{D}$ be specified by the additional restrictions

$$\begin{aligned} d_{\max} &\le n^b, \\ \sum_{\{\text{very heavy } j\}} d_j^4 &\le dn, \qquad d = 2\mathbb{E}[Z^4(\theta)]. \end{aligned} \tag{4.16}$$

Thus, we may focus our attention to an asymptotic expression for $h_1(\mathbf{w})$. Formula (4.9) along with (4.16) yield

$$\begin{aligned} h_1(\mathbf{w}) &= (1 + O(n^{-1+3b})) \frac{n!}{\prod_{j=0}^{l} v_j!} \sum_{\mathbf{d} \in \mathbf{D}_1} \hat{h}_1(\mathbf{d}), \\ \hat{h}_1(\mathbf{d}) &= \frac{(M_1 - 1)!!}{\prod_{j=1}^{n} d_j!} \exp(-\lambda/2 - \lambda^2/4), \\ \lambda &= \frac{M_2}{M_1}, \end{aligned} \tag{4.17}$$

if we restrict ourselves to $\mathbf{w}$'s such that $M_1 = 2\mu(\mathbf{w}) \ge cn$, for some $c > 0$ fixed and arbitrarily small. Actually, we restrict ourselves to $\mathbf{w}$'s such that

$$\begin{aligned} \bar{\bar{v}} &\ge an, \\ \bar{\bar{t}} &\ge (l + a)\bar{\bar{v}}, \end{aligned} \tag{4.18}$$

where $a > 0$ is fixed, arbitrarily small and it will be specified later. The double conditioning technique that was applied in Lemma 4.3.2 shows that conditions (4.18) are met asymptotically almost surely at the beginning of the deletion process, if

$$\begin{aligned} a &< \mathbb{P}[Z(\theta) \ge l], \\ \sum_{j \ge l} j \mathbb{P}[Z(\theta) = j] &\ge (l + a)\mathbb{P}[Z(\theta) \ge l]. \end{aligned}$$

Now, we introduce a family of $\bar{\bar{v}}$ independent random variables $Y_1, \ldots, Y_{\bar{\bar{v}}}$, each distributed as $Z(z)$, a Poisson random variable of mean z, conditioned on $Z(z) \ge l$. For every $1 \le j \le \bar{\bar{v}}$, we have

$$\begin{aligned}\mathbb{P}[Y_j = r] &= \frac{\mathbb{P}[Z(z) = r]}{p_l(z)}, \qquad r \geq l,\\ p_l(z) &= \mathbb{P}[Z(z) \geq l] = e^{-z} e_l(z),\\ e_l(z) &= \sum_{r \geq l} \frac{z^r}{r!}.\end{aligned}$$

The parameter $z > 0$ is defined as the solution of the equation

$$\bar{\bar{v}} \mathbb{E}[Y_1] = \bar{\bar{t}}. \tag{4.19}$$

Note that $\mathbb{E}[Y_1] = z e_l'(z)/e_l(z)$ and the expression on the right-hand side is strictly increasing. Therefore the solution is unique and bounded away from 0 and ∞ uniformly for all $\mathbf{w}$'s for which conditions (4.18) are met. Let

$$\lambda = \frac{\sum_{j=1}^{l-1} j(j-1)v_j + \sum_{j=1}^{\bar{\bar{v}}} Y_j(Y_j - 1)}{\sum_{j=1}^{l-1} j v_j + \sum_{j=1}^{\bar{\bar{v}}} Y_j}$$

and

$$\bar{\lambda} = \frac{\sum_{j=1}^{l-1} j(j-1)v_j + \sum_{j=1}^{\bar{\bar{v}}} \mathbb{E}[Y_j(Y_j - 1)]}{\sum_{j=1}^{l-1} j v_j + \sum_{j=1}^{\bar{\bar{v}}} \mathbb{E}[Y_j]}.$$

After these considerations, we obtain the following:

Lemma 4.3.3 *Uniformly for* $\mathbf{w}$ *such that* $h(\mathbf{w}) > 0$ *and the conditions (4.15) and (4.18) are met,*

$$\begin{aligned}h(\mathbf{w}) &= \left(1 + O(n^{-1+3b} + n^{-1/2}\log_2 n)\right) \frac{n!(M_1 - 1)!!}{\bar{\bar{v}}! \prod_{j=1}^{l-1} ((j!)^{v_j} v_j!)} \times\\ &\quad \frac{(e_l(z))^{\bar{\bar{v}}}}{z^{\bar{\bar{t}}}} \exp(-\bar{\lambda}/2 - \bar{\lambda}^2/4) \frac{1}{\sqrt{\bar{\bar{v}} 2\pi \, Var(Y_1)}},\end{aligned}$$

where Y_1, $\bar{\lambda}$, z, $p_l(z)$ *are defined above.*

Proof. We have

$$\begin{aligned}\sum_{\mathbf{d} \in \mathbf{D}_1} \hat{h}_1(\mathbf{d}) &= \frac{(M_1 - 1)!!}{\prod_{j=1}^{l-1} (j!)^{v_j}} \sum_{\mathbf{d} \in \mathbf{D}_1} \exp\left(-\frac{\lambda}{2} - \frac{\lambda^2}{4}\right) \frac{1}{\prod_{j=\sum_{i=0}^{l-1} v_i + 1}^{n} d_j!}\\ &= \frac{(M_1 - 1)!!}{\prod_{j=1}^{l-1} (j!)^{v_j}} (e_l(z))^{\bar{\bar{v}}} \sum_{\mathbf{d} \in \mathbf{D}_1} \frac{\exp\left(-\frac{\lambda}{2} - \frac{\lambda^2}{4}\right)}{(e_l(z))^{\bar{\bar{v}}}} \left(\prod_{j=\sum_{i=0}^{l-1} v_i + 1}^{n} \frac{e^{-z} z^{d_j}}{d_j!} \right) \times\\ &\quad \frac{1}{(e^{-z})^{\bar{\bar{v}}} \prod_{j=\sum_{i=0}^{l-1} v_i + 1}^{n} z^{d_j}}\\ &= \frac{(M_1 - 1)!!}{\prod_{j=1}^{l-1} (j!)^{v_j}} \frac{(e_l(z))^{\bar{\bar{v}}}}{z^{\bar{\bar{t}}}} \mathbb{E}\left[\exp\left(-\frac{\lambda}{2} - \frac{\lambda^2}{4}\right) ; \sum_{i=1}^{\bar{\bar{v}}} Y_i = \bar{\bar{t}}, \ \mathbf{Y} \in \mathcal{Y} \right],\end{aligned} \tag{4.20}$$

where

$$\mathcal{Y} \;=\; \left\{ \mathbf{Y} = (y_1, \ldots, y_{\bar{\bar{v}}}) \;:\; \max_l y_l \le n^b,\; \sum_l y_l^4 \le dn \right\}.$$

Note that it is sufficient to estimate $\mathbb{E}\left[\exp\left(-\frac{\lambda}{2} - \frac{\lambda^2}{4}\right)\;;\; \sum_{i=1}^{\bar{\bar{v}}} Y_i = \bar{\bar{t}}\right]$, since

$$\mathbb{P}[\mathbf{Y} \notin \mathcal{Y}] = O(e^{-n^b}),$$

by using a Chernoff-type argument.

Moreover, the distribution of λ is sharply concentrated around $\bar{\lambda}$. To see this, notice that uniformly for $\mathbf{w}$'s satisfying conditions (4.18), we have

$$\begin{aligned} \mathbb{P}\left[\left|\sum_i^{\bar{\bar{v}}} Y_i - \bar{\bar{v}}\mathbb{E}[Y_1]\right| \ge \log_2 n \sqrt{n}\right] &= O\left(\exp(-\gamma \log_2^2 n)\right) \\ \mathbb{P}\left[\left|\sum_i^{\bar{\bar{v}}} Y_i(Y_i - 1) - \bar{\bar{v}}\mathbb{E}[Y_1(Y_1 - 1)]\right| \ge \log_2 n \sqrt{n}\right] &= O\left(\exp(-\gamma \log_2^2 n)\right), \end{aligned}$$

for some $\gamma = \gamma(a) > 0$. Therefore,

$$\left|\lambda - \bar{\lambda}\right| = O_C\left(\frac{\log_2 n}{\sqrt{n}}\right)$$

with probability at least $1 - C\exp(-\gamma \log_2^2 n)$ for some constant C and for any n sufficiently large. Consequently, the expectation in (4.20) is equal to

$$\left(1 + O\left(\frac{\log_2 n}{\sqrt{n}}\right)\right) \exp(-\bar{\lambda}/2 - \bar{\lambda}^2/4)\; \mathbb{P}\left[\sum_{i=1}^{\bar{\bar{v}}} Y_i = \bar{\bar{t}}\right] + O\left(\exp(-\gamma \log_2^2 n)\right).$$

Moreover, by (4.19) and a local limit theorem for the sum of lattice-type i.i.d. random variables (see [15] - Theorem 3 p.490) we obtain:

$$\mathbb{P}\left[\sum_{i=1}^{\bar{\bar{v}}} Y_i = \bar{\bar{t}}\right] = \frac{1}{\sqrt{\bar{\bar{v}}\; 2\pi\; Var(Y_1)}} \left(1 + O\left(\frac{1}{n^{1/2}}\right)\right),$$

uniformly for $\mathbf{w}$'s subject to (4.18). Now, using (4.17) the lemma follows. ∎

This lemma along with the previous work implies the following proposition:

Proposition 4.3.4 *Suppose that* $\mathbf{w}$ *is non-terminal and* $h(\mathbf{w}) > 0$. *If* $\mathbf{w}$ *satisfies conditions (4.15), (4.18) and* $\mathbf{w}'$ *is such that* $\mathbf{u}$ *is the solution of the linear system (4.2) then*

$$\begin{aligned} p[\mathbf{w}' \mid \mathbf{w}] \;=\; & \frac{v_i(1 - \delta_{v_i 0})}{v}\; \mathbb{P}[Multin(i; \bar{p}) = \mathbf{u}] \left(1 + O\left(\sum_{j=1}^{k-1} \frac{(u_j - 1)^+}{v_j + 1}\right)\right) \times \\ & \left(1 + O\left(n^{-1+3b} + n^{-1/2}\log_2 n\right)\right), \end{aligned}$$

where Multin$(i;\bar{p})$ *is the multinomially distributed random vector* $\{X_1, \ldots, X_{l+1}\}$*, with parameters equal to* $i = \mu - \mu'$ *(the number of deleted edges), and the probability vector* $\bar{p} = \bar{p}(\mathbf{w})$ *consisting of the following elements:*

$$\begin{aligned} p_j &= \frac{j(v_j - \delta_{ij})}{2\mu - i}, \qquad 1 \le j \le l-1, \\ p_l &= \frac{z^l \bar{\bar{v}}}{e_l(z)(l-1)!(2\mu - i)}, \\ p_{l+1} &= \frac{\bar{\bar{v}} z}{(2\mu - i)}, \end{aligned} \tag{4.21}$$

and z *is the unique root of Equation (4.19).*

Proof. Note that

$$\begin{aligned} \frac{\bar{\bar{v}}!}{\bar{\bar{v}}'!} \prod_{j=0}^{l-1} \frac{v_j!}{v_j'!} \prod_{j=1}^{l-1} (j!)^{v_j - v_j'} &= \left(\prod_{j=0}^{l} \frac{v_j!}{v_j'!} \right) \left(\prod_{j=0}^{l-1} (j!)^{-u_{j+1} + u_j + \delta_{ij}} \right) \\ &= i! \left(\prod_{j=0}^{l} \frac{v_j!}{v_j'!} \right) \left(\prod_{j=1}^{l-1} j^{u_j} \right) \left(\frac{1}{(l-1)!} \right)^{u_l}, \end{aligned}$$

where $u_0 = -1$ and $v_l = \bar{\bar{v}}$, $v_l' = \bar{\bar{v}}'$, since

$$\begin{aligned} \prod_{j=1}^{l-1} (j!)^{u_j - u_{j+1} + \delta_{ij}} &= 1^{u_1 - u_2 + \delta_{i1}} \cdot 2^{u_2 - u_3 + \delta_{i2}} \cdots (j!)^{u_j - u_{j+1} + \delta_{ij}} \times \\ &\quad ((j+1)!)^{u_{j+1} - u_{j+2} + \delta_{i(j+1)}} \cdots ((l-1)!)^{u_{l-1} - u_l + \delta_{i(l-1)}} \\ &= i! \prod_{j=1}^{l-1} j^{u_j} \left(\frac{1}{(l-1)!} \right)^{u_l}. \end{aligned}$$

Now, let Y_1', z', $\bar{\lambda}'$ be for $\mathbf{w}'$ what Y_1, z and $\bar{\lambda}$ are for $\mathbf{w}$. Note that $\|\mathbf{w} - \mathbf{w}'\| = O(1)$ uniformly for all $\mathbf{w}$ and $\mathbf{w}'$ related via (4.2). Thus, conditions (4.18) imply that

$$|z' - z| = O\left(\frac{1}{n}\right),$$

and therefore,

$$\begin{aligned} Var(Y_1') &= \left(1 + O\left(\frac{1}{n}\right)\right) Var(Y_1), \\ \bar{\lambda}' &= \left(1 + O\left(\frac{1}{n}\right)\right) \bar{\lambda}. \end{aligned}$$

Now, let us consider the following function

$$f_{\bar{\bar{v}}\bar{\bar{t}}}(y) = \bar{\bar{v}} \ln e_l(y) - \bar{\bar{t}} \ln y, \qquad y > 0,$$

so that

$$\frac{(e_l(y))^{\bar{\bar{v}}}}{y^{\bar{\bar{t}}}} = \exp(f_{\bar{\bar{v}}\bar{\bar{t}}}(y)).$$

Then, by (4.19) we obtain

$$\begin{aligned} \left(\frac{d}{dy} f_{\bar{\bar{v}}\bar{\bar{t}}}(y)\right)\bigg|_{y=z} &= \bar{\bar{v}}\frac{e_l'(z)}{e_l(z)} - \frac{\bar{\bar{t}}}{z} \\ &= \frac{1}{z}\left(\bar{\bar{v}}\frac{ze_l'(z)}{e_l(z)} - \bar{\bar{t}}\right) \\ &= \frac{1}{z}(\bar{\bar{v}}\mathbb{E}[Y_1] - \bar{\bar{t}}) = 0. \end{aligned}$$

So, z is a stationary point of $f_{\bar{\bar{v}}\bar{\bar{t}}}(y)$ (meaning that z' is a stationary point of $f_{\bar{\bar{v}}'\bar{\bar{t}}'}(y)$). Thus, by Taylor's Theorem, we obtain

$$\begin{aligned} \left|f_{\bar{\bar{v}}'\bar{\bar{t}}'}(z') - f_{\bar{\bar{v}}'\bar{\bar{t}}'}(z)\right| &= \left|\frac{1}{2}\, f''_{\bar{\bar{v}}'\bar{\bar{t}}'}(z')(z'-z)^2 + o\left(\frac{1}{n^2}\right)\right| \\ &= O\left(n\frac{1}{n^2}\right) = O\left(\frac{1}{n}\right). \end{aligned}$$

Therefore,

$$\begin{aligned} f_{\bar{\bar{v}}'\bar{\bar{t}}'}(z') - f_{\bar{\bar{v}}\bar{\bar{t}}}(z) &= f_{\bar{\bar{v}}'\bar{\bar{t}}'}(z) - f_{\bar{\bar{v}}\bar{\bar{t}}}(z) + O\left(\frac{1}{n}\right) \\ &= (\bar{\bar{v}}' - \bar{\bar{v}})\ln e_l(z) - (\bar{\bar{t}}' - \bar{\bar{t}})\ln z + O\left(\frac{1}{n}\right). \end{aligned}$$

Note that

$$\bar{\bar{v}} - \bar{\bar{v}}' = u_l, \quad \bar{\bar{t}} - \bar{\bar{t}}' = lu_l + u_{l+1}.$$

Combining the above relations, we obtain

$$\frac{(e_l(z'))^{\bar{\bar{v}}'}}{(z')^{\bar{\bar{t}}'}} \div \frac{(e_l(z))^{\bar{\bar{v}}}}{(z)^{\bar{\bar{t}}}} = \left(\frac{z^l}{e_l(z)}\right)^{u_l} \; z^{u_l+1} \; \left(1 + O\left(\frac{1}{n}\right)\right).$$

Finally,

$$\frac{(M_1' - 1)!!}{(M_1 - 1)!!} = (2\mu)^{-i}(1 + O(n^{-1})) = (2\mu - i)^{-i}(1 + O(n^{-1})),$$

since i is a constant. Therefore, by Proposition 4.3.1, Equation (4.7), Lemma 4.3.3 and the previous relations, we obtain:

$$p[\mathbf{w}' \mid \mathbf{w}] \;=\; \frac{v_i(1-\delta_{v_i0})}{v}\,\frac{(M_1'-1)!!}{(M_1-1)!!}\; i!\,\prod_{j=1}^{l-1} j^{u_j}\;\left(\frac{1}{(l-1)!}\right)^{u_l}\;\left(\frac{z^l}{e_l(z)}\right)^{u_l}\; z^{u_{l+1}}\times$$

$$
\begin{aligned}
& \frac{\bar{\bar{v}}!}{u_l! u_{l+1}!(\bar{\bar{v}} - u_l - u_{l+1})!} \prod_{j=1}^{l-1} \binom{v_j - \delta_{ij}}{u_j} \left(1 + O\left(n^{-1+3b} + n^{-1/2}\log_2 n\right)\right) \\
= \quad & \frac{v_i(1-\delta_{v_i 0})}{v} \frac{(M_1' - 1)!!}{(M_1 - 1)!!} i! \prod_{j=1}^{l-1} j^{u_j} \left(\frac{1}{(l-1)!}\right)^{u_l} \left(\frac{z^l}{e_l(z)}\right)^{u_l} z^{u_{l+1}} \times \\
& \frac{\bar{\bar{v}}!}{u_l! u_{l+1}!(\bar{\bar{v}} - u_l - u_{l+1})!} \prod_{j=1}^{l-1} \frac{(v_j - \delta_{ij})^{u_j}}{u_j!} \left(1 + O\left(\sum_{j=1}^{k-1} \frac{(u_j - 1)^+}{v_j + 1}\right)\right) \times \\
& \left(1 + O\left(n^{-1+3b} + n^{-1/2}\log_2 n\right)\right) \\
= \quad & \frac{v_i(1-\delta_{v_i 0})}{v} \frac{i!}{u_1! \cdots u_{l+1}!} \prod_{j=1}^{l-1} \left(\frac{j(v_j - \delta_{ij})}{2\mu - i}\right)^{u_j} \left(\frac{z^l \bar{\bar{v}}}{e_l(z)(l-1)!(2\mu - i)}\right)^{u_l} \times \\
& \left(\frac{\bar{\bar{v}} z}{(2\mu - i)}\right)^{u_{l+1}} \times \left(1 + O\left(\sum_{j=1}^{k-1} \frac{(u_j - 1)^+}{v_j + 1}\right)\right) \left(1 + O\left(n^{-1+3b} + n^{-1/2}\log_2 n\right)\right) \\
= \quad & \frac{v_i(1-\delta_{v_i 0})}{v} \mathbb{P}[\mathrm{M}ultin(i; \bar{p}) = \mathbf{u}] \left(1 + O\left(\sum_{j=1}^{k-1} \frac{(u_j - 1)^+}{v_j + 1}\right)\right) \times \\
& \left(1 + O\left(n^{-1+3b} + n^{-1/2}\log_2 n\right)\right),
\end{aligned}
$$

where $\bar{p}$ is the probability vector defined in (4.21), and the proposition follows. ∎

4.4 Approximating $\{\mathbb{E}[\mathbf{w}(\tau)]\}$

Let us set

$$
q[\mathbf{w}' \mid \mathbf{w}] = \frac{v_i(1-\delta_{v_i 0})}{v} \mathbb{P}[\mathrm{M}ultin(i; \bar{p}) = \mathbf{u}],
$$

where $i = i(\mathbf{w}', \mathbf{w}) = \mu - \mu'$, and $\mathbf{u} = \mathbf{u}(\mathbf{w}, \mathbf{w}')$, $\bar{p} = \bar{p}(\mathbf{w})$ are defined as in Proposition 4.3.4. Now, we consider the Markov chain defined on S_l, where the transition probabilities are $q[\mathbf{w}' \mid \mathbf{w}]$. If $\mathbb{E}_q[\cdot]$ denotes the expected value in this new Markov chain, then for $j = 0, \ldots, l-1$ we obtain:

$$
\begin{aligned}
\mathbb{E}_q[v_j(\tau+1) \mid \mathbf{w}(\tau) = \mathbf{w}] &= \sum_{\mathbf{w}'} v_j' q[\mathbf{w}' \mid \mathbf{w}] \\
&= \sum_{1 \le i \le k-1} \frac{v_i}{v} \mathbb{E}[v_j + X_{j+1} - X_j - \delta_{ij}],
\end{aligned}
$$

where we set $X_0 = -1$. Since $\mathbb{E}[X_j] = i p_j$, we get:

$$
\mathbb{E}_q[v_j(\tau+1) \mid \mathbf{w}(\tau) = \mathbf{w}] = v_j + f_j(\mathbf{w}(\tau))(1 + O(n^{-1})), \tag{4.22}
$$

uniformly for those $\mathbf{w}$'s for which conditions (4.18) are met, where

$$f_j(\mathbf{w}) = \begin{cases} 1 + \frac{v_1 s}{2\mu v} & j = 0 \\ \frac{(j+1)v_{j+1}s}{2\mu v} - \frac{jv_j s}{2\mu v} - \frac{v_j}{v} & 1 \le j \le k-1 \\ \frac{(j+1)v_{j+1}s}{2\mu v} - \frac{jv_j s}{2\mu v} & k \le j \le l-2 \\ \frac{z^l \bar{\bar{v}} s}{2\mu v (l-1)! e_l(z)} - \frac{(l-1)v_{l-1}s}{2\mu v} & j = l-1 \end{cases}. \tag{4.23}$$

This is the case, because for $0 \le j \le k-1$,

$$\begin{aligned} &\sum_{1\le i\le k-1} \frac{v_i}{v}\mathbb{E}[v_j + X_{j+1} - X_j - \delta_{ij}] = \\ &= \quad v_j + (p_{j+1} - p_j)\frac{\sum_{i=1}^{k-1} iv_i}{v} - \frac{v_j}{v} = v_j + (p_{j+1} - p_j)\frac{s}{v} - \frac{v_j}{v}, \end{aligned}$$

and for $k \le j \le l-1$ the last term vanishes. On the other hand,

$$\begin{aligned} \mathbb{E}_q[\mu(\tau+1) \mid \mathbf{w}(\tau) = \mathbf{w}] &= \sum_{\mathbf{w}'} \mu' q[\mathbf{w}' \mid \mathbf{w}] \\ &= \mu - \sum_{1\le i\le k-1} i\,\frac{v_i}{v} = \mu - \frac{s}{v}. \end{aligned} \tag{4.24}$$

Equations (4.22) and (4.24) along with Proposition 4.3.4 suggest that $\{\mathbf{w}(\tau)\}$ must be quite close to the solution $\tilde{\mathbf{w}} = (\tilde{\mathbf{v}}, \tilde{\mu})$ of the following system of differential equations:

$$\begin{aligned} \frac{dv_j(\tau)}{d\tau} &= f_j(\mathbf{w}(\tau)), \quad 0 \le j \le l-1, \\ \frac{d\mu(\tau)}{d\tau} &= -\frac{s(\tau)}{v(\tau)}, \end{aligned} \tag{4.25}$$

subject to the initial conditions:

$$\begin{aligned} \tilde{v}_j(0) &= v_j(\mathcal{G}_{n,m}), \quad 0 \le j \le l-1, \\ \tilde{\mu}(0) &= m. \end{aligned} \tag{4.26}$$

Note that $z(0)$ is the solution of the Equation (4.19) with $\bar{\bar{v}} = \bar{\bar{v}}(\mathcal{G}_{n,m})$ and $\bar{\bar{t}} = \bar{\bar{t}}(\mathcal{G}_{n,m})$.

We shall see that the above system has some useful integrals. As in [37], we are proving that along the trajectory of system (4.25)

$$\frac{z^2}{\mu} \equiv \text{const.} \tag{4.27}$$

To see this, note that by (4.19), we obtain:

$$\frac{d}{d\tau}\frac{ze_l'(z)}{e_l(z)} = \left(\frac{\bar{\bar{t}}}{\bar{\bar{v}}}\right)'_\tau = \frac{1}{\bar{\bar{v}}}\frac{d\bar{\bar{t}}}{d\tau} - \frac{\bar{\bar{v}}'}{\bar{\bar{v}}^2}\,\bar{\bar{t}}, \tag{4.28}$$

where $\bar{\bar{v}}' = d\bar{\bar{v}}/d\tau$. Moreover, using (4.25) and the definition of $\bar{\bar{t}}$, we have:

$$\begin{aligned}
\frac{d\bar{\bar{t}}}{d\tau} &= 2\frac{d\mu}{d\tau} - \sum_{j=1}^{l-2} j\left(\frac{(j+1)v_{j+1}s}{2\mu v} - \frac{jv_js}{2\mu v} - \frac{v_j}{v}\,\delta_{j\le k-1}\right) \\
&\quad -(l-1)\left(\frac{z^l\bar{\bar{v}}s}{2\mu v(l-1)!e_l(z)} - \frac{(l-1)v_{l-1}s}{2\mu v}\right) \\
&= 2\frac{d\mu}{d\tau} + \sum_{j=1}^{l-1}\frac{jv_js}{2\mu v} + \sum_{j=1}^{k-1}\frac{jv_j}{v} - \frac{(l-1)z^l\bar{\bar{v}}s}{2\mu v(l-1)!e_l(z)} \\
&= 2\frac{d\mu}{d\tau} + \sum_{j=1}^{l-1}\frac{jv_js}{2\mu v} + \frac{s}{v} - \frac{(l-1)z^l\bar{\bar{v}}s}{2\mu v(l-1)!e_l(z)} \\
&= -\frac{2s}{v} + \frac{s}{v}\frac{s}{2\mu} + \frac{s}{v} + \sum_{j=k}^{l-1}\frac{jv_js}{2\mu v} - \frac{(l-1)z^l\bar{\bar{v}}s}{2\mu v(l-1)!e_l(z)} \\
&= -\frac{s}{v} + \frac{s}{v}\frac{s}{2\mu} + \sum_{j=k}^{l-1}\frac{jv_js}{2\mu v} - \frac{(l-1)z^l\bar{\bar{v}}s}{2\mu v(l-1)!e_l(z)} \\
&= -\frac{s}{v} + \frac{s}{v}\frac{s}{2\mu} + \frac{2\mu-\bar{\bar{t}}-s}{v}\frac{s}{2\mu} - \frac{(l-1)z^l\bar{\bar{v}}s}{2\mu v(l-1)!e_l(z)} \\
&= -\frac{2\mu s}{2\mu v} + \frac{s}{v}\frac{s}{2\mu} + \frac{2\mu-\bar{\bar{t}}-s}{v}\frac{s}{2\mu} - \frac{(l-1)z^l\bar{\bar{v}}s}{2\mu v(l-1)!e_l(z)} \\
&= -\frac{s}{v}\frac{\bar{\bar{t}}}{2\mu} - \frac{(l-1)z^l\bar{\bar{v}}s}{2\mu v(l-1)!e_l(z)}.
\end{aligned}$$

On the other hand,

$$\frac{d\bar{\bar{v}}}{d\tau} = -\sum_{j=0}^{l-1}\frac{dv_j(\tau)}{d\tau} = -\frac{z^l\bar{\bar{v}}s}{2\mu v(l-1)!e_l(z)}. \tag{4.29}$$

Therefore, by (4.28), we have

$$\begin{aligned}
\frac{d}{d\tau}\frac{ze_l'(z)}{e_l(z)} &= \frac{1}{\bar{\bar{v}}}\left(-\frac{s}{v}\frac{\bar{\bar{t}}}{2\mu} - \frac{(l-1)z^l\bar{\bar{v}}s}{2\mu v(l-1)!e_l(z)}\right) + \frac{\bar{\bar{v}}}{\bar{\bar{v}}^2}\frac{z^l}{(l-1)!e_l(z)}\frac{s\bar{\bar{t}}}{2\mu v} \\
&= \frac{s}{2\mu v}\left(-\frac{ze_l'(z)}{e_l(z)} - \frac{z^2\left(z^{l-2}/(l-2)!\right)}{e_l(z)} + \frac{z^2\left(z^{l-1}/(l-1)!\right)e_l'(z)}{e_l^2(z)}\right) \\
&= \frac{s}{2\mu v}\left(-\frac{ze_l'(z)}{e_l(z)} - \frac{z^2e_l''(z)}{e_l(z)} + \left(\frac{ze_l'(z)}{e_l(z)}\right)^2\right).
\end{aligned}$$

Notice that the expression in parenthesis is equal to $-z\frac{d}{dz}\frac{ze_l'(z)}{e_l(z)}$. Therefore,

$$\frac{d}{d\tau}\frac{ze_l'(z)}{e_l(z)} = -\frac{sz}{2\mu v}\frac{d}{dz}\frac{ze_l'(z)}{e_l(z)},$$

which implies that

$$\frac{dz}{d\tau} = -\frac{sz}{2\mu v} = \frac{z}{2\mu}\,\frac{d\mu}{d\tau}, \tag{4.30}$$

since $ze_l'(z)/e_l(z)$ is strictly monotone with respect to z and, therefore, its derivative with respect to z is non-vanishing. Moreover, we obtain $\frac{dz}{d\mu} = \frac{z}{2\mu}$. The last equality yields (4.27).

From (4.29) and (4.30), we obtain

$$\frac{d\bar{\bar{v}}}{dz} = \bar{\bar{v}}\,\frac{z^{l-1}/(l-1)!}{e_l(z)} = \bar{\bar{v}}\,\frac{dp_l(z)/dz}{p_l(z)}.$$

Thus, we have shown the following:

$$\frac{\bar{\bar{v}}}{p_l(z)} \equiv \text{const.} \tag{4.31}$$

Using (4.27) and (4.31) we get a third integral as follows:

$$\begin{aligned}\frac{\bar{\bar{t}}}{2\mu}\,\frac{z}{p_{l-1}(z)} &= \frac{z^2}{2\mu}\,\frac{\bar{\bar{t}}}{p_l(z)}\,\frac{p_l(z)}{zp_{l-1}(z)} = \frac{z^2}{2\mu}\,\frac{\bar{\bar{t}}}{p_l(z)}\,\frac{p_l(z)}{zp_{l-1}(z)}\\ &= \frac{z^2}{2\mu}\,\frac{\bar{\bar{t}}}{p_l(z)}\,\frac{\bar{\bar{v}}}{\bar{\bar{t}}} = \frac{z^2}{2\mu}\,\frac{\bar{\bar{v}}}{p_l(z)} \equiv \text{const.},\end{aligned}$$

since

$$\frac{1}{\mathbb{E}[Y_1]} = \frac{p_l(z)}{zp_{l-1}(z)} = \frac{\bar{\bar{v}}}{\bar{\bar{t}}}.$$

So, we have shown that

$$\frac{\bar{\bar{t}}}{2\mu}\,\frac{z}{p_{l-1}(z)} \equiv \text{const.} \tag{4.32}$$

We shall give one more equation that concerns $s(\tau)$, which we are going to use in the course of our proof. Using (4.22), for those $\mathbf{w}$'s for which conditions (4.18) are met we obtain:

$$\begin{aligned}\mathbb{E}_q[s(\tau+1) \mid \mathbf{w}(\tau)=\mathbf{w}] &= s(\tau) + \sum_{j=1}^{k-1} j\left(\frac{(j+1)v_{j+1}s}{2\mu v} - \frac{jv_js}{2\mu v} - \frac{v_j}{v}\right) + O(n^{-1})\\ &= s(\tau) + \sum_{j=1}^{k-1}\left(\frac{j(j+1)v_{j+1}s}{2\mu v} - \frac{j^2v_js}{2\mu v} - \frac{jv_j}{v}\right) + O(n^{-1})\\ &= s(\tau) - \frac{s}{v} + \sum_{j=1}^{k-1}\left(\frac{(j+1)^2v_{j+1}s}{2\mu v} - \frac{j^2v_js}{2\mu v}\right) - \sum_{j=1}^{k-1}\frac{(j+1)v_{j+1}s}{2\mu v} + O(n^{-1})\\ &= s(\tau) - \frac{s}{v} - \frac{v_1s}{2\mu v} + \frac{k^2v_ks}{2\mu v} - \left(\frac{s^2}{2\mu v} - \frac{v_1s}{2\mu v} + \frac{kv_ks}{2\mu v}\right) + O(n^{-1})\\ &= s(\tau) - \frac{s}{v} + \frac{k^2v_ks}{2\mu v} - \frac{s^2}{2\mu v} - \frac{kv_ks}{2\mu v} + O(n^{-1})\\ &= s(\tau) - \frac{s^2}{2\mu v} - \frac{s}{v}\left(1 + \frac{kv_k}{2\mu} - \frac{k^2v_k}{2\mu}\right) + O(n^{-1})\\ &= s(\tau) - \frac{s^2}{2\mu v} - \frac{s}{v}\left(1 - \frac{k(k-1)v_k}{2\mu}\right) + O(n^{-1}).\end{aligned} \tag{4.33}$$

Now, we define the following quantities:

$$J_1(\mathbf{w}) \;=\; \frac{nz^2}{\mu}, \quad J_2(\mathbf{w}) = \frac{\bar{\bar{v}}}{np_l(z)}, \quad J_3(\mathbf{w}) = \frac{t}{2\mu}\,\frac{z}{p_{k-1}(z)}, \quad J_4(\mathbf{w}) = \frac{\bar{\bar{t}}}{2\mu}\,\frac{z}{p_{l-1}(z)}. \tag{4.34}$$

4.5 Proof of Theorem 4.1.1

For given $a > 0$,we define the set $\mathbf{W} = \mathbf{W}(a)$ by

$$\mathbf{W}(a) \;=\; \left\{\mathbf{w} \in S_l \;:\; h(\mathbf{w}) > 0,\ g(\mathbf{w}) \le \frac{1}{n},\ \bar{v} \ge an,\ \bar{\bar{t}} \ge (l+a)\bar{\bar{v}}\right\}, \tag{4.35}$$

and the stopping time $\mathcal{T} = \mathcal{T}(a)$ by

$$\mathcal{T}(a) \;=\; \begin{cases} \min_\tau\{\tau < T \;:\; \mathbf{w}(\tau) \notin \mathbf{W}(a)\} & \text{if such } \tau \text{ exists} \\ T & \text{otherwise} \end{cases}. \tag{4.36}$$

Consider a subsystem of the system (4.25). Namely,

$$\begin{aligned} \frac{dv_j}{d\tau} &= \frac{(j+1)v_{j+1}s}{2\mu v} - \frac{jv_j s}{2\mu v}, && k \le j \le l-2, \\ \frac{dv_{l-1}}{d\tau} &= \frac{z^l\bar{\bar{v}}s}{2\mu v(l-1)!e_l(z)} - \frac{(l-1)v_{l-1}s}{2\mu v}, && j = l-1. \end{aligned}$$

We can express this system in terms of z using (4.30). We obtain

$$\begin{aligned} \frac{dv_j}{dz}\,\frac{dz}{d\tau} &= -\frac{sz}{2\mu v}\left(-\frac{(j+1)v_{j+1}}{z} + \frac{jv_j}{z}\right), && k \le j \le l-2, \\ \frac{dv_{l-1}}{dz}\,\frac{dz}{d\tau} &= -\frac{sz}{2\mu v}\left(-\frac{z^{l-1}\bar{\bar{v}}}{(l-1)!e_l(z)} + \frac{(l-1)v_{l-1}}{z}\right), && j = l-1, \end{aligned}$$

that is

$$\begin{aligned} \frac{dv_j}{dz} &= \frac{1}{z}\left(-(j+1)v_{j+1} + jv_j\right), && k \le j \le l-2, \\ \frac{dv_{l-1}}{dz} &= -\frac{z^{l-1}\bar{\bar{v}}}{(l-1)!e_l(z)} + \frac{(l-1)v_{l-1}}{z}, && j = l-1, \end{aligned} \tag{4.37}$$

as long as $dz/d\tau \ne 0$. Recall that $\rho_j(z)$ denotes the probability that a Poisson random variable of mean z is equal to j. We shall prove that the following quantities are integrals of the above system:

$$\begin{aligned} I_1 &= I_1(\mathbf{w}) = e^{-z}\left(\frac{v_{l-1}}{\rho_{l-1}(z)} - \frac{\bar{\bar{v}}}{p_l(z)}\right), \\ I_t &= I_t(\mathbf{w}) = e^{-z}\left(\frac{v_{l-t}}{\rho_{l-t}(z)} - \frac{\bar{\bar{v}}}{p_l(z)}\right) - \sum_{i=1}^{t-1}\frac{(-1)^i z^i}{i!}I_{t-i}, \quad \text{for } 2 \le t \le l-k. \end{aligned} \tag{4.38}$$

We prove this by induction on t. For $t = 1$, multiplying the last equation of (4.37) by the integrating factor $1/z^{l-1}$, we obtain

$$\begin{aligned}
\frac{v_{l-1}}{z^{l-1}} - \frac{v_{l-1}(0)}{z(0)^{l-1}} &= -\int_{z(0)}^{z} \frac{e^{-\mathrm{z}}\mathrm{z}^{l-1}}{\mathrm{z}^{l-1}(l-1)!}\frac{\bar{\bar{v}}}{p_l(\mathrm{z})}d\mathrm{z} = \frac{e^{-z}}{(l-1)!}\frac{\bar{\bar{v}}}{p_l(z)} - \frac{e^{-z(0)}}{(l-1)!}\frac{\bar{\bar{v}}(0)}{p_l(z(0))} \Rightarrow \\
\frac{v_{l-1}}{z^{l-1}} - \frac{e^{-z}}{(l-1)!}\frac{\bar{\bar{v}}}{p_l(z)} &= \frac{v_{l-1}(0)}{z(0)^{l-1}} - \frac{e^{-z(0)}}{(l-1)!}\frac{\bar{\bar{v}}(0)}{p_l(z(0))} \Rightarrow \\
I_1 &= e^{-z}\left(\frac{v_{l-1}}{\rho_{l-1}(z)} - \frac{\bar{\bar{v}}}{p_l(z)}\right) = \text{const.},
\end{aligned}$$

since $\bar{\bar{v}}/p_l(z)$ remains constant as the system (4.25) evolves (actually, as we have seen, its derivative with respect to z is zero). Now assume that our claim is true for some $1 \le t < l-k$. We shall prove that it is true for $t+1$. Multiplying the differential equation of v_{l-t-1} by the integrating factor $1/z^{l-t-1}$ and applying the induction hypothesis, we have

$$\begin{aligned}
\frac{1}{z^{l-t-1}}\frac{dv_{l-t-1}}{dz} - \frac{(l-t-1)v_{l-t-1}}{z^{l-t}} &= -\frac{(l-t)v_{l-t}}{z^{l-t}} \Rightarrow \\
\frac{d\left(v_{l-t-1}/z^{l-t-1}\right)}{dz} &= -\frac{(l-t)v_{l-t}}{z^{l-t}} \Rightarrow \\
\frac{d\left(\frac{v_{l-t-1}}{z^{l-t-1}/(l-t-1)!}\right)}{dz} &= -\frac{v_{l-t}}{z^{l-t}/(l-t)!} \Rightarrow \\
\frac{d\left(\frac{v_{l-t-1}}{z^{l-t-1}/(l-t-1)!}\right)}{dz} &= -e^{-z}\frac{\bar{\bar{v}}}{p_l(z)} - \sum_{i=0}^{t-1}\frac{(-1)^i z^i}{i!}I_{t-i} \Rightarrow \\
\frac{v_{l-t-1}}{z^{l-t-1}/(l-t-1)!} &= e^{-z}\frac{\bar{\bar{v}}}{p_l(z)} - \sum_{i=0}^{t-1}\frac{(-1)^i z^{i+1}}{(i+1)!}I_{t-i} + \text{const.} \Rightarrow \\
I_{t+1} &= e^{-z}\left(\frac{v_{l-t-1}}{\rho_{l-t-1}(z)} - \frac{\bar{\bar{v}}}{p_l(z)}\right) - \sum_{i=0}^{t-1}\frac{(-1)^{i+1}z^{i+1}}{(i+1)!}I_{t-i} = \text{const.}
\end{aligned}$$

since $\bar{\bar{v}}/p_l(z)$ and $I_1, \ldots, I_t$ remain constant as the system (4.25) evolves.

Now, we will present one more integral of the above system. Note that by (4.38) we have for $t = 1, \ldots, l-k$

$$v_{l-t} = \rho_{l-t}(z)\frac{\bar{\bar{v}}}{p_l(z)} + e^z\rho_{l-t}(z)\sum_{i=0}^{t-1}\frac{(-1)^i z^i}{i!}I_{t-i}.$$

Therefore,

$$\begin{aligned}
\sum_{j=k}^{l-1} jv_j &= \sum_{j=k}^{l-1}\left(j\rho_j(z)\frac{\bar{\bar{v}}}{p_l(z)} + je^z\rho_j(z)\sum_{i=0}^{l-j-1}\frac{(-1)^i z^i}{i!}I_{l-j-i}\right) \\
&= \sum_{j=k}^{l-1} j\rho_j(z)\frac{\bar{\bar{v}}}{p_l(z)} + \sum_{j=k}^{l-1} je^z\rho_j(z)\sum_{i=0}^{l-j-1}\frac{(-1)^i z^i}{i!}I_{l-j-i}
\end{aligned}$$

and, consequently,

$$
\begin{aligned}
\frac{t}{2\mu} &= \left(\bar{\bar{v}}\ \frac{ze_l'(z)}{e_l(z)} + \sum_{j=k}^{l-1} j\rho_j(z)\ \frac{\bar{\bar{v}}}{p_l(z)}\right)\ \frac{1}{2\mu} + \frac{1}{2\mu}\sum_{j=k}^{l-1} je^z\rho_j(z)\sum_{i=0}^{l-j-1}\frac{(-1)^i z^i}{i!} I_{l-j-i} \\
&= \frac{n}{2\mu}\ J_2(\mathbf{w})\ \left(e^{-z}ze_l'(z) + \sum_{j=k}^{l-1} j\rho_j(z)\right) + \frac{1}{2\mu}\sum_{j=k}^{l-1} je^z\rho_j(z)\sum_{i=0}^{l-j-1}\frac{(-1)^i z^i}{i!} I_{l-j-i} \\
&= \frac{n}{2\mu}\ J_2(\mathbf{w})\ e^{-z}\ \left(z\sum_{j\geq l} j\ \frac{z^{j-1}}{j!} + \sum_{j=k}^{l-1} j\frac{z^j}{j!}\right) + \frac{1}{2\mu}\sum_{j=k}^{l-1} je^z\rho_j(z)\sum_{i=0}^{l-j-1}\frac{(-1)^i z^i}{i!} I_{l-j-i} \\
&= \frac{nz}{2\mu}\ J_2(\mathbf{w})\ e^{-z}\ \sum_{j\geq k-1}\frac{z^j}{j!} + \frac{1}{2\mu}\sum_{j=k}^{l-1} je^z\rho_j(z)\sum_{i=0}^{l-j-1}\frac{(-1)^i z^i}{i!} I_{l-j-i} \\
&= \frac{nz}{2\mu}\ J_2(\mathbf{w})\ p_{k-1}(z) + \frac{1}{2\mu}\sum_{j=k}^{l-1} je^z\rho_j(z)\sum_{i=0}^{l-j-1}\frac{(-1)^i z^i}{i!} I_{l-j-i} \\
&= \frac{J_1(\mathbf{w})\ J_2(\mathbf{w})}{2}\ \frac{p_{k-1}(z)}{z} + \frac{1}{2\mu}\sum_{j=k}^{l-1} je^z\rho_j(z)\sum_{i=0}^{l-j-1}\frac{(-1)^i z^i}{i!} I_{l-j-i}.
\end{aligned}
$$

Hence,

$$
I \equiv I(\mathbf{w}) \equiv J_3(\mathbf{w}) - \frac{J_1(\mathbf{w})\ J_2(\mathbf{w})}{2} - \frac{z}{p_{k-1}(z)}\frac{1}{2\mu}\sum_{j=k}^{l-1} je^z\rho_j(z)\sum_{i=0}^{l-j-1}\frac{(-1)^i z^i}{i!} I_{l-j-i} = \text{const.}
$$

Now, arguing as in [37] (see Lemma 1 there) we prove the following:

Lemma 4.5.1 *Let J be any of the following: J_i, where $i = 1, 2, 4$, or I or $I_1/n, \ldots, I_{l-k}/n$. Conditioned on $\{\mathbf{w}(0) \in \mathbf{W}(a)\} \cap \{\mu(\mathbf{w}(0)) = \lceil \theta n/2 \rceil\}$, for $0 < \alpha < \min\{1/2, 1-3b\}$, we have:*

$$
\mathbb{P}\left[\max_{\tau\leq\mathcal{T}} |J(\mathbf{w}(\tau)) - J(\mathbf{w}(0))| > x\right] = O(e^{-xn^{\alpha}}), \tag{4.39}
$$

uniformly for $x > 0$.

Proof. We introduce the function

$$
Q(\mathbf{w}) = \exp\left(n^{\alpha}(J(\mathbf{w}) - J(\mathbf{w}(0)))\right).
$$

Let us evaluate

$$
\Sigma = \sum_{\mathbf{w}'} Q(\mathbf{w}')p[\mathbf{w}' \mid \mathbf{w}], \quad \mathbf{w} \in \mathbf{W} = \mathbf{W}(a),
$$

By the definition of $\mathbf{W}(a)$ and since $g(\mathbf{w})$ decreases in the course of the deletion process, we deduce that for n sufficiently large $\mathbf{w}' \in \mathbf{W}(a/2)$, whenever $\mathbf{w} \in \mathbf{W}(a)$ and $p[\mathbf{w}' \mid \mathbf{w}] > 0$. For every point that belongs to the line segment connecting $\mathbf{w}$ and $\mathbf{w}'$, the components of grad$J(\mathbf{w})$ are of order n^{-1}, while the second order derivatives are of order n^{-2}, uniformly for any $\mathbf{w} = \mathbf{w}(\tau)$, where $\tau \leq \mathcal{T}$. Therefore,

$$J(\mathbf{w}') = J(\mathbf{w}) + (\mathbf{w}' - \mathbf{w})^* \ \mathrm{grad}J(\mathbf{w}) + O(n^{-2}).$$

So, expanding the exponential function, we obtain

$$Q(\mathbf{w}') = Q(\mathbf{w}) \left(1 + n^{\alpha} \ (\mathbf{w}' - \mathbf{w})^* \ \mathrm{grad}J(\mathbf{w}) + O(n^{2(\alpha-1)})\right),$$

and consequently

$$\Sigma = Q(\mathbf{w}) \left(1 + n^{\alpha}\mathbb{E}[\mathbf{w}' - \mathbf{w} \mid \mathbf{w}]^* \ \mathrm{grad}J(\mathbf{w}) + O(n^{2(\alpha-1)})\right). \tag{4.40}$$

By its definition, $J(\tilde{\mathbf{w}}(\tau))$ remains constant along the trajectory $\tilde{\mathbf{w}}(\tau)$ of the differential equations system (4.25). This means that

$$\mathbb{E}_q[\mathbf{w}' - \mathbf{w} \mid \mathbf{w}] \cdot \mathrm{grad}J(\mathbf{w}) = 0.$$

Thus,

$$\mathbb{E}[\mathbf{w}' - \mathbf{w} \mid \mathbf{w}]^* \cdot \mathrm{grad}J(\mathbf{w}) = \sum_{\mathbf{w}'} (\mathbf{w}' - \mathbf{w})^* \mathrm{grad}J(\mathbf{w}) \ \left(p[\mathbf{w}' \mid \mathbf{w}] - q[\mathbf{w}' \mid \mathbf{w}]\right). \tag{4.41}$$

By Proposition 4.3.4,

$$\left|p[\mathbf{w}' \mid \mathbf{w}] - q[\mathbf{w}' \mid \mathbf{w}]\right| \ = \ O\left(n^{-1+3b} + n^{-1/2}\log_2 n\right) + O\left(q[\mathbf{w}' \mid \mathbf{w}] \sum_{j=1}^{k-1} \frac{(u_j - 1)^+}{v_j + 1}\right). \tag{4.42}$$

Now, for $1 \leq j \leq k-1$,

$$\sum_{\mathbf{w}'} q[\mathbf{w}' \mid \mathbf{w}] \ \frac{(u_j - 1)^+}{v_j + 1} \ = \ \sum_{i=1}^{k-1} \frac{v_i}{v} \ \mathbb{E}\left[\frac{(X_j - 1)^+}{v_j + 1}\right] = O\left(\frac{1}{\mu}\right) = O\left(\frac{1}{n}\right). \tag{4.43}$$

So, the "distance" between the two transition probabilities in (4.42) becomes

$$\left|p[\mathbf{w}' \mid \mathbf{w}] - q[\mathbf{w}' \mid \mathbf{w}]\right| = O(n^{-1+3b} + n^{-1/2}\log_2 n).$$

On the other hand, $\|\mathrm{grad}J(\mathbf{w})\| = O(n^{-1})$ and we obtain from (4.41)

$$\mathbb{E}[\mathbf{w}' - \mathbf{w} \mid \mathbf{w}]^* \cdot \mathrm{grad}J(\mathbf{w}) = O(n^{-2+3b} + n^{-3/2}\log_2 n).$$

Therefore,

$$\begin{aligned} \Sigma &= Q(\mathbf{w})(1+O(n^{-\omega}\log_2 n)), \\ \omega &= \min\{2-3b-\alpha, 3/2-\alpha, 2(1-\alpha)\} > 1, \end{aligned} \tag{4.44}$$

since $\alpha < \min\{1/2, 1-3b\}$. Probabilistically, the above equation means the following. We introduce the following random sequence:

$$\{R(\tau)\} = \{Q(\mathbf{w}(\tau))\}.$$

Then, for $\mathbf{w}(\tau) \in \mathbf{W}$, we have

$$\mathbb{E}[R(\tau+1) \mid \mathbf{w}(\tau)] = (1+O(n^{-\omega}\log_2 n))R(\tau),$$

which means that $\{R(\tau)\}$ is *almost* a martingale. Since the total number of steps is at most n, the above equality implies that the sequence

$$\{\tilde{R}(\tau)\} = \{(1+n^{-\omega}\log_2^2 n)^{-\tau}\ R(\tau)\}$$

is a *supermartingale* for n sufficiently large, for $\tau \leq \mathcal{T}$. Now, fix $x > 0$ and introduce a stopping time

$$\mathcal{T}' = \begin{cases} \min\{\tau \leq \mathcal{T}\ :\ J(\mathbf{w}(\tau)) - J(\mathbf{w}(0)) > x\} & \text{if such } \tau \text{ exists} \\ \mathcal{T}+1 & \text{otherwise} \end{cases}.$$

Applying the Optional Stopping Theorem to the supermartingale $\{\tilde{R}(\tau)\}$ and the stopping time $\mathcal{T} \wedge \mathcal{T}'$, and returning to $\{R(\tau)\}$, we obtain:

$$\begin{aligned} \mathbb{E}[Q(\mathbf{w}(\mathcal{T} \wedge \mathcal{T}'))] &\leq (1+n^{-\omega}\log_2^2 n)^n\ \mathbb{E}[Q(\mathbf{w}(0))] \\ &= (1+n^{-\omega}\log_2^2 n)^n = O(1). \end{aligned}$$

Since

$$\mathbb{E}[Q(\mathbf{w}(\mathcal{T} \wedge \mathcal{T}'))] \geq e^{xn^\alpha}\ \mathbb{P}[\mathcal{T}' \leq \mathcal{T}],$$

we obtain

$$\mathbb{P}[\max_{\tau \leq \mathcal{T}} (J(\mathbf{w}(\tau)) - J(\mathbf{w}(0))) > x] = \mathbb{P}[\mathcal{T}' \leq \mathcal{T}] = O(e^{-xn^\alpha}), \tag{4.45}$$

uniformly for $x > 0$.

If we define

$$Q(\mathbf{w}) = \exp\left(n^\alpha(J(\mathbf{w}(0)) - J(\mathbf{w}))\right),$$

then doing the same manipulations we deduce that the sequence $\{\tilde{R}(\tau)\} = \{\left(1+n^{-\omega}\log_2^2 n\right)^{-\tau} Q(\mathbf{w}(\tau))\}$ is a supermartingale for n sufficiently large, where ω is as in (4.44). Now, for some $x > 0$ we define the stopping time

$$\mathcal{T}' = \begin{cases} \min\{\tau \le \mathcal{T} \;:\; J(\mathbf{w}(\tau)) - J(\mathbf{w}(0)) < -x\} & \text{if such } \tau \text{ exists} \\ \mathcal{T}+1 & \text{otherwise} \end{cases},$$

and, applying the Optional Stopping Theorem, we obtain

$$\mathbb{E}[Q(\mathbf{w}(\mathcal{T}' \wedge \mathcal{T}))] \le \left(1+n^{-\omega}\log_2^2 n\right)^n \mathbb{E}[Q(\mathbf{w}(0))] = O(1),$$

and, moreover,

$$\mathbb{E}[Q(\mathbf{w}(\mathcal{T}' \wedge \mathcal{T}))] \ge e^{xn^\alpha}\mathbb{P}[\mathcal{T}' \le \mathcal{T}].$$

Thus, we have

$$\mathbb{P}[\min_{\tau\le\mathcal{T}} (J(\mathbf{w}(\tau)) - J(\mathbf{w}(0))) < -x] = \mathbb{P}[\mathcal{T}' \le \mathcal{T}] = O(e^{-xn^\alpha}), \tag{4.46}$$

uniformly for $x > 0$. The lemma follows from (4.45) and (4.46). ∎

The following is immediate:

Corollary 4.5.2 *For* $0 < \beta < \alpha < \min\{1/2, 1-3b\}$*, we have*

$$\mathbb{P}[A \mid \mathbf{w}(0) \in \mathbf{W}(a),\ \mu(0) = \lceil \theta n/2 \rceil] = O(e^{-n^{\alpha-\beta}}),$$

where

$$A = A(a,\beta,\alpha) = \left\{ \max_{J\in\{J_1,J_2,J_4,I,I_1/n,\ldots,I_{l-k}/n\},\ \tau\le\mathcal{T}} |J(\mathbf{w}(\tau)) - J(\mathbf{w}(0))| > n^{-\beta} \right\}.$$

Let B be the event $g(\mathbf{w}(\mathcal{G}_{n,m})) \le 1/n$. As we have already seen

$$\mathbb{P}[B] \ge 1 - O(e^{-n^b}). \tag{4.47}$$

The double-conditioning technique along with the bounded differences lemma (see Corollary 2.27 in [23] or [30]) reveal that for every $b_1 < 1/3$ there exists a constant $c_1 > 0$ such that,

$$\mathbb{P}[C] \ge 1 - O(e^{-c_1 n^{b_1}}), \tag{4.48}$$

where $C = C_1 \cap C_2 \cap_{j=k}^{l-1} V_j$ and

$$\begin{aligned} C_1 &= \left\{ |\bar{\bar{v}}(\mathcal{G}_{n,m}) - np_l(\theta)| \le n^{(1+b_1)/2} \right\}, \\ C_2 &= \left\{ |\bar{\bar{t}}(\mathcal{G}_{n,m}) - n\theta p_{l-1}(\theta)| \le n^{(1+b_1)/2} \right\}, \\ V_j &= \left\{ |v_j(\mathcal{G}_{n,m}) - n\rho_j(\theta)| \le n^{(1+b_1)/2} \right\}, \quad j = k,\ldots,l-1. \end{aligned} \tag{4.49}$$

Notice that on C

$$z(0) = z(\mathcal{G}_{n,m}) = \theta + O(n^{-(1-b_1)/2}), \tag{4.50}$$

by (4.49) and (4.19). Moreover, if C is realised then for each $j = k, \ldots, l-1$ we have

$$\left| \frac{v_j(0)}{n\rho_j(z(0))} - \frac{\bar{\bar{v}}(0)}{np_l(z(0))} \right| = O\left(n^{-\frac{1-b_1}{2}}\right).$$

Therefore, $I_1(\mathbf{w}(0))/n = O\left(n^{-\frac{1-b_1}{2}}\right)$ and, by induction, we obtain $I_t(\mathbf{w}(0))/n = O\left(n^{-\frac{1-b_1}{2}}\right)$, for $t = 2, \ldots, l-k$, as well.

We shall choose an a so that $B \cap C \subseteq \{\mathbf{w}(0) \in \mathbf{W}(a),\ \mu(0) = \lceil \theta n/2 \rceil\}$. Now, suppose that $\bar{A} \cap B \cap C$ is realised. We have for any $\tau \leq \mathcal{T}$

$$\begin{aligned}
&\left| \frac{I_1(\mathbf{w}(\tau))}{n} - \frac{I_1(\mathbf{w}(0))}{n} \right| = \\
&\qquad \left| e^{-z(\tau)} \left(\frac{v_{l-1}(\tau)}{n\rho_{l-1}(z(\tau))} - \frac{\bar{\bar{v}}(\tau)}{np_l(z(\tau))} \right) - e^{-z(0)} \left(\frac{v_{l-1}(0)}{n\rho_{l-1}(z(0))} - \frac{\bar{\bar{v}}(0)}{np_l(z(0))} \right) \right| \\
&= \quad \left| e^{-z(\tau)} \left(\frac{v_{l-1}(\tau)}{n\rho_{l-1}(z(\tau))} - 1 + O\left(n^{-\beta} + n^{-\frac{1-b_1}{2}}\right) \right) + O\left(n^{-\frac{1-b_1}{2}}\right) \right|,
\end{aligned}$$

since $\bar{\bar{v}}(0)/(np_l(z(0))) = 1 + O\left(n^{-\frac{1-b_1}{2}}\right)$ (on C). Therefore, uniformly for any $\tau \leq \mathcal{T}$ we have

$$\frac{v_{l-1}(\tau)}{n\rho_{l-1}(z(\tau))} = 1 + O\left(n^{-\beta} + n^{-\frac{1-b_1}{2}}\right), \tag{4.51}$$

since $\bar{A}$ is realised and z is uniformly bounded away from 0 and ∞ for any $\tau \leq \mathcal{T}$. Similarly, for any $t = 2, \ldots, l-k$ and for any $\tau \leq \mathcal{T}$ we have

$$\begin{aligned}
&\left| \frac{I_t(\mathbf{w}(\tau))}{n} - \frac{I_t(\mathbf{w}(0))}{n} \right| = \\
&\quad \left| e^{-z(\tau)} \left(\frac{v_{l-t}(\tau)}{n\rho_{l-t}(z(\tau))} - \frac{\bar{\bar{v}}(\tau)}{np_l(z(\tau))} \right) - \sum_{i=1}^{t-1} \frac{(-1)^i z(\tau)^i}{i!} \frac{I_{t-i}(\mathbf{w}(\tau))}{n} \right. \\
&\quad \left. -e^{-z(0)} \left(\frac{v_{l-t}(0)}{n\rho_{l-t}(z(0))} - \frac{\bar{\bar{v}}(0)}{np_l(z(0))} \right) + \sum_{i=1}^{t-1} \frac{(-1)^i z(0)^i}{i!} \frac{I_{t-i}(\mathbf{w}(0))}{n} \right| \\
&= \quad \left| e^{-z(\tau)} \left(\frac{v_{l-t}(\tau)}{n\rho_{l-t}(z(\tau))} - \frac{\bar{\bar{v}}(0)}{np_l(z(0))} + O(n^{-\beta}) \right) - \sum_{i=1}^{t-1} \frac{(-1)^i z(\tau)^i}{i!} \frac{I_{t-i}(\mathbf{w}(\tau)) - I_{t-i}(\mathbf{w}(0))}{n} \right. \\
&\quad \left. +O\left(n^{-\frac{1-b_1}{2}}\right) + \sum_{i=1}^{t-1} \frac{(-1)^i (z(0)^i - z(\tau)^i)}{i!} \frac{I_{t-i}(\mathbf{w}(0))}{n} \right|.
\end{aligned}$$

Thus, uniformly for any $\tau \leq \mathcal{T}$ we have

$$\frac{v_{l-t}(\tau)}{n\rho_{l-t}(z(\tau))} = 1 + O\left(n^{-\beta} + n^{-\frac{1-b_1}{2}}\right), \tag{4.52}$$

since $\bar{A}$ is realised and z is uniformly bounded away from 0 and ∞ for any $\tau \le \mathcal{T}$.

Using the same argument, we obtain uniformly for any $\tau \le \mathcal{T}$:

$$
\begin{aligned}
&|I(\mathbf{w}(\tau)) - I(\mathbf{w}(0))| = \\
&\quad \Big| J_3(\mathbf{w}(\tau)) - \frac{J_1(\mathbf{w}(\tau))\ J_2(\mathbf{w}(\tau))}{2} \\
&\quad - \frac{z(\tau)}{\mathbb{P}_{k-1}(z(\tau))} \frac{1}{2\mu(\tau)} \sum_{j=k}^{l-1} j e^{z(\tau)} \rho_j(z(\tau)) \sum_{i=0}^{l-j-1} \frac{(-1)^i z(\tau)^i}{i!} I_{l-j-i}(\mathbf{w}(\tau)) \\
&\quad - J_3(\mathbf{w}(0)) + \frac{J_1(\mathbf{w}(0))\ J_2(\mathbf{w}(0))}{2} \\
&\quad + \frac{z(0)}{p_{k-1}(z(0))} \frac{1}{2\mu(0)} \sum_{j=k}^{l-1} j e^{z(0)} \rho_j(z(0)) \sum_{i=0}^{l-j-1} \frac{(-1)^i z(0)^i}{i!} I_{l-j-i}(\mathbf{w}(0)) \Big| \\
= &\quad \Big| J_3(\mathbf{w}(\tau)) - \frac{J_1(\mathbf{w}(0))\ J_2(\mathbf{w}(0))}{2} + O(n^{-\beta}) \\
&\quad - \frac{z(\tau)}{p_{k-1}(z(\tau))} \frac{1}{2\mu(\tau)} \sum_{j=k}^{l-1} j e^{z(\tau)} \rho_j(z(\tau)) \sum_{i=0}^{l-j-1} \frac{(-1)^i z(\tau)^i}{i!} (I_{l-j-i}(\mathbf{w}(\tau)) - I_{l-j-i}(\mathbf{w}(0))) \\
&\quad - \frac{z(\tau)}{p_{k-1}(z(\tau))} \frac{1}{2\mu(\tau)} \sum_{j=k}^{l-1} j e^{z(\tau)} \rho_j(z(\tau)) \sum_{i=0}^{l-j-1} \frac{(-1)^i z(\tau)^i}{i!} I_{l-j-i}(\mathbf{w}(0)) - J_3(\mathbf{w}(0)) \\
&\quad + \frac{J_1(\mathbf{w}(0))\ J_2(\mathbf{w}(0))}{2} + \frac{z(0)}{p_{k-1}(z(0))} \frac{1}{2\mu(0)} \sum_{j=k}^{l-1} j e^{z(0)} \rho_j(z(0)) \sum_{i=0}^{l-j-1} \frac{(-1)^i z(0)^i}{i!} I_{l-j-i}(\mathbf{w}(0)) \Big| \\
= &\quad \Big| J_3(\mathbf{w}(\tau)) - \theta + O\left(n^{-\beta} + n^{-\frac{1-b_1}{2}}\right) \Big|,
\end{aligned}
$$

since $\bar{A}$ is realised and z, $\mu(\tau)/n$ are uniformly bounded away from 0 and ∞ for any $\tau \le \mathcal{T}$. Hence, uniformly for any $\tau \le \mathcal{T}$ we have

$$
\frac{t(\tau)}{2\mu(\tau)} \frac{z(\tau)}{p_{k-1}(z(\tau))} = \theta + O\left(n^{-\beta} + n^{-\frac{1-b_1}{2}}\right). \tag{4.53}
$$

Having fixed α, β, b, b_1, a, Corollary 4.5.2 and inequalities (4.47) and (4.48) imply that

$$
\mathbb{P}[\bar{A} \cap B \cap C] = 1 - o(1).
$$

Now, recall that for $k \ge 2$

$$
\gamma_k = \inf \left\{ \frac{\lambda}{p_{k-1}(\lambda)} \ : \ \lambda > 0 \right\}. \tag{4.54}
$$

It can be easily seen that the function $\lambda/p_{k-1}(\lambda)$ is strictly convex over $(0, \infty)$. Also $\lambda/p_1(\lambda)$ is increasing on the domain $(0, \infty)$ and $\gamma_2 = \lim_{\lambda \to 0+} \lambda/p_1(\lambda) = 1$. For $k \ge 3$, γ_k is actually the minimum of the function $\lambda/p_{k-1}(\lambda)$ over $(0,\infty)$. Thus, for $k\ge 3$, let λ_k be the unique real root of $\gamma_k=\lambda/p_{k-1}(\lambda)$ and set $\lambda_2=0$. Assume that

$$\theta \ge \gamma_k + n^{-\delta}, \tag{4.55}$$

where $\delta\in(0,1/2)$, if $k\ge 3$, and $\theta>\gamma_2=1$ (bounded away from γ_2), if $k=2$. Let α,β,b,b_1 be such that

$$\delta < \min\left\{\beta, \frac{1-b_1}{2}\right\} \quad \text{for } k\ge 3 \tag{4.56}$$

and recall that

$$\begin{aligned} \beta &< \alpha < \min\{1-3b, 1/2\}, \\ b &< 1/3, \qquad b_1 < 1/3. \end{aligned} \tag{4.57}$$

Since $\theta>\gamma_k$, for $k\ge 3$, the equation

$$\frac{\lambda}{p_{k-1}(\lambda)} = \theta$$

has two roots. However, when $k=2$, there exists a unique root. Let $\lambda_k(\theta)$ be the larger root, for $k\ge 3$, and the unique root for $k=2$. Also let $\theta_1=\min\{\beta/2,(1-b_1)/4\}$. We distinguish between two cases:

1. $k\ge 3$

   Since

   $$\left(\frac{z}{p_{k-1}(z)}\right)'\Bigg|_{z=\lambda_k} = 0, \qquad \left(\frac{z}{p_{k-1}(z)}\right)''\Bigg|_{z=\lambda_k} > 0,$$

   our hypothesis (4.55) implies that

   $$\lambda_k(\theta) - \lambda_k \ge \chi n^{-\delta/2}, \tag{4.58}$$

   for some constant $\chi>0$ (every such symbol will denote a positive constant). Now, for a fixed $\nu>0$ we introduce

   $$\hat{z} = \lambda_k(\theta) + \nu n^{-\theta_1}, \tag{4.59}$$

   where θ_1 has been defined above. Notice that, by (4.58),

   $$\begin{aligned} \frac{\hat{z}}{p_{k-1}(\hat{z})} &= \theta + \left(\frac{z}{p_{k-1}(z)}\right)'\Bigg|_{z=\lambda_k(\theta)} \nu n^{-\theta_1} + O(n^{-2\theta_1}) \\ &= \theta + O(n^{-\theta_1}), \\ \frac{\hat{z}}{p_{k-1}(\hat{z})} &\ge \theta + \left(\left(\frac{z}{p_{k-1}(z)}\right)'\Bigg|_{z=\lambda_k} + \chi_1 n^{-\delta/2}\right) \nu n^{-\theta_1} + O(n^{-2\theta_1}) \\ &\ge \theta + \chi_2 \nu n^{-\eta}, \end{aligned} \tag{4.60}$$

for n sufficiently large, where $\eta = \delta/2 + \theta_1$, since $\delta/2 < \theta_1$.

2. $k = 2$

   In this case (the very first part is a definition)

$$\left(\frac{z}{p_1(z)}\right)'\Bigg|_{z=\lambda_2} = \lim_{z\to 0+}\left(\frac{z}{p_1(z)}\right)' = \frac{1}{2}, \qquad \lim_{z\to 0+}\left(\frac{z}{p_1(z)}\right)'' > 0.$$

   Here, for a fixed $\nu > 0$ we introduce

$$\hat{z} = \lambda_2(\theta) + \nu n^{-\eta}, \tag{4.61}$$

   where, in this case, we choose $\theta_1 < \eta < 2\theta_1$. So,

$$\begin{aligned} \frac{\hat{z}}{p_1(\hat{z})} &= \theta + \left(\frac{z}{p_1(z)}\right)'\Bigg|_{z=\lambda_2(\theta)} \nu n^{-\eta} + O(n^{-2\eta}) \\ &= \theta + \Theta(n^{-\eta}). \end{aligned} \tag{4.62}$$

In both cases, we set

$$a = \min\left\{\frac{1}{1+\nu n^{-\eta}} p_l(\hat{z}), \frac{1}{1+n^{-1/2}} \frac{\hat{z} p_{l-1}(\hat{z})}{p_l(\hat{z})} - l\right\}. \tag{4.63}$$

Note that $B \cap C \subseteq \{\mathbf{w}(0) \in \mathbf{W}(a),\ \mu(0) = \lceil \theta n/2 \rceil\}$, since for n sufficiently large $z(0) > \hat{z}$ and, therefore, $\bar{\bar{v}}(\mathcal{G}_{n,m})/n > p_l(\hat{z})$ and $\bar{\bar{t}}(\mathcal{G}_{n,m})/n > (\hat{z} p_{l-1}(\hat{z})/p_l(\hat{z}))\, p_l(z(0))$.

Now, we show the following:

Lemma 4.5.3 *Asymptotically almost surely, there exists $\hat{\tau} < \mathcal{T}(a)$ such that*

$$z(\hat{\tau} - 1) > \hat{z}, \qquad z(\hat{\tau}) \le \hat{z}.$$

In fact,

$$\mathbb{P}[\exists \hat{\tau} < \mathcal{T}(a) \;:\; z(\hat{\tau}-1) > \hat{z},\ z(\hat{\tau}) \le \hat{z}] \ge 1 - O(e^{-c_2 n^{\rho}}),$$

for some $c_2 > 0$, where $\rho = \min\{\alpha - \beta, b, b_1\}$.

Proof. To see this, assume that the event $B \cap C$ occurs in the initial random graph. Thus, by Corollary 4.5.2 the event $\bar{A}$ occurs with conditional probability *at least* $1 - O(e^{-n^{\alpha-\beta}})$. Therefore, we may assume simultaneous occurrence of all three events $\bar{A}$, B and C. Also, note that for n sufficiently large we have $\theta > \hat{z}$ and $\mathbf{w}(\mathcal{G}_{n,m}) \in \mathbf{W}(a)$. Let us consider the following cases.

1. $\mathcal{T}(a) = T$. In this case at time T the deletion process delivers a k-core. Thus, $2\mu(T) = t(T)$ and by (4.53)

$$\frac{z(T)}{p_{k-1}(z(T))} = \theta(1 + O(n^{-\beta} + n^{-(1-b_1)/2})) = \theta(1 + O(n^{-2\theta_1})).$$

Now, $z(T)/z(T-1) = 1 + O(n^{-1})$, since $\|\mathbf{w}(T) - \mathbf{w}(T-1)\| = O(1)$, $\mathbf{w}(T-1) \in \mathbf{W}(a)$ and $\mathbf{w}(T) \in \mathbf{W}(a/2)$. Therefore,

$$\frac{z(T-1)}{p_{k-1}(z(T-1))} = \theta(1 + O(n^{-2\theta_1})).$$

Since $2\theta_1 > \eta$, for any $k \geq 2$, equations (4.60), (4.62) imply that for n sufficiently large,

$$\frac{\hat{z}}{p_{k-1}(\hat{z})} > \frac{z(T-1)}{p_{k-1}(z(T-1))},$$

which, along with the monotonicity of $z/p_{k-1}(z)$, imply the existence of a $\hat{\tau}$ as this is specified by the lemma. Note that $z(\hat{\tau})$ might be less than λ_k, for $k \geq 3$. We shall see in the proof of the next lemma that this *is not* the case with high probability.

2. $\mathcal{T}(a) < T$. Thus, for some $\tau < \mathcal{T}$, for the first time $\mathbf{w}(\tau) \in \mathbf{W}(a)$, but either $\bar{\bar{v}}(\tau+1) < an$ or $\bar{\bar{t}}(\tau+1) < (l+a)\bar{\bar{v}}(\tau+1)$. Note that $z(\tau) = z(\tau+1)(1 + O(n^{-1}))$, since $\mathbf{w}(\tau+1) \in \mathbf{W}(a/2)$. In the first case,

$$\begin{aligned}(1 + O(n^{-1}))\frac{\bar{\bar{v}}(\tau+1)}{p_l(z(\tau))} = \frac{\bar{\bar{v}}(\tau+1)}{p_l(z(\tau+1))} &= (1 + O(n^{-1}))\ \frac{\bar{\bar{v}}(\tau)}{p_l(z(\tau))} \\ &= (1 + O(n^{-\beta}))(1 + O(n^{-(1-b_1)/2}))n \\ &= n(1 + O(n^{-2\theta_1})).\end{aligned}$$

Either when $k = 2$ or $k \geq 3$, the definition of a implies that

$$\frac{1}{1 + \nu n^{-\eta}}\ \frac{p_l(\hat{z})}{p_l(z(\tau))} \geq 1 + O(n^{-2\theta_1}).$$

For n sufficiently large, this becomes a strict inequality since $2\theta_1 > \eta$. Therefore, we obtain $p_l(\hat{z}) > p_l(z(\tau))$, whence $\hat{z} > z(\tau)$.

In the second case, for any $k \geq 2$ and for n sufficiently large, we have

$$\begin{aligned}\frac{z(\tau)p_{l-1}(z(\tau))}{p_l(z(\tau))} &= (1 + O(n^{-1}))\frac{z(\tau+1)p_{l-1}(z(\tau+1))}{p_l(z(\tau+1))} \\ &= (1 + O(n^{-1}))\frac{\bar{\bar{t}}(\tau+1)}{\bar{\bar{v}}(\tau+1)} \\ &\leq (1 + O(n^{-1}))\ (l+a) \leq \frac{1 + O(n^{-1})}{1 + n^{-1/2}}\ \frac{\hat{z}p_{l-1}(\hat{z})}{p_l(\hat{z})} \\ &< \frac{\hat{z}p_{l-1}(\hat{z})}{p_l(\hat{z})},\end{aligned}$$

which shows that $z(\tau) < \hat{z}$, since $zp_{l-1}(z)/p_l(z)$ is strictly increasing.

On the other hand, Corollary 4.5.2 as well as (4.47), (4.48) imply the second part of the lemma. ∎

Now, let us investigate the size of various parameters at time $\hat{\tau}$. Firstly,

$$z(\hat{\tau}) = (1 + O(n^{-1}))\hat{z}.$$

Moreover conditioning on $\bar{A} \cap B \cap C$, by (4.53)

$$\begin{aligned} \frac{t(\hat{\tau})}{2\mu(\hat{\tau})} &= \left(1 + O\left(n^{-\beta} + n^{-(1-b_1)/2}\right)\right)\theta \div \frac{z(\hat{\tau})}{p_{k-1}(\hat{\tau})} \\ &= (1 + O(n^{-2\theta_1}))\theta \div \theta(1 + \Omega(n^{-\eta})) \\ &= 1 + O(n^{-\eta}), \end{aligned}$$

using (4.60) and (4.62). Therefore, for $k \geq 2$

$$s(\hat{\tau}) = 2\mu(\hat{\tau}) - t(\hat{\tau}) = O(n^{1-\eta}). \tag{4.64}$$

The number of edges at time $\hat{\tau}$ is

$$\begin{aligned} \mu(\hat{\tau}) &= \mu(0)\frac{z^2(\hat{\tau})}{z^2(0)}(1 + O(n^{-\beta})) \\ &= \frac{\theta n}{2}\frac{z^2(\hat{\tau})}{\theta^2}(1 + O(n^{-\beta}))(1 + O(n^{-(1-b_1)/2})) \\ &= \frac{n\hat{z}^2}{2\theta}(1 + O(n^{-\beta}))(1 + O(n^{-(1-b_1)/2})) \\ &= \frac{n\lambda_k^2(\theta)}{2\theta}(1 + O(n^{-\theta_1})). \end{aligned} \tag{4.65}$$

By (4.51),(4.52), the number of vertices of degree j, for $j = k, \ldots, l-1$, at time $\hat{\tau}$ is

$$\begin{aligned} v_j(\hat{\tau}) &= \rho_j(z(\hat{\tau}))\, n\left(1 + O\left(n^{-\beta} + O(n^{-(1-b_1)/2}\right)\right) \\ &= (1 + O(n^{-2\theta_1}))n\rho_j(\hat{z}) \\ &= (1 + O(n^{-\theta_1}))n\rho_j(\lambda_k(\theta)). \end{aligned} \tag{4.66}$$

Now, following [37] the next lemma shows that asymptotically almost surely the deletion process will stop within a sublinear number of steps after $\hat{\tau}$:

Lemma 4.5.4 *There exists $\sigma \in (0, 1)$, where $\sigma > \delta$ if $k \geq 3$, such that with (conditional) probability at least $1 - O(e^{-c_3 n^{\sigma-\delta}})$ for $k \geq 3$, or $1 - O(e^{-c_4 n^{\sigma}})$ for $k = 2$, there exists a $\tau \in [\hat{\tau}, \hat{\tau} + n^{\sigma}]$ for which the deletion process stops at time τ, where $c_3, c_4 > 0$ are constants.*

Proof. Firstly, note that for any such σ and for $\hat{\tau} \leq \tau \leq \hat{\tau} + n^{\sigma}$,

$$|\mu(\tau) - \mu(\hat{\tau})|, |t(\tau) - t(\hat{\tau})|, |v_j(\tau) - v_j(\hat{\tau})| = O(n^{\sigma}),$$

for any $j = k, \ldots, l-1$, and all of these parameters are of order n. Moreover,

$$s(\tau) = O(n^{\sigma_1}),$$

where $\sigma_1 = \max\{1-\eta, \sigma\}$. Note that for any $\hat{\tau} \leq \tau \leq \hat{\tau} + n^{\sigma}$ we have

$$z(\tau) = \hat{z} + O(n^{-(1-\sigma)}),$$

since $\mathrm{grad} z(\mathbf{w}(\hat{\tau})) = O(n^{-1})$. The above asymptotic equality along with (4.58) and (4.59) imply that for $k \geq 3$ and for any $\hat{\tau} \leq \tau \leq \hat{\tau} + n^{\sigma}$ we have

$$\begin{aligned} z(\tau) - \lambda_k &= \hat{z} + O(n^{-(1-\sigma)}) - \lambda_k \\ &= \lambda_k(\theta) - \lambda_k + O(n^{-(1-\sigma)}) + \nu n^{-\theta_1} \\ &\geq \chi n^{-\delta/2} + O(n^{-(1-\sigma)}) + \nu n^{-\theta_1} \\ &\geq \chi n^{-\delta/2} + O(n^{-(1-\sigma)}) \geq \chi_3 n^{-\delta/2}, \end{aligned}$$

for $1 - \sigma > \delta/2$ and for n sufficiently large. Similarly for $k = 2$, we obtain:

$$\begin{aligned} z(\tau) - \lambda_2 &= \hat{z} + O(n^{-(1-\sigma)}) - \lambda_2 \\ &= \lambda_2(\theta) + O(n^{-(1-\sigma)}) + \nu n^{-\eta} > 0, \end{aligned}$$

for n sufficiently large. Let $\Delta s(\tau) = s(\tau+1) - s(\tau)$. Equation (4.33) implies that for $\hat{\tau} \leq \tau \leq \hat{\tau} + n^{\sigma}$, if $s(\tau) > 0$ and n is sufficiently large we have

$$\begin{aligned} &\mathbb{E}_q[\Delta s(\tau) \mid \mathbf{w}(\tau) = \mathbf{w}] = \\ &\quad -\frac{s^2}{2\mu v} - \frac{s}{v}\left(1 - \frac{k(k-1)v_k(\tau)}{2\mu(\tau)}\right) + O(n^{-1}) \\ &= -\frac{s^2}{2\mu v} - \frac{s}{v}\left(1 - \frac{k(k-1)(v_k(\hat{\tau}) + O(n^{\sigma}))}{2\mu(\hat{\tau}) + O(n^{\sigma})}\right) + O(n^{-1}) \\ &= -\frac{s^2}{2\mu v} - \frac{s}{v}\left(1 - \frac{k(k-1)(n\rho_k(\lambda_k(\theta))(1 + O(n^{-\theta_1})) + O(n^{\sigma}))}{\frac{n\lambda_k^2(\theta)}{\theta}(1 + O(n^{-\theta_1})) + O(n^{\sigma})}\right) + O(n^{-1}) \\ &= -\frac{s^2}{2\mu v} - \frac{s}{v}\left(1 - \frac{k(k-1)\theta\rho_k(\lambda_k(\theta))}{\lambda_k^2(\theta)}\left(1 + O\left(n^{-\theta_1} + n^{-(1-\sigma)}\right)\right)\right) + O(n^{-1}) \\ &= -\frac{s^2}{2\mu v} - \frac{s}{v}\left(1 - \frac{\lambda_k(\theta)}{p_{k-1}(\lambda_k(\theta))}\frac{k(k-1)e^{-\lambda_k(\theta)}\lambda_k^k(\theta)}{\lambda_k^2(\theta)k!}\left(1 + O\left(n^{-\theta_1} + n^{-(1-\sigma)}\right)\right)\right) + O(n^{-1}) \end{aligned}$$

$$
\begin{aligned}
&= -\frac{s^2}{2\mu v} - \frac{s}{v}\left(1 - \frac{1}{p_{k-1}(\lambda_k(\theta))}\frac{e^{-\lambda_k(\theta)}\lambda_k^{k-1}(\theta)}{(k-2)!}\left(1 + O\left(n^{-\theta_1} + n^{-(1-\sigma)}\right)\right)\right) + O(n^{-1}) \\
&= -\frac{s^2}{2\mu v} - \frac{s}{v}\left(1 - \frac{1}{e_k'(\lambda_k(\theta))}\frac{\lambda_k^{k-1}(\theta)}{(k-2)!} + O\left(n^{-\theta_1} + n^{-(1-\sigma)}\right)\right) + O(n^{-1}) \\
&= -\frac{s^2}{2\mu v} - \frac{s}{v}\left(p_{k-1}(\lambda_k(\theta))\left(\frac{z}{p_{k-1}(z)}\right)'\Bigg|_{z=\lambda_k(\theta)} + O\left(n^{-\theta_1} + n^{-(1-\sigma)}\right)\right) + O(n^{-1}) \\
&\le -\chi_4 n^{-\delta/2}, \text{ for } k \ge 3 \text{ or } -\chi_4, \text{ for } k = 2,
\end{aligned}
$$

(the constants are not meant to be the same, but for convenience we use the same symbol), since $s/v > 1$ whenever $s > 0$, using (4.60) and (4.62) and choosing $1 - \sigma > \delta/2$ if $k \ge 3$. On the other hand, $\Delta s(\tau) \le 2(k-1)$ always. Therefore, by (4.42), (4.43) and the above inequality, we obtain:

$$
\begin{aligned}
\mathbb{E}[\Delta s(\tau) \mid \mathbf{w}(\tau) = \mathbf{w}] &= \sum_{\mathbf{w}'}\left(s(\mathbf{w}') - s(\mathbf{w})\right)\ p[\mathbf{w}' \mid \mathbf{w}] \\
&\le \sum_{\mathbf{w}'}\left(s(\mathbf{w}') - s(\mathbf{w})\right) q[\mathbf{w}' \mid \mathbf{w}] + \\
&\quad O\left(n^{-1+3b} + n^{-1/2}\log_2 n\right) + O\left(\sum_{\mathbf{w}'} q[\mathbf{w}' \mid \mathbf{w}]\sum_{j=1}^{k-1}\frac{(u_j - 1)^+}{v_j + 1}\right) \\
&= \mathbb{E}_q[\Delta s(\tau) \mid \mathbf{w}(\tau)] + O\left(n^{-1+3b} + n^{-1/2}\log_2 n\right) \\
&\quad + O\left(\sum_{j=1}^{k-1}\sum_{\mathbf{w}'} q[\mathbf{w}' \mid \mathbf{w}]\frac{(u_j - 1)^+}{v_j + 1}\right) \\
&\le -\chi_5 n^{-\delta/2} \text{ for } k \ge 3 \text{ or } -\chi_5, \text{ for } k = 2, \qquad (4.67)
\end{aligned}
$$

(we make the same assumption for the constants as above). We distinguish between two cases:

1. $k \ge 3$

   For $y > 0$, it follows from the previous inequality that

$$
\begin{aligned}
&\mathbb{E}[e^{ys(\tau+1)} \mid \mathbf{w}(\tau)] = \\
&\qquad e^{ys(\tau)}\exp\left(y\mathbb{E}[\Delta s(\tau) \mid \mathbf{w}(\tau)]\right)\ \mathbb{E}\left[\exp(y(\Delta s(\tau) - \mathbb{E}[\Delta s(\tau) \mid \mathbf{w}(\tau)])) \mid \mathbf{w}(\tau)\right] \\
&\le e^{ys(\tau)}\exp\left(-y\chi_5 n^{-\delta/2} + 2y^2 l^2\right),
\end{aligned}
$$

   since $\mathbb{E}[e^{yY}] \le e^{y^2 d^2/2}$, if $|Y| \le d$ and $\mathbb{E}[Y] = 0$. We set

$$
y = y_n = \frac{\chi_5 n^{-\delta/2}}{4l^2},
$$

since this minimises the second exponent of the right-hand side of the above inequality. Letting $\chi = \chi_5^2/8l^2$, we obtain

$$\mathbb{E}[e^{y_n s(\tau+1)} \mid \mathbf{w}(\tau)] \le e^{y_n s(\tau)}\ e^{-\chi n^{-\delta}}.$$

Therefore, the sequence

$$\{S(\tau)\}_{\tau \ge \hat{\tau}} = \{\exp(y_n s(\tau) + (\tau - \hat{\tau})\chi n^{-\delta})\}_{\tau \ge \hat{\tau}}$$

is a supermartingale, as long as $s(\tau) > 0$. Now, applying the Optional Stopping Theorem (note that $\hat{\tau}$ is a stopping time), for n sufficiently large we obtain

$$\begin{aligned} \mathbb{E}[S(\hat{\tau} + n^\sigma \wedge T) \mid \mathbf{w}(\hat{\tau})] &\le S(\hat{\tau}) = \exp(y_n s(\hat{\tau})) \\ &\le \exp\left(\chi^* n^{1-\eta-\delta/2}\right), \end{aligned}$$

by (4.64). By the definition of S, we have

$$\mathbb{E}[S(\hat{\tau} + n^\sigma \wedge T) \mid \mathbf{w}(\hat{\tau})] \ \ge\ \mathbb{P}[T \ge \hat{\tau} + n^\sigma]\ \exp\left(\chi n^{\sigma-\delta}\right),$$

if we choose $\sigma > \delta$. Therefore,

$$\mathbb{P}[T - \hat{\tau} \ge n^\sigma \mid \mathbf{w}(\hat{\tau})] \le \exp\left(\chi^* n^{1-\eta-\delta/2} - \chi n^{\sigma-\delta}\right) \le \exp\left(-\chi^{**} n^{\sigma-\delta}\right) = o(1),$$

if

$$\sigma - \delta > 1 - \eta - \delta/2.$$

Note that this constraint is satisfied if and only if $\sigma > 1 - \theta_1$. So, it suffices to choose $1-\delta/2 > \sigma > \max\{\delta, 1-\theta_1\}$. Note that $1-\delta/2 > \delta$, since $\delta \in (0, 1/2)$ and $1-\delta/2 > 1-\theta_1$, since $\delta/2 < \theta_1$.

2. $k = 2$

   This case is almost identical to the above one: the term $n^{-\delta/2}$ disappears and the subsequent calculations follow without alterations. Here, we have to choose $\sigma > 1 - \eta$. For the sake of completeness, we present the proof. For $y > 0$, using inequality (4.67) we have

$$\begin{aligned} &\mathbb{E}[e^{ys(\tau+1)} \mid \mathbf{w}(\tau)] = \\ &\qquad e^{ys(\tau)} \exp\left(y\mathbb{E}[\Delta s(\tau) \mid \mathbf{w}(\tau)]\right)\ \mathbb{E}\left[\exp(y(\Delta s(\tau) - \mathbb{E}[\Delta s(\tau) \mid \mathbf{w}(\tau)])) \mid \mathbf{w}(\tau)\right] \\ &\le\ e^{ys(\tau)} \exp\left(-y\chi_5 + 2y^2 l^2\right), \end{aligned}$$

since $\mathbb{E}[e^{yY}] \leq e^{y^2d^2/2}$, if $|Y| \leq d$ and $\mathbb{E}[Y] = 0$. Here we set

$$y = \frac{\chi_5}{4l^2},$$

since this minimises the second exponent of the right-hand side of the above inequality. Letting $\chi = \chi_5^2/8l^2$, we obtain

$$\mathbb{E}[e^{ys(\tau+1)} \mid \mathbf{w}(\tau)] \leq e^{ys(\tau)}\ e^{-\chi}.$$

Therefore, the sequence

$$\{S(\tau)\}_{\tau \geq \hat{\tau}} = \{\exp(ys(\tau) + (\tau - \hat{\tau})\chi)\}_{\tau \geq \hat{\tau}}$$

is a supermartingale, as long as $s(\tau) > 0$. Now, applying the Optional Sampling Theorem, for n sufficiently large we obtain

$$\begin{aligned} \mathbb{E}[S(\hat{\tau} + n^\sigma \wedge T) \mid \mathbf{w}(\hat{\tau})] &\leq S(\hat{\tau}) = \exp(ys(\hat{\tau})) \\ &\leq \exp\left(\chi^* n^{1-\eta}\right), \end{aligned}$$

by (4.64). By the definition of S, we have

$$\mathbb{E}[S(\hat{\tau} + n^\sigma \wedge T) \mid \mathbf{w}(\hat{\tau})] \geq \mathbb{P}[T \geq \hat{\tau} + n^\sigma]\ \exp\left(\chi n^\sigma\right).$$

Therefore,

$$\mathbb{P}[T - \hat{\tau} \geq n^\sigma \mid \mathbf{w}(\hat{\tau})] \leq \exp\left(\chi^* n^{1-\eta} - \chi n^\sigma\right) \leq \exp\left(-\chi^{**} n^\sigma\right) = o(1),$$

if $\sigma > 1 - \eta$.

Thus, with conditional probability at least $1 - O(e^{-c_3 n^{\sigma-\delta}})$ for $k \geq 3$, or at least $1 - O(e^{-c_4 n^\sigma})$ for $k = 2$, the deletion process stops at time τ, where $\hat{\tau} \leq \tau \leq \hat{\tau} + n^\sigma$ and $c_3, c_4 > 0$ are some constants. ∎

The above lemma along with equation (4.66) imply that with probability at least $1 - O(e^{-cn^\zeta})$, where $\zeta = \min\{\alpha - \beta, b, b_1, \sigma - \delta\}$ for $k \geq 3$ or $\zeta = \min\{\alpha - \beta, b, b_1, \sigma\}$ for $k = 2$ and $c > 0$ is a constant, at the end of the deletion process applied to a $\mathcal{G}_{n,m}$ random graph for any $j = k, \ldots, l-1$ we have

$$v_j(k - \mathrm{cr}(\mathcal{G}_{n,m})) = n\rho_j(\lambda_k(\theta)) + O_C(n^\gamma), \tag{4.68}$$

where $\gamma = \max\{1 - \theta_1, \sigma\}$, thus giving the number of vertices of degree j of the k-core of the random graph. The parameter γ has been chosen this way because after time $\hat{\tau}$ and until the end of the deletion process the number of vertices of degree j can be changed by $O(n^{\sigma})$, as we have seen in the proof of the previous lemma. Similarly, using (4.65) the number of edges of the k-core at the end of the deletion process is with the same probability as above

$$\mu(k-\mathrm{cr}(\mathcal{G}_{n,m})) = n\frac{\lambda_k^2(\theta)}{2\theta} + O_C(n^{\gamma}), \tag{4.69}$$

where $\gamma = \max\{1-\theta_1, \sigma\}$. The following remark concludes the proof of Theorem 1.1.

Remark. Note that if $k = 2$ and $\theta = 1 + n^{-\delta}$, where $\delta \in (0, 1/3)$, Taylor's Theorem implies that $\lambda_2(\theta) = 2n^{-\delta} + o(n^{-\delta})$. If $m(n) = \lceil \theta n/2 \rceil = \lceil n/2 + (1/2)n^{1-\delta} \rceil = \lceil n/2 + s \rceil$, then the number of vertices of degree d in the 2-core of a $\mathcal{G}_{n,m}$ random graph is (see [27] or [23])

$$\begin{aligned} & 4^d s^d/(n^{d-1}d!)(1+o_p(1)) \\ = \ & 4^d \left(\frac{1}{2}n^{1-\delta}\right)^d /(n^{d-1}d!)(1+o_p(1)) \\ = \ & \exp\left(2n^{-\delta}\right) 4^d \left(\frac{1}{2}n^{1-\delta}\right)^d /(n^{d-1}d!)(1+o_p(1)) \\ = \ & n\exp\left(2n^{-\delta}\right) \frac{\left(2n^{-\delta}\right)^d}{d!}(1+o_p(1)) \\ = \ & n\rho_d(\lambda_2(\theta))(1+o_p(1)). \end{aligned}$$

for any constant $d \geq 2$. Thus, we have re-expressed the results that appear in [27], concerning the degree sequence of the 2-core of a $\mathcal{G}_{n,m}$ random graph, where $m = \lceil n/2 + s \rceil$ and $n^{2/3} \ll s \ll n$.

Chapter 5

On the structure of the core of sparse random graphs

5.1 Introduction

In this chapter, we present some features of the core of sparse $\mathcal{G}_{n,m}$ random graphs. Recall from the previous chapter that for a graph $G = (V, E)$ and a natural number $k \geq 2$, the k-core of G is the maximal subgraph of minimum degree at least k (and it is empty if there is no such graph). We are interested in the 2-core of a $\mathcal{G}_{n,m}$ random graph, which we simply call core and denote by $\mathrm{cr}(\mathcal{G}_{n,m})$. Note that, by the definition of the core, any graph is the union of its core and a collection of trees either disjoint from the core or rooted at a vertex of it (but having no other vertex in common with it).

It is well-known that a sparse $\mathcal{G}_{n,m}$ random graph, having $\theta > 1$ fixed, typically consists of a unique "giant" (of linear order) complex component and a few "small" (of logarithmic order at most) unicyclic components, as well as some "small" tree-components (see [6] or [23], where these facts are presented in full). The typical picture of $\mathrm{cr}(\mathcal{G}_{n,m})$ is similar and was described in [36]. Namely, it consists of a giant component, which is a subgraph of the giant component of the random graph and whose order is a certain proportion of the order of the latter, along with a few isolated cycles which are the cycles of the unicyclic components of $\mathcal{G}_{n,m}$. Very precise results concerning the distribution of them as well as the distributions of the cycles that are not isolated in $\mathrm{cr}(\mathcal{G}_{n,m})$ when $\theta = \theta(n)$ is near the critical point were obtained very recently by S. Janson in [21]. The first theorem we prove achieves this (partially), but for any fixed $\theta > 1$. For such a θ, let $\lambda_2(\theta)$ be the unique root of the equation $\lambda/(1 - e^{-\lambda}) = \theta$. Also, for any real number $z > 0$

and a natural number $k \geq 2$, we define $p_k(z) = \mathbb{P}[X \geq k]$, where $X = \mathrm{Po}(z)$ is a Poisson random variable of mean z. We have:

Theorem 5.1.1 *For any fixed $\theta > 1$ and $m = \lceil \theta n/2 \rceil$ the following hold:*

1. *The random graph $\mathrm{cr}(\mathcal{G}_{n,m})$ consists a.a.s. of a unique greatest component of order $np_2(\lambda_2(\theta)) + o_p(n)$ and size $n\frac{\lambda_2^2(\theta)}{2\theta} + o_p(n)$, that has more than one cycle, with the remaining components being cycles with total order $O_p(1)$.*

2. *The number of isolated cycles of $\mathrm{cr}(\mathcal{G}_{n,m})$ converges in distribution as $n \to \infty$ to a Poisson random variable of mean*
$$-\frac{1}{2}\ln(1 - \theta\exp(-\lambda_2(\theta))) - \frac{1}{2}\theta\exp(-\lambda_2(\theta)) - \frac{1}{4}\left(\theta\exp(-\lambda_2(\theta))\right)^2.$$

3. *For any integer $k \geq 3$, the number of cycles having length k that are isolated in $\mathrm{cr}(\mathcal{G}_{n,m})$ converges in distribution as $n \to \infty$ to a Poisson random variable of mean $\left(\theta e^{-\lambda_2(\theta)}\right)^k/2k$. The number of cycles having length k that are not isolated there converges in distribution as $n \to \infty$ to a Poisson random variable of mean $\theta^k(1 - e^{-k\lambda_2(\theta)})/2k$. Moreover, any finite collection of the above random variables are asymptotically independent.*

The first part of the preceding theorem was also proved using other methods by B. Pittel in [36]. In that paper, the order of the core of a $\mathcal{G}_{n,m}$ random graph beyond the critical point was given by a different formula. Numerical evidence suggests that this formula and the formula we give coincide. Though we believe that there is some identity that transforms one onto the other, we have not been able to prove this rigorously.

The second part of the present work focuses on the giant component of $\mathrm{cr}(\mathcal{G}_{n,m})$. Though there is a very precise and clear picture of the birth of the giant component of a $\mathcal{G}_{n,m}$ random graph as well as of its structure close to the critical point (see for example [22], [26], [27] - alternatively [6] or [23] for a complete description), the first attempt to give a picture of it for any fixed $\theta > 1$ was made by B.Pittel in [36]. Firstly, recall that a *2-edge-connected component* of a graph is a maximal connected subgraph having at least 3 vertices with no cutedges and a *2-vertex-connected component* of a graph is a maximal connected subgraph which has at least 3 vertices and has no cutvertices. B. Pittel proved (see Theorem 5.4.1 below) that for any fixed $\theta > 1$ the giant component of a $\mathcal{G}_{n,m}$ random graph typically consists of a 2-vertex-connected component of linear order, a collection of unicyclic components each sprouting from a different vertex of the 2-vertex-connected component with total order that is bounded in probability, as well as a collection of small trees each rooted at a vertex of the 2-vertex-connected component whose total order is linear. Thus, the giant component of $\mathrm{cr}(\mathcal{G}_{n,m})$ consists of a huge 2-vertex-connected component, whereas the remaining 2-vertex-connected components are small cycles and their total order is bounded in probability. Moreover, each 2-edge-connected component not containing the giant 2-vertex-connected component is joined to the giant 2-vertex-connected component by a unique path, whose internal vertices are all of degree 2 and its length is bounded in probability as well.

Our goal is to make these results more precise. For a graph G on V_n, we define $L_i(G)$ to be the i-th largest component (if there is more than one component of the same order, then we assume increasing lexicographic ordering) and let $|L_i(G)|$ denote its order. Furthermore, we define the *essential core* of G to be the largest 2-vertex-connected component of $L_1(\mathrm{cr}(G))$ (if there is more than one of them, then again we assume increasing lexicographic ordering). We denote it by $\mathrm{ess}-\mathrm{cr}(G)$. The second theorem we prove is as follows:

Theorem 5.1.2 *For any fixed $\theta > 1$, and $m = \lceil \theta n/2 \rceil$, we have:*

1. *The number of 2-edge-connected components which are cycles and belong to $L_1(\mathrm{cr}(\mathcal{G}_{n,m}))$ converges in distribution as $n \to \infty$ to a Poisson random variable of mean*

$$\left(1-\theta e^{-\lambda_2(\theta)}\right) \frac{\theta e^{-\lambda_2(\theta)}\lambda_2(\theta)}{2} \left(\frac{3\left(\theta e^{-\lambda_2(\theta)}\right)^2 - 2\left(\theta e^{-\lambda_2(\theta)}\right)^3}{(1-\theta e^{-\lambda_2(\theta)})^2} - \frac{2\left(\theta e^{-\lambda_2(\theta)}\right)^2}{1-\theta e^{-\lambda_2(\theta)}} \right).$$

   The total number of 2-vertex-connected components which are cycles and belong to the same 2-edge-connected component as $\mathrm{ess}-\mathrm{cr}(\mathcal{G}_{n,m})$ converges in distribution as $n \to \infty$ to a Poisson random variable of mean

$$\frac{\theta}{2}(1-e^{-\lambda_2(\theta)} - \lambda_2(\theta)e^{-\lambda_2(\theta)})\frac{\left(\theta e^{-\lambda_2(\theta)}\right)^2}{1-\theta e^{-\lambda_2(\theta)}}.$$

   Furthermore, these random variables are asymptotically independent.

2. *For any integers $k_1 \geq 3$ and $k_2 \geq 0$, the number of 2-edge-connected components of $L_1(\mathrm{cr}(\mathcal{G}_{n,m}))$, which are cycles of length k_1 and are joined to $\mathrm{ess}-\mathrm{cr}(\mathcal{G}_{n,m})$ by a path having k_2 internal vertices converges in distribution as $n \to \infty$ to a Poisson random variable of mean*

$$\frac{\theta e^{-\lambda_2(\theta)}\lambda_2(\theta)(1-\theta e^{-\lambda_2(\theta)})}{2}\left(\theta e^{-\lambda_2(\theta)}\right)^{k_1+k_2-1}.$$

   The number of 2-vertex-connected components which are cycles of length $k \geq 3$ and belong to the same 2-edge-connected component as $\mathrm{ess}-\mathrm{cr}(\mathcal{G}_{n,m})$ converges in distribution as $n \to \infty$

to a Poisson random variable of mean

$$\frac{\theta}{2}\left(\theta e^{-\lambda_2(\theta)}\right)^{k-1}\left(1 - e^{-\lambda_2(\theta)} - e^{-\lambda_2(\theta)}\lambda_2(\theta)\right).$$

Moreover, any finite collection of the above random variables are asymptotically independent.

Both of these theorems are deduced from Theorem 4.1.1 in the previous chapter and follow from the observation that if we condition on the degree sequence of the core of $\mathcal{G}_{n,m}$, then this is uniformly distributed over the set of graphs having this degree sequence. In particular, we define a probability space which is the product of the probability spaces of the degree sequences of $\mathrm{cr}(\mathcal{G}_{n,m})$: each element of this space is a sequence of degree sequences on n vertices, for each $n \in \mathbb{Z}^+$. Using Theorem 4.1.1, we determine a subspace of this space of measure 1 (see Section 5.2 for a precise definition), and for an arbitrary "asymptotic" degree sequence in this subspace we apply the method of moments on the space of uniform random graphs having this degree sequence in order to determine the asymptotic distributions of the graph functionals mentioned in Theorems 5.1.1 and 5.1.2 there. Finally, we use Proposition 5.2.4 to show that in fact these are the asymptotic distributions in the $\mathcal{G}_{n,m}$ model. Some technical but standard results of the theory of sparse random graphs are also used in the course of our proofs - see Section 5.6.

For a graph G, the *kernel* $\ker(G)$ is the multigraph (possibly with loops) obtained from those components of the 2-core of G that have more than one cycle by replacing each path whose internal vertices are all of degree two by a single edge. T. Łuczak in [27] (see also [23]) obtained results concerning the structure of $\ker(\mathcal{G}_{n,m})$, when θ is close to the critical point. In Section 5.5, we obtain from Theorems 5.1.1 and 5.1.2 some structural properties of $\ker(\mathcal{G}_{n,m})$, for any constant $\theta > 1$. Namely, we give precise estimates for the degree sequence of the kernel, its order and its size as well as its number of loops.

5.2 Some preliminary results and definitions

We begin this section presenting a key-result concerning the degree sequence of the k-core of a $\mathcal{G}_{n,m}$ random graph, with $m = \lceil \theta n/2 \rceil$. For $k \geq 2$, let k–$\mathrm{cr}(\mathcal{G}_{n,m})$ denote the k-core of a $\mathcal{G}_{n,m}$ random graph and, therefore, $\mathrm{cr}(\mathcal{G}_{n,m}) = 2$–$\mathrm{cr}(\mathcal{G}_{n,m})$.

Recall that for a simple graph G on $V \subseteq V_n$, the sequence $(v_0(G), \ldots, v_{n-1}(G))$, where for $j = 0, \ldots, n-1$, $v_j(G)$ is the number of vertices of G of degree j, is said to be the *degree sequence* of G. We now describe briefly the deletion process which was introduced in [37] and described in the previous chapter. Given a graph on V_n and an integer $k \geq 2$, at each step, we choose a vertex uniformly at random amongst the non-isolated vertices of degree less than k and delete all the edges incident to it, thus making it isolated. This step is repeated so long as there are edges to be deleted and the current set of vertices of degree at least k, say H, is non-empty. At the end, either $H \neq \emptyset$ and so H is the vertex set of the k-core of the initial graph, or $H = \emptyset$ and so the k-core there is empty. We denote by $\mathrm{DP}_k(G)$ the graph on V_n which is the output of the deletion process with parameter k taking as input a graph G on V_n. Here, we apply the deletion process to a $\mathcal{G}_{n,m}$ random graph. Note that for any integer $k \geq 2$, the deletion process finds the k-core of the input graph if and only if this is non-empty. We consider S_n to be the state space consisting of n-tuples of positive integers, where the deletion process induces a Markov chain on it - see [37] or the previous chapter. Namely, it encodes the degree sequence of the underlying graph in the course of the deletion process. Let $\mathbf{w} = (v_0, \ldots, v_{n-1}) \in S_n$. If $\mathcal{H}(\mathbf{w})$ denotes the set of simple graphs on V_n whose degree sequence is $\mathbf{w}$ and $h(\mathbf{w})$ is its cardinality, then for $G \in \mathcal{H}(\mathbf{w})$ Proposition 2.1 (b) in [37] (see also Proposition 4.3.1 in the previous chapter) yields

$$\mathbb{P}[\mathrm{DP}_k(\mathcal{G}_{n,m}) = G \mid \mathbf{w}(T) = \mathbf{w}] = \frac{1}{h(\mathbf{w})},$$

where T is the stopping time of the deletion process, that is $\mathbf{w}(T)$ is the degree sequence of $\mathrm{DP}_k(\mathcal{G}_{n,m})$, for any $k \geq 2$. Now, let $\mathbf{d} = (d_1, \ldots, d_n)$ be a sequence of non-negative integers, such that $\sum_{i \geq 1} d_i$ is even and for every $1 \leq i < n$ we have $d_i \leq d_{i+1}$. Let $\hat{\mathcal{H}}(\mathbf{d})$ be the set of simple graphs on V_n, where the vertex i, for every $i \in V_n$, has degree equal to d_i and let $\hat{h}(\mathbf{d})$ be the cardinality of this set. It is immediate that for each sequence $\mathbf{w} = (v_0, \ldots, v_{n-1})$ such that $\sum_{i=1}^{n-1} i v_i$ is even, we can construct such a sequence $\mathbf{d} = \mathbf{d}(\mathbf{w})$, and vice versa. More specifically, for a simple graph G on V_n if $\mathbf{w} = (v_0, \ldots, v_{n-1}) = (v_0(G), \ldots, v_{n-1}(G))$, then we call the corresponding $\mathbf{d}(\mathbf{w})$ the *labelled degree sequence* of G and we denote it by $\mathbf{d}(G)$. For a vector $\mathbf{d}_n = (d_1, d_2, \ldots, d_n)$ (or $(d_1, d_2, \ldots, d_n, 0, \ldots)$ - we will be using both notations interchangeably) such that its (n first) entries are in non-decreasing order and they have even sum, we let $G(\mathbf{d}_n)$ be a random graph on V_n uniformly distributed over $\hat{\mathcal{H}}(\mathbf{d}_n)$. Note that if $\mathcal{I}$ is an isomorphism class in $\hat{\mathcal{H}}(\mathbf{d}(\mathbf{w}))$ and $\mathcal{I}'$ is the corresponding class in $\mathcal{H}(\mathbf{w})$, then $|\mathcal{I}'|/|\mathcal{I}| = n!/\prod_{i=0}^{n-1} v_i!$. Thus, the probability of a property closed under automorphisms is the same in the uniform probability spaces $\mathcal{H}(\mathbf{w})$ and $\hat{\mathcal{H}}(\mathbf{d}(\mathbf{w}))$. In what follows, we assume that $\theta > 1$ is fixed and $m = \lceil \theta n/2 \rceil$.

For each $n \in \mathbb{Z}^+$, let $\mathbf{D}_n$ be the space of all infinite integer vectors of the form $(d_1, \ldots, d_n, 0, \ldots)$ for which there exists a graph G on V_n having m edges so that $(d_1, \ldots, d_n) = \mathbf{d}(\mathrm{DP}_2(G))$, endowed with the natural probability measure inherited from the $\mathcal{G}_{n,m}$ space, which we denote by μ_n. It is convenient to put these probability spaces for n=1,2,.. together to form one probability space.

In particular, let $\mathbf{D} = \prod_{n=1}^{\infty} \mathbf{D}_n$ be the product of these spaces and μ be the product probability measure on the product σ-algebra. If an event E that belongs to this algebra is such that $\mu(E) = 1$, then we say that E occurs *almost surely* (a.s.). An element of the space $\mathbf{D}$ is denoted by $(\mathbf{d}_n)$ and note that this is a sequence of infinite vectors indexed by the set of positive integers. For a given $(\mathbf{d}_n)$, if $\mathbf{d}_n = (d_1, \ldots, d_n, 0, \ldots)$ for $n \in \mathbb{Z}^+$, we set $D_i = D_i(n) = |\{j \in V_n \ : \ d_j = i\}|$, for $i \in \mathbb{N}$ and $\Delta = \Delta(n) = \max_{1 \le i \le n}\{d_i\} = d_n$. Also, we let $M = M(n) = \frac{1}{2} \sum_{i=1}^{n} d_i$. For any $n \in \mathbb{Z}^+$, we denote by π_n the projection of $\Omega = \prod_{i=1}^{\infty} \Omega_i$ onto Ω_n. We state and prove the following propositions which will be used in the sequel:

Proposition 5.2.1 *Let $\{(\Omega_n, \mathcal{F}_n, \mathbb{P}_n)\}_{n \in \mathbb{Z}^+}$ be a family of probability spaces and let $\Omega = \prod_{n \in \mathbb{Z}^+} \Omega_n$ be the product space endowed with the product measure, say μ. Let $\{\mathcal{E}_n \in \mathcal{F}_n\}_{n \in \mathbb{Z}^+}$ be a family of measurable sets such that $\sum_{n \in \mathbb{Z}^+} \mathbb{P}_n[\mathcal{E}_n^c] < \infty$ and let $G = \{\omega \in \Omega \ : \ \exists N \in \mathbb{N}$ s.t. $\forall n > N \ \pi_n(\omega) \in \mathcal{E}_n\}$. Then $\mu(G) = 1$.*

Proof. For each $n \in \mathbb{Z}^+$, let $\mathcal{E}'_n = \{\omega \in \Omega \ : \ \pi_n(\omega) \in \mathcal{E}_n\}$ and note that these events are independent. Then $G^c = \cap_{N \in \mathbb{Z}^+} \cup_{n \ge N} \mathcal{E}'^c_n$ and by the second Borel-Cantelli Lemma we have $\mu(G^c) = 0$, since $\sum_{n \in \mathbb{Z}^+} \mu(\mathcal{E}'^c_n) = \sum_{n \in \mathbb{Z}^+} \mathbb{P}_n[\mathcal{E}_n^c] < \infty$. ∎

Proposition 5.2.2 *Under the assumptions of Proposition 5.2.1, for each $n \in \mathbb{Z}^+$, let $X_n : \Omega_n \to \mathbb{R}$ be a random variable and suppose that $X_n = x + o_p(1)$, for some $x \in \mathbb{R}$, and for every $\varepsilon > 0$ we have $\sum_{n \in \mathbb{Z}^+} \mathbb{P}_n[|X_n - x| > \varepsilon] < \infty$. Let $L = \{\omega \in \Omega \ : \ \lim_{n \to \infty} X_n(\pi_n(\omega)) = x\}$. Then $\mu(L) = 1$.*

Proof. For each $n \in \mathbb{Z}^+$ and for every $\varepsilon > 0$, we set $\mathcal{E}_n(\varepsilon) = \{\omega_n \in \Omega_n \ : \ |X_n(\omega_n) - x| \le \varepsilon\}$ and let $L(\varepsilon) = \{\omega \in \Omega \ : \ \exists N \in \mathbb{N}$ s.t. $\forall n > N \ \pi_n(\omega) \in \mathcal{E}_n(\varepsilon)\}$. By the previous proposition, we have $\mu(L(\varepsilon)) = 1$, for every $\varepsilon > 0$. It is immediate that $\{L(1/t)\}_{t \in \mathbb{Z}^+}$ is a decreasing family of sets and $L = \cap_{t \in \mathbb{Z}^+} L(1/t)$. Therefore $\mu(L) = \lim_{t \to \infty} \mu(L(1/t)) = 1$. ∎

For some $c > 0$ let $\mathcal{D} = \mathcal{D}(c)$ be the set of $(\mathbf{d}_n) \in \mathbf{D}$ with the property that for every $0 < \varepsilon < 1$, there exist $k_0, N \in \mathbb{N}$ such that

$$Y(\mathbf{d}_n) = \sum_{i=2}^{n-1} i^4 D_i \le cn, \ X_{n,k_0}(\mathbf{d}_n) = \sum_{i=k_0}^{\Delta} \binom{i}{2} \frac{D_i}{n} < \varepsilon \text{ and } \Delta \le \lceil \ln n \rceil,$$

for any $n > N$.

Claim 5.2.3 *There exists $c > 0$ for which $\mu(\mathcal{D}) = 1$.*

Proof. For every real $\varepsilon > 0$ and $k \in \mathbb{N}$, we let

$$L(\varepsilon, k) = \{(\mathbf{d}_n) \in \mathbf{D} \ : \ \exists N \in \mathbb{N} \text{ s.t. } \forall n > N \ \mathbf{d}_n \in \mathcal{E}_n(\varepsilon, k)\}$$

where $\mathcal{E}_n(\varepsilon, k) = \{\mathbf{d}_n \in \mathbf{D}_n : Y(\mathbf{d}_n) \leq cn,\ X_{n,k}(\mathbf{d}_n) < \varepsilon,\ \Delta \leq \lceil \ln n \rceil\}$. If c is as in Lemma 5.6.1 in Section 5.6, then this implies that for every $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that $\sum_{n \in \mathbb{Z}^+} \mu_n(\mathcal{E}_n^c(\varepsilon, k)) < \infty$. Therefore, by Proposition 5.2.1 we deduce that $\mu(L(\varepsilon, k)) = 1$. Thus, if we let $L(\varepsilon) = \cup_{k \in \mathbb{N}} L(\varepsilon, k)$, we have $\mu(L(\varepsilon)) = 1$ as well. Now notice that $\mathcal{D} = \cap_{0<\varepsilon<1} L(\varepsilon) = \cap_{t=1}^{\infty} L(1/t)$ and that $\{L(1/t)\}_{t \in \mathbb{Z}^+}$ forms a decreasing family of sets as t increases. Therefore, $\mu(\mathcal{D}) = \lim_{t \to \infty} \mu(L(1/t)) = 1$. ∎

Note that Theorem 4.1.1 in the previous chapter along with Proposition 5.2.2 imply that for every $i \geq 2$, we have $\lim_{n\to\infty} D_i/n = \rho_i(\lambda_2(\theta))$ and $\lim_{n\to\infty} M/n = \lim_{n\to\infty} \sum_{i=2}^{\Delta} iD_i/n = \lambda_2^2(\theta)/2\theta$ a.s.. For c as in the previous claim, let G ("good") be the set of those $(\mathbf{d}_n) \in \mathcal{D}$ for which the preceding statement is true. We have $\mu(G) = 1$. Now, it is easy to see the following:

Proposition 5.2.4 *For a fixed $k \in \mathbb{Z}^+$, let X_n be a function on the set of graphs on V_n taking values in $\mathbb{R}^k$, which depends on the isomorphism type of the core, and let $x \in \mathbb{R}^k$. If there exists $p \in [0,1]$ such that for every $(\mathbf{d}_n) \in G$, we have $\lim_{n\to\infty} \mathbb{P}[X_n(G(\mathbf{d}_n)) \leq x] = p$, then $\lim_{n\to\infty} \mathbb{P}[X_n(\mathcal{G}_{n,m}) \leq x] = p$.*

Proof. The proof is almost straightforward:

$$
\begin{aligned}
\mathbb{P}[X_n(\mathcal{G}_{n,m}) \leq x] &= \sum_{\mathbf{d}_n \in \mathbf{D}_n} \mathbb{P}[X_n(\mathcal{G}_{n,m}) \leq x \mid \mathbf{d}(\mathrm{DP}_2(\mathcal{G}_{n,m})) = \mathbf{d}_n]\ \mathbb{P}[\mathbf{d}(\mathrm{DP}_2(\mathcal{G}_{n,m})) = \mathbf{d}_n] \\
&= \sum_{\mathbf{d}_n \in \mathbf{D}_n} \int_{\{(\mathfrak{d}_n)\ :\ \pi_n((\mathfrak{d}_n)) = \mathbf{d}_n\}} \mathbb{P}[X_n(G(\mathbf{d}_n)) \leq x]\ \mu(d(\mathfrak{d}_n)) \\
&= \int \mathbb{P}[X_n(G(\pi_n((\mathfrak{d}_n)))) \leq x]\ \mu(d(\mathfrak{d}_n)).
\end{aligned}
$$

Since the integrand is bounded below by 0, applying Fatou's Lemma, we obtain:

$$
\begin{aligned}
\liminf_{n\to\infty} \mathbb{P}[X_n(\mathcal{G}_{n,m}) \leq x] &= \liminf_{n\to\infty} \int \mathbb{P}[X_n(G(\pi_n((\mathfrak{d}_n)))) \leq x]\ \mu(d(\mathfrak{d}_n)) \\
&\geq \int \liminf_{n\to\infty} \mathbb{P}[X_n(G(\pi_n((\mathfrak{d}_n)))) \leq x]\ \mu(d(\mathfrak{d}_n)) \\
&= \int_G \liminf_{n\to\infty} \mathbb{P}[X_n(G(\pi_n((\mathfrak{d}_n)))) \leq x]\ \mu(d(\mathfrak{d}_n)) \\
&= p \int_G \mu(d(\mathfrak{d}_n)) = p.
\end{aligned}
$$

Now, applying the Reverse Fatou's Lemma (since the integrand is bounded above by 1), we have

$$
\limsup_{n\to\infty} \mathbb{P}[X_n(\mathcal{G}_{n,m}) \leq x] = \limsup_{n\to\infty} \int \mathbb{P}[X_n(G(\pi_n((\mathfrak{d}_n)))) \leq x]\ \mu(d(\mathfrak{d}_n))
$$

$$\begin{aligned}
&\leq \int \limsup_{n\to\infty} \mathbb{P}[X_n(G(\pi_n((\mathfrak{d}_n)))) \leq x] \; \mu(d(\mathfrak{d}_n)) \\
&= \int_G \limsup_{n\to\infty} \mathbb{P}[X_n(G(\pi_n((\mathfrak{d}_n)))) \leq x] \; \mu(d(\mathfrak{d}_n)) \\
&= p \int_G \mu(d(\mathfrak{d}_n)) = p,
\end{aligned}$$

which concludes the proof of the proposition. ∎

We state without proof the following lemma, which follows from the main theorem in [31] (see (4.9) in the previous chapter):

Lemma 5.2.5 *If* $(\mathbf{d}_n) \in G$, *then*

$$\left|\hat{\mathcal{H}}(\mathbf{d}_n)\right| = \left(1 + O\left(\frac{\ln^3 n}{n}\right)\right) e^{-\lambda-\lambda^2} \frac{(2M)!}{M! 2^M \prod_{i=1}^n d_i!},$$

where $\lambda = \sum_{i=1}^n \binom{d_i}{2}/(2M)$.

Now, we are ready to proceed with the proofs of Theorems 5.1.1 and 5.1.2.

5.3 The distribution of cycles in the core of a $\mathcal{G}_{n,m}$ random graph

In this section, we mainly investigate the asymptotic distribution of the total number of isolated cycles as well as of the number of cycles of fixed length which are either isolated or not in $\mathrm{cr}(\mathcal{G}_{n,m})$, thus proving Theorem 5.1.1.

Firstly, we prove parts (i) and (ii) of Theorem 5.1.1. We state without proof a part of a theorem which was proved by T. Łuczak in [29] (Theorem 12.2(ii) there), slightly adapted to our context:

Theorem 5.3.1 *Let* $(\mathbf{d}_n = (d_1, \dots, d_n))_{n\in\mathbb{Z}^+}$ *be such that for* $i = 1, \dots, D_0$ *we have* $d_i = 0$ *and* $2 \leq \min_{i>D_0}\{d_i\}$ *as well as* $\max_i\{d_i\} \leq n^{0.01}$, *where* $D_0 = D_0(n)$ *is the number of zeros in* $\mathbf{d}_n$ *with* $n - D_0 \to \infty$ *as* $n \to \infty$. *Also let* $D_2 = D_2(n)$ *be the number of twos in* $\mathbf{d}_n$, $2M = 2M(n) = \sum_{i=1}^n d_i$, $L_i(G(\mathbf{d}_n))$ *denote the i-th largest component of the random graph* $G(\mathbf{d}_n)$, *and* $\omega = \omega(n)$ *be a function that tends to infinity with n. If* $\lim_{n\to\infty} D_2/M = b < 1$, *then* $\lim_{n\to\infty} \mathbb{P}[|L_1(G(\mathbf{d}_n))| \geq N - \omega] = 1$, *where* $N = n - D_0$, *all of the smaller* ($i \geq 2$) *non-trivial components are cycles with probability tending to 1 as* $n \to \infty$, *and for* $t = 0, 1, \dots$,

$$\lim_{n\to\infty} \mathbb{P}[G(\mathbf{d}_n) \text{ consists of } t+1 \text{ non-trivial components}] = e^{-\mu}\frac{\mu^t}{t!},$$

where $\mu = -\frac{1}{2}\ln(1-b) - \frac{1}{2}b - \frac{1}{4}b^2$.

Suppose that $(\mathbf{d}_n) \in G$. Firstly, note that the second part of Lemma 5.7.1 in Section 5.7 implies that $\lim_{n\to\infty} \sum_{i=2}^{\Delta} D_i/n = p_2(\lambda_2(\theta))$. Now, by the definition of G we obtain $\lim_{n\to\infty} D_2/n = \rho_2(\lambda_2(\theta))$, $\lim_{n\to\infty} M/n = \frac{\lambda_2^2(\theta)}{2\theta}$ and, therefore, $\lim_{n\to\infty} D_2/M = \theta e^{-\lambda_2(\theta)} < 1$. Moreover, we have $\Delta < n^{0.01}$, for n sufficiently large. Also notice that $\frac{\lambda_2^2(\theta)}{2\theta} > p_2(\lambda_2(\theta))$, which along with the above observations and Theorem 5.3.1 imply that $|L_1(G(\mathbf{d}_n))|/n = p_2(\lambda_2(\theta)) + o_p(1)$ and $L_1(G(\mathbf{d}_n))$ has more than one cycle with probability tending to 1 as $n \to \infty$. On the other hand, the number of vertices that belong to smaller non-trivial components of $G(\mathbf{d}_n)$ is $O_p(1)$, with the latter being cycles with probability tending to 1 as $n \to \infty$ and finally the distribution of their total number is given by the above theorem, with $b = \theta e^{-\lambda_2(\theta)}$. Therefore, applying Proposition 5.2.4, we deduce Theorem 5.1.1(i) and (ii).

We proceed with the proof of the final part of Theorem 5.1.1. For any natural number $k \geq 3$ and a graph G on V_n, we let $Z_{nk}(G)$, $Z'_{nk}(G)$ be the number of isolated cycles of length k and the number of cycles of the same length that are not isolated in G, respectively. We first prove the following lemma:

Lemma 5.3.2 *For integers $t, s \geq 0$, let $k_1, \ldots, k_{s+t} \geq 3$ and $l_1, \ldots, l_{s+t} \geq 1$ be sequences of natural numbers. Let $(\mathbf{d}_n) \in G$ and for any integer $k \geq 3$, we set $Z_{nk} = Z_{nk}(G(\mathbf{d}_n))$ and $Z'_{nk} = Z'_{nk}(G(\mathbf{d}_n))$. Then*

$$\lim_{n\to\infty} \mathbb{E}[(Z'_{nk_1})_{l_1} \cdots (Z'_{nk_t})_{l_t} (Z_{nk_{t+1}})_{l_{t+1}} \cdots (Z_{nk_{t+s}})_{l_{t+s}}] = \prod_{j=1}^{t} \left(\frac{\theta^{k_j}(1 - e^{-\lambda_2(\theta)k_j})}{2k_j} \right)^{l_j} \prod_{j=1}^{s} \left(\frac{(\theta e^{-\lambda_2(\theta)})^{k_{t+j}}}{2k_{t+j}} \right)^{l_{t+j}}.$$

(Here we assume that the empty product is equal to 1.)

Proof. The number on the left hand side of the above equation expresses the total number of families of collections of cycles indexed by the set $\{1, \ldots, t+s\}$, where for any $i = 1, \ldots, t$, we have ordered l_i-tuples of cycles of length k_i which are not isolated, and for any $i = t+1, \ldots, t+s$ we have ordered l_i-tuples of isolated cycles of length k_i. Moreover, every such family induces a subgraph of order at most $\sum_i^{t+s} l_i k_i$. So, the expected number of those families with at least two overlapping non-isolated cycles tends to 0 as $n \to \infty$, by Lemma 5.6.2, since they induce subgraphs of bounded order which have more edges than vertices and $(\mathbf{d}_n) \in G$. Therefore, the above expectation is asymptotically equal to the expected number of those families consisting of pairwise disjoint cycles. Let X_n be the total number of the latter.

Let $D_j^{(i)}$ denote the number of those vertices amongst the D_j vertices of degree j in the labelled degree sequence $(d_1, \ldots, d_n)$ that are left if we remove the edges of the first $i-1$ cycles of a given family and $\mathbf{d}'_n = (d'_1, \ldots, d'_n)$ is the labelled degree sequence that remains from $(d_1, \ldots, d_n)$ after the removal of all the edges of these cycles. Also, note that $D_j^{(i)} = D_j(1 + O(1/n))$. On the other hand, if vertex i participates in a cycle, then $\mathbf{d}'_n$ has one more vertex of degree $d_i - 2$ and one less vertex of degree d_i.

Given positive integers k and i, we use the following notation

$$\sum_k^{(i)} \equiv \sum_{p_i=1}^{k} \sum_{\substack{\text{ordered} \\ p_i - \text{partitions of } k \\ \left(k_1^{(i)}, \ldots, k_{p_i}^{(i)}\right)}} \sum_{\substack{p_i - \text{tuples} \\ 2 \le i_1^{(i)} < \cdots < i_{p_i}^{(i)} \le \Delta, \\ \text{such that if } p_i = 1, \text{ then } i_1^{(i)} \neq 2}} .$$

Therefore, by Lemma 5.2.5, we have

$$\begin{aligned}
\mathbb{E}[X_n(G(\mathbf{d}_n))] &= \prod_{i=1}^{t+s} \left(\frac{(k_i-1)!}{2}\right)^{l_i} \times \\
&\sum_{k_1}^{(1)} \prod_{j=1}^{p_1} \binom{D_{i_j^{(1)}}^{(1)}}{k_j^{(1)}} \left(\cdots \left(\sum_{k_1}^{(l_1)} \prod_{j=1}^{p_{l_1}} \binom{D_{i_j^{(l_1)}}^{(l_1)}}{k_j^{(l_1)}} \right.\right. \\
&\vdots \\
&\sum_{k_t}^{(\sum_{i=1}^{t-1} l_i+1)} \prod_{j=1}^{p_{\sum_{i=1}^{t-1} l_i+1}} \binom{D_{i_j^{(\sum_{i=1}^{t-1} l_i+1)}}^{(\sum_{i=1}^{t-1} l_i+1)}}{k_j^{(\sum_{i=1}^{t-1} l_i+1)}} \left(\cdots \left(\sum_{k_t}^{(\sum_{i=1}^{t} l_i)} \prod_{j=1}^{p_{\sum_{i=1}^{t} l_i}} \binom{D_{i_j^{(\sum_{i=1}^{t} l_i)}}^{(\sum_{i=1}^{t} l_i)}}{k_j^{(\sum_{i=1}^{t} l_i)}} \right.\right. \\
&\binom{D_2^{(\sum_{i=1}^{t} l_i+1)}}{k_{t+1}} \cdots \binom{D_2^{(\sum_{i=1}^{t} l_i+1)} - (l_{t+1}-1)k_{t+1}}{k_{t+1}} \\
&\vdots \\
&\binom{D_2^{(\sum_{i=1}^{t+s} l_i)}}{k_{t+s}} \cdots \binom{D_2^{(\sum_{i=1}^{t+s} l_i)} - (l_{t+s}-1)k_{t+s}}{k_{t+s}} \\
&\frac{(2M - 2(l_1k_1 + \cdots + l_{t+s}k_{t+s}))!}{(M - (l_1k_1 + \cdots + l_{t+s}k_{t+s}))! 2^{M-(l_1k_1+\cdots+l_{t+s}k_{t+s})} \prod_{i=1}^{n} d'_i!} \frac{M! 2^M \prod_{i=1}^{n} d_i!}{(2M)!} \times \\
&\left.\left. \exp\left(-\lambda' + \lambda - (\lambda')^2 + \lambda^2\right) \right) \cdots \right) \left(1 + O\left(\frac{\ln^3 n}{n}\right)\right) \\
&= \prod_{i=1}^{t+s} \left(\frac{k_i!}{2k_i}\right)^{l_i} \sum_{k_1}^{(1)} \prod_{j=1}^{p_1} \frac{D_{i_j^{(1)}}^{k_j^{(1)}}}{k_j^{(1)}!} \left(\cdots \left(\sum_{k_1}^{(l_1)} \prod_{j=1}^{p_{l_1}} \frac{D_{i_j^{(l_1)}}^{k_j^{(l_1)}}}{k_j^{(l_1)}!} \right.\right. \\
&\vdots
\end{aligned}$$

$$\sum_{k_t}^{(\sum_{i=1}^{t-1} l_i+1)} \prod_{j=1}^{p_{\sum_{i=1}^{t-1} l_i+1}} \frac{D_{i_j^{(\sum_{i=1}^{t-1} l_i+1)}}^{k_j^{(\sum_{i=1}^{t-1} l_i+1)}}}{k_j^{(\sum_{i=1}^{t-1} l_i+1)}!} \Bigg(\cdots \Bigg(\sum_{k_t}^{(\sum_{i=1}^{t} l_i)} \prod_{j=1}^{p_{\sum_{i=1}^{t} l_i}} \frac{D_{i_j^{(\sum_{i=1}^{t} l_i)}}^{k_j^{(\sum_{i=1}^{t} l_i)}}}{k_j^{(\sum_{i=1}^{t} l_i)}!}$$

$$\left(\frac{D_2^{k_{t+1}}}{k_{t+1}!}\right)^{l_{t+1}} \cdots \left(\frac{D_2^{k_{t+s}}}{k_{t+s}!}\right)^{l_{t+s}}$$

$$\frac{M^{l_1k_1+\cdots+l_{t+s}k_{t+s}}}{(2M)^{2(l_1k_1+\cdots+l_{t+s}k_{t+s})}} \; 2^{l_1k_1+\cdots+l_tk_t} \; 2^{2(l_{t+1}k_{t+1}+\cdots+l_{t+s}k_{t+s})} \times$$

$$\prod_{i=1}^{l_1+\cdots+l_t} \prod_{j=1}^{p_i} \binom{i_j^{(i)}}{2}^{k_j^{(i)}} 2^{k_j^{(i)}} \Bigg) \cdots \Bigg) \left(1 + O\left(\frac{\ln^3 n}{n}\right)\right).$$

$$= \prod_{i=1}^{t} \left(\frac{1}{2k_i} \left(\left(\sum_{j=2}^{\Delta} \binom{j}{2} \frac{D_j}{M} \right)^{k_i} - \left(\frac{D_2}{M} \right)^{k_i} \right) \right)^{l_i} \prod_{i=1}^{s} \left(\frac{1}{2k_{t+i}} \left(\frac{D_2}{M} \right)^{k_{t+i}} \right)^{l_{t+i}} \times$$

$$\left(1 + O\left(\frac{\ln^3 n}{n}\right)\right).$$

The error term is deduced from the following observations. Let $C \subset V_n$ be the set those vertices that belong to the cycles of a given family. Notice that for any integer $d \geq 2$, we have

$$-\binom{d-2}{2} + \binom{d}{2} = 2d - 3 \text{ and } -\binom{d-1}{2} + \binom{d}{2} = d - 1.$$

(The second identity will be used in the next section.) For the two subsequent calculations, we shall assume that $(d_1', \ldots, d_n')$ is the sequence that is obtained from $\mathbf{d}$, where for each $i \in V_n$ d_i' is the degree of vertex i after the removal of the edges of C. Clearly, this is a permutation of $\mathbf{d}'$ as it was defined above. Therefore, for n sufficiently large,

$$\begin{aligned} -\lambda' + \lambda &= -\frac{\sum_{i=1}^n \binom{d_i'}{2}}{2(M - (l_1k_1 + \cdots + l_{t+s}k_{t+s}))} + \frac{\sum_{i=1}^n \binom{d_i}{2}}{2M} \\ &< \frac{1}{2M} \sum_{i \in C} \left(-\binom{d_i'}{2} + \binom{d_i}{2} \right) = \frac{1}{2M} \sum_{i \in C} (2d_i - 3) = O\left(\frac{\Delta}{M}\right) = O\left(\frac{\ln n}{n}\right), \end{aligned}$$

since the order of the union of the cycles (i.e. $|C|$) is bounded and $(\mathbf{d}_n) \in G$. Similarly,

$$\begin{aligned} -(\lambda')^2 + \lambda^2 &= -\frac{\left(\sum_{i=1}^n \binom{d_i'}{2}\right)^2}{4(M - (l_1k_1 + \cdots + l_{t+s}k_{t+s}))^2} + \frac{\left(\sum_{i=1}^n \binom{d_i}{2}\right)^2}{4M^2} \\ &< \frac{1}{4M^2} \Bigg(\sum_{i \notin C} \binom{d_i}{2} \left(\sum_{i=1}^n \binom{d_i}{2} \right) - \sum_{i \notin C} \binom{d_i'}{2} \left(\sum_{i=1}^n \binom{d_i'}{2} \right) + \\ &\quad \sum_{i \in C} \binom{d_i}{2} \left(\sum_{i=1}^n \binom{d_i}{2} \right) - \sum_{i \in C} \binom{d_i'}{2} \left(\sum_{i=1}^n \binom{d_i'}{2} \right) \Bigg) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4M^2}\left(\sum_{i\notin C}\binom{d_i}{2}\sum_{i\in C}(2d_i-3)+\sum_{i\in C}\left(\binom{d_i}{2}-\binom{d_i'}{2}\right)\left(\sum_{i\notin C}\binom{d_i}{2}\right)+\right.\\
&\left.\sum_{i\in C}\binom{d_i}{2}\left(\sum_{i\in C}\binom{d_i}{2}\right)-\sum_{i\in C}\binom{d_i'}{2}\left(\sum_{i\in C}\binom{d_i}{2}-2d_i+3\right)\right)\\
&= \frac{1}{4M^2}\left(\sum_{i\notin C}\binom{d_i}{2}\sum_{i\in C}(2d_i-3)+\sum_{i\in C}\left(\binom{d_i}{2}-\binom{d_i'}{2}\right)\left(\sum_{i\notin C}\binom{d_i}{2}\right)+\right.\\
&\left.\sum_{i\in C}\left(\binom{d_i}{2}-\binom{d_i'}{2}\right)\left(\sum_{i\in C}\binom{d_i}{2}\right)+\sum_{i\in C}\binom{d_i'}{2}\sum_{i\in C}(2d_i-3)\right)\\
&= \frac{1}{4M^2}\left(2\sum_{i\in C}(2d_i-3)\left(\sum_{i\notin C}\binom{d_i}{2}\right)+\sum_{i\in C}(2d_i-3)\left(\sum_{i\in C}\binom{d_i}{2}\right)\right.\\
&\left.+\sum_{i\in C}\binom{d_i'}{2}\sum_{i\in C}(2d_i-3)\right)=\frac{nO(\Delta^3)+O(\Delta^3)}{4M^2}=O\left(\frac{\ln^3 n}{n}\right),
\end{aligned}$$

since $|C|$ is bounded and $(\mathbf{d}_n)\in G$. Now, by using the two previous relations, we obtain

$$\exp\left(-\lambda'+\lambda-(\lambda')^2+\lambda^2\right)=\exp\left(O\left(\frac{\ln^3 n}{n}\right)\right)=1+O\left(\frac{\ln^3 n}{n}\right). \tag{5.1}$$

The first part of Lemma 5.7.1 concludes the proof. ∎

We are ready to proceed with the proof of the final part of Theorem 5.1.1. We still assume that $(\mathbf{d}_n)\in G$. If we set $t=0$ and $s=1$ in the above lemma, then for any integers $k\geq 3$ and $l\geq 1$, we have

$$\lim_{n\to\infty}(Z_{nk})_l=\left(\frac{\left(\theta e^{-\lambda_2(\theta)}\right)^k}{2k}\right)^l,$$

and this implies that $Z_{nk}\xrightarrow{d}\mathrm{Po}\left((\theta e^{-\lambda_2(\theta)})^k/2k\right)$ as $n\to\infty$ (see Corollary 6.8 in [23]). Similarly, setting $t=1$ and $s=0$ in the above lemma, for any integers $k\geq 3$ and $l\geq 1$, we have

$$\lim_{n\to\infty}\left(Z'_{nk}\right)_l=\left(\frac{\theta^k\left(1-e^{-k\lambda_2(\theta)}\right)}{2k}\right)^l,$$

which implies that $Z'_{nk}\xrightarrow{d}\mathrm{Po}\left((\theta^k(1-e^{-k\lambda_2(\theta)})/2k)\right)$, as $n\to\infty$. Now, the above lemma along with Theorem 6.10 in [23] imply that the joint distribution of any finite collection of pair-wise distinct random variables $(Z_{nk_1},\dots Z_{nk_t},Z'_{nk_{t+1}},\dots,Z'_{nk_{t+s}})$ converges to the distribution of $(Z_{k_1},\dots Z_{k_t},Z'_{k_{t+1}},\dots,Z'_{k_{t+s}})$, where, for $i=1,\dots,t$, we have $Z_{k_i}=\mathrm{Po}\left((\theta e^{-\lambda_2(\theta)})^{k_i}/2k_i\right)$, and for $i=t+1,\dots,t+s$ $Z'_{k_i}=\mathrm{Po}\left(\theta^{k_i}(1-e^{-k_i\lambda_2(\theta)})/2k_i\right)$ are independent. Finally, Proposition 5.2.4 concludes the proof of Theorem 5.1.1(iii).

Remark. Note that what we have just shown yields that for any integer $k \geq 3$ the total number of k-cycles in the core of a $\mathcal{G}_{n,m}$ random graph (i.e. the total number of k-cycles in a $\mathcal{G}_{n,m}$ random graph) is asymptotically Poisson distributed having mean equal to $\theta^k/2k$ (see [23] or [6]).

5.4 The asymptotic distribution of the cyclic 2-vertex(-edge) -connected components of $\mathrm{cr}(\mathcal{G}_{n,m})$

We begin with a general result concerning the structure of $L_1(\mathcal{G}_{n,m})$. We state without proof the following theorem from [36] (Theorem 3(a) there):

Theorem 5.4.1 *For any fixed $\theta > 1$, $L_1(\mathcal{G}_{n,m})$ consists a.a.s. of a giant 2-vertex-connected subgraph of order $\Theta_p(n)$, a collection of trees and a collection of unicyclic components, each of them sprouting from a different vertex of the 2-vertex-connected component, whose total number and order is $O_p(1)$.*

The above theorem implies that $L_1(\mathrm{cr}(\mathcal{G}_{n,m}))$ consists of an essential core of linear order while each of the remaining non-trivial blocks contains precisely one cutvertex of $L_1(\mathrm{cr}(\mathcal{G}_{n,m}))$ and is a cycle. Moreover, the total number and order of the blocks of $L_1(\mathrm{cr}(\mathcal{G}_{n,m}))$ apart from the essential core is $O_p(1)$.

Let $n \in \mathbb{Z}^+$. For a graph G on V_n with no vertices of degree 1 and integers $k, k_1 \geq 3$ and $k_2 \geq 0$, let $X_k^v(G)$ be the number of the 2-vertex-connected components of G which are k-cycles, contain exactly one cutvertex of G and are not 2-edge-connected components and let $X_{k_1,k_2}^e(G)$ be the number of the 2-edge-connected components of G which are k_1-cycles and there is a unique path with k_2 internal vertices of degree 2 in G having exactly one endvertex in the cycle and the other endvertex has degree at least 3 in G (uniqueness is meant with respect to the existence of a different path which has possibly different length). We call these paths the *attaching paths* of the 2-edge-connected components. Moreover, let $X_k^e(G)$ be the number of the 2-edge-connected components of G which are cycles, contain precisely one cutvertex of G and the total number of vertices on the cycle together with the internal vertices of the attaching path is equal to k.

For a function $\omega : \mathbb{Z}^+ \to \mathbb{Z}^+$, which will be specified later, such that $\omega(n) \to \infty$ as $n \to \infty$, and $(\mathbf{d}_n) \in \mathbf{D}$ we let $X^e(G(\mathbf{d}_n)) = \sum_{k=3}^{\omega(n)\vee 3} X_k^e(G(\mathbf{d}_n))$ and $X^v(G(\mathbf{d}_n)) = \sum_{k=3}^{\omega(n)\vee 3} X_k^v(G(\mathbf{d}_n))$. Similarly, we define $X^e(\mathcal{G}_{n,m}) = \sum_{k=3}^{\omega(n)\vee 3} X_k^e(\mathrm{cr}(\mathcal{G}_{n,m}))$ and $X^v(\mathcal{G}_{n,m}) = \sum_{k=3}^{\omega(n)\vee 3} X_k^v(\mathrm{cr}(\mathcal{G}_{n,m}))$.
Remark. Note that the above theorem along with Theorem 5.1.1(i) immediately implies that these random variables are within $o_p(1)$ of the random variables that count the numbers of the

2-edge-connected components of $L_1(\mathrm{cr}(\mathcal{G}_{n,m}))$ as well as the 2-vertex-connected components that belong to the same 2-edge-connected component as $\mathrm{ess}-\mathrm{cr}(\mathcal{G}_{n,m})$ in $L_1(\mathrm{cr}(\mathcal{G}_{n,m}))$, respectively.

In this section, we investigate the asymptotic distributions of the above random variables, aiming towards the proof of Theorem 5.1.2. Firstly, we present the following calculations which will be used in the present section. Let G be a graph on V_n with no vertices of degree 1 and $\mathbf{d} = (d_1, \ldots, d_n)$ be its labelled degree sequence. For some $r \in \mathbb{Z}^+$, let C be a collection of subgraphs of G consisting of r vertex disjoint cyclic 2-vertex-connected components of G which are not 2-edge-connected components and cyclic 2-edge-connected components of G including their attaching paths, with each cycle containing precisely one cutvertex of G. Let t, s be the total number of vertices of degree 2 and 3 respectively of G that totally participate in these subgraphs (that is those vertices which become isolated if we remove the edges of these subgraphs). Also, let S_1 be the multiset of degrees of those cutvertices in G that belong to the 2-vertex-connected components of C which are not 2-edge-connected components and S_2 be the multiset of the degrees in G of those endvertices of the attaching paths of the 2-edge-connected components in C that do not belong to their cycles. Let $\mathbf{d}'$ be the labelled degree sequence of G with the edges that belong to the elements of C removed and note that this is a permutation of the sequence $(d'_1, \ldots, d'_n)$ that is obtained from $\mathbf{d}$, where for each $i \in V_n$ d'_i is the degree of vertex i after the removal of the edges of C. Thus with λ as it was defined in Lemma 5.2.5 and λ' being the corresponding quantity for $\mathbf{d}'$, we have, for n sufficiently large,

$$\begin{aligned}
-\lambda' + \lambda &< \frac{1}{2M}\left(-\sum_{j=1}^{n}\binom{d'_j}{2} + \sum_{j=1}^{n}\binom{d_j}{2}\right) \\
&= \frac{1}{2M}\left(t + 3s + \sum_{i\in S_1}\left(\binom{i}{2} - \binom{i-2}{2}\right) + \sum_{i\in S_2}\left(\binom{i}{2} - \binom{i-1}{2}\right)\right) \\
&= \frac{1}{2M}\left(t + 3s + \sum_{i\in S_1}(2i-3) + \sum_{i\in S_2}(i-1)\right). \qquad (5.2)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&-\lambda'^2 + \lambda^2 < \\
&\qquad \frac{1}{4M^2}\left(-\left(\sum_{j=1}^{n}\binom{d'_j}{2}\right)^2 + \left(\sum_{j=1}^{n}\binom{d_j}{2}\right)^2\right) \\
&= \frac{1}{4M^2}\left(-\sum_{j\notin V(C)}\binom{d'_i}{2}\left(\sum_{j=1}^{n}\binom{d'_j}{2}\right) - \sum_{j\in V(C)}\binom{d'_i}{2}\left(\sum_{j=1}^{n}\binom{d'_j}{2}\right) + \right.
\end{aligned}$$

$$\sum_{j\notin V(C)}\binom{d_j}{2}\left(\sum_{j=1}^{n}\binom{d_j}{2}\right)+\sum_{j\in V(C)}\binom{d_j}{2}\left(\sum_{j=1}^{n}\binom{d_j}{2}\right)\Bigg)$$

$$= \frac{1}{4M^2}\Bigg(\sum_{j\in V(C)}\binom{d_j}{2}\left(\sum_{j=1}^{n}\binom{d_j}{2}\right)+\sum_{j\notin V(C)}\binom{d_j}{2}\left(\sum_{j=1}^{n}\binom{d_j}{2}-\binom{d'_j}{2}\right)$$

$$-\sum_{j\in V(C)}\binom{d'_j}{2}\left(\sum_{j=1}^{n}\binom{d'_j}{2}\right)\Bigg)$$

$$= \frac{1}{4M^2}\Bigg(\left(t+3s+\sum_{i\in S_1\cup S_2}\binom{i}{2}\right)\left(\sum_{j=1}^{n}\binom{d_j}{2}\right)$$

$$+(t+3s+\sum_{i\in S_1}(2i-3)+\sum_{i\in S_2}(i-1))\sum_{j\notin V(C)}\binom{d_j}{2}$$

$$-\left(\sum_{i\in S_1}\binom{i-2}{2}+\sum_{i\in S_2}\binom{i-1}{2}\right)\left(\sum_{j=1}^{n}\binom{d'_j}{2}\right)\Bigg)$$

$$= \frac{1}{4M^2}\Bigg(\Bigg(t+3s+\sum_{i\in S_1}\left(\binom{i}{2}-\binom{i-2}{2}\right)+\sum_{i\in S_2}\left(\binom{i}{2}-\binom{i-1}{2}\right)$$

$$+t+3s+\sum_{i\in S_1}(2i-3)+\sum_{i\in S_2}(i-1)\Bigg)\left(\sum_{j\notin V(C)}\binom{d_j}{2}\right)$$

$$+\left(t+3s+\sum_{i\in S_1\cup S_2}\binom{i}{2}\right)\left(\sum_{j\in V(C)}\binom{d_j}{2}\right)$$

$$-\left(\sum_{i\in S_1}\binom{i-2}{2}+\sum_{i\in S_2}\binom{i-1}{2}\right)\left(\sum_{j\in V(C)}\binom{d'_j}{2}\right)\Bigg)$$

$$= \frac{1}{4M^2}\Bigg(2\left(t+3s+\sum_{i\in S_1}(2i-3)+\sum_{i\in S_2}(i-1)\right)\left(\sum_{j\notin V(C)}\binom{d_j}{2}\right)$$

$$+\left(t+3s+\sum_{i\in S_1\cup S_2}\binom{i}{2}\right)\left(\sum_{j\in V(C)}\binom{d_j}{2}\right)-\left(\sum_{i\in S_1}\binom{i-2}{2}+\sum_{i\in S_2}\binom{i-1}{2}\right)^2\Bigg)$$

$$= \frac{1}{4M^2}\Bigg(2\left(t+3s+\sum_{i\in S_1}(2i-3)+\sum_{i\in S_2}(i-1)\right)\left(\sum_{j\notin V(C)}\binom{d_j}{2}\right)+\left(t+3s+\sum_{i\in S_1\cup S_2}\binom{i}{2}\right)^2$$

$$-\left(\sum_{i\in S_1}\binom{i-2}{2}+\sum_{i\in S_2}\binom{i-1}{2}\right)^2\Bigg). \tag{5.3}$$

We begin with the calculation of

$$\mathbb{E}[(X^e(G(\mathbf{d}_n)))_{l_1}(X^v(G(\mathbf{d}_n)))_{l_2}],$$

for some positive integers l_1, l_2 and some $(\mathbf{d}_n) \in G$, where we set $\omega(n) = \lfloor \ln\ln n / \ln\ln\ln n \rfloor$ (for any $n \geq 16$ - otherwise we set $\omega(n) = 1$) in the definitions of $X^e(G(\mathbf{d}_n))$ and $X^v(G(\mathbf{d}_n))$. Note that the above random variable counts the number of pairs of ordered l_1-tuples of "small" 2-edge-connected cyclic components and ordered l_2-tuples of "small" 2-vertex-connected cyclic components that are not 2-edge-connected components in $G(\mathbf{d}_n)$, each of the above containing exactly one cutvertex of $G(\mathbf{d}_n)$. Since $(\mathbf{d}_n) \in G$, by Lemma 5.6.2 it is sufficient to consider in the calculation of the above expectation only those pairs that consist of graphs that are vertex disjoint. For two functions $f, g : \mathbb{N} \to \mathbb{N}$, we write $f(n) \approx g(n)$ if and only if $|f(n) - g(n)| = o(1)$.

In the following calculation, $D_j^{(i)}$ will denote the number of those vertices amongst the D_j vertices of degree j in the labelled degree sequence $(d_1, \ldots, d_n)$ that are left after the first $i-1$ subgraphs have been chosen and their edges removed and $\mathbf{d}'_n = (d'_1, \ldots, d'_n)$ be the labelled degree sequence after the removal of all the edges of the subgraphs. The indices j_l, i_l will indicate the degree of that endvertex of the attaching path of the l-th cyclic 2-edge-connected component which does not belong to its cycle and the degree of the cutvertex of the l-th cyclic 2-vertex-connected component, for $l = 1, \ldots, l_1$ and $l = 1, \ldots, l_2$ respectively. The next two observations will clarify the intermediate steps of the following calculation. Firstly, note that $D_j^{(i)} = D_j(1 + O(\omega(n)/n))$, since $(\mathbf{d}_n) \in G$ and each subgraph of a given pair is of order $\omega(n)$ at most. Secondly, if vertex i totally participates in one of the cycles or in one of the attaching paths, then $\mathbf{d}'_n$ has one more vertex of degree 0 and one less vertex of degree d_i. Otherwise, in the case of 2-edge-connected cyclic components, if the vertex i is the endvertex of the attaching path of such a component which does not belong to its cycle, then $\mathbf{d}'_n$ has one more vertex of degree $d_i - 1$ and one less vertex of degree d_i. In the case of 2-vertex-connected cyclic components which are not 2-edge-connected components, if vertex i is the unique vertex that joins the cycle with the rest of the graph, then $\mathbf{d}'_n$ has one more vertex of degree $d_i - 2$ and one less vertex of degree d_i. With the convention that $\sum_{k_1=a,k_2=b}^{c} = \sum_{k_1=a,k_2=b,k_1+k_2\leq c}^{c}$ for $a + b \leq c$, we have for n sufficiently large:

$$\mathbb{E}[(X^e(G(\mathbf{d}_n)))_{l_1}(X^v(G(\mathbf{d}_n)))_{l_2}] \approx \sum_{k_1^{(1)}=3,k_1^{(2)}=0}^{\omega(n)} \binom{D_2}{k_1^{(1)}-1} \frac{(k_1^{(1)}-1)!}{2} \binom{D_2-(k_1^{(1)}-1)}{k_1^{(2)}} k_1^{(2)}! D_3^{(1)} \sum_{j_1=3}^{\Delta} (D_{j_1}^{(1)} - \delta_{j_1,3}) \Bigg(\cdots$$

$$
\sum_{k_{l_1}^{(1)}=3,k_{l_1}^{(2)}=0}^{\omega(n)} \binom{D_2-\sum_{i=1}^{l_1-1}(k_i^{(1)}+k_i^{(2)}-1)}{k_{l_1}^{(1)}-1} \frac{(k_{l_1}^{(1)}-1)!}{2} \times
$$

$$
\binom{D_2-\sum_{i=1}^{l_1-1}(k_i^{(1)}+k_i^{(2)}-1)-(k_{l_1}^{(1)}-1)}{k_{l_1}^{(2)}} k_{l_1}^{(2)}! D_3^{(l_1)} \sum_{j_{l_1}=3}^{\Delta} (D_{j_{l_1}}^{(l_1)}-\delta_{j_{l_1},3}) \Bigg(
$$

$$
\sum_{k_1=3}^{\omega(n)} \binom{D_2-\sum_{i=1}^{l_1}(k_i^{(1)}+k_i^{(2)}-1)}{k_1-1} \frac{(k_1-1)!}{2} \sum_{i_1=4}^{\Delta} D_{i_1}^{(l_1+1)} \Bigg(\cdots
$$

$$
\sum_{k_{l_2}=3}^{\omega(n)} \binom{D_2-\sum_{i=1}^{l_1}(k_i^{(1)}+k_i^{(2)}-1)-\sum_{i=1}^{l_2-1}(k_i-1)}{k_{l_2}-1} \frac{(k_{l_2}-1)!}{2} \sum_{i_{l_2}=4}^{\Delta} D_{i_{l_2}}^{(l_1+l_2)} \times
$$

$$
\frac{(2M-2(\sum_{i=1}^{l_1}(k_i^{(1)}+k_i^{(2)}+1)+\sum_{i=1}^{l_2}k_i))!}{(M-(\sum_{i=1}^{l_1}(k_i^{(1)}+k_i^{(2)}+1)+\sum_{i=1}^{l_2}k_i))! 2^{M-(\sum_{i=1}^{l_1}(k_i^{(1)}+k_i^{(2)}+1)+\sum_{i=1}^{l_2}k_i)} \prod_{j=1}^{n} d_j'!} \times
$$

$$
\frac{M! 2^M \prod_{j=1}^{n} d_j!}{(2M)!} \exp\left(-\lambda'+\lambda-(\lambda')^2+\lambda^2\right) \Bigg) \cdots \Bigg) \left(1+O\left(\frac{\ln^3 n}{n}\right)\right)
$$

$$
= \sum_{k_1^{(1)}=3,k_1^{(2)}=0}^{\omega(n)} \frac{D_2^{k_1^{(1)}+k_1^{(2)}-1}}{(k_1^{(1)}-1)! k_1^{(2)}!} \frac{(k_1^{(1)}-1)! k_1^{(2)}!}{2} D_3 \sum_{j_1=3}^{\Delta} D_{j_1} \Bigg(\cdots
$$

$$
\sum_{k_{l_1}^{(1)}=3,k_{l_1}^{(2)}=0}^{\omega(n)} \frac{D_2^{k_{l_1}^{(1)}+k_{l_1}^{(2)}-1}}{(k_{l_1}^{(1)}-1)! k_{l_1}^{(2)}!} \frac{(k_{l_1}^{(1)}-1)! k_{l_1}^{(2)}!}{2} D_3 \sum_{j_{l_1}=3}^{\Delta} D_{j_{l_1}} \Bigg(
$$

$$
\sum_{k_1=3}^{\omega(n)} \frac{D_2^{k_1-1}}{(k_1-1)!} \frac{(k_1-1)!}{2} \sum_{i_1=4}^{\Delta} D_{i_1} \Bigg(\cdots \sum_{k_{l_2}=3}^{\omega(n)} \frac{D_2^{k_{l_2}-1}}{(k_{l_2}-1)!} \frac{(k_{l_2}-1)!}{2} \sum_{i_{l_2}=4}^{\Delta} D_{i_{l_2}} \times
$$

$$
\frac{(M)_{\sum_{i=1}^{l_1}(k_i^{(1)}+k_i^{(2)}+1)+\sum_{i=1}^{l_2}k_i} 2^{\sum_{i=1}^{l_1}(k_i^{(1)}+k_i^{(2)}+1)+\sum_{i=1}^{l_2}k_i}}{(2M)_{2(\sum_{i=1}^{l_1}(k_i^{(1)}+k_i^{(2)}+1)+\sum_{i=1}^{l_2}k_i)}}
$$

$$
\frac{\prod_{j=1}^{n} d_j!}{\prod_{j=1}^{n} d_j'!} \exp\left(-\lambda'+\lambda-(\lambda')^2+\lambda^2\right) \Bigg) \cdots \Bigg) \left(1+O\left(\frac{\omega(n)^2}{n}+\frac{\ln^3 n}{n}\right)\right)
$$

$$
= (M-D_2)^{l_1} \sum_{k_1^{(1)}=3,k_1^{(2)}=0}^{\omega(n)} \frac{3D_2^{k_1^{(1)}+k_1^{(2)}-1}}{2} D_3 \Bigg(\cdots \sum_{k_{l_1}^{(1)}=3,k_{l_1}^{(2)}=0}^{\omega(n)} \frac{3D_2^{k_{l_1}^{(1)}+k_{l_1}^{(2)}-1}}{2} D_3 \Bigg(
$$

$$
\sum_{k_1=3}^{\omega(n)} \frac{D_2^{k_1-1}}{2} \sum_{i_1=4}^{\Delta} \binom{i_1}{2} D_{i_1} \Bigg(\cdots \sum_{k_{l_2}=3}^{\omega(n)} \frac{D_2^{k_{l_2}-1}}{2} \sum_{i_{l_2}=4}^{\Delta} \binom{i_{l_2}}{2} D_{i_{l_2}} \times
$$

$$
\frac{M^{\sum_{i=1}^{l_1}(k_i^{(1)}+k_i^{(2)}+1)+\sum_{i=1}^{l_2}k_i} 2^{2(\sum_{i=1}^{l_1}(k_i^{(1)}+k_i^{(2)}+1)+\sum_{i=1}^{l_2}k_i)}}{(2M)^{2(\sum_{i=1}^{l_1}(k_i^{(1)}+k_i^{(2)}+1)+\sum_{i=1}^{l_2}k_i)}} \times
$$

$$\exp\left(-\lambda' + \lambda - (\lambda')^2 + \lambda^2\right)\Bigg)\cdots\Bigg)\left(1 + O\left(\frac{\omega(n)^2}{n} + \frac{\ln^3 n}{n}\right)\right)$$

$$= (M - D_2)^{l_1} \sum_{k_1^{(1)}=3,k_1^{(2)}=0}^{\omega(n)} \frac{3D_2^{k_1^{(1)}+k_1^{(2)}-1}}{2} D_3\Bigg(\cdots \sum_{k_{l_1}^{(1)}=3,k_{l_1}^{(2)}=0}^{\omega(n)} \frac{3D_2^{k_{l_1}^{(1)}+k_{l_1}^{(2)}-1}}{2} D_3\Bigg($$

$$\sum_{k_1=3}^{\omega(n)} \frac{D_2^{k_1-1}}{2} \sum_{i_1=4}^{\Delta} \binom{i_1}{2} D_{i_1}\Bigg(\cdots \sum_{k_{l_2}=3}^{\omega(n)} \frac{D_2^{k_{l_2}-1}}{2} \sum_{i_{l_2}=4}^{\Delta} \binom{i_{l_2}}{2} D_{i_{l_2}} \times$$

$$\frac{1}{M^{\sum_{i=1}^{l_1}(k_i^{(1)}+k_i^{(2)}+1)+\sum_{i=1}^{l_2} k_i}}\Bigg)\cdots\Bigg)\left(1 + O\left(\frac{\ln^3 n}{n}\right)\right)$$

$$= \left(\frac{M - D_2}{M} \frac{3}{2} \sum_{k=3}^{\omega(n)} (k-2)\left(\frac{D_2}{M}\right)^{k-1} \frac{D_3}{M}\right)^{l_1} \left(\frac{1}{2}\sum_{k=3}^{\omega(n)} \left(\frac{D_2}{M}\right)^{k-1} \sum_{i=4}^{\Delta} \binom{i}{2} \frac{D_i}{M}\right)^{l_2} \times$$

$$\left(1 + O\left(\frac{\ln^3 n}{n}\right)\right).$$

Using (5.2) and (5.3), we obtain the final error term of the above relation.

Note that

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{2}\sum_{k=3}^{\omega(n)} \left(\frac{D_2}{M}\right)^{k-1} \sum_{i=4}^{\Delta} \binom{i}{2} \frac{D_i}{M} &= \frac{\theta}{2}(1 - e^{-\lambda_2(\theta)} - \lambda_2(\theta)e^{-\lambda_2(\theta)}) \sum_{k=3}^{\infty} \left(\theta e^{-\lambda_2(\theta)}\right)^{k-1} \\
&= \frac{\theta}{2}(1 - e^{-\lambda_2(\theta)} - \lambda_2(\theta)e^{-\lambda_2(\theta)}) \frac{\left(\theta e^{-\lambda_2(\theta)}\right)^2}{1 - \theta e^{-\lambda_2(\theta)}},
\end{aligned}$$

and

$$\lim_{n\to\infty} \frac{M - D_2}{M} \frac{3}{2} \sum_{k=3}^{\omega(n)} (k-2)\left(\frac{D_2}{M}\right)^{k-1} \frac{D_3}{M} =$$

$$\left(1 - \theta e^{-\lambda_2(\theta)}\right) \frac{\theta e^{-\lambda_2(\theta)}\lambda_2(\theta)}{2} \sum_{k=3}^{\infty} (k-2)\left(\theta e^{-\lambda_2(\theta)}\right)^{k-1}$$

$$= \left(1 - \theta e^{-\lambda_2(\theta)}\right) \frac{\theta e^{-\lambda_2(\theta)}\lambda_2(\theta)}{2} \left(\frac{3\left(\theta e^{-\lambda_2(\theta)}\right)^2 - 2\left(\theta e^{-\lambda_2(\theta)}\right)^3}{\left(1 - \theta e^{-\lambda_2(\theta)}\right)^2} - \frac{2\left(\theta e^{-\lambda_2(\theta)}\right)^2}{1 - \theta e^{-\lambda_2(\theta)}}\right),$$

by the first and the third part of Lemma 5.7.1. From Corollary 6.8, Theorem 6.10 in [23] and Proposition 5.2.4, we deduce that $X^v(\mathcal{G}_{n,m}) \xrightarrow{d} X^v$, $X^e(\mathcal{G}_{n,m}) \xrightarrow{d} X^e$ and $(X^v(\mathcal{G}_{n,m}), X^e(\mathcal{G}_{n,m})) \xrightarrow{d} (X^v, X^e)$, where

$$X^v = \mathrm{Po}\left(\frac{\theta}{2}(1 - e^{-\lambda_2(\theta)} - \lambda_2(\theta)e^{-\lambda_2(\theta)}) \frac{\left(\theta e^{-\lambda_2(\theta)}\right)^2}{1 - \theta e^{-\lambda_2(\theta)}}\right)$$

and

$$X^e = \mathrm{Po}\left(\left(1 - \theta e^{-\lambda_2(\theta)}\right) \frac{\theta e^{-\lambda_2(\theta)}\lambda_2(\theta)}{2} \left(\frac{3\left(\theta e^{-\lambda_2(\theta)}\right)^2 - 2\left(\theta e^{-\lambda_2(\theta)}\right)^3}{\left(1 - \theta e^{-\lambda_2(\theta)}\right)^2} - \frac{2\left(\theta e^{-\lambda_2(\theta)}\right)^2}{1 - \theta e^{-\lambda_2(\theta)}}\right)\right)$$

are independent. This yields that

$$X^v(\mathcal{G}_{n,m}) + X^e(\mathcal{G}_{n,m}) \xrightarrow{d}$$
$$\mathrm{Po}\Bigg(\frac{\theta}{2}(1 - e^{-\lambda_2(\theta)} - \lambda_2(\theta)e^{-\lambda_2(\theta)})\frac{\left(\theta e^{-\lambda_2(\theta)}\right)^2}{1-\theta e^{-\lambda_2(\theta)}}$$
$$+\left(1-\theta e^{-\lambda_2(\theta)}\right)\frac{\theta e^{-\lambda_2(\theta)}\lambda_2(\theta)}{2}\left(\frac{3\left(\theta e^{-\lambda_2(\theta)}\right)^2 - 2\left(\theta e^{-\lambda_2(\theta)}\right)^3}{\left(1-\theta e^{-\lambda_2(\theta)}\right)^2} - \frac{2\left(\theta e^{-\lambda_2(\theta)}\right)^2}{1-\theta e^{-\lambda_2(\theta)}}\right)\Bigg).$$

These along with Theorem 5.4.1 conclude the proof of Theorem 5.1.2 (a).

Now, focusing on such components according to the size of the cycles and their attaching paths (if they have), we present and prove the analogue of Lemma 5.3.2:

Lemma 5.4.2 *Assume that* $(\mathbf{d}_n) \in G$. *Let* $s, t \in \mathbb{N}$ *and* $k_1^{(1)}, \ldots, k_s^{(1)}, k_{s+1}, \ldots, k_{s+t} \geq 3$, $k_1^{(2)} \ldots, k_s^{(2)} \geq 0$, $l_1, \ldots, l_{s+t} \geq 1$ *be also natural numbers. For any integers* $k, k_1 \geq 3$ *and* $k_2 \geq 0$, *we set* $X^e_{k_1k_2n} = X^e_{k_1k_2}(G(\mathbf{d}_n))$ *and* $X^v_{kn} = X^v_k(G(\mathbf{d}_n))$. *Then*

$$\lim_{n\to\infty} \mathbb{E}[(X^e_{k_1^{(1)}k_1^{(2)}n})_{l_1}\cdots(X^e_{k_s^{(1)}k_s^{(2)}n})_{l_s}(X^v_{k_{s+1}n})_{l_{s+1}}\cdots(X^v_{k_{s+t}n})_{l_{s+t}}] =$$
$$\prod_{j=1}^{s}\left(\frac{\theta e^{-\lambda_2(\theta)}\lambda_2(\theta)(1-\theta e^{-\lambda_2(\theta)})}{2}\left(\theta e^{-\lambda_2(\theta)}\right)^{k_j^{(1)}+k_j^{(2)}-1}\right)^{l_j} \times$$
$$\prod_{j=1}^{t}\left(\frac{\theta}{2}\left(\theta e^{-\lambda_2(\theta)}\right)^{k_{s+j}-1}\left(1-e^{-\lambda_2(\theta)} - e^{-\lambda_2(\theta)}\lambda_2(\theta)\right)\right)^{l_{s+j}}$$

(Here, we assume that the empty product is equal to 1.)

Proof. Note that the number on the left hand side of the above equation expresses the total number of collections of ordered tuples of cyclic 2-edge-connected components including their attaching paths and 2-vertex-connected components which are not 2-edge-connected components indexed by the set $\{1, \ldots, s+t\}$, where for any $i = 1, \ldots, s$, we have l_i-tuples of 2-edge-connected components which are cycles of length $k_i^{(1)}$ and have a unique attaching path with $k_i^{(2)}$ internal vertices and for $i = s+1, \ldots, s+t$, we have l_i-tuples of 2-vertex-connected components which are cycles of order k_i and are not 2-edge-connected components, each containing exactly one cutvertex of $G(\mathbf{d}_n)$. Moreover, every such family induces a subgraph of order at most $\sum_{i=1}^{s} l_i(k_i^{(1)} + k_i^{(2)} + 1) + \sum_{i=s+1}^{s+t} k_i$. So, the expected number of those families with at least two overlapping such cycles (whose overlap of course is a vertex of degree greater than 2) tends to 0, by Lemma 5.6.2, since they induce subgraphs of bounded order which have more edges than vertices and $(\mathbf{d}_n) \in G$. Therefore, the above expectation is asymptotically equal to the expected number of those families consisting of pairwise disjoint subgraphs. Let X_n be the total number of the latter.

As in the proof of Lemma 5.3.2, $D_j^{(i)}$ will denote the number of those vertices amongst the D_j vertices of degree j that are left in the labelled degree sequence $(d_1,\ldots,d_n)$ after the first $i-1$ subgraphs have been chosen and their edges removed and $\mathbf{d}'_n=(d'_1,\ldots,d'_n)$ be the labelled degree sequence after the removal of the edges of the subgraphs. The indices j_l, i_l will indicate the degree of that endvertex of the attaching path of the l-th cyclic 2-edge-connected component which does not belong to its cycle and the degree of the cutvertex of the l-th cyclic 2-vertex-connected component, for $l=1,\ldots,l_1+\cdots+l_s$ and $l=1,\ldots,l_{s+1}+\cdots+l_{s+t}$ respectively. Also, note that $D_j^{(i)}=D_j(1+O(1/n))$, as $(\mathbf{d}_n)\in G$ and all the subgraphs contained in a given pair have bounded size. Now, if vertex i totally participates in one of the subgraphs, then $\mathbf{d}'_n$ has an extra vertex of degree 0 and one less vertex of degree d_i. Otherwise, in the case of 2-edge-connected cyclic components, if the vertex i is the endvertex of the attaching path of a 2-edge-connected cyclic component of a given pair that does not belong to its cycle, then $\mathbf{d}'_n$ has an extra vertex of degree d_i-1, whereas it has one less vertex of degree d_i. In the case of 2-vertex-connected cyclic components that are not 2-edge-connected components, for each vertex i which is the unique vertex that joins such a component with the rest of the graph, $\mathbf{d}'_n$ has an extra vertex of degree d_i-2, whereas it has one less vertex of degree d_i. After these preliminaries, Lemma 5.2.5 implies:

$$\mathbb{E}[X_n]=$$

$$\binom{D_2^{(1)}}{k_1^{(1)}-1}\frac{(k_1^{(1)}-1)!}{2}\binom{D_2^{(1)}-(k_1^{(1)}-1)}{k_1^{(2)}}k_1^{(2)}!D_3^{(1)}\sum_{j_1=3}^{\Delta}(D_{j_1}^{(1)}-\delta_{j_1 3})\Bigg(\cdots\binom{D_2^{(l_1)}}{k_1^{(1)}-1}$$

$$\frac{(k_1^{(1)}-1)!}{2}\binom{D_2^{(l_1)}-(k_1^{(1)}-1)}{k_1^{(2)}}k_1^{(2)}!D_3^{(l_1)}\sum_{j_{l_1}=3}^{\Delta}(D_{j_{l_1}}^{(l_1)}-\delta_{j_{l_1}3})\Bigg($$

$$\vdots$$

$$\binom{D_2^{(\sum_{i=1}^{s-1}l_i+1)}}{k_s^{(1)}-1}\frac{(k_s^{(1)}-1)!}{2}\binom{D_2^{(\sum_{i=1}^{s-1}l_i+1)}-(k_s^{(1)}-1)}{k_s^{(2)}}k_s^{(2)}!D_3^{(\sum_{i=1}^{s-1}l_i+1)}\cdots$$

$$\binom{D_2^{(\sum_{i=1}^{s}l_i)}}{k_s^{(1)}-1}\frac{(k_s^{(1)}-1)!}{2}\binom{D_2^{(\sum_{i=1}^{s}l_i)}-(k_s^{(1)}-1)}{k_s^{(2)}}k_s^{(2)}!D_3^{(\sum_{i=1}^{s}l_i)}\sum_{j_{\sum_{i=1}^{s}l_i}=3}^{\Delta}(D_{j_{\sum_{i=1}^{s}l_i}}^{(\sum_{i=1}^{s}l_i)}-\delta_{j_{\sum_{i=1}^{s}l_i}3})$$

$$\binom{D_2^{(\sum_{i=1}^{s}l_i+1)}}{k_{s+1}-1}\frac{(k_{s+1}-1)!}{2}\sum_{i_1=4}^{\Delta}D_{i_1}^{(\sum_{i=1}^{s}l_i+1)}\Bigg(\cdots\binom{D_2^{(\sum_{i=1}^{s}l_i+l_{s+1})}}{k_{s+1}-1}\frac{(k_{s+1}-1)!}{2}$$

$$\sum_{i_{l_{s+1}}=4}^{\Delta}D_{i_{l_{s+1}}}^{(\sum_{i=1}^{s}l_i+l_{s+1})}\Bigg($$

$$\vdots$$

$$
\binom{D_2^{(\sum_{i=1}^{s+t-1} l_i+1)}}{k_{s+t}-1}\frac{(k_{s+t}-1)!}{2}\sum_{i_{\sum_{i=1}^{t-1} l_{s+i}+1}=4}^{\Delta} D_{i_{\sum_{i=1}^{t-1} l_{s+i}+1}}^{(\sum_{i=1}^{s+t-1} l_i+1)}\Bigg(\cdots\binom{D_2^{(\sum_{i=1}^{s+t} l_i)}}{k_{s+t}-1}\frac{(k_{s+t}-1)!}{2}
$$

$$
\sum_{i_{\sum_{i=1}^{t} l_{s+i}}=4}^{\Delta} D_{i_{\sum_{i=1}^{t} l_{s+i}}}^{(\sum_{i=1}^{s+t} l_i)}\Bigg(
$$

$$
\frac{(2M-2(\sum_{i=1}^{s} l_i(k_i^{(1)}+k_i^{(2)}+1)+\sum_{i=s+1}^{s+t} l_ik_i))!}{(M-(\sum_{i=1}^{s} l_i(k_i^{(1)}+k_i^{(2)}+1)+\sum_{i=s+1}^{s+t} l_ik_i))!2^{M-(\sum_{i=1}^{s} l_i(k_i^{(1)}+k_i^{(2)}+1)+\sum_{i=s+1}^{s+t} l_ik_i)}\prod_{i=1}^{n} d_i'!}\times
$$

$$
\frac{M!2^M\prod_{i=1}^{n} d_i!}{(2M)!}\exp\left(-\lambda'+\lambda-(\lambda')^2+\lambda^2\right)\Bigg)\cdots\Bigg)\left(1+O\left(\frac{\ln^3 n}{n}\right)\right)
$$

$$
=\left(\frac{D_2^{k_1^{(1)}+k_1^{(2)}-1}}{(k_1^{(1)}-1)!k_1^{(2)}!}\frac{(k_1^{(1)}-1)!k_1^{(2)}!}{2}D_3\right)^{l_1}\cdots\left(\frac{D_2^{k_s^{(1)}+k_s^{(2)}-1}}{(k_s^{(1)}-1)!k_s^{(2)}!}\frac{(k_s^{(1)}-1)!k_s^{(2)}!}{2}D_3\right)^{l_s}
$$

$$
\sum_{j_1=3}^{\Delta} D_{j_1}\Bigg(\cdots\sum_{j_{\sum_{i=1}^{s} l_i}=3}^{\Delta} D_{j_{\sum_{i=1}^{s} l_i}}\Bigg(
$$

$$
\frac{D_2^{k_{s+1}-1}}{(k_{s+1}-1)!}\frac{(k_{s+1}-1)!}{2}\sum_{i_1=4}^{\Delta} D_{i_1}\Bigg(\cdots\frac{D_2^{k_{s+1}-1}}{(k_{s+1}-1)!}\frac{(k_{s+1}-1)!}{2}\sum_{i_{l_{s+1}}=4}^{\Delta} D_{i_{l_{s+1}}}\Bigg(
$$

$$
\vdots
$$

$$
\frac{D_2^{k_{s+t}-1}}{(k_{s+t}-1)!}\frac{(k_{s+t}-1)!}{2}\sum_{i_{\sum_{i=1}^{t-1} l_{s+i}+1}=4}^{\Delta} D_{i_{\sum_{i=1}^{t-1} l_{s+i}+1}}\Bigg(\cdots\frac{D_2^{k_{s+t}-1}}{(k_{s+t}-1)!}\frac{(k_{s+t}-1)!}{2}
$$

$$
\sum_{i_{\sum_{i=1}^{t} l_{s+i}}=4}^{\Delta} D_{i_{\sum_{i=1}^{t} l_{s+i}}}\left(\frac{(M)_{\sum_{i=1}^{s} l_i(k_i^{(1)}+k_i^{(2)}+1)+\sum_{i=s+1}^{s+t} l_ik_i}2^{\sum_{i=1}^{s} l_i(k_i^{(1)}+k_i^{(2)}+1)+\sum_{i=s+1}^{s+t} l_ik_i}}{(2M)_{2(\sum_{i=1}^{s} l_i(k_i^{(1)}+k_i^{(2)}+1)+\sum_{i=s+1}^{s+t} l_ik_i)}}\right.
$$

$$
\left.\frac{\prod_{i=1}^{n} d_i!}{\prod_{i=1}^{n} d_i'!}\exp\left(-\lambda'+\lambda-(\lambda')^2+\lambda^2\right)\right)\cdots\Bigg)\quad\left(1+O\left(\frac{\ln^3 n}{n}\right)\right)
$$

$$
=\left(2(M-D_2)\frac{3D_2^{k_1^{(1)}+k_1^{(2)}-1}}{2}D_3\frac{2^{k_1^{(1)}+k_1^{(2)}+1}}{2}\right)^{l_1}\cdots\left(2(M-D_2)\frac{3D_2^{k_s^{(1)}+k_s^{(2)}-1}}{2}D_3\frac{2^{k_s^{(2)}+k_s^{(1)}+1}}{2}\right)^{l_s}
$$

$$
\frac{D_2^{k_{s+1}-1}}{2}\sum_{i_1=4}^{\Delta} 2^{k_{s+1}}\binom{i_1}{2}D_{i_1}\Bigg(\cdots\frac{D_2^{k_{s+1}-1}}{2}\sum_{i_{l_{s+1}}=4}^{\Delta} 2^{k_{s+1}}\binom{i_{l_{s+1}}}{2}D_{i_{l_{s+1}}}\Bigg(
$$

$$
\vdots
$$

$$
\frac{D_2^{k_{s+t}-1}}{2}\sum_{i_{\sum_{i=1}^{t-1} l_{s+i}+1}=4}^{\Delta} 2^{k_{s+t}}\binom{i_{\sum_{i=1}^{t-1} l_{s+i}+1}}{2}D_{i_{\sum_{i=1}^{t-1} l_{s+i}+1}}\Bigg(\cdots\frac{D_2^{k_{s+t}-1}}{2}\times
$$

$$\sum_{i_{\sum_{i=1}^{t} l_{s+i}}=4}^{\Delta} 2^{k_{s+t}} \binom{i_{\sum_{i=1}^{t} l_{s+i}}}{2} D_{i_{\sum_{i=1}^{t} l_{s+i}}} \Bigg(\frac{M^{\sum_{i=1}^{s} l_i(k_i^{(1)}+k_i^{(2)}+1)+\sum_{i=s+1}^{s+t} l_i k_i} 2^{\sum_{i=1}^{s} l_i(k_i^{(1)}+k_i^{(2)}+1)+\sum_{i=s+1}^{s+t} l_i k_i}}{(2M)^{2(\sum_{i=1}^{s} l_i(k_i^{(1)}+k_i^{(2)}+1)+\sum_{i=s+1}^{s+t} l_i k_i)}} \times$$

$$\left(1+O\left(\frac{\ln^3 n}{n}\right)\right)\Bigg)\cdots\Bigg)$$

$$= \left(\frac{M-D_2}{M}\,\frac{3}{2}\left(\frac{D_2}{M}\right)^{k_1^{(1)}+k_1^{(2)}-1}\frac{D_3}{M}\right)^{l_1}\cdots\left(\frac{M-D_2}{M}\,\frac{3}{2}\left(\frac{D_2}{M}\right)^{k_s^{(1)}+k_s^{(2)}-1}\frac{D_3}{M}\right)^{l_s}\times$$

$$\left(\frac{1}{2}\left(\frac{D_2}{M}\right)^{k_{s+1}-1}\sum_{i=4}^{\Delta}\binom{i}{2}\frac{D_i}{M}\right)^{l_{s+1}}\cdots\left(\frac{1}{2}\left(\frac{D_2}{M}\right)^{k_{s+t}-1}\sum_{i=4}^{\Delta}\binom{i}{2}\frac{D_i}{M}\right)^{l_{s+t}}\times$$

$$\left(1+O\left(\frac{\ln^3 n}{n}\right)\right).$$

Using (5.2) and (5.3), we obtain the error term of the above relation. The first part of Lemma 5.7.1 in Section 5.7 implies the lemma. ∎

If we set $s=1$ and $t=0$ in the above lemma, then for any integers $k_1 \geq 3$, $k_2 \geq 0$ and $l \geq 1$, we have

$$\lim_{n\to\infty}\mathbb{E}\left[\left(X^e_{k_1k_2n}(G(\mathbf{d}_n))\right)_l\right]=\left(\frac{\theta e^{-\lambda_2(\theta)}\lambda_2(\theta)(1-\theta e^{-\lambda_2(\theta)})}{2}\left(\theta e^{-\lambda_2(\theta)}\right)^{k_1+k_2-1}\right)^l,$$

and this implies that $X^e_{k_1k_2n}(\mathrm{cr}(\mathcal{G}_{n,m})) \xrightarrow{d} \mathrm{Po}\left(\frac{\theta e^{-\lambda_2(\theta)}\lambda_2(\theta)(1-\theta e^{-\lambda_2(\theta)})}{2}\left(\theta e^{-\lambda_2(\theta)}\right)^{k_1+k_2-1}\right)$, as $n\to\infty$, using Corollary 6.8 in [23] and Proposition 5.2.4. If we set $s=0$ and $t=1$ in the above lemma, then for any integers $k\geq 3$ and $l\geq 1$, we have

$$\lim_{n\to\infty}\mathbb{E}\left[(X^v_{kn}(G(\mathbf{d}_n)))_l\right]=\left(\frac{\theta}{2}\left(\theta e^{-\lambda_2(\theta)}\right)^{k-1}\left(1-e^{-\lambda_2(\theta)}-\lambda_2(\theta)e^{-\lambda_2(\theta)}\right)\right)^l,$$

and, as above, we deduce that $X^v_{kn}(\mathrm{cr}(\mathcal{G}_{n,m})) \xrightarrow{d} \mathrm{Po}\left(\frac{\theta}{2}\left(\theta e^{-\lambda_2(\theta)}\right)^{k-1}\left(1-e^{-\lambda_2(\theta)}-\lambda_2(\theta)e^{-\lambda_2(\theta)}\right)\right)$, as $n\to\infty$. Finally, Lemma 5.4.2 along with Theorem 6.10 in [23] and Proposition 5.2.4 imply that the joint distribution of any finite collection of random variables

$$(X^e_{k_1^{(1)}k_1^{(2)}n}(\mathrm{cr}(\mathcal{G}_{n,m})),\ldots,X^e_{k_s^{(1)}k_s^{(2)}n}(\mathrm{cr}(\mathcal{G}_{n,m})),X^v_{k_{s+1}n}(\mathrm{cr}(\mathcal{G}_{n,m})),\ldots,X^v_{k_{s+t}n}(\mathrm{cr}(\mathcal{G}_{n,m})))$$

converges to the joint distribution of $(X^e_{k_1^{(1)}k_1^{(2)}},\ldots,X^e_{k_s^{(1)}k_s^{(2)}},X^v_{k_{s+1}},\ldots,X^v_{k_{s+t}})$, where, for $i=1,\ldots,s$, we have $X^e_{k_i^{(1)}k_i^{(2)}}=\mathrm{Po}\left(\frac{\theta e^{-\lambda_2(\theta)}\lambda_2(\theta)(1-\theta e^{-\lambda_2(\theta)})}{2}\left(\theta e^{-\lambda_2(\theta)}\right)^{k_i^{(1)}+k_i^{(2)}-1}\right)$, and for $i=s+1,\ldots,s+t$

$$X^v_{k_i}=\mathrm{Po}\left(\frac{\theta}{2}\left(\theta e^{-\lambda_2(\theta)}\right)^{k_i-1}\left(1-e^{-\lambda_2(\theta)}-\lambda_2(\theta)e^{-\lambda_2(\theta)}\right)\right),$$

are independent. The remark at the beginning of this section concludes the proof of Theorem 5.1.2.

5.5 On the structure of the kernel of a sparse random graph

In this section, we investigate the implications of Theorems 5.1.1, 5.1.2 and 4.1.1 of the previous chapter to the asymptotic structure of the kernel of a $\mathcal{G}_{n,m}$ random graph. As we have defined it in the introduction, for a graph G the graph $\ker(G)$ is the multigraph (possibly with loops) obtained from the components of the core of G which have more than one cycles, by replacing each path whose internal vertices are all of degree two by a single edge. We now give some properties of $\ker(\mathcal{G}_{n,m})$, which follow from Theorems 5.1.1, 5.1.2 and 4.1.1. The fact that the number of vertices in $\mathrm{cr}(\mathcal{G}_{n,m})$ that do not belong to its giant component is $O_p(1)$ and all of them are of degree 2 a.a.s. implies that the order of $\ker(\mathcal{G}_{n,m})$ is equal to $n(p_2(\lambda_2(\theta)) - \rho_2(\lambda_2(\theta))) + o_p(n) = np_3(\lambda_2(\theta)) + o_p(n)$, by Theorems 5.1.1(a) and 4.1.1. Furthermore, since the difference between the number of edges and the number of vertices in the kernel is equal to this difference in the core, we deduce that the number of edges in $\ker(\mathcal{G}_{n,m})$ is equal to $n(\frac{\lambda_2^2(\theta)}{2\theta} - \rho_2(\lambda_2(\theta))) + o_p(n)$. The degree sequence of $\ker(\mathcal{G}_{n,m})$ is the degree sequence of $\mathrm{cr}(\mathcal{G}_{n,m})$, restricted to degrees greater than 2. Finally, the number of loops is asymptotically a Poisson random variable of mean equal to the sum of the two means of the two Poisson random variables that correspond to the number of cyclic 2-edge and 2-vertex-connected components of $L_1(\mathrm{cr}(\mathcal{G}_{n,m}))$ and are given in the first part of Theorem 5.1.2.

5.6 Some results from the theory of random graphs

We show the following lemma which is used mainly in Section 5.2.

Lemma 5.6.1 *Let $\Delta(n) = \lceil \ln n \rceil$. Also, let $Y(\mathcal{G}_{n,m}) = \sum_{i=2}^{n-1} i^4 v_i(\mathrm{DP}_2(\mathcal{G}_{n,m}))$ and, for $k \in \mathbb{N}$, $X_{n,k} = X_{n,k}(\mathcal{G}_{n,m}) = \sum_{i=k}^{n-1} \binom{i}{2} \frac{v_i(\mathrm{DP}_2(\mathcal{G}_{n,m}))}{n}$. Then, there exists a $c > 0$ such that for every $\varepsilon > 0$, there exists a $k_0 \in \mathbb{N}$ so that*

$$\mathbb{P}[Y(\mathcal{G}_{n,m}) > cn \text{ or } X_{n,k_0} \geq \varepsilon \text{ or } \mathit{max.deg.}(\mathrm{DP}_2(\mathcal{G}_{n,m})) > \Delta(n)] = o\left(\frac{1}{n^2}\right).$$

Proof. Let us fix $\varepsilon > 0$. Note that

$$Y(\mathcal{G}_{n,m}) \leq \sum_{i=1}^{n-1} i^4 v_i(\mathcal{G}_{n,m}) \text{ and } X_{n,k} \leq \sum_{i=k}^{n-1} \binom{i}{2} \frac{v_i(\mathcal{G}_{n,m})}{n} \leq \sum_{i=k}^{n-1} i^2 \frac{v_i(\mathcal{G}_{n,m})}{n}, \tag{5.4}$$

always for $k \geq 2$. Let $Y'(\mathcal{G}_{n,m})$, $X'_{n,k}(\mathcal{G}_{n,m})$ be the quantities on the right hand sides of the above inequalities. We shall focus on $Y'(\mathcal{G}_{n,m})$ and $X'_{n,k}(\mathcal{G}_{n,m})$ rather than $Y(\mathcal{G}_{n,m})$ and $X_{n,k}$.

Also, note that max.deg.$(\mathrm{DP}_2(\mathcal{G}_{n,m})) \leq$ max.deg.$(\mathcal{G}_{n,m})$. For some $k \geq 2$ and some $c > 0$ which will be specified later we set $E_1 = \{\text{max.deg.}(\mathcal{G}_{n,m}) > \Delta(n)\}$, $E_2 = \{X'_{n,k}(\mathcal{G}_{n,m}) \geq \varepsilon\}$ and $E_3 = \{Y'(\mathcal{G}_{n,m}) > cn\}$ and $P_1 = \mathbb{P}[E_1]$, $P_2 = \mathbb{P}[E_2]$, $P_3 = \mathbb{P}[E_3]$.

Recall from Lemma 4.3.2 from the previous chapter that a $\mathcal{G}_{n,m}$ random graph can be viewed as the multigraph process $MG(n,m)$ conditioned on the event

$$A_n = \{MG(n,m) \text{ has no loops and no multiple edges}\}.$$

(At each stage of the multigraph process we form an edge by picking uniformly at random, independently and with replacement two vertices, i and j, ignoring the previous choices; if $i = j$ then the multigraph gets a loop at i.) Using $\mathbb{P}[U \mid V] \leq \mathbb{P}[U]/\mathbb{P}[V]$ (this is the first conditioning), and denoting by P'_i the analogous probabilities in $MG(n,m)$, for $i = 1, 2, 3$, we have

$$P_i \leq \frac{P'_i}{A_n} = O(P'_i), \tag{5.5}$$

since

$$\lim_{n\to\infty} \mathbb{P}[A_n] = \lim_{n\to\infty} \frac{\binom{\binom{n}{2}}{m} m! 2^m}{n^{2m}} = \exp(-\theta/2 - \theta^2/4) > 0.$$

As we have already seen in the proof of Lemma 4.3.2 in the previous chapter, the second conditioning comes from the fact that the degree sequence of the random multigraph produced by the multigraph process can be viewed as the random sequence of occupancy numbers when we throw $2m$ distinguishable balls into n bins. These are jointly distributed as n independent copies, say $\{Z_j(\theta)\}_{j=1,\dots,n}$, of $Z(\theta)$ (a Poisson random variable with mean θ) conditioned on $S_n = \sum_{j=1}^n Z_j(\theta) = 2m$. Since S_n is $Z(2m)$, we have

$$\mathbb{P}[S_n = 2m] = \mathbb{P}[Z(2m) = 2m] = e^{-2m} \frac{(2m)^{2m}}{(2m)!} \geq \text{const.} \times m^{-1/2}.$$

Also, let k be such that $\mathbb{E}\left[\sum_{i=k}^{\infty} i^2 v_i\right] < \varepsilon n/2$, where $v_i = |\{j \in [n] \; : \; Z_j(\theta) = i\}|$ and note that $\mathbb{E}\left[\sum_{i=k}^{\infty} i^2 v_i\right] = \Theta(n)$. If $X''_{n,k} = \sum_{i=k}^{\infty} i^2 v_i$, then

$$\mathbb{P}\left[X''_{n,k} \geq \varepsilon n\right] \leq \mathbb{P}\left[X''_{n,k} > 2\mathbb{E}\left[X''_{n,k}\right]\right].$$

Moreover, we set $c = 2\mathbb{E}[Z^4(\theta)]$. Therefore,

$$P'_i = O(n^{1/2} P''_i), \tag{5.6}$$

where

$$P''_1 \;=\; \mathbb{P}\left[\max_{1\leq j\leq n} Z_j(\theta) > \Delta(n)\right],$$

$$\begin{aligned} P_2'' &= \mathbb{P}\left[X_{n,k}'' > 2\mathbb{E}\left[X_{n,k}''\right]\right], \\ P_3'' &= \mathbb{P}\left[\sum_{j=1}^{n} Z_j^4(\theta) > 2\mathbb{E}[Z^4(\theta)].\right] \end{aligned}$$

We have

$$P_1'' \leq n\mathbb{P}[Z(\theta) > \Delta(n)] = n \sum_{r>\Delta(n)} e^{-\theta}\frac{\theta^r}{r!} = o\left(\frac{1}{n^4}\right). \tag{5.7}$$

Now, for each $j = 1, \ldots, n$ we define

$$Y_j = \begin{cases} Z_j(\theta) & k \leq Z_j(\theta) \leq \Delta(n) \\ 0 & \text{otherwise} \end{cases},$$

and note that $\{Y_j\}_{j\in[n]}$ is independent. Then, $X_{n,k}'' = \sum_{j=1}^n Y_j^2$, if $Z_j(\theta) \leq \Delta(n)$ for each $j = 1, \ldots, n$. Also, note that $X_{n,k}'' \geq \sum_{j=1}^n Y_j^2$ always. Using the bounded differences lemma (see Corollary 2.27 in [23] or [30]) we obtain:

$$\begin{aligned} &\mathbb{P}\left[X_{n,k}'' > 2\mathbb{E}[X_{n,k}''] \ \cap \ \left(\cap_{j=1}^n Z_j(\theta) \leq \Delta(n)\right)\right] = \\ &\quad \mathbb{P}\left[\sum_{j=1}^n Y_j^2 > 2\mathbb{E}[X_{n,k}''] \ \cap \ \left(\cap_{j=1}^n Z_j(\theta) \leq \Delta(n)\right)\right] \\ &\leq \mathbb{P}\left[\sum_{j=1}^n Y_j^2 > 2\mathbb{E}\left[\sum_{j=1}^n Y_j^2\right] \ \cap \ \left(\cap_{j=1}^n Z_j(\theta) \leq \Delta(n)\right)\right] \\ &\leq \mathbb{P}\left[\sum_{j=1}^n Y_j^2 > 2\mathbb{E}\left[\sum_{j=1}^n Y_j^2\right]\right] \\ &\leq \exp\left(-\frac{\mathbb{E}\left[\sum_{j=1}^n Y_j^2\right]^2}{2n\Delta^4(n)}\right) = \exp\left(-\Theta\left(\frac{n}{\ln^4 n}\right)\right). \end{aligned}$$

Thus, by (5.7) we obtain

$$\begin{aligned} P_2'' &\leq \mathbb{P}\left[X_{n,k}'' > 2\mathbb{E}[X_{n,k}''] \ \cap \ \left(\cap_{j=1}^n Z_j(\theta) \leq \Delta(n)\right)\right] + \mathbb{P}\left[\max_{1\leq j\leq n} Z_j(\theta) > \Delta(n)\right] \\ &\leq \exp\left(-\Theta\left(\frac{n}{\ln^4 n}\right)\right) + o\left(\frac{1}{n^4}\right) = o\left(\frac{1}{n^4}\right). \end{aligned} \tag{5.8}$$

Finally, using Chernoff's bound we obtain

$$P_3'' \leq \exp(-dn), \tag{5.9}$$

for some $d > 0$. The estimates (5.5)-(5.9) along with (5.4) imply the statement of the lemma. ∎

Now, for $n \geq 16$ we let $\omega(n) = \lfloor \ln \ln n \rfloor$ and for $1 \leq n \leq 15$ we let $\omega(n) = 2$. For a graph G on V_n, let $S(G)$ be the number of subgraphs of order at most $\max\{\omega(n), 2\}$ with their size exceeding their order. Then the following holds:

Lemma 5.6.2 *For any* $(\mathbf{d}_n) \in G$*, we have*

$$\mathbb{E}[S(G(\mathbf{d}_n))] = o(1).$$

Proof. For n sufficiently large, if $l \in \mathbb{Z}^+$, then for every $2 \leq k \leq \omega(n)$ the expected number of subgraphs of $G(\mathbf{d}_n)$ of order k and size $k+l$ is, by Lemma 5.2.5, at most

$$\begin{aligned}
& \binom{n}{k} \binom{\binom{k}{2}}{k+l} \frac{(2M-2k-2l)!}{(M-k-l)! 2^{M-k-l}} \frac{M! 2^M}{(2M)!} \max_{\{\text{graphs on } [k]\}} \max_{[V_n]^k} \frac{\prod_{i=1}^n d_i!}{\prod_{i=1}^n d_i'!} \exp(-\lambda' - \lambda'^2 + \lambda + \lambda^2) \\
\leq\ & \frac{n^k}{k!} \left(\frac{k^2 e}{k+l}\right)^{k+l} \frac{2^{k+l} (M)_{k+l}}{(2M)_{2k+2l}} \max_{\{\text{graphs on } [k]\}} \max_{[V_n]^k} \frac{\prod_{i=1}^n d_i!}{\prod_{i=1}^n d_i'!} \exp(\lambda + \lambda^2) \\
=\ & \frac{n^k}{k!} \left(\frac{k^2 e}{k+l}\right)^{k+l} \frac{2^{k+l} M^{k+l}}{(2M)^{2k+2l}} \max_{\{\text{graphs on } [k]\}} \max_{[V_n]^k} \frac{\prod_{i=1}^n d_i!}{\prod_{i=1}^n d_i'!} \exp(\lambda + \lambda^2) \ (1+o(1)) \\
=\ & \frac{n^k}{k!} \left(\frac{k^2 e}{k+l}\right)^{k+l} \frac{2^{-k-l}}{M^{k+l}} \max_{\{\text{graphs on } [k]\}} \max_{[V_n]^k} \frac{\prod_{i=1}^n d_i!}{\prod_{i=1}^n d_i'!} \exp(\lambda + \lambda^2) \ (1+o(1)) \\
\leq\ & \Theta(1) \left(\frac{e^2}{2}\right)^k \left(\frac{ke}{2}\right)^{k^2} \frac{1}{M} \left(\frac{(\lceil \ln n \rceil)!}{(\lceil \ln n \rceil - \lceil \ln \ln n \rceil)!}\right)^k \\
\leq\ & \Theta(1) \left(\frac{e^2 (\ln n)^{\ln \ln n}}{2}\right)^k \left(\frac{e \ln \ln n}{2}\right)^{k \ln \ln n} \frac{1}{M},
\end{aligned}$$

since $(\mathbf{d}_n) \in G$, where λ is defined as in Lemma 5.2.5 and λ' is the corresponding quantity for the sequence $(d_i')_{i=1,\ldots,n}$, which is the labelled degree sequence that is left after the removal of the edges of the subgraph on the chosen k vertices. Therefore, since $M = \Theta(n)$ we obtain:

$$\begin{aligned}
\mathbb{E}[S(G(\mathbf{d}_n))] &\leq \frac{\ln^2 \ln n}{M} \sum_{k=2}^{\lfloor \ln \ln n \rfloor} \left(\frac{e^2 (\ln n)^{\ln \ln n}}{2}\right)^k \left(\frac{e \ln \ln n}{2}\right)^{k \ln \ln n} \\
&= O\left(\left(\frac{e^2 (\ln n)^{\ln \ln n}}{2}\right)^{\ln \ln n + 1} \left(\frac{e \ln \ln n}{2}\right)^{\ln^2 \ln n + \ln \ln n} \frac{\ln^2 \ln n}{M}\right) = o(1).
\end{aligned}$$

∎

5.7 Limiting properties of the degree sequences in G

Finally, we prove the following technical statement which was used in Sections 5.3, 5.4:

Lemma 5.7.1 *For any* $(\mathbf{d}_n) \in G$ *the following hold:*

1. *For any integer* $k \geq 2$*, we have*

$$\lim_{n\to\infty} \sum_{i=k}^{\Delta} \binom{i}{2} \frac{D_i}{M} \quad = \quad \theta \left(1 - \exp(-\lambda_2(\theta)) \sum_{i=0}^{k-3} \lambda_2^i(\theta) \right).$$

If $k = 2$*, then we assume that the sum on the right hand side is equal to 0.*

2. *We have*

$$\lim_{n\to\infty} \sum_{i=2}^{\Delta} \frac{D_i}{n} = \sum_{i=2}^{\infty} \rho_i(\lambda_2(\theta)) = p_2(\lambda_2(\theta)).$$

3. *For every function* $\omega(n)$ *that tends to infinity as* $n \to \infty$*, we have*

$$\lim_{n\to\infty} \sum_{k=2}^{\omega(n)} \left(\frac{D_2}{M}\right)^k = \sum_{k=2}^{\infty} \left(\theta e^{-\lambda_2(\theta)}\right)^k = \frac{\left(\theta e^{-\lambda_2(\theta)}\right)^2}{1 - \theta e^{-\lambda_2(\theta)}},$$

and

$$\lim_{n\to\infty} \sum_{k=3}^{\omega(n)} (k-2) \left(\frac{D_2}{M}\right)^{k-1} = \sum_{k=3}^{\infty} (k-2) \left(\theta e^{-\lambda_2(\theta)}\right)^{k-1}.$$

Proof. We prove the first part. For every integer $k_0 \geq k$, we have for n sufficiently large

$$\sum_{i=k}^{k_0} \binom{i}{2} \frac{D_i}{M} \leq \sum_{i=k}^{\Delta} \binom{i}{2} \frac{D_i}{M}.$$

Thus,

$$\lim_{n\to\infty} \sum_{i=k}^{k_0} \binom{i}{2} \frac{D_i}{M} \leq \liminf_{n\to\infty} \sum_{i=k}^{\Delta} \binom{i}{2} \frac{D_i}{M},$$

where

$$\lim_{n\to\infty} \sum_{i=k}^{k_0} \binom{i}{2} \frac{D_i}{M} = \sum_{i=k}^{k_0} \binom{i}{2} \frac{2\theta e^{-\lambda_2(\theta)} \lambda_2^i(\theta)}{i! \lambda_2^2(\theta)} = \theta e^{-\lambda_2(\theta)} \sum_{i=k-2}^{k_0-2} \frac{\lambda_2(\theta)^i}{i!}.$$

Since this is true for any integer $k_0 \geq 2$, we deduce that

$$\theta \left(1 - \exp(-\lambda_2(\theta)) \sum_{i=0}^{k-3} \lambda_2^i(\theta) \right) = \theta e^{-\lambda_2(\theta)} \sum_{i=k-2}^{\infty} \frac{\lambda_2(\theta)^i}{i!} \leq \liminf_{n\to\infty} \sum_{i=k}^{\Delta} \binom{i}{2} \frac{D_i}{M}.$$

On the other hand, since $(\mathbf{d}_n) \in G$, for every $\varepsilon > 0$ there exist $k_0, N \in \mathbb{N}$ such that for every $n > N$, we have

$$\sum_{i=k}^{\Delta} \binom{i}{2} \frac{D_i}{M} \leq \sum_{i=k}^{k_0} \binom{i}{2} \frac{D_i}{M} + \varepsilon.$$

Therefore,

$$\begin{aligned}\limsup_{n\to\infty}\sum_{i=k}^{\Delta}\binom{i}{2}\frac{D_i}{M} &\le \lim_{n\to\infty}\sum_{i=k}^{k_0}\binom{i}{2}\frac{D_i}{M}+\varepsilon=\sum_{i=k}^{k_0}\binom{i}{2}\frac{2\theta e^{-\lambda_2(\theta)}\lambda_2^i(\theta)}{i!\lambda_2^2(\theta)}+\varepsilon\\ &\le \sum_{i=k}^{\infty}\binom{i}{2}\frac{2\theta e^{-\lambda_2(\theta)}\lambda_2^i(\theta)}{i!\lambda_2^2(\theta)}+\varepsilon=\theta\left(1-\exp(-\lambda_2(\theta))\sum_{i=0}^{k-3}\lambda_2^i(\theta)\right)+\varepsilon.\end{aligned}$$

Since ε is arbitrary, we deduce that

$$\begin{aligned}\theta\left(1-\exp(-\lambda_2(\theta))\sum_{i=0}^{k-3}\lambda_2^i(\theta)\right) &\le \liminf_{n\to\infty}\sum_{i=k}^{\Delta}\binom{i}{2}\frac{D_i}{M}\\ &\le \limsup_{n\to\infty}\sum_{i=k}^{\Delta}\binom{i}{2}\frac{D_i}{M}\le\theta\left(1-\exp(-\lambda_2(\theta))\sum_{i=0}^{k-3}\lambda_2^i(\theta)\right),\end{aligned}$$

which concludes the proof of the first part of the lemma.

The proof of the second part of the lemma is nearly identical to the above one. For every integer $k_0 \ge 2$ and for n sufficiently large, we have

$$\sum_{i=2}^{k_0}\frac{D_i}{n}\le\sum_{i=2}^{\Delta}\frac{D_i}{n}.$$

Thus,

$$\lim_{n\to\infty}\sum_{i=2}^{k_0}\frac{D_i}{n}\le\liminf_{n\to\infty}\sum_{i=2}^{\Delta}\frac{D_i}{n},$$

where

$$\lim_{n\to\infty}\sum_{i=2}^{k_0}\frac{D_i}{n}=\sum_{i=2}^{k_0}\rho_i(\lambda_2(\theta)).$$

Since this is true for any integer $k_0 \ge 2$, we deduce that

$$\sum_{i=2}^{\infty}\rho_i(\lambda_2(\theta))\le\liminf_{n\to\infty}\sum_{i=k}^{\Delta}\frac{D_i}{n}.$$

Now, since $(\mathbf{d}_n)\in G$, for every $\varepsilon>0$ there exist $k_0, N\in\mathbb{N}$ such that for every $n>N$, we have

$$\sum_{i=2}^{\Delta}\frac{D_i}{n}\le\sum_{i=2}^{k_0}\frac{D_i}{n}+\sum_{k_0+1}^{\Delta}\binom{i}{2}\frac{D_i}{n}\le\sum_{i=2}^{k_0}\frac{D_i}{n}+\varepsilon.$$

Hence,

$$\begin{aligned}\limsup_{n\to\infty}\sum_{i=2}^{\Delta}\frac{D_i}{n} &\le \lim_{n\to\infty}\sum_{i=2}^{k_0}\frac{D_i}{n}+\varepsilon=\sum_{i=2}^{k_0}\rho_i(\lambda_2(\theta))+\varepsilon\\ &\le \sum_{i=2}^{\infty}\rho_i(\lambda_2(\theta))+\varepsilon,\end{aligned}$$

for every $\varepsilon > 0$. Therefore,

$$\limsup_{n\to\infty} \sum_{i=2}^{\Delta} \frac{D_i}{n} \leq \sum_{i=2}^{\infty} \rho_i(\lambda_2(\theta)),$$

which along with the previous analysis yields

$$\sum_{i=2}^{\infty} \rho_i(\lambda_2(\theta)) \leq \liminf_{n\to\infty} \sum_{i=2}^{\Delta} \frac{D_i}{n} \leq \limsup_{n\to\infty} \sum_{i=2}^{\Delta} \frac{D_i}{n} \leq \sum_{i=2}^{\infty} \rho_i(\lambda_2(\theta)),$$

and this concludes the proof of the second part of the lemma.

We now proceed with the third part of the lemma. We shall prove the first statement, as the proof of the second one is identical. For every integer $k_0 \geq 2$ and for n sufficiently large, we have

$$\sum_{k=2}^{k_0} \left(\frac{D_2}{M}\right)^k \leq \sum_{k=2}^{\omega(n)} \left(\frac{D_2}{M}\right)^k.$$

Thus,

$$\liminf_{n\to\infty} \sum_{k=2}^{\Delta} \left(\frac{D_2}{M}\right)^k \geq \lim_{n\to\infty} \sum_{k=2}^{k_0} \left(\frac{D_2}{M}\right)^k = \sum_{k=2}^{k_0} \left(\frac{2\theta e^{-\lambda_2(\theta)}\lambda_2^2(\theta)}{2\lambda_2^2(\theta)}\right)^k = \sum_{k=2}^{k_0} \left(\theta e^{-\lambda_2(\theta)}\right)^k.$$

Since this is true for any integer $k_0 \geq 2$, we deduce that

$$\sum_{k=2}^{\infty} \left(\theta e^{-\lambda_2(\theta)}\right)^k \leq \liminf_{n\to\infty} \sum_{k=2}^{\omega(n)} \left(\frac{D_2}{M}\right)^k.$$

On the other hand, there exists a real $0 < \lambda < 1$ and $N \in \mathbb{N}$, such that for every $n > N$, we have $D_2/M < \lambda$. Therefore, for any $\varepsilon > 0$, there exists an integer $k_0 \geq 2$ such that

$$\sum_{k=2}^{\omega(n)} \left(\frac{D_2}{M}\right)^k \leq \sum_{k=2}^{k_0} \left(\frac{D_2}{M}\right)^k + \sum_{k=k_0+1}^{\omega(n)} \lambda^k \leq \sum_{k=2}^{k_0} \left(\frac{D_2}{M}\right)^k + \sum_{k=k_0+1}^{\infty} \lambda^k \leq \sum_{k=2}^{k_0} \left(\frac{D_2}{M}\right)^k + \varepsilon,$$

for every $n > N$. Therefore,

$$\limsup_{n\to\infty} \sum_{k=2}^{\omega(n)} \left(\frac{D_2}{M}\right)^k \leq \lim_{n\to\infty} \sum_{k=2}^{k_0} \left(\frac{D_2}{M}\right)^k + \varepsilon \leq \sum_{k=2}^{k_0} \left(\theta e^{-\lambda_2(\theta)}\right)^k + \varepsilon \leq \sum_{k=2}^{\infty} \left(\theta e^{-\lambda_2(\theta)}\right)^k + \varepsilon.$$

Since ε is arbitrary, we deduce that

$$\limsup_{n\to\infty} \sum_{k=2}^{\omega(n)} \left(\frac{D_2}{M}\right)^k \leq \sum_{k=2}^{\infty} \left(\theta e^{-\lambda_2(\theta)}\right)^k,$$

which concludes the proof of the first formula of the third part of the lemma. ∎

5.8 Discussion

This chapter continues the study of the structure of the core of a $\mathcal{G}_{n,m}$ random graph, that started in [36], giving the asymptotic distributions of the 2-vertex and 2-edge-connected components of it as well as the distributions of small cycles which are either isolated or not there. Thus, it is natural to turn our attention to the essential core, since this is the dominant 2-vertex-connected component of the giant component. It seems that there is a series of natural questions concerning the essential core of a $\mathcal{G}_{n,m}$ random graph above the critical point that remain to be answered. For example, is it a.a.s. Hamiltonian? What is the exact connectivity of its kernel? Does it have a perfect matching a.a.s.? Also, there are questions concerning the k-core, for any $k \geq 3$, after its appearance. It is known that it is a.a.s. k-vertex-connected [28], but for example is it a.a.s. Hamiltonian? How does it look like in general? All these questions are open for further research.

Appendix A

Some calculations from subsection 2.6.3

Recall that in subsection 2.6.3 we set $x_0(u) = x_0(r,t)$ to be the root of the following equation:

$$E_u(x) = 1 - e^{-x} - ux = 0, \tag{A.1}$$

for $0 < u = u(r,t) \leq 1$.

Suppose that y is either r or t. We have

$$e^{-x_0} \cdot \frac{\partial x_0}{\partial y} - \frac{\partial u}{\partial y} \cdot x_0 - u \cdot \frac{\partial x_0}{\partial y} = 0.$$

Therefore,

$$\frac{\partial x_0}{\partial y} = \frac{(\partial u/\partial y)x_0}{e^{-x_0} - u}. \tag{A.2}$$

Again, using (A.1) we have

$$e^{x_0} - 1 - e^{x_0}ux_0 = 0.$$

So,

$$u = e^{-x_0}(e^{x_0} - 1)x_0^{-1}.$$

Thus, (A.2) becomes

$$\begin{aligned} \frac{\partial x_0}{\partial y} &= \frac{(\partial u/\partial y)x_0}{e^{-x_0} - e^{-x_0}(e^{x_0} - 1)x_0^{-1}} \\ &= \frac{e^{x_0}(\partial u/\partial y)x_0^2}{1 + x_0 - e^{x_0}}. \end{aligned} \tag{A.3}$$

The denominator is negative, since $1 + x_0 < e^{x_0}$. Thus, the sign of this expression depends upon the sign of $\partial u/\partial y$.

Now, we shall give a lower bound on $f(t,r)$ and we will study its monotonicity. In fact we shall work with

$$f(t,r)^2 = f(u)^2 = \frac{1-u}{x_0 - \frac{1}{u} + 1},$$

where $u = r/t$. We have

$$\begin{aligned} f(u)^2 &= \frac{1-u}{x_0 - \frac{1}{u} + 1} = \frac{1-u}{x_0 - \frac{1-u}{u}} \\ &= \frac{u(1-u)}{ux_0 - (1-u)} > \frac{u(1-u)}{ux_0} = \frac{1-u}{x_0}. \end{aligned}$$

Thus

$$f(u)^2 > \frac{1-u}{x_0}. \tag{A.4}$$

In what follows, we are trying to investigate the monotonicity of the latter function. The derivative of this with respect to u is

$$\left(\frac{1-u}{x_0}\right)' = \frac{-x_0 - (1-u)\frac{\partial x_0}{\partial u}}{x_0^2}.$$

We have to determine the sign of the numerator. Note that

$$\frac{\partial x_0}{\partial u} = \frac{x_0}{e^{-x_0} - u}.$$

Therefore, we have

$$\begin{aligned} -x_0 - (1-u)\frac{\partial x_0}{\partial u} &= -x_0\left(1 + (1-u)\frac{1}{e^{-x_0} - u}\right) = -x_0\left(1 + \frac{1-u}{1 - ux_0 - u}\right) \\ &= -x_0\left(1 + \frac{1}{1 - \frac{u}{1-u}x_0}\right). \end{aligned}$$

The expression on the right hand side is 0 if and only if,

$$\begin{aligned} \frac{1}{1 - \frac{u}{1-u}x_0} &= -1 \quad \Leftrightarrow \\ 1 = x_0\frac{u}{1-u} &- 1 \quad \Leftrightarrow \\ 2 = \frac{x_0 u}{1-u} \Leftrightarrow x_0 &= \frac{2(1-u)}{u}. \end{aligned}$$

This yields

$$1 - e^{-\frac{2(1-u)}{u}} - 2(1-u) = 0.$$

But the function $1 - e^{-\frac{2(1-u)}{u}} - 2(1-u)$ is strictly less than zero for $u < 1$, and its only root is $u = 1$. This follows from the monotonicity of $1 - e^{-\frac{2(1-u)}{u}} - 2(1-u)$ with respect to u - this is increasing with respect to u. Thus, the function $((1-u)/x_0)'$ is either positive or negative for any $0 < u < 1$.

Note, that for $u = (1-e^{-2})/2$, we have $x_0 = 2$. Since $u < 1/2$, we have $x_0 < 2(1-u)/u$, which implies that $1 + \frac{1}{1-\frac{u}{1-u}x_0} < 0$, for $0 < u < 1$. Hence, the derivative of $(1-u)/x_0$ with respect to u is positive, and, therefore, this is a strictly increasing function with respect to u. Thus, for $0 < u \leq 1$, this is an increasing function. We will use this fact to obtain a lower bound for $f(t, r)$ for specific t, r.

Appendix B

Some calculations from subsection 2.6.4

In this appendix we present the elementary but tedious calculations for the function $P'(k,\alpha,\lambda,\theta)$ on $\mathcal{D}$, as it was defined in subsection 2.6.4:

$$\frac{P'(k,\alpha,\lambda,\theta)}{f\left(\frac{2(1-\alpha)}{\theta\lambda}\right)\ f\left(\frac{2(1-k)}{\theta(1-\lambda)}\right)} =$$

$$= \quad \frac{n^{(2-\alpha-k)}(1-\alpha)^{(1-\alpha)}(1-k)^{(1-k)}e^{-(2-\alpha-k)n}}{n^{\frac{\theta}{2}}(1-k)^{\frac{\theta}{2}}} \cdot 2^{\frac{\theta}{2}H(\lambda)} \cdot \left(\frac{1-k}{1-\alpha}\right)^{\frac{\lambda\theta}{2}} \times$$

$$n^{\left(\frac{\theta\lambda}{2}+\alpha-1\right)} \cdot t(1-\alpha,\lambda\theta/2) \cdot n^{\left(\frac{\theta}{2}-\frac{\theta\lambda}{2}-1+k\right)} \cdot t(1-k,(1-\lambda)\theta/2) \times$$

$$\left(\frac{(1-k)(k-\alpha)}{\phi(k,\alpha)}\right)^{\frac{\theta}{2}(1-\lambda)} \cdot \left(1-\frac{(1-k)(k-\alpha)}{\phi(k,\alpha)}\right)^{\frac{\lambda\theta}{2}}$$

$$= \quad \frac{(1-\alpha)^{(1-\alpha)}(1-k)^{(1-k)}e^{-(2-\alpha-k)}}{(1-k)^{\frac{\theta}{2}}} \cdot 2^{\frac{\theta}{2}H(\lambda)} \cdot \left(\frac{1-k}{1-\alpha}\right)^{\frac{\lambda\theta}{2}} \times$$

$$t(1-\alpha,\lambda\theta/2) \cdot t(1-k,(1-\lambda)\theta/2) \cdot \left(\frac{(1-k)(k-\alpha)}{\phi(k,\alpha)}\right)^{\frac{\theta}{2}(1-\lambda)} \cdot \left(1-\frac{(1-k)(k-\alpha)}{\phi(k,\alpha)}\right)^{\frac{\lambda\theta}{2}},$$

where

$$t(x,y) = (e^{x_0}-1)^x \cdot (x_0)^{-y} \cdot e^{-(y-x)} \cdot (y-x)^{(y-x)} \cdot \left(\frac{y}{x}\right)^x \cdot \left(\frac{y}{y-x}\right)^{(y-x)},$$

x_0 being the positive root of the equation $1-e^{-t}-(x/y)t=0$, whenever $0<x/y\leq 1$. Thus, for $(k,\alpha,\lambda)\in\mathcal{D}$ we have

$$\frac{P'(k,\alpha,\lambda,\theta)}{f\left(\frac{2(1-\alpha)}{\theta\lambda}\right)\ f\left(\frac{2(1-k)}{\theta(1-\lambda)}\right)} =$$

$$
\begin{aligned}
= \quad & \frac{(1-\alpha)^{(1-\alpha)}(1-k)^{(1-k)}e^{-(2-\alpha-k)}}{(1-k)^{\frac{\theta}{2}}} \cdot 2^{\frac{\theta}{2}H(\lambda)} \cdot \left(\frac{1-k}{1-\alpha}\right)^{\frac{\lambda\theta}{2}} \times \\
& (e^{x_1}-1)^{(1-\alpha)} \cdot (x_1)^{-\frac{\lambda\theta}{2}} \cdot e^{-\left(\frac{\theta\lambda}{2}+\alpha-1\right)} \cdot \left(\frac{\theta\lambda}{2}+\alpha-1\right)^{\left(\frac{\theta\lambda}{2}+\alpha-1\right)} \times \\
& \left(\frac{\lambda\theta}{2(1-\alpha)}\right)^{(1-\alpha)} \cdot \left(\frac{\lambda\frac{\theta}{2}}{\frac{\theta\lambda}{2}+\alpha-1}\right)^{\left(\frac{\theta\lambda}{2}+\alpha-1\right)} \times \\
& (e^{x_2}-1)^{(1-k)} \cdot (x_2)^{-\frac{\theta}{2}(1-\lambda)} \cdot e^{-\left(\frac{\theta}{2}-\frac{\theta\lambda}{2}-1+k\right)} \cdot \times \\
& \left(\frac{\theta}{2}-\frac{\theta\lambda}{2}-1+k\right)^{\left(\frac{\theta}{2}-\frac{\theta\lambda}{2}-1+k\right)} \cdot \left(\frac{\theta(1-\lambda)}{2(1-k)}\right)^{(1-k)} \cdot \left(\frac{\frac{\theta(1-\lambda)}{2}}{\left(\frac{\theta}{2}-\frac{\theta\lambda}{2}-1+k\right)}\right)^{\left(\frac{\theta}{2}-\frac{\theta\lambda}{2}-1+k\right)} \times \\
& \left(\frac{(1-k)(k-\alpha)}{\phi(k,\alpha)}\right)^{\frac{\theta}{2}(1-\lambda)} \cdot \left(1-\frac{(1-k)(k-\alpha)}{\phi(k,\alpha)}\right)^{\frac{\lambda\theta}{2}}, \\
= \quad & (1-k)^{-\frac{\theta}{2}} \cdot 2^{\frac{\theta}{2}H(\lambda)} \cdot \left(\frac{1-k}{1-\alpha}\right)^{\frac{\lambda\theta}{2}} \times \\
& (e^{x_1}-1)^{(1-\alpha)} \cdot (e^{x_2}-1)^{(1-k)} \cdot (x_1)^{-\frac{\lambda\theta}{2}} \cdot (x_2)^{-\frac{\theta}{2}(1-\lambda)} \cdot e^{-\frac{\theta}{2}} \times \\
& \left(\frac{\lambda\theta}{2}\right)^{\frac{\theta\lambda}{2}} \cdot \left(\frac{\theta(1-\lambda)}{2}\right)^{\frac{\theta(1-\lambda)}{2}} \cdot \left(\frac{(1-k)(k-\alpha)}{\phi(k,\alpha)}\right)^{\frac{\theta}{2}(1-\lambda)} \cdot \left(1-\frac{(1-k)(k-\alpha)}{\phi(k,\alpha)}\right)^{\frac{\lambda\theta}{2}},
\end{aligned}
$$

where x_1 is the positive root of the equation

$$
E_1(x) = 1 - e^{-x} - \frac{2(1-\alpha)}{\theta\lambda}x = 0 \tag{B.1}
$$

and x_2 is the positive root of the equation

$$
E_2(x) = 1 - e^{-x} - \frac{2(1-k)}{\theta(1-\lambda)}x = 0. \tag{B.2}
$$

Appendix C

The concavity of $h(k, \alpha, \lambda)$ over $\mathcal{D}$

In this appendix, we show that the function $h(k, \alpha, \lambda)$ as it was defined in Section 2.8 is strictly concave over the interior of $\mathcal{D}$, where $\mathcal{D}$ was defined in (2.26). (Recall that the function $h(k, \alpha, \lambda, \theta)$ was defined in (2.27), and we have fixed $\theta = \theta_2 = 4.9895$ to obtain $h(k, \alpha, \lambda)$.) This treatment was inspired by the correction [19] to [24]. We split the function $h(k, \alpha, \lambda)$ (multiplied by $\ln 2$ to change to natural logarithms) into four parts. Namely, for any $(k, \alpha, \lambda) \in \mathcal{D}$, we have

$$\ln 2 \; h(k, \alpha, \lambda) = h_1(k, \alpha, \lambda) + h_2(k, \alpha, \lambda) + h_3(k, \alpha, \lambda) + \frac{\theta_2}{2} h_4(k, \alpha, \lambda),$$

where

$$\begin{aligned}
h_1(k, \alpha, \lambda) &= -(1-k)\ln(1-k) - (k-\alpha)\ln(k-\alpha) - \alpha \ln \alpha \\
&\quad + \left(\frac{\theta_2}{2} - \frac{\theta_2}{2}\log_2(e) + \frac{\theta_2}{2}\log_2\left(\frac{\theta_2}{2}\right)\right)\ln 2, \\
h_2(k, \alpha, \lambda) &= (1-k)\ln(e^{x_2} - 1) - \frac{\theta_2}{2}(1-\lambda)\ln x_2 - \frac{\theta_2(1-\lambda)}{2}\ln(1-k) + \frac{\theta_2(1-\lambda)}{2}\ln(1-\lambda), \\
h_3(k, \alpha, \lambda) &= (1-\alpha)\ln(e^{x_1} - 1) - \frac{\lambda\theta_2}{2}\ln x_1 - \frac{\lambda\theta_2}{2}\ln(1-\alpha) + \frac{\lambda\theta_2}{2}\ln \lambda, \\
h_4(k, \alpha, \lambda) &= -\lambda \ln \lambda - (1-\lambda)\ln(1-\lambda) + (1-\lambda)\ln(1-k) \\
&\quad + (1-\lambda)\ln(k-\alpha) + \lambda \ln \alpha + \lambda \ln(1-\alpha),
\end{aligned}$$

where x_1 and x_2 were defined in (B.1) and (B.2), respectively. For $i = 1, \ldots, 4$, we set $h_i = h_i(k, \alpha, \lambda)$. We prove that each of these functions is concave over the interior of $\mathcal{D}$, with h_4 being strictly concave over the interior of $\mathcal{D}$. To prove that a function is concave (strictly concave, respectively) over an open domain, we have to prove that its Hessian matrix (i.e. the matrix of the second partial derivatives) is negative semidefinite (definite, respectively) over this domain (see for example [40], Theorem 5.5.5 p. 230). By, for example, Theorem 6E in [38] (p.339), to check negative semidefiniteness (definiteness, respectively) of a real symmetric matrix it is necessary and sufficient to show that the principal minors have alternating signs (and are non-zero in the case of definiteness), the first one being non-positive (strictly negative, respectively). Thus, we may deal with $h_1, \ldots, h_4$ as follows.

1. Observe that h_1 does not depend on λ. It can be easily checked that h_1 is concave over the interior of $\mathcal{D}$. Its elements are the following:

$$\begin{aligned}
\frac{\partial^2 h_1}{\partial k^2} &= -\frac{1}{(1-k)} - \frac{1}{(k-\alpha)} < -\frac{1}{(k-\alpha)}, \\
\frac{\partial^2 h_1}{\partial k \partial \alpha} &= \frac{1}{k-\alpha}, \\
\frac{\partial^2 h_1}{\partial \alpha^2} &= -\frac{1}{\alpha} - \frac{1}{k-\alpha} < -\frac{1}{(k-\alpha)}.
\end{aligned}$$

Thus $\frac{\partial^2 h_1}{\partial k^2} < 0$ and $\frac{\partial^2 h_1}{\partial k^2}\frac{\partial^2 h_1}{\partial \alpha^2} - \left(\frac{\partial^2 h_1}{\partial k \partial \alpha}\right)^2 > 0$, and so is concave over $\mathcal{D}$.

2. Next we consider the function h_2, which does not depend on α. Using (B.2), we have:

$$\begin{aligned}
\frac{\partial h_2}{\partial k} &= -\ln(e^{x_2}-1) + \frac{(1-k)e^{x_2}}{e^{x_2}-1}\frac{\partial x_2}{\partial k} - \frac{\theta_2}{2}(1-\lambda)\frac{1}{x_2}\frac{\partial x_2}{\partial k} + \frac{\theta_2(1-\lambda)}{2(1-k)} \\
&= -\ln(e^{x_2}-1) + \frac{\theta_2(1-\lambda)}{2(1-k)},
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial h_2}{\partial \lambda} &= \frac{(1-k)e^{x_2}}{e^{x_2}-1}\frac{\partial x_2}{\partial \lambda} - \frac{\theta_2}{2}(1-\lambda)\frac{1}{x_2}\frac{\partial x_2}{\partial \lambda} + \frac{\theta_2}{2}\ln x_2 + \frac{\theta_2}{2}\ln(1-k) - \frac{\theta_2}{2}\ln(1-\lambda) - \frac{\theta_2}{2} \\
&= \frac{\theta_2}{2}\ln x_2 + \frac{\theta_2}{2}\ln(1-k) - \frac{\theta_2}{2}\ln(1-\lambda) - \frac{\theta_2}{2}.
\end{aligned}$$

Therefore, by (A.3) we obtain:

$$\begin{aligned}
\frac{\partial^2 h_2}{\partial k^2} &= -\frac{e^{x_2}}{e^{x_2}-1}\frac{\partial x_2}{\partial k} + \frac{1-\lambda}{(1-k)^2}\frac{\theta_2}{2} \\
&= \frac{e^{x_2}}{e^{x_2}-1}\frac{e^{x_2}x_2^2}{1+x_2-e^{x_2}}\frac{2}{\theta_2(1-\lambda)} + \frac{1-\lambda}{(1-k)^2}\frac{\theta_2}{2} \\
&= \frac{1}{1-k}\left(\frac{e^{x_2}}{e^{x_2}-1}\frac{e^{x_2}x_2^2}{1+x_2-e^{x_2}}\frac{2(1-k)}{\theta_2(1-\lambda)} + \frac{\theta_2(1-\lambda)}{2(1-k)}\right) \\
&= \frac{1}{1-k}\left(\frac{e^{x_2}}{e^{x_2}-1}\frac{e^{x_2}x_2^2}{1+x_2-e^{x_2}}\frac{e^{x_2}-1}{x_2e^{x_2}} + \frac{x_2e^{x_2}}{e^{x_2}-1}\right) \\
&= \frac{1}{1-k}\left(\frac{e^{x_2}x_2}{1+x_2-e^{x_2}} + \frac{x_2e^{x_2}}{e^{x_2}-1}\right) \\
&= \frac{1}{1-k}\frac{e^{x_2}}{e^{x_2}-1}\frac{x_2^2}{1+x_2-e^{x_2}} = \frac{\theta_2}{2}\frac{1-\lambda}{(1-k)^2}\frac{x_2}{1+x_2-e^{x_2}}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{\partial^2 h_2}{\partial \lambda^2} &= \frac{\theta_2}{2}\frac{1}{x_2}\frac{\partial x_2}{\partial \lambda} + \frac{\theta_2}{2(1-\lambda)} = \frac{\theta_2}{2}\frac{1}{x_2}\frac{x_2^2 e^{x_2}}{1+x_2-e^{x_2}}\frac{2(1-k)}{\theta_2(1-\lambda)^2} + \frac{\theta_2}{2(1-\lambda)} \\
&= \frac{\theta_2}{2}\frac{1}{x_2}\frac{e^{x_2}x_2^2}{1+x_2-e^{x_2}}\frac{e^{x_2}-1}{x_2 e^{x_2}}\frac{1}{1-\lambda} + \frac{\theta_2}{2(1-\lambda)} \\
&= \frac{\theta_2}{2(1-\lambda)}\frac{e^{x_2}-1}{1+x_2-e^{x_2}} + \frac{\theta_2}{2(1-\lambda)} \\
&= \frac{\theta_2}{2(1-\lambda)}\frac{e^{x_2}-1+1+x_2-e^{x_2}}{1+x_2-e^{x_2}} = \frac{\theta_2}{2(1-\lambda)}\frac{x_2}{1+x_2-e^{x_2}},
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^2 h_2}{\partial k \partial \lambda} &= \frac{\partial^2 h_2}{\partial \lambda \partial k} = \frac{\theta_2}{2}\frac{1}{x_2}\frac{\partial x_2}{\partial k} - \frac{\theta_2}{2(1-k)} = -\frac{\theta_2}{2}\frac{1}{x_2}\frac{e^{x_2}x_2^2}{1+x_2-e^{x_2}}\frac{2}{\theta_2(1-\lambda)} - \frac{\theta_2}{2(1-k)} \\
&= \frac{\theta_2}{2}\frac{1}{1-k}\left(-\frac{1}{x_2}\frac{e^{x_2}x_2^2}{1+x_2-e^{x_2}}\frac{2(1-k)}{\theta_2(1-\lambda)} - 1\right) \\
&= \frac{\theta_2}{2}\frac{1}{1-k}\left(-\frac{e^{x_2}x_2}{1+x_2-e^{x_2}}\frac{e^{x_2}-1}{x_2 e^{x_2}} - 1\right) \\
&= -\frac{\theta_2}{2}\frac{1}{1-k}\frac{e^{x_2}-1+1+x_2-e^{x_2}}{1+x_2-e^{x_2}} = -\frac{\theta_2}{2}\frac{1}{1-k}\frac{x_2}{1+x_2-e^{x_2}}.
\end{aligned}$$

Indeed, we have $\frac{\partial^2 h_2}{\partial k^2} < 0$ over $\mathcal{D}$. To complete the proof of the concavity of h_2, it suffices to show that the following determinant is non-negative over the interior of $\mathcal{D}$ (note that this is sufficient since h_2 does not depend on α):

$$\Delta = \begin{vmatrix} \frac{\theta_2}{2}\frac{1-\lambda}{(1-k)^2}\frac{x_2}{1+x_2-e^{x_2}} & -\frac{\theta_2}{2}\frac{1}{1-k}\frac{x_2}{1+x_2-e^{x_2}} \\ -\frac{\theta_2}{2}\frac{1}{1-k}\frac{x_2}{1+x_2-e^{x_2}} & \frac{\theta_2}{2(1-\lambda)}\frac{x_2}{1+x_2-e^{x_2}} \end{vmatrix}.$$

We have

$$\Delta = \left(\frac{\theta_2}{2}\right)^2 \frac{1}{(1-k)^2}\frac{x_2^2}{(1+x_2-e^{x_2})^2} - \left(\frac{\theta_2}{2}\right)^2 \frac{1}{(1-k)^2}\frac{x_2^2}{(1+x_2-e^{x_2})^2} = 0,$$

which shows that h_2 is concave over the interior of $\mathcal{D}$.

3. The function h_3 has precisely the same form as h_2, where $1-\lambda$ has been replaced by λ. Thus, we deduce that h_3 is also concave over the interior of $\mathcal{D}$.

4. We show that h_4 is strictly concave over the interior of $\mathcal{D}$. Its first partial derivatives are the following:

$$\begin{aligned}
\frac{\partial h_4}{\partial k} &= -\frac{1-\lambda}{1-k} + \frac{1-\lambda}{k-\alpha}, \\
\frac{\partial h_4}{\partial \alpha} &= -\frac{1-\lambda}{k-\alpha} + \frac{\lambda}{\alpha} - \frac{\lambda}{1-\alpha}, \\
\frac{\partial h_4}{\partial \lambda} &= -\ln\lambda + \ln(1-\lambda) - \ln(1-k) - \ln(k-\alpha) + \ln\alpha + \ln(1-\alpha).
\end{aligned}$$

Thus, the second partial derivatives are:

$$\begin{aligned}
\frac{\partial^2 h_4}{\partial k^2} &= -\frac{1-\lambda}{(1-k)^2} - \frac{1-\lambda}{(k-\alpha)^2}, \\
\frac{\partial^2 h_4}{\partial k \partial \alpha} &= \frac{1-\lambda}{(k-\alpha)^2}, \\
\frac{\partial^2 h_4}{\partial k \partial \lambda} &= \frac{1}{1-k} - \frac{1}{k-\alpha}, \\
\frac{\partial^2 h_4}{\partial \alpha^2} &= -\frac{1-\lambda}{(k-\alpha)^2} - \frac{\lambda}{\alpha^2} - \frac{\lambda}{(1-\alpha)^2}, \\
\frac{\partial^2 h_4}{\partial \alpha \partial \lambda} &= \frac{1}{k-\alpha} + \frac{1}{\alpha} - \frac{1}{1-\alpha}, \\
\frac{\partial^2 h_4}{\partial \lambda^2} &= -\frac{1}{\lambda} - \frac{1}{1-\lambda}.
\end{aligned}$$

To show strict concavity, it suffices to show that the principal minors of the following matrix are non-zero and have alternating signs with the first one being negative:

$$\begin{bmatrix}
-\frac{1-\lambda}{(1-k)^2} - \frac{1-\lambda}{(k-\alpha)^2} & \frac{1-\lambda}{(k-\alpha)^2} & \frac{1}{1-k} - \frac{1}{k-\alpha} \\
\frac{1-\lambda}{(k-\alpha)^2} & -\frac{1-\lambda}{(k-\alpha)^2} - \frac{\lambda}{\alpha^2} - \frac{\lambda}{(1-\alpha)^2} & \frac{1}{k-\alpha} + \frac{1}{\alpha} - \frac{1}{1-\alpha} \\
\frac{1}{1-k} - \frac{1}{k-\alpha} & \frac{1}{k-\alpha} + \frac{1}{\alpha} - \frac{1}{1-\alpha} & -\frac{1}{\lambda} - \frac{1}{1-\lambda}
\end{bmatrix}.$$

The first principal minor is as required. The second principal minor is also as required, since $\frac{\partial^2 h_4}{\partial k^2}$ and $\frac{\partial^2 h_4}{\partial \alpha^2}$ are strictly less than $-\frac{1-\lambda}{(k-\alpha)^2}$ and therefore,

$$\left(\frac{1-\lambda}{(1-k)^2} + \frac{1-\lambda}{(k-\alpha)^2}\right)\left(\frac{1-\lambda}{(k-\alpha)^2} + \frac{\lambda}{\alpha^2} + \frac{\lambda}{(1-\alpha)^2}\right) - \left(\frac{1-\lambda}{(k-\alpha)^2}\right)^2 >$$
$$\left(\frac{1-\lambda}{(k-\alpha)^2}\right)^2 - \left(\frac{1-\lambda}{(k-\alpha)^2}\right)^2 = 0.$$

The third one, which is the determinant Δ of the matrix, satisfies

$$\begin{aligned}
\Delta \;=\; & -\frac{1-\lambda}{(k-\alpha)^2}\Bigg(-\frac{1}{\lambda(k-\alpha)^2} - \frac{1}{(1-k)(k-\alpha)} - \frac{1}{\alpha(1-k)} + \frac{1}{(1-k)(1-\alpha)} \\
& + \frac{1}{(k-\alpha)^2} + \frac{1}{\alpha(k-\alpha)} - \frac{1}{(1-\alpha)(k-\alpha)}\Bigg) \\
& - \left(\frac{1-\lambda}{(k-\alpha)^2} + \frac{\lambda}{\alpha^2} + \frac{\lambda}{(1-\alpha)^2}\right)\Bigg(\frac{1}{\lambda(1-k)^2} + \frac{1}{\lambda(k-\alpha)^2} - \frac{1}{(1-k)^2} - \frac{1}{(k-\alpha)^2} \\
& + \frac{1}{(1-k)(k-\alpha)} + \frac{1}{(1-k)(k-\alpha)}\Bigg) \\
& - \left(\frac{1}{k-\alpha} + \frac{1}{\alpha} - \frac{1}{1-\alpha}\right)\Bigg(-\frac{1-\lambda}{(1-k)^2(k-\alpha)} - \frac{1-\lambda}{(k-\alpha)^3} - \frac{1-\lambda}{\alpha(1-k)^2}
\end{aligned}$$

$$
\begin{aligned}
& -\frac{1-\lambda}{\alpha(k-\alpha)^2}+\frac{1-\lambda}{(1-\alpha)(1-k)^2}+\frac{1-\lambda}{(k-\alpha)^2(1-\alpha)}-\frac{1-\lambda}{(k-\alpha)^2(1-k)}+\frac{1-\lambda}{(k-\alpha)^3}\Bigg) \\
= \quad & \frac{1-\lambda}{\lambda(k-\alpha)^4}+\frac{1-\lambda}{(1-k)(k-\alpha)^3}+\frac{1-\lambda}{\alpha(1-k)(k-\alpha)^2}-\frac{1-\lambda}{(1-k)(1-\alpha)(k-\alpha)^2} \\
& -\frac{1-\lambda}{(k-\alpha)^4}-\frac{1-\lambda}{\alpha(k-\alpha)^3}+\frac{1-\lambda}{(1-\alpha)(k-\alpha)^3} \\
& -\frac{1-\lambda}{\lambda(k-\alpha)^2(1-k)^2}-\frac{1-\lambda}{\lambda(k-\alpha)^4}+\frac{1-\lambda}{(k-\alpha)^2(1-k)^2}+\frac{1-\lambda}{(k-\alpha)^4} \\
& -\frac{1-\lambda}{(1-k)(k-\alpha)^3}-\frac{1-\lambda}{(1-k)(k-\alpha)^3}-\frac{1}{\alpha^2(1-k)^2}-\frac{1}{\alpha^2(k-\alpha)^2} \\
& +\frac{\lambda}{\alpha^2(1-k)^2}+\frac{\lambda}{\alpha^2(k-\alpha)^2}-\frac{\lambda}{\alpha^2(1-k)(k-\alpha)}-\frac{\lambda}{\alpha^2(1-k)(k-\alpha)} \\
& -\frac{1}{(1-\alpha)^2(1-k)^2}-\frac{1}{(1-\alpha)^2(k-\alpha)^2}+\frac{\lambda}{(1-\alpha)^2(1-k)^2}+\frac{\lambda}{(1-\alpha)^2(k-\alpha)^2} \\
& -\frac{\lambda}{(1-\alpha)^2(1-k)(k-\alpha)}-\frac{\lambda}{(1-\alpha)^2(1-k)(k-\alpha)} \\
& +\frac{1-\lambda}{(1-k)^2(k-\alpha)^2}+\frac{1-\lambda}{(k-\alpha)^4}+\frac{1-\lambda}{\alpha(1-k)^2(k-\alpha)}+\frac{1-\lambda}{\alpha(k-\alpha)^3} \\
& -\frac{1-\lambda}{(1-\alpha)(1-k)^2(k-\alpha)}-\frac{1-\lambda}{(k-\alpha)^3(1-\alpha)}+\frac{1-\lambda}{(k-\alpha)^3(1-k)}-\frac{1-\lambda}{(k-\alpha)^4} \\
& +\frac{1-\lambda}{\alpha(1-k)^2(k-\alpha)}+\frac{1-\lambda}{\alpha(k-\alpha)^3}+\frac{1-\lambda}{\alpha^2(1-k)^2}+\frac{1-\lambda}{\alpha^2(k-\alpha)^2} \\
& -\frac{1-\lambda}{\alpha(1-\alpha)(1-k)^2}-\frac{1-\lambda}{\alpha(k-\alpha)^2(1-\alpha)}+\frac{1-\lambda}{\alpha(k-\alpha)^2(1-k)}-\frac{1-\lambda}{\alpha(k-\alpha)^3} \\
& -\frac{1-\lambda}{(1-k)^2(k-\alpha)(1-\alpha)}-\frac{1-\lambda}{(1-\alpha)(k-\alpha)^3}-\frac{1-\lambda}{\alpha(1-k)^2(1-\alpha)} \\
& -\frac{1-\lambda}{\alpha(1-\alpha)(k-\alpha)^2}+\frac{1-\lambda}{(1-\alpha)^2(1-k)^2}+\frac{1-\lambda}{(k-\alpha)^2(1-\alpha)^2} \\
& -\frac{1-\lambda}{(k-\alpha)^2(1-k)(1-\alpha)}+\frac{1-\lambda}{(k-\alpha)^3(1-\alpha)} \\
= \quad & \frac{1-\lambda}{\alpha(1-k)(k-\alpha)^2}-\frac{1-\lambda}{(1-k)(1-\alpha)(k-\alpha)^2}-\frac{1-\lambda}{\lambda(k-\alpha)^2(1-k)^2} \\
& +\frac{1-\lambda}{(k-\alpha)^2(1-k)^2}-\frac{\lambda}{\alpha^2(1-k)(k-\alpha)}-\frac{\lambda}{\alpha^2(1-k)(k-\alpha)} \\
& -\frac{\lambda}{(1-\alpha)^2(1-k)(k-\alpha)}-\frac{\lambda}{(1-\alpha)^2(1-k)(k-\alpha)}-\frac{1-\lambda}{(1-k)^2(k-\alpha)(1-\alpha)} \\
& +\frac{1-\lambda}{(1-k)^2(k-\alpha)^2}+\frac{1-\lambda}{\alpha(1-k)^2(k-\alpha)}-\frac{1-\lambda}{(1-\alpha)(1-k)^2(k-\alpha)} \\
& +\frac{1-\lambda}{\alpha(1-k)^2(k-\alpha)}-\frac{1-\lambda}{\alpha(1-\alpha)(1-k)^2}-\frac{1-\lambda}{\alpha(k-\alpha)^2(1-\alpha)}+\frac{1-\lambda}{\alpha(k-\alpha)^2(1-k)} \\
& -\frac{1-\lambda}{\alpha(1-k)^2(1-\alpha)}-\frac{1-\lambda}{\alpha(1-\alpha)(k-\alpha)^2}-\frac{1-\lambda}{(k-\alpha)^2(1-k)(1-\alpha)}
\end{aligned}
$$

$$
\begin{aligned}
= \; & \frac{1-\lambda}{\alpha^2(1-\alpha)^2(1-k)^2(k-\alpha)^2}\Bigg(\alpha(1-k)(1-\alpha)^2-(1-k)(1-\alpha)\alpha^2+\alpha^2(1-\alpha)^2 \\
& +\alpha(k-\alpha)(1-\alpha)^2-\alpha^2(1-\alpha)(k-\alpha)+\alpha(k-\alpha)(1-\alpha)^2 \\
& -\alpha(1-\alpha)(k-\alpha)^2-\alpha(1-\alpha)(1-k)^2+\alpha(1-k)(1-\alpha)^2-(k-\alpha)(1-\alpha)\alpha^2 \\
& -\alpha(1-\alpha)(k-\alpha)^2-\alpha(1-\alpha)(1-k)^2-(1-k)(1-\alpha)\alpha^2\Bigg) \\
& +\frac{(1-\lambda)\alpha^2(1-\alpha)^2}{\alpha^2(1-\alpha)^2(1-k)^2(k-\alpha)^2}-\frac{1-\lambda}{\lambda(k-\alpha)^2(1-k)^2} \\
& -\frac{2\lambda(1-\alpha)^2(1-k)(k-\alpha)}{\alpha^2(1-\alpha)^2(1-k)^2(k-\alpha)^2}-\frac{2\lambda\alpha^2(1-k)(k-\alpha)}{\alpha^2(1-\alpha)^2(1-k)^2(k-\alpha)^2} \\
= \; & \frac{1-\lambda}{\alpha^2(1-\alpha)^2(1-k)^2(k-\alpha)^2}\Bigg(\alpha(1-\alpha)(1-\alpha)^2+\alpha^2(1-\alpha)(k-\alpha) \\
& +\alpha(1-k)(1-\alpha)(k-\alpha)+\alpha(k-\alpha)(1-\alpha)(1-k)-\alpha^2(1-\alpha)(k-\alpha) \\
& -\alpha(1-\alpha)(1-k)^2-(k-\alpha)(1-\alpha)\alpha^2-\alpha(1-\alpha)(k-\alpha)^2-(1-k)(1-\alpha)\alpha^2\Bigg) \\
& +\frac{(1-\lambda)\alpha^2(1-\alpha)^2}{\alpha^2(1-\alpha)^2(1-k)^2(k-\alpha)^2}-\frac{(1-\lambda)\alpha^2(1-\alpha)^2}{\lambda\alpha^2(1-\alpha)^2(k-\alpha)^2(1-k)^2} \\
& -\frac{\lambda^2(2\alpha^2+2(1-\alpha)^2)(1-k)(k-\alpha)}{\lambda\alpha^2(1-\alpha)^2(1-k)^2(k-\alpha)^2} \\
= \; & \frac{1-\lambda}{\alpha^2(1-\alpha)^2(1-k)^2(k-\alpha)^2}\Bigg(\alpha(1-\alpha)(1-\alpha)^2+2\alpha(1-k)(1-\alpha)(k-\alpha) \\
& -\alpha(1-\alpha)(1-k)^2-(1-\alpha)^2\alpha^2-\alpha(1-\alpha)(k-\alpha)^2\Bigg) \\
& +\frac{(1-\lambda)\alpha^2(1-\alpha)^2}{\alpha^2(1-\alpha)^2(1-k)^2(k-\alpha)^2}-\frac{(1-\lambda)\alpha^2(1-\alpha)^2}{\lambda\alpha^2(1-\alpha)^2(k-\alpha)^2(1-k)^2} \\
& -\frac{\lambda^2(2\alpha^2+2(1-\alpha)^2)(1-k)(k-\alpha)}{\lambda\alpha^2(1-\alpha)^2(1-k)^2(k-\alpha)^2} \\
= \; & \frac{1-\lambda}{\lambda\alpha^2(1-\alpha)^2(1-k)^2(k-\alpha)^2}\Bigg(\lambda\alpha(1-\alpha)\Big((1-\alpha)^2+2(1-k)(k-\alpha)-(1-k)^2 \\
& -(1-\alpha)\alpha-(k-\alpha)^2\Big)\Bigg) \\
& +\frac{(1-\lambda)\alpha^2(1-\alpha)^2}{\alpha^2(1-\alpha)^2(1-k)^2(k-\alpha)^2}-\frac{(1-\lambda)\alpha^2(1-\alpha)^2}{\lambda\alpha^2(1-\alpha)^2(k-\alpha)^2(1-k)^2} \\
& -\frac{\lambda^2(2\alpha^2+2(1-\alpha)^2)(1-k)(k-\alpha)}{\lambda\alpha^2(1-\alpha)^2(1-k)^2(k-\alpha)^2} \\
= \; & \frac{(1-\lambda)\left(\lambda\alpha(1-\alpha)\left((1-\alpha)^2-(1-k-k+\alpha)^2-(1-\alpha)\alpha\right)\right)}{\lambda\alpha^2(1-\alpha)^2(1-k)^2(k-\alpha)^2}
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{(1-\lambda)\alpha^2(1-\alpha)^2}{\alpha^2(1-\alpha)^2(1-k)^2(k-\alpha)^2} - \frac{(1-\lambda)\alpha^2(1-\alpha)^2}{\lambda\alpha^2(1-\alpha)^2(k-\alpha)^2(1-k)^2} \\
& -\frac{\lambda^2(2\alpha^2+2(1-\alpha)^2)(1-k)(k-\alpha)}{\lambda\alpha^2(1-\alpha)^2(1-k)^2(k-\alpha)^2} \\
= \; & \frac{(1-\lambda)\left(\lambda\alpha(1-\alpha)\Big(4(1-k)(k-\alpha)-(1-\alpha)\alpha+(1-\alpha)\alpha\Big)\right)}{\lambda\alpha^2(1-\alpha)^2(1-k)^2(k-\alpha)^2} \\
& -\frac{(1-\lambda)\alpha^2(1-\alpha)^2}{\lambda\alpha^2(1-\alpha)^2(k-\alpha)^2(1-k)^2} - \frac{\lambda^2(2\alpha^2+2(1-\alpha)^2)(1-k)(k-\alpha)}{\lambda\alpha^2(1-\alpha)^2(1-k)^2(k-\alpha)^2} \\
= \; & \frac{4(1-\lambda)\lambda\alpha(1-\alpha)(1-k)(k-\alpha)-2\lambda^2(1-k)(k-\alpha)((1-\alpha)^2+\alpha^2)}{\lambda\alpha^2(1-\alpha)^2(1-k)^2(k-\alpha)^2} \\
& -\frac{(1-\lambda)\alpha^2(1-\alpha)^2}{\lambda\alpha^2(1-\alpha)^2(k-\alpha)^2(1-k)^2} \\
= \; & \frac{2\lambda(1-k)(k-\alpha)\Big(2(1-\lambda)\alpha(1-\alpha)-\lambda(1-\alpha)^2-\lambda\alpha^2\Big)}{\lambda\alpha^2(1-\alpha)^2(1-k)^2(k-\alpha)^2} \\
& -\frac{(1-\lambda)\alpha^2(1-\alpha)^2}{\lambda\alpha^2(1-\alpha)^2(k-\alpha)^2(1-k)^2} \\
= \; & \frac{2\lambda(1-k)(k-\alpha)\Big(2\alpha(1-\alpha)-\lambda\Big)}{\lambda\alpha^2(1-\alpha)^2(1-k)^2(k-\alpha)^2} - \frac{(1-\lambda)\alpha^2(1-\alpha)^2}{\lambda\alpha^2(1-\alpha)^2(k-\alpha)^2(1-k)^2} \\
= \; & \frac{-(1-\lambda)\alpha^2(1-\alpha)^2+4\lambda\alpha(1-\alpha)(k-\alpha)(1-k)-2\lambda^2(k-\alpha)(1-k)}{\lambda\alpha^2(1-\alpha)^2(k-\alpha)^2(1-k)^2},
\end{aligned}
$$

and this can verified e.g. by using Maple. The denominator is always strictly positive, so it is sufficient to show that

$$
-(1-\lambda)\alpha^2(1-\alpha)^2+4\lambda\alpha(1-\alpha)(k-\alpha)(1-k)-2\lambda^2(k-\alpha)(1-k)<0,
$$

for any $(k,\alpha,\lambda)\in\mathcal{D}$. Let us fix $\alpha,\lambda<1$ and consider the above expression as a function of k, for $\alpha\le k\le 1$. Note that

$$
\begin{aligned}
& -(1-\lambda)\alpha^2(1-\alpha)^2+4\lambda\alpha(1-\alpha)(k-\alpha)(1-k)-2\lambda^2(k-\alpha)(1-k) \\
= \; & -k^2\eta+k(\alpha+1)\eta-((1-\lambda)\alpha^2(1-\alpha)^2+\alpha\eta),
\end{aligned}
$$

where $\eta=(4\alpha(1-\alpha)-2\lambda)\lambda$. If $\eta=0$, then obviously this expression is negative. On the other hand, for $k=\alpha$ or $k=1$, the above expression is equal to $-(1-\lambda)\alpha^2(1-\alpha)^2<0$. So, to show that this quadratic is negative it is sufficient to prove that at its stationary point $k=(\alpha+1)/2$ this is negative (since the stationary point will be necessarily between α and 1). Thus, the value of the function at this point is

$$
-(1-\lambda)\alpha^2(1-\alpha)^2+\lambda\alpha(1-\alpha)^3-\frac{\lambda^2(1-\alpha)^2}{2}
$$

$$
\begin{aligned}
&= (1-\alpha)^2 \left(-(1-\lambda)\alpha^2 + \lambda\alpha(1-\alpha) - \frac{\lambda^2}{2} \right) \\
&= -(1-\alpha)^2 \left(\frac{\alpha^2}{2} + \frac{1}{2}(\alpha-\lambda)^2 \right) < 0,
\end{aligned}
$$

over the interior of $\mathcal{D}$ (since $0 < \alpha < 1$), and this concludes the proof of the strict concavity of h_4.

Appendix D

Section 2.10: the Maple code

In this appendix, we present the Maple codes of the procedures that are used in Section 2.10 in order to count the number of rigid 3-colourings of small trees.

The procedure `Prufer` takes as inputs the Prüfer encoding L of a tree and the order n of it and constructs its adjacency matrix.

```
> Prufer:=proc(L,n) local i, j, T, Fin, Ap,
> found, l1, l2;
> T:=[seq([seq(0,i=1..n)],j=1..n)];
> Fin:=[seq(0,i=1..n)];
> Ap:=[seq(0,i=1..n)];
> for i from 1 to n-2 do
> for j from i+1 to n-2 do
> Ap[L[j]]:=1;
> od;
> found:=0;
> j:=1;
> while found=0 do
> if Ap[j]=0 and Fin[j]=0 and j<>L[i] then
> found:=1; fi;
> j:=j+1;
> od;
> Fin[j-1]:=1;
```

```
> T[L[i],j-1]:=1; T[j-1,L[i]]:=1;
> Ap:=[0,0,0,0,0,0];
> od;
> i:=1;found:=0;l1:=0;l2:=0;
> while found=0 do
> if Fin[i]=0 and l1=0 then l1:=i; fi;
> if Fin[i]=0 and l1<>i then l2:=i; found:=1; fi;
> i:=i+1;
> od;
> T[l1,l2]:=1; T[l2,l1]:=1;
> RETURN(T);
> end;
```

The procedure `Rigid` takes as inputs the Prüfer encoding L of a tree, a 3-colouring *col* of n elements as well as the order of the tree n and tests whether or not this is a rigid 3-colouring.

```
> Rigid:=proc(L,col,n) local i, j, T, found1, found2;
> T:=Prufer(L,n);found1:=0; found2:=0;
> for i from 1 to n do found1:=0; found2:=0;
> if col[i]=3 then
> for j from 1 to n do
> if i<>j then if T[i,j]=1 and col[j]=3 then RETURN(false)
> elif T[i,j]=1 and col[j]=1 then found1:=1;
> elif T[i,j]=1 and col[j]=2 then found2:=1;
> fi;
> fi;
> od;
> if found1=0 or found2=0 then RETURN(false); fi;
> fi;
> if col[i]=2 then
> for j from 1 to n do
> if i<>j then if T[i,j]=1 and col[j]=2 then RETURN(false)
```

```
> elif T[i,j]=1 and col[j]=1 then found1:=1;
> fi;
> fi;
> od;
> if found1=0 then RETURN(false); fi;
> fi;
> if col[i]=1 then
> for j from 1 to n do
> if i<>j then if T[i,j]=1 and col[j]=1 then RETURN(false) fi;
> fi;
> od; fi; od; RETURN(true);
> end;
```

The procedure `conv` converts a natural number n into its t-ary form (over the alphabet $1,\dots,t$) of length l.

```
> conv:=proc(n,t,l) local s, i, j, col;
> s:=n;
> i:=1; col:=[seq( 0, j=1..l )];
> while i <= l do
> col[l-i+1]:=(s mod t)+1;
> s:=floor(s/t);
> i:=i+1;
> od;
> RETURN(col);
> end;
```

The procedure `Test` outputs the product of the numbers of rigid 3-colourings over all possible trees on n vertices.

```
> Test:=proc(n) local i, j, T, count, count1,
> MAX, MIN, Prd;
```

```
> MAX:=0; MIN:=3^n; count1:=0;Prd:=1;
> for i from 0 to (n^(n-2))-1 do
> print(i);
> count:=0;
> for j from 0 to 3^n-1 do
> if Rigid(conv(i,n,n-2),conv(j,3,n),n)
> then count:=count+1; fi;
> od;
> print(count);
> if count=2 then
> count1:=count1+1; fi;
> if count <= MIN and count
> > 2 then MIN:=count; fi;
> if count >= MAX then MAX:=count; fi; Prd:=Prd*count;
> od;
> RETURN(MAX,MIN,count1,Prd);
> end;
```

Bibliography

[1] D. Achlioptas and E. Friedgut. A sharp threshold for k-colourability. *Random Structures & Algorithms*, 14(1):63–70, 1999.

[2] D. Achlioptas and M. Molloy. The analysis of a list-colouring algorithm on a random graph. *38th Annual Symposium on Foundations of Computer Science*, pages 204–212, 1997.

[3] D. Achlioptas and M. Molloy. Almost all graphs with $2.522n$ edges are not 3-colorable. *Electronic Journal of Combinatorics*, 6 (1), 1999.

[4] D. Achlioptas and C. Moore. Almost all graphs of average degree 4 are 3-colourable. *34th Annual ACM Symposium on Theory of Computing*, pages 199–208, 2002.

[5] N. Alon, J.H. Spencer, and P. Erdös. *The Probabilistic Method.* J. Wiley & Sons, New York, 1992.

[6] B. Bollobás. *Random Graphs.* Cambridge University Press, Cambridge, 2001.

[7] B. Bollobás and A. Thomason. Threshold functions. *Combinatorica*, 7:35–38, 1986.

[8] M.J. Box. A new method of constrained optimisation and a comparison with other methods. *The Comp. Journal*, 8:42–52, 1965.

[9] B. D. Bunday. *Basic Optimisation Methods.* Edward Arnold, London, 1984.

[10] V. Chvátal. Almost all graphs with $1.44n$ edges are 3-colourable. *Random Structures & Algorithms*, 2:11–28, 1991.

[11] O. Dubois and Y. Boufkhad. A general upper bound for the satisfiability threshold of random r-sat formulae. *Journal of Algorithms*, 24:395–420, 1997.

[12] O. Dubois and J. Mandler. On the non-3-colourability of random graphs. Preprint.

[13] P. E. Dunne and M. Zito. An improved upper bound on the non-3-colourability threshold. *Information Processing Letters*, 65:17–23, 1998.

[14] P. Erdös and A. Rényi. On the evolution of random graphs. *Publication of the Mathematical Institute of the Hungarian Academy of Sciences*, 5:17–61, 1960.

[15] W. Feller. *An Introduction to Probability Theory and Its Applications, II.* Wiley, New York, 1971.

[16] N. Fountoulakis. On the upper bound of the non-3-colourability threshold of random graphs. MSc Dissertation, Mathematical Institute, University of Oxford, September 2000.

[17] E. Friedgut. Sharp thresholds for graph properties, and the k-sat problem. *Journal of the American Mathematical Society*, 12:1017–1054, 1999.

[18] A. M. Frieze and T. Łuczak. On the independence and chromatic numbers of random regular graphs. *Journal of Combinatorial Theory Series B*, 54:123–132, 1992.

[19] I. Giotis, A. C. Kaporis, and L. M. Kirousis. Corrigendum to "A note on the non-colourability threshold of a random graph". *Electronic Journal of Combinatorics*, 7 (1), 2000.

[20] T. Hogg and C. P. Williams. The hardest constrained problems: a double phase transition. *Artificial Inteligence*, 69:359–377, 1994.

[21] S. Janson. Cycles and unicyclic components in random graphs. To appear in *Combinatorics, Probability and Computing.*

[22] S. Janson, D.E. Knuth, T. Łuczak, and B. Pittel. The birth of the giant component. *Random Structures & Algorithms*, 3:233–358, 1993.

[23] S. Janson, T. Łuczak, and A. Ruciński. *Random Graphs.* Wiley-Interscience, New York, 2000.

[24] A. C. Kaporis, L. M. Kirousis, and Y. C. Stamatiou. A note on the non-colourability threshold of a random graph. *Electronic Journal of Combinatorics*, 7 (1), 2000.

[25] L. M. Kirousis, E. Kranakis, D. Krizanc, and Y. C. Stamatiou. Approximating the unsatisfiability threshold of random formulas. *Random Structures & Algorithms*, 12(3):253–269, 1998.

[26] T. Łuczak. Component behavior near the critical point of the random graph process. *Random Structures & Algorithms*, 1:287–310, 1990.

[27] T. Łuczak. Cycles in random graphs near the critical point. *Random Structures & Algorithms*, 2:421–440, 1991.

[28] T. Łuczak. Size and connectivity of the k-core of a random graph. *Discrete Mathematics*, 91:61–68, 1991.

[29] T. Łuczak. Sparse random graphs with a given degree sequence. In A. Frieze and T. Łuczak, editors, *Random Graphs*, pages 165–182. Jonh Wiley & Sons, Inc., 1992.

[30] C.J.H. McDiarmid. Concentration. In M. Habib, C. McDiarmid, J. Ramirez-Alfonsin, and B. Reed, editors, *Probabilistic Methods for Algorithmic Discrete Mathematics*, pages 195–248. Springer, 1991.

[31] B. D. McKay and N.C. Wormald. Asymptotic enumeration by degree sequence of graphs with degrees $o(n^{1/2})$. *Combinatorica*, 11:369–382, 1991.

[32] M. Molloy. The chromatic number of sparse random graphs. Master's thesis, University of Waterloo, 1992.

[33] M. Molloy. A gap between the appearances of a k-core and a $(k+1)$-chromatic graph. *Random Structures & Algorithms*, 6:159–160, 1996.

[34] M. Molloy. Thresholds for colourability and satisfiability in random graphs and boolean formulae. In J. Hirschfeld, editor, *Surveys in Combinatorics*, pages 165–197. Cambridge University Press, 2001.

[35] M. Molloy and B. Reed. A critical point for random graphs with a given degree sequence. *Random Structures & Algorithms*, 6:161–179, 1995.

[36] B. Pittel. On tree census and the giant component in sparse random graphs. *Random Structures & Algorithms*, 1:311–342, 1990.

[37] B. Pittel, J.H. Spencer, and N.C. Wormald. Sudden emergence of a giant k-core in a random graph. *Journal of Combinatorial Theory Series B*, 67:111–151, 1996.

[38] G. Strang. *Linear Algebra and its Applications*. Harcourt Brace Jovanovich, San Diego, 1988.

[39] N. M. Temme. Asymptotic estimates for Stirling numbers. *Stud. Appl. Math.*, 89:223–243, 1993.

[40] R. Webster. *Convexity.* Oxford University Press, New York, 1994.

[41] D.J.A. Welsh. *Codes and Cryptography.* Oxford University Press, Oxford, 1989.