Twistor Theory and the K.P. Equations

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Abstract

In this thesis, we discuss a geometric construction analogous to the Ward correspondence for the KP equations. We propose a Dirac operator based on the inverse scattering transform for the KP-II equation and discuss the similarities and differences to the Ward correspondence.

We also consider the KP-I equation, describing a geometric construction for a certain class of solutions. We also discuss the general inverse scattering of the equation, how this is related to the KP-II equation and the problems with describing a single geometric construction that incorporates both equations.

We also consider the Davey-Stewartson equations, which have a similar behaviour. We demonstrate explicitly the problems of localising the theory with generic boundary conditions. We also present a reformulation of the Dirac operator and demonstrate a duality between the Dirac operator and the first Lax operator for the DS-II equations.

We then proceed to generalise the Dirac operator construction to generate other integrable systems. These include the mKP and Ishimori equations, and an extension to the KP and mKP hierarchies.

Acknowledgements

“$E = mc^2$? I had $F = mc^2$!”

Eddie Izzard, 1997

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This thesis is dedicated to Jim Mitchison.

Simply the best.

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Part I

INTRODUCTION

Chapter 1

Introduction

Roger Penrose first introduced the mathematical world to twistor theory in 1967 in [21] as a geometric framework with which to rewrite the theory of General Relativity in a way that would also allow Quantum Theory to be incorporated into the same theory. Since then, twistor theory has been utilised in a variety of fields both expected and unexpected. The application that concerns this work is in the field of integrable systems. Richard Ward first described what has become known as the Ward construction in [24]. It is a geometrical construction, describing a correspondence between solutions of the anti-self-dual Yang-Mills (ASDYM) equations and holomorphic vector bundles over twistor space.

The ASDYM equations are extremely important in the field of integrable systems. Almost all known integrable systems can be generated as reductions of the ASDYM equations or the ASDYM hierarchy. It has been shown that these systems also have a correspondence that is a symmetry reduction of the Ward correspondence. This theory is comprehensively documented by Mason and Woodhouse in [20].

Possibly the most pertinent phrase in the preceding paragraph is *almost all*. The definition of an integrable system is a matter of debate amongst people in the field, so what would be desirable is to develop one definition that everyone can agree on. What would be a rather elegant definition would be that an integrable system is one that possesses a Ward correspondence. Unfortunately that definition would exclude, amongst others, the Kadomtsev-Petviashvili (KP) equations: a (2+1)-dimensional system, derived to describe two-dimensional surface water waves.

These equations are considered integrable. They arise as the commutation condition of a Lax pair and there is an extremely elegant theory concerning them, as developed by Segal and Wilson, [23] and in Japan by, amongst others, Date, Jimbo, Kashiwara and Miwa, [5]. However, the equations are not reductions of the ASDYM equations in a satisfactory way (there is a reduction involving a choice of an infinite-dimensional gauge group, as described by Ablowitz et al, [1] and a similar one given by Caudrey, [4] but this is unsuitable for our needs) and as such do not possess a standard Ward correspondence. The behaviour of the equations is rather different than even the closest reduction of the ASDYM equations. The key difference is that the Lax pair consistency for the KP equations cannot be expressed as a geometric integrability condition, unlike the condition for the ASDYM equation - the Lax pair consists of operators that are constant on α-planes.

The purpose of this thesis is to investigate whether there exists a suitable generalisation of the Ward correspondence to describe solutions of the KP equations, which other integrable systems would also possess such a correspondence and whether the correspondence could be used to derive other integrable systems.

In Chapter 2, we give some background information on the KP equations and on some other integrable systems that we are interested in. We also provide the necessary geometric theory for the Ward correspondence. After detailing the ASDYM equations and twistor space, we describe the Ward correspondence for both twistor space and minitwistor space. As our proposed construction for the KP equation involves minitwistor space, we prove

Ward's result for minitwistor space.

In Chapter 3, we describe the Dirac operator that we are proposing as the alternative to the Ward correspondence. We give an example, as far as possible, of a solution of the KP-II equation. Finally we discuss the reasons behind the existence of the Dirac operator and why there are problems with localising the theory - a desirable step as the Ward correspondence is a local theory.

In Chapter 4, we detail the alternative theory to the Dirac operator for the KP-I equation, the nonlocal Riemann-Hilbert problem. We also describe another important difference between the KP-I and KP-II equations, lump solutions, and give a geometric description of them.

In Chapter 5, we discuss the relationship between the KP equations and the Korteweg de Vries (KdV) equation (which does possess a Ward correspondence) and also the relationships between KP-I and KP-II, demonstrating the passage from the Dirac operator to the nonlocal Riemann-Hilbert problem directly.

In Chapter 6, we look at another integrable system, the Davey-Stewartson (DS) equations. This has some similarities to the KP equations, but some important differences too. We describe the Dirac operator and the links between the equations. We also consider the possibilities of a local theory for the DS-II equations, and the apparent duality between the Lax operator and the Dirac operator.

In Chapter 7, we present several generalisations of the Dirac operator construction and show that these generate other integrable systems, both known (such as the modified KP equation) and unknown. We also describe what could be described as a *worst case scenario* - the Veselov-Novikov (NVN-I) equation. This is an integrable system for which the inverse scattering (on which the Dirac operator construction is based) is of a type that would be extremely difficult to deal with - the Dirac operator proposed for the KP-II equation would need to be modified by a large amount.

Finally, we present our conclusions, as to the suitability of the Dirac operator as the alternative to the Ward correspondence for the KP equations. We propose an alternative that incorporates all the equations we are interested in, although it is a formal construction that is of little practical use. We then briefly discuss the integrability of the KP equations and propose a conjecture linking the Dirac operator to the theory of tau functions.

Contributions Made in this Thesis

To summarise, in this thesis, working with the Dirac operator for the KP equation first proposed by Mason in [17], we

- describe the relationship between the Dirac operator and the Segal-Wilson theory for the KP equation;
- present a (partly) worked example of a solution to the Dirac operator; and
- discuss the problems arising when we attempt to localise the Dirac operator.

Moving on include the KP-I equation, we

- present a geometric construction for the lump solutions of the KP-I equation;
- demonstrate the links between the KP-I and the KP-II equations through linearisation and the so-called KP-σ equation, and briefly discuss the links between the KP equations and the KdV equation.

We then consider the Davey-Stewartson-II equation. We

- demonstrate that there exists the same analytic continuation from DS-I to DS-II as there is for the KP equations;

- demonstrate explicitly the problems of localising the Dirac construction for the DS-II equation; and
- show a alternative to the Dirac equation for DS-II and show that, despite appearances, gives no new information.

We then consider how the Dirac operator can be generalised, and

- demonstrate how the modified KP (mKP) equation can be obtained;
- describe a number of new integrable systems obtained from generalising the Dirac operator; and
- present the arguments for and against the nonlocal d-bar problem as an alternative to the Dirac operator.

Notation

Mention must be made of a notational convention used throughout this thesis. Where a function is written as $\phi(\lambda)$, this does not imply that it is holomorphic in λ. It is used as shorthand for $\phi(\lambda, \bar{\lambda})$ and $\phi(\bar{\lambda})$ is used as shorthand for $\phi(\bar{\lambda}, \lambda)$. This notation is essential for reasons of space. It only applies to functions of the spectral parameter λ (or ξ if it is in an integrand).

Chapter 2

Background Material

In this section, we provide some of the background necessary for the work that follows. We discuss the relevant $(2+1)$-dimensional integrable systems and the notion of inverse scattering. Following this, we describe the Ward correspondence for twistor space, after giving some basic information on holomorphic vector bundles and twistor space. We then prove the existence of the Ward construction over minitwistor space (a reduction of twistor space), as this is the space than concerns us when discussing the KP equations.

2.1 Inverse Scattering for (2+1)-dimensional Integrable Systems

Inverse scattering is a general method for solving nonlinear partial differential equations. The word "general" is used because the method is really little more than a framework, as the specifics vary greatly from equation to equation. Nevertheless, the inverse scattering techniques developed for the (2+1)-dimensional integrable systems we wish to study will form the basis for the geometric construction that we propose. The framework is as follows. Suppose we are given some initial data for a partial differential equation.

Rather than consider the time evolution of the initial data directly, we first map this initial condition to what is known as *scattering data*. This is the *direct scattering transform*. We then calculate the time-evolution for the scattering data, and then reconstruct the solution of the differential equation from the time-independent scattering data. This is what is known as the *inverse scattering transform*.

We first describe the KP equations and the associated inverse scattering transforms. Following this, there is a brief description of some other integrable systems that will be discussed in later sections, notably the Davey-Stewartson equations.

2.1.1 The Kadomtsev-Petviashvili Equations

The KP equation is first described in [11]. It was derived as a two-dimensional model for surface water waves (indeed, if $u_y = 0$, then the KdV equation - a model for one-dimensional surface waves - arises) and also has applications in plasma physics. The form of the equation that we will consider is given by

$$(4u_t + 12uu_x - u_{xxx})_x - 3\sigma^2 u_{yy} = 0, \tag{2.1}$$

where $\sigma^2 = \pm 1$. The choice of σ^2 is extremely important. The choice $\sigma = i$ is referred to as the KP-I equation, while $\sigma = 1$ is referred to as the KP-II equation. This choice has a significant effect on the behaviour of the equation, and as such, we will consider each choice as a separate equation.

Whilst the equations resembles the Korteweg de Vries equation, they are seemingly unobtainable as symmetry reductions of the anti-self-dual Yang-Mills (ASDYM) equations (see section 2.3) as argued by Mason in [17]. As such, the KP equations cannot be expressed by the standard twistor correspondence, which is based on the ASDYM equations or a reduction thereof. The KP equations are, however, integrable systems (they exist as the compatibility condition of a Lax pair for example) and as such we shall look for a more general twistor construction for them, based on their inverse scattering transforms.

The Lax pair for the equations is given by

$$\begin{aligned} L_2 &= \sigma\partial_y - \partial_x^2 + 2u, \\ L_3 &= \partial_t - \partial_x^3 + 3u\partial_x + v. \end{aligned} \qquad (2.2)$$

In fact each equation is part of a whole hierarchy of equations, generated by Lax operators of the general form

$$L_n = \sigma\partial_{t_n} - \partial_x^n + \sum_{r=0}^{n-2} u^r \partial_x^r, \quad n = 2, 3, \dots . \qquad (2.3)$$

For most integrable systems, the Lax operators have a spectral parameter included, one that does not appear in the commutation relation. This is not the case for the $(2+1)$-dimensional systems that we wish to consider, and so the spectral parameter is introduced by putting $\psi = \phi e^{\mu}$, where $L_n\psi = 0$ and μ is given by $\mu = \lambda x + \lambda^2 y + \lambda^3 t + \cdots + \lambda^n t_n$ for KP-II and $i\lambda x + i\lambda^2 y - i\lambda^3 t + \cdots + i^{n+1}\lambda^n t_n$ for KP-I. If we define the operators $\hat{L}_n$ by

$$\hat{L}_n = e^{-\mu} L_n e^{\mu}, \qquad (2.4)$$

then we see that $\hat{L}_n\phi = 0$ and $\hat{L}_n$ contains the spectral parameter λ. The choice of σ^2 is crucial to the inverse scattering transform. The choice $\sigma^2 = -1$ is referred to as the KP-I equation whilst the choice $\sigma^2 = 1$ is the KP-II equation. The inverse scattering is quite different for the two equations, and as such we will detail these separately.

The KP-II Equation and the D-bar Problem

The inverse scattering for the KP-II ($\sigma = 1$) equation centres around what is known as a semi-local d-bar problem. By semi-local, we mean that if we wish to define the d-bar problem on a region of the complex plane (rather than the whole of $\mathbb{CP}^1$) we are restricted to regions symmetric about the real λ axis, as if $\phi(\lambda)$ is defined, so will be $\phi(\bar{\lambda})$.

The d-bar equation is given by the equation

$$\partial_{\bar{\lambda}}\phi(x, y, t; \lambda, \bar{\lambda}) = \alpha(\lambda, \bar{\lambda})e^{\bar{\mu}-\mu}\phi(x, y, t; \bar{\lambda}, \lambda), \tag{2.5}$$

where $\mu = \lambda x + \lambda^2 y + \lambda^3 t$, together with the boundary condition that $\phi \to 1$ as both λ and $x^2 + y^2 \to \infty$, where ϕ is a function of the real spatial variables x, y and t and a complex spectral parameter λ. This relation can be derived from the equations $\hat{L}_2\phi = 0 = \hat{L}_3\phi$ (or equivalently $L_2\psi = 0 = L_3\psi = 0$). We describe this process in the following chapter. It is this relation that we base the Dirac operator on.

The KP-I equation and the Nonlocal Riemann-Hilbert problem

The inverse scattering transform for the KP-I equation at first looks quite different to the case for the KP-II. Where the function ϕ for the KP-II case is holomorphic nowhere in the spectral parameter, the equivalent function for the KP-I equation is holomorphic everywhere except on the real λ-axis. Here it is discontinuous, and the jump is described by a nonlocal Riemann-Hilbert (NLRH) problem, given by

$$\phi_+(\lambda) - \phi_-(\lambda) = \int_{\mathbb{R}} \alpha(\lambda, \kappa)\phi_-(\kappa)e^{\mu_\kappa - \mu_\lambda}d\kappa, \quad \lambda, \kappa \in \mathbb{R}, \tag{2.6}$$

where ϕ_+ (ϕ_-) is holomorphic on the upper (lower) half-plane and $\mu_\lambda = i\lambda x + i\lambda^2 y - i\lambda^3 t$..

A link between KP-I and KP-II

The difference in the forms of the inverse scattering would appear to pose a problem, as ideally, we do not wish to have a twistor construction for the KP-II equation that does not incorporate the KP-I equation and vice versa.

In the case of the KdV equation (indeed, for any reduction of the self-dual Yang-Mills equations) there is a choice of how to describe the holomorphic vector bundles central to the Ward construction, corresponding to the Čech description or the Dolbeault description (see Section 2.2). This choice corresponds to a choice between solving a (local) Riemann-Hilbert problem or solving an equivalent local d-bar problem, of the form $(\partial_{\bar{\lambda}} + \alpha)\phi = 0$. The similarities between the descriptions and the inverse scattering for (respectively) KP-I (a *nonlocal* Riemann-Hilbert problem) and KP-II (a *semi-local* d-bar problem) are striking so we would expect that there are close links between the two inverse scattering transforms that we may exploit. These links do exist and will be discussed in Chapter 5.

2.1.2 Other Relevant Integrable Systems

There are a number of other 2+1-dimensional integrable systems that we wish to consider. We first will set our notation. Each of these equations includes a σ term, where $\sigma^2 = \pm 1$, which makes a difference to the nature of the inverse scattering. There is some disagreement in the literature on the subject as to which numbering (I or II) relates to which equation. We will adopt the more popular choice and label any equation giving rise to a nonlocal Riemann-Hilbert problem as -I and any equation that gives a d-bar relation as -II. For the equations that we shall consider, this labelling is unambiguous.

(The alternative labelling, adopted primarily by Konopelchenko is to label any equation arising from the choice $\sigma = i$ as -I and from $\sigma = 1$ as -II.)

The Davey-Stewartson Equations

The Davey-Stewartson equations are first described in [6] as a model for a physical water wave problem. The form in which they are more generally studied is a slight generalisation of the formulation presented therein.

The DS equations arise from a Lax pair acting on matrix-valued functions, rather than the scalar-valued functions of the KP equations, but in many ways they are simpler, primarily due to the spatial coordinates x and y being on an equal footing. There are several equivalent ways of stating these equations - we will use the following;

$$\begin{aligned} \beta_t - \sigma^{-1}\beta_{xy} + \rho\beta &= 0, \\ \gamma_t + \sigma^{-1}\gamma_{xy} - \rho\gamma &= 0, \\ \left(\partial_x^2 - \sigma^2\partial_y^2\right)\rho + 8\sigma^{-1}\left(\beta\gamma\right)_{xy} &= 0, \end{aligned} \tag{2.7}$$

where $\sigma^2 = \pm 1$. The reason for this is that it gives itself to the coordinate transformation $\xi = x + \sigma y$ and $\tilde{\xi} = x - \sigma y$, under which the equations become

$$\begin{aligned} \beta_t + \left(\partial_{\tilde{\xi}}^2 - \partial_\xi^2 + \rho\right)\beta = 0 \\ \gamma_t - \left(\partial_{\tilde{\xi}}^2 - \partial_\xi^2 + \rho\right)\gamma = 0 \\ \rho_{\xi\tilde{\xi}} = 2(\partial_\xi^2 - \partial_{\tilde{\xi}}^2)(\beta\gamma). \end{aligned} \tag{2.8}$$

The Lax pair is given by

$$L_1 = \begin{pmatrix} \partial_{\tilde{\xi}} & 0 \\ 0 & \partial_\xi \end{pmatrix} - \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}, \tag{2.9}$$

and

$$L_2 = \partial_t - \begin{pmatrix} \partial_\xi^2 & 0 \\ 0 & \partial_{\tilde{\xi}}^2 \end{pmatrix} - \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}\begin{pmatrix} \partial_\xi & 0 \\ 0 & \partial_{\tilde{\xi}} \end{pmatrix} + A, \tag{2.10}$$

where A is a matrix-valued function defined by the commutation condition, which is symmetric in ξ and $\tilde{\xi}$. Note that this is not the most commonly presented formulation of the Lax pair. We derive this expression in Chapter 3 for the DS-II equations specifically to provide the above symmetry. The form of the inverse scattering for all of these integrable systems that we are considering is based on the first Lax operator. The first operator for the KP equations is parabolic, which causes problems when we try to localise the structure. We consider the DS equations as the first Lax operator is hyperbolic (DS-I) or elliptic (DS-II) which makes it somewhat easier to deal with than the KP case.

The Modified Kadomtsev-Petviashvili Equations

The modified Kadomtsev-Petviashvili (mKP) equation is given by

$$u_t = u_{xxx} - 6u^2 u_x - 6\sigma u_x \left(\partial_x^{-1} u_y\right) + 3\sigma^2 \left(\partial_x^{-1} u_{yy}\right), \tag{2.11}$$

where again $\sigma^2 = \pm 1$ and $(\partial_x^{-1} f)(x) := \int_{-\infty}^{x} f(s)ds$. Its similarities to the KP equations are most easily noted by comparison of the Lax pair for the KP equation, given in equation (2.2) with the Lax pair for the mKP equation

$$\begin{aligned} \hat{L}_2 &= \sigma\partial_y - \partial_x^2 + 2u\partial_x, \\ \hat{L}_3 &= \partial_t - \partial_x^3 + 3u\partial_x^2 + v\partial_x. \end{aligned} \tag{2.12}$$

2.2 Geometrical Background

Before we proceed with discussion of the KP equations, there are certain geometrical ideas that need to be introduced.

2.2.1 Holomorphic Vector Bundles

We are interested in the form of holomorphic vector bundles over complex manifolds, which can be described by a d-bar operator (the Dolbeault picture) or by patching functions (the Čech picture). We detail each description as they both have relevance to our work on (2+1)-dimensional systems.

The Dolbeault picture

Suppose we have a complex manifold M of dimension n. By taking real and imaginary parts of holomorphic coordinates, it can be represented as a real manifold of dimension $2n$. It is distinguished from a general real manifold of even dimension by the existence of an additional structure, the $\bar{\partial}$-operator, $\bar{\partial} : f \to \bar{\partial} f$, where $\bar{\partial} f$ is the $(0,1)$-part of $\mathrm{d}f$.

Similarly, we can think of a holomorphic vector bundle E as a smooth complex vector bundle with an operator $\bar{\partial}_E$. In a local holomorphic trivialisation, $\bar{\partial}_E$ is defined component by component. In a general smooth trivialisation, $\bar{\partial}_E = \bar{\partial} + \Phi$ where Φ is a matrix-valued $(0,1)$-form that undergoes gauge transformations of the form

$$\Phi \to g^{-1}\Phi g + g^{-1}\bar{\partial} g$$

under a change of smooth trivialisation.

The characteristic property of $\bar{\partial}_E$ is the partial flatness condition,

$$\bar{\partial}\Phi + \Phi \wedge \Phi = 0, \tag{2.13}$$

where $\bar{\partial}$ is applied entry by entry.

Holomorphic sections of the bundle are solutions to $\bar{\partial}_E \psi = 0$; these equations are over-determined, but the partial flatness condition is the consistency condition for the existence of enough local solutions for there to exist a holomorphic trivialisation.

The Čech picture

The patching data of a holomorphic vector bundle consists of equivalence classes of the patching matrices between local holomorphic trivialisations of the bundle.

Let U_α be an open covering of the manifold, M. On each U_α, there is given a holomorphic frame field $e_{\alpha i}$, $i = 1, \ldots, n$, where n is the rank of the bundle. On the non-empty intersections $U_\alpha \cap U_\beta$, there exists a holomorphic map

$$P_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(n, \mathbb{C}),$$

such that

$$(e_{\beta 1}, \ldots, e_{\beta n}) = (e_{\alpha 1}, \ldots, e_{\alpha n}) P_{\alpha\beta}.$$

$P_{\alpha\beta}$ is called the *patching matrix* from U_α to U_β. The patching data satisfies three conditions;

1. each patching matrix is holomorphic and nonsingular;
2. for each nonempty $U_\alpha \cap U_\beta$, $P_{\alpha\beta} = P_{\beta\alpha}^{-1}$; and
3. for each nonempty $U_\alpha \cap U_\beta \cap U_\gamma$, $P_{\alpha\beta} P_{\beta\gamma} P_{\gamma\alpha} = 1$.

Any such collection of patching matrices satisfying the above conditions determines a holomorphic bundle.

Two holomorphic bundles E and E' are equivalent if there exists a biholomorphic map $E \to E'$ that sends the fibres of E linearly onto the corresponding fibres of E'. Such a map exists if and only if there exist systems of local trivialisations for E and E', with the same open sets U_α, such that their patching matrices are related by

$$P_{\alpha\beta} = h_\alpha^{-1} P'_{\alpha\beta} h_\beta \tag{2.14}$$

for some family of holomorphic maps $h_\alpha : U_\alpha \to \mathrm{GL}(n, \mathbb{C})$. In particular, E is trivial (equivalent to a product bundle) if and only if its patching matrices can be factorised as $P_{\alpha\beta} = h_\alpha^{-1} h_\beta$. Hence we can classify holomorphic vector bundles by maps $P_{\alpha\beta}$ satisfying the above conditions, subject to the equivalence relation (2.14). It should be noted that the Dolbeault and the Čech pictures are equivalent formulations, that is, there exists a one-to-one correspondence between the gauge equivalence classes in the two formulations.

The Čech-Dolbeault Isomorphism

As there are two equally valid descriptions of holomorphic vector bundles over a manifold $\mathcal{M}$, there must be an equivalence between them. Suppose $\mathcal{M}$ is covered by two sets U and V.

Suppose we have a patching function F_{UV}, holomorphic on $U \cap V$. By the Dolbeault-Poincaré lemma, we can write this as $F = f_U^{-1} f_V$, for some f_U, smooth on U and f_V, smooth on V. This factorisation is unique up to a globally smooth factor g, say, in f_U and f_V. We define $(0,1)$-forms on U and V by

$$\alpha_U = \left(\bar{\partial} f_U\right) f_U^{-1}, \quad \alpha_V = \left(\bar{\partial} f_V\right) f_V^{-1}$$

respectively. F is holomorphic on the intersection $U \cap V$, hence, on $U \cap V$,

$$\begin{aligned} 0 &= \bar{\partial} F \\ &= \bar{\partial}(f_U^{-1} f_V) \\ &= f_U^{-1}(\left(\bar{\partial} f_V\right) f_V^{-1} - \left(\bar{\partial} f_U\right) f_U^{-1}) f_V \\ &= f_U^{-1}(\alpha_V - \alpha_U) f_V. \end{aligned} \tag{2.15}$$

Hence $\alpha_U = \alpha_V$ on $U \cap V$ and we have defined a global $(0,1)$-form. Also, on U,

$$\begin{aligned} \bar{\partial}\alpha &= \bar{\partial}\left[\left(\bar{\partial} f_U\right) f_U^{-1}\right] \\ &= \left(\bar{\partial} f_U\right) \bar{\partial}\left(f_U^{-1}\right) \\ &= -\left(\bar{\partial} f_U\right) f_U^{-1} \left(\bar{\partial} f_U\right) f_U^{-1} \\ &= -\alpha \wedge \alpha, \end{aligned} \tag{2.16}$$

and similarly on V. Hence the $(0,1)$-form satisfies the partial flatness condition and defines a holomorphic vector bundle.

Now suppose instead we have a $(0,1)$-form α satisfying $\bar{\partial}\alpha + \alpha \wedge \alpha = 0$. Then by partial flatness, there exists a function f_U that is smooth on U and such that $\alpha = f_U^{-1}\left(\bar{\partial} f_U\right)$ on U. Similarly, there exists f_V with the same properties on V. Hence on $U \cap V$, defining F by $F = f_U^{-1} f_V$, by a similar calculation to that above, it follows that $\bar{\partial} F = 0$, and F is holomorphic on $U \cap V$. This result has an obvious generalisation for a covering of $\mathcal{M}$ consisting of more than two sets.

2.2.2 Twistor Space and Minitwistor Space

The Ward construction that is central to the connections between twistor theory and integrable systems was first described in [24] with respect to twistor space. However, in looking at the majority of 2+1-dimensional integrable systems, we find that it is more useful to consider minitwistor space as a starting point. As such, we describe both twistor space and minitwistor space.

Twistor Space

Take M to be complex space-time, with double-null coordinates w, $\tilde{w}$, z and $\tilde{z}$ so that the metric is given by

$$ds^2 = dwd\tilde{w} - dzd\tilde{z}.$$

An α-plane in M is a plane with tangent vectors $L = \partial_w - \zeta\partial_{\tilde{z}}$ and $M = \partial_z - \zeta\partial_{\tilde{w}}$ where $\zeta \in \mathbb{CP}^1$. Hence the space of α-planes can be coordinatised by three complex coordinates ζ, $l = \zeta w + \tilde{z}$ and $m = \zeta z + \tilde{w}$. The entire space of α-planes is $\mathbb{CP}^3$ (including those at infinity) and this is (projective) twistor space, $\mathbb{PT}$.

Given a region U in space-time, we can define the twistor space of U to be

$$\mathcal{P} = \{Z \in \mathbb{PT} \,|\, Z \cap U \neq \emptyset\}, \tag{2.17}$$

i.e. the space of α-planes that intersect U. Note that we will only consider regions where the intersection between U and any α-plane is either empty or simply connected. The correspondence space $\mathcal{F}$ is defined as the five-dimensional complex space, coordinatised by $w, z, \tilde{w}, \tilde{z}$ and ζ and defined as the set of pairs (x, Z) where x is a point in U and Z is an α-plane through x. This space is fibred over $\mathcal{P}$ and U by the projections p and q given explicitly by

$$\begin{aligned} p : (w, z, \tilde{w}, \tilde{z}, \zeta) &\rightarrow (l, m, \zeta) = (\zeta w + \tilde{z}, \zeta z + \tilde{w}, \zeta) \\ q : (w, z, \tilde{w}, \tilde{z}, \zeta) &\rightarrow (w, z, \tilde{w}, \tilde{z}). \end{aligned} \tag{2.18}$$

Minitwistor Space

Minitwistor space is obtained by quotienting by a symmetry on twistor space, that symmetry corresponding to a translation along a non-null vector X. This gives that minitwistor space ($\mathbb{MT}$) is the total space of the line bundle $\mathcal{O}(2)$, the line bundle of Chern class 2 over $\mathbb{CP}^1$. The Riemann sphere is viewed stereographically as $\mathbb{C} \cup \{\infty\}$. It will be given coordinates λ on $\mathbb{C}$ and $\lambda' = \lambda^{-1}$ on $\mathbb{C}' = \mathbb{C} \cup \{\infty\} - \{0\}$. $\mathcal{O}(2)$ can then be given coordinates (μ, λ) for the region fibred over $\mathbb{C}$ and $(\mu', \lambda') = (\mu\lambda^{-2}, \lambda^{-1})$ on the region over $\mathbb{C}'$.

Space-time will be defined as Γ, the space of global holomorphic sections of $\mathcal{O}(2)$. An element σ of Γ, $\sigma : \mathbb{CP}^1 \to \mathcal{O}(2)$ is represented by $\mu = \sigma(t_i, \lambda)$, a polynomial in λ of degree 2, with complex coefficients t_i,

$$\sigma(\lambda) = \sum_{i=0}^{2} t_i \lambda^i. \tag{2.19}$$

On $\mathbb{C}'$,

$$\mu' = \sum_{i=0}^{2} t_i (\lambda')^{2-i}, \tag{2.20}$$

which is regular at $\lambda = \infty$, so the sections are global as desired. If further positive or negative powers of λ are included in equation (2.19), then there will be a pole at either $\lambda = 0$ or $\lambda = \infty$. Hence equation (2.19) describes all the global sections and therefore $\Gamma = \mathbb{C}^3$.

For the construction of hierarchies of Lax operators and integrable systems, we need to consider $\mathcal{O}(n)$ instead of $\mathcal{O}(2)$, which is an extended form of minitwistor space. Space-time will then be $\mathcal{C}^{n+1}$. From this point onwards, when we refer to minitwistor space, we in fact mean $\mathcal{O}(n)$. Let Z be a point in $\mathcal{O}(n)$, $Z = (\mu, \lambda)$. This corresponds in Γ to the hyperplane Σ_Z of all holomorphic sections passing through Z; Σ_Z consists of those $t_i \in \mathbb{C}^{n+1}$ satisfying equation (2.19) for fixed $(\sigma(\lambda), \lambda) = (\mu, \lambda)$.

2.3 The Ward Correspondence

As the intention of this work is to find a geometric construction that generalises the Ward construction for the ASDYM equations and reductions of those equations, we describe the equations and briefly describe the Ward construction over twistor space for those equations. As the Dirac operator that we propose acts on sections of a bundle over minitwistor space, we also present and prove in detail the corresponding construction for minitwistor space and the Bogomolny hierarchy.

2.3.1 The Anti-Self-Dual Yang-Mills Equations

Suppose D is a connection on a complex rank n vector bundle E over a region U in space-time. In a local trivialisation, where $D = d + \Phi$, the curvature is given by $F = F_{ab} dx^a \wedge dx^b$, where

$$F_{ab} = \partial_a \Phi_b - \partial_b \Phi_a + [\Phi_a, \Phi_b] \,. \tag{2.21}$$

The ASDYM equations are given by

$$F = - \star F \tag{2.22}$$

where $\star$ is the Hodge duality operator, and are equivalent to the condition that the curvature should vanish when restricted to α-planes. In terms of the above formulation and working in double-null coordinates, with metric

$$ds^2 = dw d\tilde{w} - dz d\tilde{z},$$

the equations are given by

$$F_{wz} = 0, \quad F_{z\tilde{z}} - F_{w\tilde{w}} = 0, \quad \text{and} \quad F_{\tilde{w}\tilde{z}} = 0. \tag{2.23}$$

Equivalently, the equations can be considered as the commutation relation between a Lax pair of operators. Denoting $D_a = d_a + \Phi_a$, it is straightforward to show that $F_{ab} = [D_a, D_b]$. Hence the ASDYM equations can be formulated as the condition that the operators

$$L_1 = D_w - \zeta D_{\tilde{z}} \quad \text{and} \quad L_2 = D_z - \zeta D_{\tilde{w}} \tag{2.24}$$

commute for every value of ζ. It is this formulation that is central to the theory of integrability and twistor theory.

The ASDYM equations are probably the most important set of equations in the theory of integrability, due to the fact that most other integrable systems can be found as reductions of them. There are a number of forms that these reductions can take. For information on these, a thorough account is presented by Mason and Woodhouse in [20]. Through these reductions, one can obtain a large class of integrable systems, including the Bogomolny equations, the Korteweg de Vries (KdV) equation and the nonlinear Schrödinger equation (NLS). By utilising these reductions, one can generate a Ward transform for each of these integrable systems.

2.3.2 The Ward Correspondence for Twistor Space

Theorem 2.1 *Let U be an open set in complexified space-time such that the intersection of U with every α-plane that meets U is connected and simply-connected. Then there is a one-to-one correspondence between solutions of the ASDYM equation on U, with gauge group $GL(n,\mathbb{C})$ and holomorphic vector bundles $E' \to \mathcal{P}$ such that $E'|_{\hat{x}}$ is trivial for every x in U, where $\hat{x}$ is the line in $\mathcal{P}$ corresponding to all α-planes containing x.*

This proof is well-documented in the literature but we will outline one half of the proof. More detail, and the second half can be found in Mason and Woodhouse, [20].

Outline Proof: Starting with an ASD connection D on a rank n bundle $E \to U$, the fibre of the holomorphic vector bundle over $\mathcal{P}$ at $Z \in \mathcal{P}$ is given by the space of sections of E over $Z \cap U$ that are covariantly constant. D has zero curvature when restricted to an α-plane, and $Z \cap U$ is simply connected and connected, thus the sections are single-valued on $Z \cap U$. Thus $E'_Z \sim \mathbb{C}^n$. $\square$

2.3.3 The Ward Correspondence for Minitwistor Space

Before we state the theorem for the Ward construction for mini-twistor space, we first give a definition.

Definition 2.2 *For an open region $R \in \Gamma$, define $\mathcal{O}(n)_R$ by*

$$\mathcal{O}(n)_R = \{Z \in \mathcal{O}(n) : \Sigma_Z \cap R \neq \varnothing\}. \tag{2.25}$$

R will be said to be suitable if each $\Sigma_Z \cap R$ is homotopically and analytically trivial and if the subset of Γ consisting of sections lying in $\mathcal{O}(n)_R$ coincides with R itself.

Theorem 2.3 *For R suitable, let E be a vector bundle over $\mathcal{O}(n)_R$ with structure group $SL(m, \mathbb{C})$ such that its restriction to any holomorphic section of $\mathcal{O}(n)_R$ over $\mathbb{CP}^1$ is trivial. Each such E determines and is determined by a gauge equivalence class of a system of nonlinear differential equations on R, the nth level of the Bogomolny hierarchy, as defined below.*

Proof (Dolbeault Picture): Suppose E is a holomorphic vector bundle over $\mathcal{O}(n)$. Choose a globally smooth frame for the bundle. The fibres of $\mathcal{O}(n)$ are Stein, so the frame can be chosen so that it is holomorphic on the fibres of $\mathcal{O}(n)$. Hence

$$\bar{\partial}_E = \bar{\partial} + \alpha(\mu, \lambda, \bar{\lambda}) d\bar{\lambda},$$

where $\bar{\partial}$ is the ordinary $\bar{\partial}$-operator on functions, α is holomorphic in the coordinate μ and μ and λ are as defined previously for $\mathcal{O}(n)$.

To recover a solution of the Bogomolny hierarchy on $\mathbb{C}^{n+1}$ from the vector bundle, we must construct global holomorphic frames on each section given by

$$\mu = t_0 + t_1\lambda + \ldots + t_n\lambda^n, \quad t_1 \in \mathbb{R}.$$

This requires the solution of the $\bar{\partial}$-equation

$$\left(\bar{\partial} + \alpha\left(t_0 + t_1\lambda + \ldots t_n\lambda^n, \lambda, \bar{\lambda}\right)\right) f(\mathbf{t}, \lambda, \bar{\lambda}) = 0, \tag{2.26}$$

where $\mathbf{t} = (t_i) \in \mathbb{C}^{n+1}$, and f is smooth and non-degenerate on $\mathbb{CP}^1$. We now use an index theorem and a genericity argument to show that for generic $\mathbf{t}$ and a, such a solution exists and is unique up to left multiplication by a constant matrix.

Suppose E is a line bundle. The index of an operator is a topological invariant, so the index of $\bar{\partial}_E$ is the same as that of $\bar{\partial}$ of a trivial line bundle. The index of $\bar{\partial}$ is equal to the dimension of the kernel of $\bar{\partial}$ minus the dimension of the cokernel of $\bar{\partial}$. The kernel consists of the functions that are holomorphic over the entire Riemann sphere, that is the constant functions. If we consider the first cohomology group of a line bundle on the sphere, it is trivial. This implies that the operator $\bar{\partial}$ is onto and hence its cokernel is zero. Therefore the index of $\bar{\partial}$, and hence of $\bar{\partial}_E$ is one. In general, E can be viewed as a collection of r line bundles, so it follows that the index of $\bar{\partial}_E$ is equal to the rank of the bundle.

For $k = 1, \ldots, n$, define the operators L_k by

$$L_k = f^{-1} \circ (\partial_k - \lambda\partial_{k-1}) \circ f. \tag{2.27}$$

It follows from this definition that

$$\begin{aligned} L_k L_m &= f^{-1} \circ (\partial_k - \lambda\partial_{k-1}) \circ f \circ f^{-1} \circ (\partial_m - \lambda\partial_{m-1}) \circ f \\ &= f^{-1} \circ (\partial_k - \lambda\partial_{k-1}) \circ (\partial_m - \lambda\partial_{m-1}) \circ f \\ &= f^{-1} \circ (\partial_m - \lambda\partial_{m-1}) \circ (\partial_k - \lambda\partial_{k-1}) \circ f \\ &= L_m L_k. \end{aligned}$$

These operators commute with each other and, as a result of equation (2.26), commute with $\partial_{\bar{\lambda}}$, and are regular over $\mathbb{CP}^1$ except for a simple pole at $\lambda = \infty$.

Consider a function $f(z)$, defined on $\mathbb{C} \cup \infty$ such that f is regular over $\mathbb{C}$, and has a simple pole at $z = \infty$. Let the residue of the pole at $z = \infty$ be given by $a \in \mathbb{C}$. Then the function $g(z)$, given by

$$g(z) = f(z) - az,$$

is holomorphic on $\mathbb{C} \cup \{\infty\}$. Hence, by Liouville's theorem, $g(z)$ is constant. Hence f is of the form

$$f(z) = az + b,$$

where a and b are constants.

By the same argument, the matrices L_k are linear functions of λ. L_k can be written in the form

$$L_k s \equiv (\Delta_k - \lambda D_{k-1})\, s = 0, \quad k = 1, \ldots, n \tag{2.28}$$

where

$$\Delta_k = \partial_k - A_k, \quad D_{k-1} = \partial_{k-1} - B_{k-1} \quad k = 1, \ldots, n.$$

The integrability conditions for the above linear system define the nth level of the Bogomolny hierarchy;

$$\begin{aligned} [\Delta_k, \Delta_j] &= 0, \\ [D_{k-1}, D_{j-1}] &= 0, \\ [\Delta_k, D_{j-1}] - [\Delta_j, D_{k-1}] &= 0, \quad 1 \leq k, j \leq n. \end{aligned} \tag{2.29}$$

These equations are gauge invariant under the transformation

$$\begin{aligned} A_k &\rightarrow h^{-1} A_k h - h^{-1} \partial_k h, \\ B_k &\rightarrow h^{-1} B_k h - h^{-1} \partial_k h, \end{aligned} \tag{2.30}$$

where h is an $\mathrm{SL}(n, \mathbb{C})$-valued function of the t_i alone – it is global and holomorphic in λ and is thus independent of λ by Liouville's theorem.

Therefore, given E we have obtained a gauge equivalence class of solutions of the Bogomolny equations, which can be seen to be independent, modulo gauge, of the choices made.

Suppose we have a solution of equation (2.29). Define the differential operator V_k by $V_k = \partial_k - \lambda \partial_{k-1}$ and let Z be any fixed twistor (μ, λ). Then Σ_Z is defined by the equation $\mu = \sigma(t_i, \lambda)$, where $\sigma(t_i, \lambda)$ is defined by equation (2.19). The vector fields V_k kill all $\sigma(t_i, \lambda)$, so they are tangent to any such twistor plane, for any fixed μ and λ. Given a solution to equation (2.29) on R, we construct a vector bundle on $\mathcal{O}(n)_R$ by defining the fibre at $Z \in \mathcal{O}(n)_R$ by

$$E_Z = \{s : L_k s = (V_k - A_k + \lambda B_{k-1})s = 0 \quad \text{on} \quad \Sigma_Z \cap R\}. \tag{2.31}$$

The linear system given in equation (2.28) defines a linear connection on every $\Sigma_Z \cap R$ that has zero curvature. As $\Sigma_Z \cap R$ is homotopically trivial, there is no holonomy, so the solution space to equation (2.31) can be taken to be the fibre of a holomorphic vector bundle over $\mathcal{O}(n)_R$.

It is straightforward to see that E is trivial when restricted to any section of $\mathcal{O}(n)_R$ over $\mathbb{CP}^1$ and that this construction is the inverse of the construction from the first half of the proof. $\square$

Whilst the Dolbeault picture is more immediately relevant to the Dirac picture, it is useful to present the proof in the Čech picture as well, to illustrate the Čech-Dolbeault isomorphism and to compare to the nonlocal Riemann-Hilbert problem.

Proof (Čech Picture): Let $\{U_\alpha\}$ be an open Stein cover of $\mathcal{O}(n)_R$, where the index α ranges over the number of sets in the cover. By definition, a holomorphic rank-m vector bundle, E, when restricted to a Stein open set

$\{U_\alpha\}$, is holomorphically trivialised by a frame f_α of E over $\{U_\alpha\}$, that is

$$f_\alpha \left(E|_{U_\alpha} \right) \simeq U_\alpha \times \mathbb{C}^m.$$

We now restrict the patching functions to any section $\mu = \sigma(t_i, \lambda)$, corresponding to a point (t_i) of R. By the assumption, E is trivial on R, so there exist $\mathrm{SL}(n, \mathbb{C})$-valued functions $s_\alpha(t_i, \lambda)$ satisfying, on each $\{\mu = \sigma\} \cap U_\alpha \cap U_\beta$,

$$s_\alpha(t_i, \lambda) = P_{\alpha\beta}(\sigma(t_i, \lambda), \lambda) s_\beta(t_i, \lambda). \tag{2.32}$$

(This triviality follows from the Birkhoff factorisation theorem and the assumption of topological triviality of the bundle.) Let V_k describe the same differential operator as in the previous section. The polynomials $\sigma(t_i, \lambda)$ are annihilated by each V_k and hence so are the functions $P_{\alpha\beta}(\sigma(t_i, \lambda), \lambda)$.

Define the quantities $\gamma_k^\alpha(t_i, \lambda)$ by the formula

$$V_k s_\alpha(t_i, \lambda) = s_\alpha(t_i, \lambda) \gamma_k^\alpha(t_i, \lambda). \tag{2.33}$$

Applying V_k to equation (2.32) gives the result $\gamma_k^\alpha(t_i, \lambda) = \gamma_k^\beta(t_i, \lambda)$ on each overlap region $\sigma \cap U_\alpha \cap U_\beta$. Therefore there is a function $\gamma_k(t_i, \lambda)$ such that

$$\gamma_k(t_i, \lambda)|_{\sigma \cap U_\alpha} = \gamma_k^\alpha(t_i, \lambda).$$

$\gamma_k(t_i, \lambda)$ is regular for all $\lambda \in \mathbb{C}$ and $\lambda^{-1} \gamma_k(t_i, \lambda)$ is regular as $\lambda \to \infty$, so, by the extension of Liouville's theorem presented in the previous section, it is linear in λ. Put $\gamma_k(t_i, \lambda) = A_k(t_i) + \lambda B_{k-1}(t_i)$. Put $s = s_\alpha(t_i, \lambda)$ for some α. Then equation (2.33) shows that s is a solution of the linear system

$$L_k s \equiv \left(\Delta_k - \lambda D_{k-1} \right) s = 0, \quad k = 1, \ldots, n \tag{2.34}$$

where

$$\Delta_k = \partial_k - A_k, \quad D_{k-1} = \partial_{k-1} - B_{k-1} \quad k = 1, \ldots, n.$$

The result then follows as described in the Dolbeault picture. □

Part II

THE KP EQUATIONS

Chapter 3

The Dirac Operator and the KP-II Equation

The twistor-like construction that we propose makes use of a Dirac operator, which is derived from the semi-local d-bar operator from the inverse scattering of the type-II equations. In this chapter, we first prove a one-to one correspondence between the set of Dirac operators of appropriate form and the set of solutions to the KP-II equation with appropriate boundary conditions. Note that proof of Proposition 3.4 first appeared in [17], while the proof of Proposition 3.5 appears in [2]. They are presented here partly for information, but also to demonstrate the proof of Theorem 3.3. We then present a partly worked example of a solution to the KP-II equation as a power series, and demonstrate the links to the theory of Grassmannians in this case. We conclude with a discussion of the source of the nonholomorphy that gives rise to the Dirac operator.

3.1 The Dirac operator for the KP Hierarchy

We consider the total space of $\mathcal{O}(n)$, coordinatised by (λ, μ). λ is taken to be an affine coordinate on $\mathbb{CP}^1$, and μ is a linear coordinate up the fibre.

Recall that global holomorphic sections of $\mathcal{O}(n)$ are given by

$$\mu = \sum_{i=0}^{n} t_i \lambda^i. \tag{3.1}$$

The coordinates (t_i) are considered as coordinates on space-time $\mathbb{C}^{n+1}$, but for the KP-II case, we will assume that the coordinates are restricted to the real slice of the space. Note also that the variables t_1, t_2,t_3 will generally be relabelled as x, y and t, to match the standard notation for the KP equations. We define a Dirac operator as follows.

Definition 3.1 *A Dirac operator is an operator of the form*

$$D_\alpha = \begin{pmatrix} \partial_{\bar{\lambda}} & \alpha \\ \bar{\alpha} & \partial_\lambda \end{pmatrix}, \tag{3.2}$$

which acts on a two-component vector

$$\Phi = \begin{pmatrix} \phi \\ \bar{\phi} \end{pmatrix}, \tag{3.3}$$

where $\alpha(\mu, \bar{\mu}; \lambda, \bar{\lambda})$ is taken to be a smooth function with the form

$$\alpha(\lambda, \bar{\lambda}, \mu, \bar{\mu}) = \exp \bar{\mu}\, \alpha_0(\lambda, \bar{\lambda}) \exp -\mu \tag{3.4}$$

We replace the d-bar equation describing a holomorphic vector bundle

$$(\partial_{\bar{\lambda}} + \alpha(\lambda, \bar{\lambda}, \mu))\phi = 0, \tag{3.5}$$

with the equation

$$D_\alpha \Phi = 0, \tag{3.6}$$

Note that in general, ϕ and μ can be matrix-valued rather than scalar-valued functions, but for the KP theory, they are scalar-valued functions.

The Dirac Operator and the Semi-Local D-Bar Operator

We should make clear why we use the Dirac operator of the form given above, rather than the semi-local d-bar operator it is based on, as given by equation (2.5). We have replaced the term $\phi(\bar{\lambda}, \lambda)$ with the term $\overline{\phi(\lambda, \bar{\lambda})}$. For the d-bar operator to be defined on a subset U of $\mathbb{CP}^1$, then if $\lambda \in U$, it is necessary that $\bar{\lambda} \in U$ as well (hence the term semi-local). This is not the case with the form of the Dirac operator. It should be noted, however, that with the KP-II equation (and the other equations we consider) that $\phi(\bar{\lambda}, \lambda) = \overline{\phi(\lambda, \bar{\lambda})}$.

Definition 3.2 *A Dirac operator for the KP-II hierarchy is as defined above such that $\alpha(\mu, \bar{\mu}; \lambda, \bar{\lambda})$ is taken to be a smooth scalar-valued function on $\mathcal{O}(n)$, restricted to a section $\mu = \sum t_i \lambda^i$, where*

$$\alpha_0(\lambda, \bar{\lambda}) \sim O(\exp(-|\lambda|^{n+1})) \quad \text{as} \quad \lambda \to \infty, \tag{3.7}$$

and with the reality condition $\alpha_0(\lambda, \bar{\lambda}) = \overline{\alpha_0(\bar{\lambda}, \lambda)}$.

We will prove the following theorem.

Theorem 3.3 *There is a one-to-one correspondence between Dirac operators of the form defined above, and solutions to the KP-II hierarchy with boundary condition*

$$u(x, y, t) \to 0, \quad \text{as } x^2 + y^2 \to \infty, \quad x, y, t \in \mathbb{R}.$$

We shall prove this theorem by proving separately two propositions. First, we use the Dirac operator to derive the Lax operators that make up the KP-II hierarchy. We then reverse the process, deriving the Dirac operator from the hierarchy.

Proposition (Mason) 3.4 *Given a Dirac operator as in Definition 3.2, there exists an associated sequence of Lax operators for the KP-II hierarchy and hence there exists an associated solution u to the KP-II hierarchy with the boundary condition $u(x, y, t) \to 0$ as $x^2 + y^2 \to \infty$.*

Proof: By the index theorem and a genericity argument, the solution space of equation (3.6) has real dimension 2 for almost all $\mathbf{t}$. If we assume that it is two dimensional at $\mathbf{t} = 0$, then it will be two dimensional in a neighbourhood of $\mathbf{t} = 0$. Thus we can identify the solution space with $\mathbb{C}$ by taking the value of ϕ at $\lambda = \infty$. We specify the solution by adding the condition that $\phi = 1$ at $\lambda = \infty$. Given that the equation

$$\partial_{\bar{\lambda}}\phi + \alpha\bar{\phi} = 0,$$

together with a boundary condition at $\lambda = \infty$ has a unique solution, it is straightforward to show that, given the reality condition on α_0 given in Definition 3.2, then

$$\overline{\phi(\lambda, \bar{\lambda})} = \phi(\bar{\lambda}, \lambda).$$

It should be made clear at this point that when we refer to $\phi(\lambda)$ or $\phi(\bar{\lambda})$, we do not mean to imply that ϕ is holomorphic in λ or $\bar{\lambda}$. It is merely convenient shorthand, adopted for considerations of space, for $\phi(\lambda) = \phi(\lambda, \bar{\lambda})$ and $\phi(\bar{\lambda}) = \phi(\bar{\lambda}, \lambda)$.

At $\lambda = \infty$,

$$\partial_{\bar{\lambda}}\phi = O(\exp -|\lambda|^{n+1}). \tag{3.8}$$

This implies that at $\lambda = \infty$, ϕ has a Taylor expansion

$$\phi(\mathbf{t}, \lambda) \sim 1 + \phi^{(1)}\lambda^{-1} + \phi^{(2)}\lambda^{-2} + \cdots \tag{3.9}$$

where the functions $\phi^{(i)}$ are independent of λ. Let Φ be as defined in equation (3.3) and define

$$\Psi = \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix} = \begin{pmatrix} e^{\mu}\phi(\lambda) \\ e^{\bar{\mu}}\phi(\bar{\lambda}) \end{pmatrix} \tag{3.10}$$

Then

$$D_{\alpha_0}\Psi = 0. \tag{3.11}$$

Note that as $[D_{\alpha_0}, \partial_{t_i}] = 0$, then $\partial_{t_i}\Psi$ is also a solution of equation (3.11). We now construct operators of the form

$$L_k = \partial_{t_k} - \partial_x^k - \sum_{i=0}^{k-2} u_i \partial_x^i, \tag{3.12}$$

such that $e^{-\mu}L_k\psi$ has a Taylor series at $\lambda = \infty$ containing only strictly negative powers of λ. The first two such operators are of the form

$$\begin{aligned} L_2 &= \partial_y - \partial_x^2 + 2u, \\ L_3 &= \partial_t - \partial_x^3 + 3u\partial_x + v, \end{aligned} \tag{3.13}$$

where $u = \phi_x^{(1)}$ and $v = 3(\phi^{(1)} - \phi_{xx}^{(1)} + \phi_x^{(2)})$. Define the action of L_k on Ψ by $\mathrm{diag}(L_k, \overline{L_k})$ and that of e^{μ} by $\mathrm{diag}(e^{\mu}, e^{\bar{\mu}})$. Then $D_{\alpha_0}(L_k\Psi) = 0$, and hence

$$D_\alpha\left(\left(e^{-\mu}L_k\right)\Psi\right) = 0. \tag{3.14}$$

However $e^{-\mu}L_k\psi$ is smooth on all of $\mathbb{CP}^1$ and vanishes at $\lambda = \infty$, so therefore must vanish everywhere. Since $L_k\psi = 0$ for all values of k, this implies that

$$[L_i, L_j] = 0 \quad \text{for} \quad \text{all} \quad 1 \leq i, j \leq n. \tag{3.15}$$

These compatibility conditions give rise to the (truncated) KP hierarchy, where u satisfies all of the equations in the hierarchy. □

Proposition (Ablowitz and Clarkson) 3.5 *Given a solution to the KP-II equation $u(x, y, t)$, there exists*

- *a unique function ϕ such that $L_2(e^{\mu}\phi) = 0 = L_3(e^{\mu}\phi)$, where $\phi \to 1$ as $\lambda, x^2 + y^2 \to \infty$;*
- *a unique function $\alpha_0(\lambda, \bar{\lambda})$ such that*

$$\partial_{\bar{\lambda}}\phi(\lambda) = \alpha_0(\lambda, \bar{\lambda})e^{\bar{\mu}}\phi(\bar{\lambda})e^{-\mu}.$$

Furthermore, if $u(x, y, t)$ is a real-valued function of real variables, then α_0 satisfies the reality condition $\alpha_0(\lambda, \bar{\lambda}) = \overline{\alpha_0(\bar{\lambda}, \lambda)}$.

Proof: Suppose $\psi(x, y, t; \lambda) = e^{\mu}\phi(x, y, t; \lambda)$ is a solution of the linear system for L_2 and L_3, where $\lambda \in \mathbb{CP}^1$ and $x, y, t \in \mathbb{R}$. We will consider solutions to $L_i\psi = 0$ of the form $\psi(x, y, t; \lambda) = \phi(x, y, t; \lambda)\exp\mu$, where $\mu = \lambda x + \lambda^2 y + \lambda^3 t$, and $\phi \to 1$ as $|\lambda|$, $x^2 + y^2 \to \infty$. The form of μ is determined by the leading order terms in the Lax pair. For the first section of this result, we suppress the t-dependence of ϕ for convenience. Denote by μ_0 the t-independent part of μ, namely $\mu_0 = \lambda x + \lambda^2 y$.

Consider $L_2\psi = 0$. Then

$$\phi_y - \phi_{xx} - 2\lambda\phi_x = -2u\phi. \tag{3.16}$$

We wish $\phi \to 1$ as $|\lambda| \to \infty$, so solutions will be of the form

$$\phi = 1 + \tilde{G}(2u\phi), \tag{3.17}$$

where

$$\tilde{G}f = \int_{\mathbb{R}^2} G(x - x', y - y'; \lambda) f(x', y') dx'dy' \tag{3.18}$$

and the Green's function G satisfies

$$G_y - G_{xx} - 2\lambda G_x = -\delta(x)\delta(y), \tag{3.19}$$

with the boundary condition

$$G \to 0 \quad \text{as}\, x, y \to \infty. \tag{3.20}$$

By taking Fourier transforms of equation (3.19) we find that

$$G(x, y; \lambda) = -\frac{1}{4\pi^2}\int_{\mathbb{R}^2} \frac{\exp i(\xi x + \eta y)}{\xi^2 - 2i\xi\lambda + i\eta} d\xi d\eta. \tag{3.21}$$

To continue, we need to make the assumption that equation (3.16) together with a suitable boundary condition as $x^2 + y^2 \to \infty$, has a unique solution. This result is proved by Fokas and Sung in [10]. Using this assumption, we proceed by finding two expressions that satisfy equation (3.16) with the same boundary condition which hence must be equal.

We first consider $\partial_{\bar{\lambda}}\phi\,(x,y;\lambda)$. To do this, we shall require an expression for $\partial_{\bar{\lambda}}G\,(x,y;\lambda)$. This is obtained by a two stage process. First, the η-integration in equation (3.21) is performed. This is a straightforward contour integration and gives the result

$$G\,(x,y;\lambda) = -\frac{1}{2\pi}\mathrm{sgn}(y)\int_{-\infty}^{\infty}\exp(i\xi x - \xi(\xi-2i\lambda)y)\theta((\xi^2+2\xi\lambda_I)y)d\xi \quad (3.22)$$

where $\lambda = \lambda_R + i\lambda_I,\ \lambda_R, \lambda_I \in \mathbb{R}$ and $\theta(x)$ is the Heaviside function, defined by

$$\theta(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}. \quad (3.23)$$

Differentiating equation (3.22) with respect to $\bar{\lambda}$ gives the result

$$\partial_{\bar{\lambda}}G\,(x,y;\lambda) = \frac{i}{2\pi}\mathrm{sgn}(-\lambda_I)\exp\{\bar{\mu}_0 - \mu_0\}. \quad (3.24)$$

We now seek an expression for $\partial_{\bar{\lambda}}\phi$ by differentiating equation (3.17);

$$\begin{aligned}\frac{\partial\phi}{\partial\bar{\lambda}} &= \int_{\mathbb{R}^2}\frac{\partial G}{\partial\bar{\lambda}}\,(x-x',y-y';\lambda)\,2u(x',y')\phi\,(x',y';\lambda)\,dx'dy' \\ &+ \int_{\mathbb{R}^2}G\,(x-x',y-y';\lambda)\,2u(x',y')\frac{\partial\phi}{\partial\bar{\lambda}}\,(x',y';\lambda)\,dx'dy'.\end{aligned}$$

Substituting this expression into equation (3.24) gives

$$\begin{aligned}\frac{\partial\phi}{\partial\bar{\lambda}} &= \exp\{\bar{\mu}_0 - \mu_0\}\alpha(\lambda,\bar{\lambda}) \\ &+ \int_{\mathbb{R}^2}G\,(x-x',y-y';\lambda)\,2u(x',y')\frac{\partial\phi}{\partial\bar{\lambda}}\,(x',y';\lambda)\,dx'dy', \quad (3.25)\end{aligned}$$

where

$$\alpha(\lambda,\bar{\lambda}) = \frac{i}{2\pi}\mathrm{sgn}(-\lambda_I)\int_{\mathbb{R}^2}2u(x',y')\phi\,(x',y';\lambda)\exp(\mu_0' - \bar{\mu}_0')dx'dy', \quad (3.26)$$

and $\mu_0' = \lambda x' + \lambda^2 y'$. Hence $\partial_{\bar{\lambda}}\phi$ satisfies equation (3.16) with boundary condition

$$\partial_{\bar{\lambda}}\phi \to \exp\{\bar{\mu}_0 - \mu_0\}\alpha(\lambda,\bar{\lambda}).$$

Next, we consider $\phi(x,y;\bar{\lambda})$. For this, we shall need an expression for $G\left(x,y;\bar{\lambda}\right)$.

$$
\begin{aligned}
G\left(x,y;\bar{\lambda}\right) &= -\frac{1}{4\pi^2}\int_{\mathbb{R}^2}\frac{\exp i(\xi x+\eta y)}{\xi^2-2i\xi\bar{\lambda}+i\eta}d\xi d\eta \\
&= -\frac{1}{4\pi^2}\exp i(2\lambda_I x+4\lambda_I\lambda_R y) \\
&\times \int_{\mathbb{R}^2}\frac{\exp i((\xi-2\lambda_I)x+(\eta-4\lambda_R\lambda_I)y)}{(\xi-2\lambda_I)^2-2i(\xi-2\lambda_I)\lambda+i(\eta-4\lambda_R\lambda_I))}d\xi d\eta \\
&= \exp\{\mu_0-\bar{\mu}_0\}\,G\left(x,y;\lambda\right). \qquad (3.27)
\end{aligned}
$$

Hence $\phi(\bar{\lambda})$ satisfies

$$
\begin{aligned}
\phi\left(x,y;\bar{\lambda}\right) &= 1+\int_{\mathbb{R}^2}G\left(x-x',y-y';\bar{\lambda}\right)2u(x',y')\phi(x',y';\bar{\lambda})dx'dy'. \\
&= 1+e^{\mu_0-\bar{\mu}_0}\int_{\mathbb{R}^2}e^{\bar{\mu}_0'-\mu_0'}G(x-x',y-y';\lambda)\phi(x',y';\bar{\lambda})dx'dy'.
\end{aligned}
$$

Multiplying this expression by $\exp\{\bar{\mu}_0-\mu_0\}\alpha(\lambda,\bar{\lambda})$ and defining

$$
\tilde{\phi}=\phi(\bar{\lambda})\exp\{\bar{\mu}_0-\mu_0\}\alpha(\lambda,\bar{\lambda}),
$$

we see that $\tilde{\phi}$ satisfies

$$
\begin{aligned}
\tilde{\phi} &= \exp\{\bar{\mu}_0-\mu_0\}\alpha(\lambda,\bar{\lambda}) \\
&+ \int_{\mathbb{R}^2}G\left(x-x',y-y';\lambda\right)2u(x',y')\tilde{\phi}(x',y')dx'dy'. \qquad (3.28)
\end{aligned}
$$

Comparing this result to equation (3.25) and using the uniqueness assumption, we see that

$$
\begin{aligned}
\frac{\partial\phi}{\partial\bar{\lambda}} &= \tilde{\phi}(x,y) \\
&= \phi(x,y;\bar{\lambda})\alpha(\lambda,\bar{\lambda})\exp\{\bar{\mu}_0-\mu_0\}, \qquad (3.29)
\end{aligned}
$$

the semi-local d-bar relation for the KP-equation.

We now reintroduce the t coordinate and multiply equation (3.29) by e^{μ}, where $\mu=\lambda x+\lambda^2 y+\lambda^3 t$. This gives

$$
e^{\mu}\frac{\partial\phi}{\partial\bar{\lambda}}\left(x,y,t;\lambda\right)=\alpha(\lambda,\bar{\lambda},t)e^{\bar{\mu}}\phi\left(x,y,t;\bar{\lambda}\right)\exp\left(\lambda^3-\bar{\lambda}^3\right)t \qquad (3.30)
$$

or, alternatively,

$$\frac{\partial \psi}{\partial \bar{\lambda}}(x,y,t;\lambda) = \alpha\left(\lambda,\bar{\lambda},t\right)\psi\left(x,y,t;\bar{\lambda}\right)\exp\left(\lambda^3-\bar{\lambda}^3\right)t \tag{3.31}$$

We now consider the t dependence of ψ. $L_3\psi = 0$ and L_3 commutes with $\partial_{\bar{\lambda}}$, hence $L_3\left(\partial_{\bar{\lambda}}\psi\right) = 0$. Applying L_3 to both sides of equation (3.31) yields

$$\partial_t\left(\alpha(\lambda,\bar{\lambda},t)\exp(\lambda^3-\bar{\lambda}^3)t\right) = 0. \tag{3.32}$$

Hence α satisfies

$$\alpha(\lambda,\bar{\lambda},t) = \alpha(\lambda,\bar{\lambda},0)\exp\{-(\lambda^3-\bar{\lambda}^3)t\}. \tag{3.33}$$

Writing $\alpha(\lambda,\bar{\lambda},0)$ as $\alpha_0(\lambda,\bar{\lambda})$, and substituting into equation (3.31) yields the result

$$\frac{\partial \psi}{\partial \bar{\lambda}}(x,y,t;\lambda) = \alpha_0(\lambda,\bar{\lambda})\psi\left(x,y,t;\bar{\lambda}\right), \tag{3.34}$$

or

$$\frac{\partial \phi}{\partial \bar{\lambda}}(x,y,t;\lambda) = \exp\left(\bar{\mu}-\mu\right)\alpha_0(\lambda,\bar{\lambda})\phi\left(x,y,t;\bar{\lambda}\right). \tag{3.35}$$

By utilising the similar result obtained by the transformation $\lambda \to \bar{\lambda}$, this can be structured as a Dirac operator

$$\begin{pmatrix} \partial_{\bar{\lambda}} & \alpha \\ \bar{\alpha} & \partial_{\lambda} \end{pmatrix}\begin{pmatrix} \phi \\ \tilde{\phi} \end{pmatrix} = 0$$

where $\alpha = -\exp(\bar{\mu}-\mu)\alpha_0(\lambda,\bar{\lambda})$, and $\tilde{\phi} = \phi\left(x,y,t;\bar{\lambda}\right)$, provided that the symmetry $\overline{\alpha_0(\lambda,\bar{\lambda})} = \alpha_0(\bar{\lambda},\lambda)$ holds.

By examining the formula for $\alpha_0(\lambda,\bar{\lambda})$, it follows that $\overline{\alpha_0(\lambda,\bar{\lambda})} = \alpha_0(\bar{\lambda},\lambda)$ if and only if $\phi\left(x,y;\bar{\lambda}\right) = \overline{\phi\left(x,y;\lambda\right)}$, which in turn holds if and only if $G\left(x,y;\bar{\lambda}\right) = \overline{G\left(x,y;\lambda\right)}$.

$$\overline{G\left(x,y;\lambda\right)} = -\frac{1}{4\pi^2}\int_{\mathbb{R}^2}\frac{\exp -i(\xi x+\eta y)}{\xi^2+2i\xi\bar{\lambda}-i\eta}d\xi d\eta.$$

Making the substitution $\xi' = -\xi$ and $\eta' = -\eta$ gives

$$\begin{aligned}\overline{G\left(x,y;\lambda\right)} &= -\frac{1}{4\pi^2}\int_{\mathbb{R}^2}\frac{\exp i(\xi' x+\eta' y)}{\xi'^2-2i\xi'\bar{\lambda}+i\eta'}d\xi' d\eta' \\ &= G\left(x,y;\bar{\lambda}\right).\end{aligned}$$

Hence $\overline{\alpha_0(\lambda,\bar{\lambda})} = \alpha_0(\bar{\lambda},\lambda)$, and therefore the Dirac operator is of the required form. □

It should be pointed out that these propositions do not quite prove the theorem stated. The difficulty is in the asymptotic behaviour of α_0 for large $|\lambda|$ – we have not proved that the α_0 derived in the proof of Proposition 3.5 has the behaviour stated in Definition 3.2. The reason for this is in the properties of u. It can be shown that the function u derived in Proposition 3.4 not only satisfies the KP-II equation, but is also entire in x, y and t. To derive the asymptotic behaviour of α_0 as given in the Definition 3.2, we need to make the assumption that u is entire and use this to derive the behaviour.

This result is beyond the scope of this thesis. However, for a detailed discussion of the asymptotics of the functions involved in the KP-II inverse scattering, see Wickerhauser, [26].

3.2 The Connection to Segal-Wilson Theory

In this section, we explain the connections between the theory of the KP equations given by Segal and Wilson in [23] and the Dirac operator from the preceding section. We first give a brief summary of the relevant parts of the theory, explain the connections, and then give a partially worked example of a solution to the KP-II equation.

Segal-Wilson Theory

Let $\mathcal{H}$ be the Hilbert space $L^2(S^1,\mathbb{C})$. This has a basis given by $\{\lambda^k : k \in \mathbb{Z}\}$, and a natural decomposition $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$, where $\mathcal{H}^+$ is the subspace spanned by $\{\lambda^k : k \geq 0\}$ and $\mathcal{H}^-$ is spanned by $\{\lambda^k : k < 0\}$. We define the Grassmannian Gr of $\mathcal{H}$ as follows; a subspace W of $\mathcal{H}$ is an element of the Grassmannian if

1. the orthogonal projection $W \to \mathcal{H}^+$ has finite dimensional kernel and cokernel; and

2. the orthogonal projection $W \to \mathcal{H}^-$ is a compact operator.

The group Γ^+ of nowhere-vanishing holomorphic maps on the unit disc acts freely by multiplication on Gr. An element of this group can be written as

$$g = \exp\left\{\sum_{i=0}^{\infty} t_i \lambda^i\right\}.$$

Given an element W of the Grassmannian, we define the group Γ_+^W by

$$\Gamma_+^W = \left\{g \in \Gamma_+ : g^{-1}W \,\text{is transverse to}\, \mathcal{H}_-\right\}.$$

The following result then holds (and is proved in [23]).

Proposition 3.6 *Given $W \in Gr$, there exists a function $\psi_W(g, \lambda)$ for $g \in \Gamma_+^W$ and $\lambda \in S^1$ such that*

1. *$\psi_W(g, \cdot) \in W$ for each $g \in \Gamma_+^W$,*

2. *ψ_W has the form $\psi_W = g\,(1 + \sum_{i=1}^{\infty} a_i(g)\lambda^{-1})$, where the functions a_i are analytic on Γ_+^W and extend to meromorphic functions on Γ_+.*

Furthermore, there exists a hierarchy of operators L_n of the form

$$L_n = \partial_{t_n} - \sum_{i=0}^{n-2} u^i \partial_{t_1}^i$$

such that $L_n \psi_W = 0$ for all n.

This hierarchy of operators is the KP hierarchy.

Segal-Wilson Theory and the Dirac operator

Clearly, there must be a connection between the two theories of the KP equation. Consider the d-bar equation

$$\partial_{\bar{\lambda}}\psi + \alpha_0(\lambda, \bar{\lambda})\bar{\psi} = 0. \tag{3.36}$$

For simplicity, we assume that α_0 has compact support on the unit disc. We claim that the solution space to this equation is an element of the Grassmannian.

Let W be the space given by the solution space of the equation, restricted to $|\lambda| = 1$. Now consider the kernel K of the orthogonal projection $W \to \mathcal{H}^+$. If $\psi \in K$, then ψ satisfies $\partial_{\bar{\lambda}}\psi + \alpha_0\bar{\psi} = 0$ on the interior of the unit disc. Also, $\psi \in \mathcal{H}^-$, so ψ satisfies

$$\begin{aligned} 0 &= \partial_{\bar{\lambda}}\psi \\ &= \partial_{\bar{\lambda}}\psi + \alpha_0\bar{\psi} \end{aligned}$$

on the exterior of the unit disc (as α_0 has compact support on the disc). Hence ψ satisfies an elliptic equation (3.36) on the whole of $\mathbb{CP}^1$. The space of such solutions is finite-dimensional. Now suppose that ψ^+ is an element of $\mathcal{H}^+$. Then, as equation (3.36) is elliptic, there exists a solution of the equation such that $\psi = \psi^+$ on $|\lambda| = 1$. Hence the cokernel of the orthogonal projection is empty (and hence finite-dimensional), and the orthogonal projection onto $\mathcal{H}^+$ is Fredholm. The fact that the orthogonal projection onto $\mathcal{H}^-$ is compact is similar, relying again on the fact that the equation is elliptic.

The element of the group Γ^+ is simply the function e^μ from the previous section. The choice of the form of ψ_W (i.e. choosing an element of W) in the Proposition is equivalent to the choice of boundary condition for $\phi = e^{-\mu}\psi$.

Hence the Dirac operator defines an element W of the Grassmannian, the boundary condition then fixes an element ψ of W which then defines the KP hierarchy.

A Power Series Solution

For our example, we consider $\alpha(\lambda, \bar{\lambda}) = \theta(1 - |\lambda|^2)$, θ being the Heaviside function. We consider power series solutions for ψ. We will denote by ψ^+ the power series for $|\lambda| > 1$ and by ψ^- the power series for $|\lambda| < 1$. The coefficients of these series will be real-valued functions of x, y and t.

In the region $|\lambda| < 1$, ψ is continuous and real-analytic, being a solution to an elliptic equation with analytic coefficients, so therefore it has the power series

$$\psi^-(\lambda, \bar{\lambda}) = \sum_{i,j=0}^{\infty} a_{ij} \lambda^i \bar{\lambda}^j. \tag{3.37}$$

The reality condition $\psi(\lambda, \bar{\lambda}) = \overline{\psi(\bar{\lambda}, \lambda)}$ gives that the coefficients a_{ij} must be real. The d-bar relation

$$\partial_{\bar{\lambda}} \psi^-(\lambda, \bar{\lambda}) = \psi^-(\bar{\lambda}, \lambda), \tag{3.38}$$

gives the relationship

$$(j+1) a_{i,j+1} = a_{ji}. \tag{3.39}$$

From this relation, it follows that the space of solutions for ψ^- is spanned by $\{\psi_n : n = 0, 1, 2, \ldots\}$, where ψ_n is given by

$$\psi_n = a_{n0}(x, y, t) \left\{ \sum_{k=0}^{\infty} \frac{n!}{k!(n+k)!} \left(\lambda^{n+k} \bar{\lambda}^k + \frac{1}{n+k+1} \lambda^k \bar{\lambda}^{n+k+1} \right) \right\}, \tag{3.40}$$

where $a_{n0}(x, y, t)$ are arbitrary functions of x, y and t. On the boundary, this gives the element of the Grassmannian as the subspace of $L^2(S^1; \mathbb{C})$ spanned by

$$\tilde{\psi}_n = \lambda^n + \frac{J_n}{\lambda^{n+1}}, \quad n = 0, 1, 2, \ldots \tag{3.41}$$

where

$$J_n = \left(\sum_{k=0}^{\infty} \frac{1}{k!(n+k)!} \right)^{-1} \sum_{k=0}^{\infty} \frac{1}{k!(n+k+1)!}.$$

Hence

$$\psi^- = \sum_{n=0}^{\infty} a_n(x, y, t) \tilde{\psi}_n(\lambda) \quad \text{on } |\lambda| = 1. \tag{3.42}$$

ψ^+ satisfies

$$\partial_{\bar{\lambda}}\psi^+ = 0, \quad |\lambda| \geq 1, \tag{3.43}$$

and as such, being holomorphic in a region containing $\lambda = \infty$, and such that $\psi \exp(-\{\lambda x + \lambda^2 y + \lambda^3 t\}) \to 1$ as $\lambda \to \infty$, can be given the expression

$$\psi^+(\lambda, \bar{\lambda}) = e^{\lambda x + \lambda^2 y + \lambda^3 t} \sum_{i=0}^{\infty} b_i \lambda^{-i}, \tag{3.44}$$

where b_i are functions of x, y and t, and $b_0 = 1$.

To find which element of this space is the required solution, we need to impose the boundary condition on ψ as $\lambda \to \infty$. In other words, we need the expressions for ψ_- in equation (3.42) and for ψ_+ in equation (3.44) to be equal. For ease of calculation, we multiply both expressions by $e^{-\mu}$, giving

$$\left(\sum_{k=0}^{\infty} c_k \lambda^k\right)\left(\sum_{k=0}^{\infty} a_k \tilde{\psi}_k\right) = \sum_{k=0}^{\infty} b_k \lambda^{-k}, \tag{3.45}$$

where

$$\sum_{i=0}^{\infty} c_i \lambda^i = \exp\{-(\lambda x + \lambda^2 y + \lambda^3 t)\}. \tag{3.46}$$

This gives an infinite number of equations that can be divided into two sets. The first, derived by considering the coefficients on λ^k for $k \geq 0$ give an semi-infinite array of equations in a_n which do not involve the coefficients b_n. These equations can be expressed by the matrix equation

$$(C + \tilde{C}J)\mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \tag{3.47}$$

where C, $\tilde{C}$ and J are all semi-infinite matrices given by

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ c_1 & 1 & 0 & 0 & \cdots \\ c_2 & c_1 & 1 & 0 & \cdots \\ c_3 & c_2 & c_1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \tilde{C} = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 & \cdots \\ c_2 & c_3 & c_4 & c_5 & \cdots \\ c_3 & c_4 & c_5 & c_6 & \cdots \\ c_4 & c_5 & c_6 & c_7 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{3.48}$$

$J = \mathrm{diag}(J_0, J_1, \ldots)$, and $\mathbf{a} = (a_0, a_1, \ldots)^T$.

The c_n are known functions of x, y and t, and therefore we can (in theory) determine the unknowns a_n from this equation. Note that doing this in practise is somewhat difficult. However we would expect that this equation would have a unique solution. We can use the coefficients of λ^k for $k < 0$ to calculate the coefficients b_n once the a_n have been found. However as we are seeking to determine $u(x, y, t)$, the corresponding solution to the KP-II equation, this is unnecessary. ψ satisfies the first Lax operator $L_2 = \partial_y - \partial_x^2 + 2u$ for all values of λ. Setting $\lambda = 0$ gives that

$$(\partial_y - \partial_x^2 + 2u)a_{00} = 0, \tag{3.49}$$

from which it follows that

$$\begin{aligned} u &= -\frac{(\partial_y - \partial_x^2)a_{00}}{2a_{00}} \\ &= -\frac{(\partial_y - \partial_x^2)a_0}{2a_0}. \end{aligned} \tag{3.50}$$

As we cannot determine a_0 explicitly, this is as far as we can take this calculation. We proceed to try and get an idea of the nature of the solution by considering a linearisation of the problem.

The linearised case

We linearise the problem by setting

$$\phi^{\pm}(x, y; \lambda) = \phi_0^{\pm}(x, y; \lambda) + \epsilon\phi_1^{\pm}(x, y; \lambda) + O(\epsilon^2),$$

$$\begin{aligned}\alpha(\lambda,\bar{\lambda}) &= \epsilon\theta(1-|\lambda|^2),\\ u(x,y,t) &= \epsilon u_1(x,y,t)+O(\epsilon^2).\end{aligned} \tag{3.51}$$

Considering the terms in zeroth and first order in ϵ gives

$$\begin{aligned}\partial_{\bar{\lambda}}\phi_0^+ &= 0, \qquad |\lambda|\geq 1,\\ \partial_{\bar{\lambda}}\phi_0^- &= 0, \qquad |\lambda|\leq 1,\\ \partial_{\bar{\lambda}}\phi_1^+ &= 0, \qquad |\lambda|\geq 1,\\ \partial_{\bar{\lambda}}\phi_1^- &= \exp\{\bar{\mu}-\mu\}, \qquad |\lambda|\leq 1\end{aligned} \tag{3.52}$$

We need $\phi^+=\phi^-$ on the boundary, meaning that

$$\phi_0^+=\phi_0^- \quad \text{and} \quad \phi_1^+=\phi_1^-, \quad |\lambda|=1, \tag{3.53}$$

and $\phi^+\to 1$ as $\lambda\to\infty$, meaning

$$\phi_0^+\to 1 \quad \text{and} \quad \phi_1^+\to 0, \quad |\lambda|\to\infty. \tag{3.54}$$

We can see immediately that $\phi_0^\pm=1$. Now consider $\phi_1^-(\lambda,\bar{\lambda})$;

$$\phi_1^-(\lambda,\bar{\lambda})=\frac{1}{2\pi i}\int_{|\xi|=1}\frac{\phi_1^-(\xi,\bar{\xi})}{\xi-\lambda}d\xi+\frac{1}{2\pi i}\int_{|\xi|\leq 1}\frac{\partial_{\bar{\xi}}\phi_1^-(\xi,\bar{\xi})}{\xi-\lambda}d\xi d\bar{\xi}. \tag{3.55}$$

ϕ_1^- is equal to ϕ_1^+ on the boundary and ϕ_1^+ is holomorphic on $|\lambda|>1$ and tends to zero, hence has the expansion

$$\phi_1^+=\frac{a_1(x,y,t)}{\lambda}+\frac{a_2(x,y,t)}{\lambda^2}+\cdots.$$

Hence the contour integral part of equation (3.55) gives

$$\begin{aligned}\frac{1}{2\pi i}\int_{|\xi|=1}\frac{\phi_1^-(\xi,\bar{\xi})}{\xi-\lambda}d\xi &= \frac{1}{2\pi i}\int_{|\xi|=1}\sum_{n=1}^{\infty}\frac{a_n}{\xi^n(\xi-\lambda)}d\xi\\ &= \sum_{n=1}^{\infty}\left(\frac{1}{2\pi i}\int_{|\xi|=1}\frac{a_n}{\xi^n(\xi-\lambda)}d\xi\right)\\ &= 0\end{aligned} \tag{3.56}$$

by basic contour integration. Hence

$$\begin{aligned}\phi_1^-(\lambda,\bar{\lambda}) &= \frac{1}{2\pi i}\int_{|\xi|\le 1}\frac{\partial_{\bar{\xi}}\phi_1^-(\xi,\bar{\xi})}{\xi-\lambda}d\xi d\bar{\xi} \\ &= \frac{1}{2\pi i}\int_{|\xi|\le 1}\frac{e^{\mu_{\bar{\xi}}-\mu_{\xi}}}{\xi-\lambda}d\xi d\bar{\xi},\end{aligned} \tag{3.57}$$

where $\mu_\xi = \xi x + \xi^2 y + \xi^3 t$. Now, putting the expression for ϕ^- into the Lax equation $L_2\psi = 0$ and rearranging, this gives

$$\epsilon u_1(x,y,t) = \frac{(\partial_x^2 - \partial_y)\left[e^{\mu}(\phi_0^- + \epsilon\phi_1^-)\right]}{2e^{\mu}(\phi_0^- + \epsilon\phi_1^-)}. \tag{3.58}$$

Simplifying this expression, we find that

$$\begin{aligned}u_1(x,y,t) &= \frac{1}{2\pi i}\int_{|\xi|\le 1}\left(\xi-\bar{\xi}\right)e^{\mu_{\bar{\xi}}-\mu_{\xi}}d\xi d\bar{\xi} \\ &= \partial_x\left(-\frac{1}{2\pi i}\int_{|\xi|\le 1}e^{\mu_{\bar{\xi}}-\mu_{\xi}}d\xi d\bar{\xi}\right).\end{aligned} \tag{3.59}$$

This is the simplest form that the solution can be presented in. One thing to note is that, in common with all solutions of the full KP-II equation that we are considering, it has a natural expression as a derivative with respect to x of another well-behaved function - the solutions we have looked at for the full KP equation are all of the form $u = \phi_x^{(1)}$.

Solutions of the KP-II Equation

The fact that we have failed to express the solution to the Dirac operator and hence of the KP-II equation in closed form is not something that is particularly surprising. There is a distinct lack of real explicit solutions to the KP-II equation satisfying the given boundary conditions in the literature, which would seem to contrast the fact that the equation is deemed integrable. This fact will be discussed further in the conclusion.

3.3 The Problems of Localisation

One of the most important points about the Ward correspondence is the fact that it is a local construction, i.e. that it exists over finite (connected) regions of space-time. Thus far the construction we have proposed is a global one, relying on the boundary conditions as x and y tend to infinity. In this section we explain how the boundary conditions at infinity give rise to the nonholomorphy of the Green's function and hence ϕ and explain some of the problems preventing us from deducing a local version of the Dirac operator.

The Nonholomorphy of the Green's Function

Consider the first Lax operator for ϕ in the KP-II Lax pair. It is defined by

$$\hat{L}_2 = e^{-\mu} L_2 e^{\mu}, \tag{3.60}$$

where L_2 has no dependence on λ and $\mu = \lambda x + \lambda^2 y$ is holomorphic in λ. ϕ satisfies $\hat{L}_2 \phi = 0$ and $\phi \to 1$ as $x^2 + y^2 \to \infty$ and $\lambda \to \infty$. Given these conditions, there appears to be no immediate reason why ϕ should be nonholomorphic everywhere in λ. The Green's function for the differential part of L_2 is known, and is given by

$$G_\psi(x, y; \lambda) = -\frac{1}{2\pi}\sqrt{\frac{\pi}{y}}\theta(y) e^{-\frac{x^2}{4y}}. \tag{3.61}$$

From equation (3.60) it would seem that a suitable Green's function for ϕ would be $\tilde{G}_\phi = e^{-\mu} G_\psi$ which would be holomorphic in λ. The reason why $\tilde{G}_\phi$ is not a suitable Green's function is due to the boundedness condition on ϕ. For a bounded solution to the Lax pair, we require that the Green's function is bounded for all x, y, and in the case of $\tilde{G}_\phi$, λ. The Green's function G_ψ is bounded, but the suggested function $\tilde{G}_\phi$ clearly is not. To obtain the required Green's function for ϕ, we could either calculate it directly (as we have done in the proof to Proposition 3.5) or we could add to the above $\tilde{G}_\phi$ a solution of $(\partial_y - \partial_x^2 - 2\lambda\partial_x) = 0$ which has the "opposite" behaviour for large x, y or

λ - in other words, such that the sum is bounded. This additional term is given by $\mathcal{G}_\phi$, where

$$\mathcal{G}_\phi(x,y) = -\frac{1}{2\pi}e^{-\frac{x^2}{4y}}e^{-\lambda x-\lambda^2 y}\mathrm{sgn}\lambda_I \int_{-i\bar{\lambda}-\frac{ix}{2y}}^{-i\lambda-\frac{ix}{2y}} e^{-y\rho^2}d\rho. \tag{3.62}$$

Hence it is from this term that the nonholomorphy derives and this term is required due to the boundary condition on ϕ as $x^2+y^2 \to \infty$. Even if we work locally in x and y, we still require that ϕ be bounded as $|\lambda| \to \infty$. This may not be enough to generate a suitable local d-bar relation.

A Possible Localised Dirac Operator

In this section, we consider the possibility of deriving a d-bar operator of a smaller region of the x, y-plane.

Suppose we consider a closed, bounded region of $\mathbb{R}^2$, with ϕ determined on the boundary. $\hat{L}_2$ is a parabolic operator, similar to the heat operator, and as such, the problem is overdetermined, and hence will not have a solution. Instead, we shall consider a closed but not bounded region of $\mathbb{R}^2$, for example the semi-infinite strip, with boundary conditions for ϕ on $x=0$, $x=1$ and $y=0$.

The problem now lies in the calculation of the Green's function for ϕ. The Green's function G_ψ is known and as with the global case previously, its conjugation with $e^{-\mu}$ is unbounded as $\lambda \to \infty$. We have not been able to calculate solutions in the desired region either to

$$\hat{L}_2 G_\phi = \delta(x)\delta(y), \quad \text{or} \quad \hat{L}_2\tilde{G}_\phi = 0.$$

Until one of these can be done, a local d-bar relation, if it exists, cannot be constructed.

The issue of localising the Dirac operator for the Davey-Stewartson equations is discussed at length in Chapter 6.

Chapter 4

The Nonlocal Riemann-Hilbert Problem and the KP-I equation

Any discussion of the KP equations would be incomplete without considering the KP-I equation. Although the KP-I and KP-II equations have markedly different inverse scattering transforms, both are integrable systems and any analogous geometric construction to the Ward construction for the KP-II equation would be incomplete if it did not in some way also incorporate the KP-I equation.

In this chapter, we consider in detail the inverse scattering of the KP-I equation, deriving the Lax pair from the nonlocal Riemann-Hilbert problem as first presented in [2] and vice versa.

Following this, we then consider a special class of solutions (soliton, or lump solutions) which does have an associated geometric construction, which we describe in detail.

4.1 Derivation of the Lax Pair

The dressing method was introduced by Zakharov and Shabat in [27]as a tool for obtaining integrable nonlinear equations. We will use the method to derive the Lax pair of the KP-I equation from a scalar nonlocal Riemann-Hilbert problem with certain symmetries imposed.

The nonlocal Riemann-Hilbert problem we consider is as follows. ϕ is taken to be holomorphic everywhere in λ except the real axis, where it has a continuous limit $\phi_\pm$ and the jump in that limit is given by

$$(\phi_+ - \phi_-)(x, y, t; \lambda) = \int_{\mathbb{R}} \alpha(\lambda, \kappa) e^{\mu_\kappa - \mu_\lambda} \phi_-(x, y, t) d\kappa, \tag{4.1}$$

where $\mu_\lambda = i(\lambda x + \lambda^2 y - \lambda^3 t)$ and μ_κ is similarly defined. ϕ_+ is holomorphic in the upper half-plane and ϕ_- is holomorphic in the lower half-plane. We assume that α vanishes for large λ and κ and that $\phi \to 1$ as $\lambda \to \infty$. We assume that this problem has a unique solution for suitable boundary conditions, i.e. $\phi_\pm$ bounded as $|\lambda| \to \infty$ on their respective domains.

Proposition 4.1 *If ϕ satisfies the nonlocal Riemann-Hilbert problem (4.1), then ϕ satisfies the Lax pair for the KP-I equation for some $u(x, y, t)$.*

Proof: As $\alpha = 0$ for large λ, for such λ, $\phi_+ = \phi_-$. Hence, as ϕ_+ and ϕ_- are both holomorphic away from the real λ-axis, we can assume that for large λ, ϕ has the Taylor series

$$\phi \sim 1 + \lambda^{-1}\phi^{(1)} + O(\lambda^{-2}) \quad \text{as} \quad \lambda \to \infty, \tag{4.2}$$

regardless of which region of the complex plane λ lies in.

Define the operator D_x by

$$D_x = e^{-\mu_\lambda} \partial_x e^{\mu_\lambda} = \partial_x + i\lambda,$$

and define operators D_y and D_t similarly. These operators commute with the integral operator in the nonlocal Riemann-Hilbert problem, and hence if ϕ satisfies the nonlocal Riemann-Hilbert problem, so does $D_x\phi$. As for the d-bar problem, we now seek combinations of these operators that, acting on ϕ as given in equation (4.2) has terms of highest order λ^{-1}. These expressions will also satisfy the nonlocal Riemann-Hilbert problem, whilst tending to zero as $\lambda \to \infty$. By uniqueness of solution to equation (4.1), this gives that the expressions must be equal to zero. Thus we find that

$$\begin{aligned} \hat{L}_2\phi &= (iD_y - D_x^2 + 2u)\phi &= 0, \\ \hat{L}_3\phi &= (D_t - D_x^3 + 3uD_x + v)\phi &= 0. \end{aligned} \tag{4.3}$$

Writing $\psi = \phi e^{\mu_\lambda}$ gives the KP-I Lax pair

$$\begin{aligned} L_2 &= i\partial_y - \partial_x^2 + 2u, \\ L_3 &= \partial_t - \partial_x^3 + 3u\partial_x + v, \end{aligned} \tag{4.4}$$

where

$$L_2\psi = 0 = L_3\psi, \tag{4.5}$$

and the KP-I equation is given by the compatibility condition $[L_2, L_3] = 0$.

□

4.2 Derivation of the Nonlocal Riemann-Hilbert Problem

As per the Dirac operator for the KP-II equation, we want this procedure to be reversible, so that the relationship between solutions of KP-I and potentials α for the nonlocal Riemann-Hilbert problem is one-to-one.

Proposition (Ablowitz and Clarkson) 4.2 *If the function $\phi(x, y, t; \lambda)$ uniquely satisfies equations*

$$L_2(e^{\mu}\phi) = 0 = L_3(e^{\mu}\phi),$$

where L_2 and L_3 are the Lax pair for the KP-I equation, together with the boundary condition that $\phi \to 1$ as $\lambda, x^2 + y^2 \to \infty$, then ϕ satisfies a nonlocal Riemann-Hilbert problem of the form (4.1) for some α.

Proof: We initially consider the equation $L_2(e^{\mu}\phi) = 0$ and as such will suppress the t dependence for the time being. We will define μ_λ^0 by

$$\mu_\lambda^0 = i\left(\lambda x + \lambda^2 y\right).$$

Then $L_2(e^{\mu}\phi) = 0$ implies that

$$i\phi_y - \phi_{xx} - 2i\lambda\phi_x = -2u\phi \tag{4.6}$$

As in the case for the KP-II equation, we look for solutions such that ϕ is bounded for all $x, y \in \mathbb{R}$ and $\lambda \in \mathbb{C}$. Such solutions can be written

$$\phi = 1 + \tilde{G}(\phi), \tag{4.7}$$

where

$$\tilde{G}f = \int_{\mathbb{R}^2} G\left(x - x', y - y'; \lambda\right) 2u(x', y')f(x', y')dx'dy' \tag{4.8}$$

and the Green's function G satisfies

$$iG_y - G_{xx} - 2i\lambda G_x = -\delta(x)\delta(y) \tag{4.9}$$

Taking Fourier transforms of (4.9) in both x and y gives

$$G\left(x, y; \lambda\right) = \frac{i}{2\pi}\int_{\mathbb{R}} e^{i\xi x} g(y; \lambda, \xi)d\xi, \tag{4.10}$$

where

$$g(y; \lambda, \xi) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{e^{i\eta y}}{\eta - \xi(\xi + 2\lambda)}d\eta. \tag{4.11}$$

In the KP-II equation, the singularities in the integral expression of the Green's function were discrete and integrable. This expression has singularities for all $\lambda_I = 0$, and as such g is undefined on the real axis.

Define

$$g_\pm(y; \lambda_R, \xi) = \lim_{\lambda_I \to 0_\pm} g(y; \lambda, \xi). \tag{4.12}$$

These functions can be analytically continued for $\lambda_I > 0$ and $\lambda_I < 0$ respectively.

The integral in the expression for $g_{\pm}(y;\lambda,\xi)$ can be evaluated by a simple contour integral. To calculate $g_{+}(y;\lambda,\xi)$, put $\lambda = \lambda_R + i\epsilon$, where $0 < \epsilon \ll 1$. If $y > 0$, use the semicircular contour, radius R, in the upper half-plane. If $y < 0$, use the semicircular contour, radius R, in the lower half-plane. The only pole of the integrand is at $\eta = \xi(\xi + 2\lambda) = \xi^2 + 2\lambda_R\xi + 2i\epsilon\xi$, which is in the upper half-plane if $\xi > 0$ and the lower half-plane if $\xi < 0$. By the theory of residues, it follows that

$$g_{\pm}(y;\lambda,\xi) = e^{i\xi(\xi+2\lambda)y}\{\theta(y)\theta(\pm\xi) - \theta(-y)\theta(\mp\xi)\}, \tag{4.13}$$

where $\theta(x)$ is the Heaviside step function, defined in (3.23). We define $G_{\pm}$ by

$$G_{\pm}(x,y;\lambda) = \frac{i}{2\pi}\int_{\mathbb{R}} e^{i\xi x} g_{\pm}(y;\lambda,\xi)d\xi, \tag{4.14}$$

and $\phi_{\pm}$ as the solutions to

$$\phi_{\pm} = 1 + \tilde{G_{\pm}}(\phi_{\pm}). \tag{4.15}$$

It follows immediately that $G_{\pm}$ can be written

$$G_{\pm}(x,y;\lambda) = \int_{\mathbb{R}} e^{i\xi x + i\xi(\xi+2\lambda)y}\{\theta(y)\theta(\pm\xi) - \theta(-y)\theta(\mp\xi)\}d\xi. \tag{4.16}$$

We now consider, for λ real, the function $\Delta(x,y;\lambda)$, defined by

$$\Delta(x,y;\lambda) = \phi_{+}(x,y;\lambda) - \phi_{-}(x,y;\lambda) \tag{4.17}$$

Then

$$\begin{aligned}\Delta(x,y;\lambda) = &\int_{\mathbb{R}^2} [G_{+} - G_{-}](x-x', y-y';\lambda)\, 2u(x',y')\phi_{+}(x',y';\lambda)\, dx'dy' \\ &+ \int_{\mathbb{R}^2} G_{-}(x-x', y-y';\lambda)\, 2u(x',y')\Delta(x',y';\lambda)dx'dy'.\end{aligned}$$

It follows that

$$[G_{+} - G_{-}](x,y;\lambda) = \frac{i}{2\pi}\int_{\mathbb{R}} \operatorname{sgn}\xi \exp\{i(\xi x + \xi(\xi+2\lambda)y)\}d\xi. \tag{4.18}$$

Define μ'_λ (and similarly μ'_κ) by

$$\mu'_\lambda = i(\lambda x' + \lambda^2 y'),$$

and $T(\lambda, \kappa)$ by

$$T(\lambda,\kappa) = \frac{i}{2\pi}\mathrm{sgn}(\lambda-\kappa)\int_{\mathbb{R}^2} 2u(x',y')\phi_+(x',y';\lambda)\, e^{\mu'_\lambda - \mu'_\kappa} dx'dy' \tag{4.19}$$

where $\lambda, \kappa \in \mathbb{R}$. Then

$$\Delta(x,y;\lambda) = \int_{\mathbb{R}} T(\lambda,\kappa)\, e^{\mu^0_\kappa - \mu^0_\lambda} d\kappa + \tilde{G}_-(\Delta). \tag{4.20}$$

Now consider the function $\Gamma(x, y; \lambda, \kappa)$ which satisfies equation (4.6) and also satisfies

$$\Gamma(x,y;\lambda,\kappa) = e^{\mu^0_\kappa - \mu^0_\lambda} + \tilde{G}_-(\Gamma) \tag{4.21}$$

This is simply the solution to $L_2\Gamma = 0$ with the boundary condition on Γ that $\Gamma \to e^{\mu^0_\kappa - \mu^0_\lambda}$ as $|\lambda| \to \infty$.

Multiplying the above equation by $T(\lambda, \kappa)$ and integrating with respect to κ gives the result

$$\phi_+(x,y;\lambda) - \phi_-(x,y;\lambda) = \int_{\mathbb{R}} T(\lambda,\kappa)\,\Gamma(x,y;\lambda,\kappa) d\kappa. \tag{4.22}$$

To express this as a nonlocal Riemann-Hilbert problem, we must express Γ in terms of ϕ_-. As for the KP equation, we look for expressions that satisfy the same integral equation as each other, where that integral equation has a unique solution. Hence the expressions will be equivalent. In this case, we define an integral operator K by

$$Kf = f - e^{\mu_\lambda}\tilde{G}_-\left(e^{-\mu_\lambda} f\right). \tag{4.23}$$

From the uniqueness of solution to equation (4.7), it follows that the equation $Kf = g$, for suitably well-behaved g, will have a unique solution. It can be shown that the expressions $\partial_\lambda(\Gamma e^{\mu_\lambda})$ and $\phi_-(\lambda)e^{\mu_\lambda}F(\lambda,\kappa)$, where

$$F(\lambda,\kappa) = \frac{i}{2\pi}\int_{\mathbb{R}^2} 2u(x',y')\Gamma(x',y';\lambda,\kappa) dx'dy', \tag{4.24}$$

both satisfy $Kf = F(\lambda,\kappa)e^{\mu_\lambda}$ and therefore

$$\partial_\lambda(\Gamma e^{\mu_\lambda}) = \phi_-(\lambda)e^{\mu_\lambda}F(\lambda,\kappa). \tag{4.25}$$

We wish to replace Γ in equation (4.22) with an expression in terms of $\phi_-(\lambda)$. Hence we need to integrate equation (4.25). We can see from equation (4.21) than $\Gamma(\lambda,\lambda)$ satisfies the same integral equation as $\phi_-(\lambda)$ and hence $\Gamma(\lambda,\lambda) = \phi_-(\lambda)$. Using this, we can integrate equation (4.25) to obtain

$$\Gamma(\lambda,\kappa) = \phi_-(\lambda)e^{\mu^0_\kappa-\mu^0_\lambda} + \int_\kappa^\lambda F(\rho,\kappa)\phi_-(\rho)e^{\mu^0_\kappa-\mu^0_\lambda}d\rho. \tag{4.26}$$

Substituting this expression into equation (4.22), gives that

$$\begin{aligned}
(\phi_+ - \phi_-)(\lambda) &= \int_{\mathbb{R}} T(\lambda,\kappa)e^{\mu_\kappa-\mu_\lambda}\phi_-(\kappa)d\kappa \\
&+ \int_\infty^\lambda \left(\left(\int_\infty^\kappa + \int_\lambda^\infty\right) T(\lambda,\kappa)F(\kappa,\rho)e^{\mu_\kappa-\mu_\lambda}d\rho\right)\phi_-(\kappa)d\kappa \\
&- \int_\lambda^\infty \left(\left(\int_\kappa^\infty + \int_\infty^\lambda\right) T(\lambda,\kappa)F(\kappa,\rho)e^{\mu_\kappa-\mu_\lambda}d\rho\right)\phi_-(\kappa)d\kappa.
\end{aligned}$$

Hence we have on the right-hand side, an integral expression including ϕ_- in each term, which is what we are looking for. Therefore, defining $\alpha(\lambda,\kappa)$ as

$$\begin{aligned}
\alpha(\lambda,\kappa) &= T(\lambda,\kappa) + \operatorname{sgn}(\lambda-\kappa)\int_{\mathbb{R}} T(\lambda,\rho)F(\kappa,\rho)d\rho \\
&- \theta(\lambda-\kappa)\int_\kappa^\lambda T(\lambda,\rho)F(\kappa,\rho)d\rho \\
&+ \theta(\kappa-\lambda)\int_\lambda^\kappa T(\lambda,\rho)F(\kappa,\rho)d\rho,
\end{aligned} \tag{4.27}$$

we obtain that

$$(\phi_+ - \phi_-)(x,y,t;\lambda) = \int_{\mathbb{R}} \alpha(\lambda,\kappa)e^{\mu^0_\kappa-\mu^0_\lambda}\phi_-(x,y,t;\kappa)d\kappa, \tag{4.28}$$

We now must consider the t-dependence of ϕ. Reintroducing the t coordinate gives

$$(\phi_+ - \phi_-)(x,y,t;\lambda) = \int_{\mathbb{R}} \alpha(\lambda,\kappa,t)e^{\mu^0_\kappa-\mu^0_\lambda}\phi_-(x,y,t;\kappa)\,d\kappa. \tag{4.29}$$

Multiplying this equation by e^{μ_λ} and substituting in the expression for ψ gives

$$(\psi_+ - \psi_-)(x, y, t; \lambda) = \int_{\mathbb{R}} \alpha(\lambda, \kappa, t)\psi_-(x, y, t; \kappa)\, d\kappa. \tag{4.30}$$

Now applying the operator L_3 to this expression, the fact that $L_3\psi_\pm = 0$ gives that

$$\begin{aligned} 0 &= L_3\Big\{\int_{\mathbb{R}} \alpha(\lambda, \kappa, t)\exp\{i(\kappa^3 - \lambda^3)t\}\psi_-(x, y, t; \kappa)\, d\kappa\Big\} \\ &= \int_{\mathbb{R}} L_3\{\alpha(\lambda, \kappa, t)\exp\{i(\kappa^3 - \lambda^3)t\}\psi_-(x, y, t; \kappa)\,\}d\kappa \\ &= \int_{\mathbb{R}} \partial_t\{\alpha(\lambda, \kappa, t)\exp\{i(\kappa^3 - \lambda^3)t\}\psi_-(x, y, t; \kappa)\, d\kappa. \end{aligned}$$

It follows from this that

$$\alpha(\lambda, \kappa, t) = \alpha(\lambda, \kappa, 0)\exp\{-i(\kappa^3 - \lambda^3)t\}. \tag{4.31}$$

Hence, this gives the expression

$$\phi_+(x, y, t; \lambda) = \phi_-(x, y, t; \lambda) + \int_{\mathbb{R}} \alpha(\lambda, \kappa)e^{\mu_\kappa - \mu_\lambda}\phi_-(x, y, t; \kappa)d\kappa \tag{4.32}$$

the desired nonlocal Riemann-Hilbert problem. □

It should be noted that there is a more attractive expression for α than that given in equation (4.27). It can be shown without too much difficulty that

$$\alpha(\lambda, \kappa) = \frac{i}{2\pi}\text{sgn}(\kappa - \lambda)\int_{\mathbb{R}^2} 2u\Gamma(\lambda, \kappa)dx'dy'. \tag{4.33}$$

4.3 Lump Solutions of the KP-I Equation

In Proposition 4.2 we have made the assumption that the Lax equations with boundary condition have a unique solution. While this is true for the KP-II equations, it is not true for the KP-I. In this section, we take a closer look at the soliton, or lump, solutions of the KP-I equation, with a geometrical construction.

4.3.1 The One-Lump Solution

A lump solution is a solution of the KP-I equation where the Lax pair admits solutions that tend to 1 as $\lambda \to \infty$ and are holomorphic everywhere in λ except at $2n$ simple poles, $\{\lambda_1, \ldots, \lambda_n, \bar{\lambda}_1, \ldots, \bar{\lambda}_n\}$, where $\operatorname{Im} \lambda_i > 0$. The solution to the Lax pair ϕ is given by

$$\phi(x, y, t; \lambda) = 1 + i \sum_{i=1}^{n} \left\{ \frac{\phi_i^+(x, y, t)}{\lambda - \lambda_i} + \frac{\phi_i^-(x, y, t)}{\bar{\lambda} - \bar{\lambda}_i} \right\}. \tag{4.34}$$

The factor of i guarantees that for real solutions $u(x, y, t)$ to the KP-I equation, $\phi_i^{\pm}(x, y, t)$ will be real. We will assume this form of the solution, and hence derive the properties of the functions $\phi_i^{\pm}(x, y, t)$. For ease of calculation, we will first consider the one-lump solution generated by

$$\phi(x, y, t; \lambda) = 1 + i \frac{\phi_1^+(x, y, t)}{\lambda - \lambda_1} + i \frac{\phi_1^-(x, y, t)}{\lambda - \bar{\lambda}_1}. \tag{4.35}$$

For this to be a solution of

$$L_2 \phi = (i\partial_y - \partial_x^2 - 2i\lambda\partial_x + 2u)\phi = 0, \tag{4.36}$$

we require that

$$u(x, y, t) = \frac{(i\partial_y - \partial_x^2 - 2i\lambda\partial_x)\phi}{-2\phi} \tag{4.37}$$

is independent of λ. This is true if and only if

$$(i\partial_y - \partial_x^2 - 2i\lambda_1\partial_x + 2u)\phi^+ = 0,$$

$$(i\partial_y - \partial_x^2 - 2i\bar{\lambda}_1\partial_x + 2u)\phi^- = 0,$$

and

$$u(x, y, t) = -\left\{\phi_x^+ + \phi_x^-\right\}. \tag{4.38}$$

Recall that $\phi \to 1$ as $x^2 + y^2 \to \infty$, hence $\phi^{\pm} \to 0$ as $x^2 + y^2 \to \infty$. Thus $\phi^{\pm}$ are required to be homogeneous solutions of $L_2 \phi^{\pm} = 0$ for $\lambda = \lambda_1$ and $\lambda = \bar{\lambda}_1$ respectively. Note also that for $u(x, y, t)$ to be real, it follows from equation (4.38) that $\phi^{\pm}$ must be real as well.

We can derive a solvable algebraic relationship between ϕ^+ and ϕ^- by insisting that the expression (4.36) holds for all values of λ, notably $\lambda = \lambda_1$ and $\lambda = \bar{\lambda}_1$. We also impose boundary conditions on $\phi^\pm$ as $x^2 + y^2 \to \infty$. The functions are chosen such that $\phi^+(x + 2\lambda_1 y) \to 1$ and $\phi^-(x + 2\bar{\lambda}_1 y) \to 1$ as $x^2 + y^2 \to \infty$. Writing $\psi = \phi e^{\mu_\lambda}$ and $\psi^\pm = \phi^\pm e^{\mu_\lambda}$, consider the equation $\hat{L}\psi = 0$, where $\hat{L}$ is the first KP-I Lax operator.

$$\begin{aligned} \hat{L}\psi &= \hat{L}\left(e^{\mu_\lambda} + i\frac{\psi_1^+(\lambda)}{\lambda - \lambda_1} + i\frac{\psi_1^-(\lambda)}{\lambda - \bar{\lambda}_1}\right), \\ &= \hat{L}\left(e^{\mu_\lambda} + i\frac{\psi_1^-(\lambda)}{\lambda - \bar{\lambda}_1}\right) + i\frac{\hat{L}\psi_1^+(\lambda)}{\lambda - \lambda_1}. \end{aligned} \tag{4.39}$$

Taking the limit as $\lambda \to \lambda_1$ in the above expression, and noting that $\hat{L}\psi^+ = 0$ when $\lambda = \lambda_1$ gives that

$$\begin{aligned} 0 &= \hat{L}\left(e^{\mu_{\lambda_1}} + i\frac{\psi_1^-(\lambda_1)}{\lambda_1 - \bar{\lambda}_1}\right) + i\frac{\partial}{\partial\lambda}\hat{L}\psi_1^+(\lambda)\bigg|_{\lambda=\lambda_1}, \\ &= \hat{L}\left(e^{\mu_{\lambda_1}} + i\frac{\psi_1^-(\lambda_1)}{\lambda_1 - \bar{\lambda}_1}\right) + i\hat{L}\left(\frac{\partial}{\partial\lambda}\psi_1^+(\lambda)\right)\bigg|_{\lambda=\lambda_1}, \\ &= \hat{L}\left(e^{\mu_{\lambda_1}} + i\frac{\psi_1^-(\lambda_1)}{\lambda_1 - \bar{\lambda}_1} - \rho(x, y, \lambda_1)\psi_1^+(\lambda_1)\right), \\ &= \hat{L}\left(e^{\mu_{\lambda_1}}\left[1 + i\frac{\phi_1^-}{\lambda_1 - \bar{\lambda}_1} - \rho(x, y, \lambda_1)\phi_1^+\right]\right), \end{aligned} \tag{4.40}$$

where

$$\rho(x, y, \lambda_1) = -i\frac{\partial\mu}{\partial\lambda}\bigg|_{\lambda=\lambda_1} = x + 2\lambda_1 y - 3\lambda_1^2 t.$$

Thus

$$\tilde{\phi} = \left[1 + i\frac{\phi_1^-}{\lambda - \bar{\lambda}_1} - \rho(x, y, \lambda)\phi_1^+\right]$$

satisfies $L\tilde{\phi} = 0$ at $\lambda = \lambda_1$ and $\tilde{\phi} \to 0$ as $x^2 + y^2 \to \infty$, from the boundary conditions on ϕ^+, hence $\tilde{\phi}(\lambda_1) = \gamma^+\phi_1^+$ for some constant $= \gamma^+$, i.e.

$$1 + i\frac{\phi_1^-}{\lambda_1 - \bar{\lambda}_1} - (\rho(x, y, \lambda_1) + \gamma^+)\phi_1^+ = 0. \tag{4.41}$$

Similarly it follows that

$$1 - i\frac{\phi_1^+}{\lambda_1 - \bar{\lambda}_1} - (\rho(x, y, \bar{\lambda}_1) + \gamma^-)\phi_1^- = 0. \tag{4.42}$$

Denoting $\rho(x, y, \lambda_1)$ by ρ and noting that $\rho(x, y, \lambda_1) = \bar{\rho}$, we find that

$$\begin{pmatrix} \phi^+ \\ \phi^- \end{pmatrix} = \frac{1}{(\rho + \gamma^+)(\bar{\rho} + \gamma^-) + c^2} \begin{pmatrix} (\bar{\rho} + \gamma^-) + c \\ (\rho + \gamma^+) - c \end{pmatrix}, \tag{4.43}$$

where $c = i/(\lambda_1 - \bar{\lambda}_1) \in \mathbb{R}$. Hence, using equation (4.38), we obtain a solution to the KP-I equation given by

$$u(x, y, t) = -2 \left(\frac{2c^2 - (\rho + \gamma^+)^2 - (\bar{\rho} + \gamma^-)^2}{\left[(\rho + \gamma^+)(\bar{\rho} + \gamma^-) + c^2\right]^2} \right). \tag{4.44}$$

The condition that u be real gives that $\gamma^- = \overline{\gamma^+}$ and by moving the origin in the x, y plane, we can set $\gamma^+ = 0$. We are also free to add a real constant to λ_1 without affecting the above calculations, therefore we can take λ_1 to be purely imaginary. Thus, if we let $\lambda_1 = i\kappa$, we obtain a one-lump solution

$$u(x, y, t) = -4 \left(\frac{\frac{1}{4\kappa^2} + 4\kappa^2 y^2 - (x + 3\kappa^2 t)^2}{\left[\frac{1}{4\kappa^2} + 4\kappa^2 y^2 + (x + 3\kappa^2 t)^2\right]^2} \right). \tag{4.45}$$

4.3.2 A Geometric Construction for Lump Solutions

There is a geometric construction that can be linked to the lump solutions of the KP-I equations as follows.

Theorem 4.3 *Suppose that $\mathbb{L} \to \mathbb{MT}$ is a line bundle over minitwistor space and there exists a set of points $\{\lambda_i : i = 1, \ldots, 2n\}$ such that*

1. $\mathbb{L} \simeq \mathcal{O}(2n)$*;*
2. *there exists a lift of ∂_μ such that $\mathcal{L}_{\partial_\mu} = -1$ at $\lambda = \infty$;*
3. *there exists a connection ∇ on $\mathbb{L}$ at each fibre $\lambda = \lambda_i$ in $\mathbb{MT}$ such that $\left[\mathcal{L}_{\partial_\mu}, \nabla\right] = 0$ and $\nabla_{\partial_\mu} = \mathcal{L}_{\partial_\mu}$;*
4. *there exists a trivialisation of the bundle at $\lambda = \infty$.*

Then generically in x, y, t, there exists a unique section of the bundle $\phi \in \Gamma(\mathbb{MT}, \mathbb{L})$ such that

- *$\phi(\lambda = \infty) = 1$;*
- *$(\nabla_{\partial_\lambda}\phi)|_{\lambda=\lambda_i} = 0$, where ∂_λ is the vector along the $\mathbb{CP}^1$ in $\mathbb{MT}$ corresponding to (x, y, t),*

and there exists a function $u(x, y, t)$ independent of λ such that

- *$L_i\phi = 0$ for each L_i in the Lax system associated with the KP-I hierarchy; and hence*
- *u satisfies the KP-I equation.*

Before we prove this theorem, we take a moment to consider the line bundle $\mathbb{L}$. We cover minitwistor space with the sets $U_0 = \{\lambda : |\lambda| < \infty\}$ and $U_\infty = \{\lambda : |\lambda| > \max|\lambda_i|\}$. Let $P(\lambda, \mu)$ be the patching function such that if f_0 is a frame on U_0 and f_∞ is a frame on U_∞, then $f_0 = Pf_\infty$. We have a lift of ∂_μ such that on U_∞, $\mathcal{L}_{\partial_\mu} = \partial_\mu - 1$ and on U_0, $\mathcal{L}_{\partial_\mu} = \partial_\mu$. Thus

$$\mathcal{L}_{\partial_\mu} f_\infty = f_\infty, \quad \text{and} \quad \mathcal{L}_{\partial_\mu} f_0 = 0.$$

Hence $\partial_\mu P = P$ and therefore

$$P(\lambda, \mu) = e^\mu P_0(\lambda).$$

$\mathbb{L}$ is a bundle of degree $2n$, therefore $P_0(\lambda)$ is a function of λ of degree $2n$, unique up to multiplication by a function of degree zero. We will take P_0 to be given by

$$P_0(\lambda) = \prod_{i=1}^{2n}(\lambda - \lambda_i).$$

We are now in a position to prove the theorem.

Proof of Theorem: Let ϕ_0 be the representation of the section ϕ in the frame on U_0 and ϕ_∞ that on U_∞. There exists a connection on the fibres $\lambda = \lambda_i$, say, $\nabla|_{\lambda=\lambda_i} = d + \gamma_i d\lambda$, where γ_i is constant from the assumptions, hence, at $\lambda = \lambda_i$,

$$\frac{\partial\phi_0}{\partial\lambda}(\lambda_i) = \gamma_i\phi_0(\lambda_i),$$

or, alternatively

$$\phi_0(\lambda) - \phi_0(\lambda_i) \sim (\lambda - \lambda_i)\gamma_i \phi_0(\lambda_i) \quad \text{near } \lambda = \lambda_\text{i}. \tag{4.46}$$

Given the patching function

$$P(\lambda, \mu) = e^{\mu} \prod_{i=1}^{2n} (\lambda - \lambda_i),$$

this gives that the extension of ϕ_∞ over U_0 is required to have a simple pole at $\lambda = \lambda_i$. Given the boundary condition on ϕ, this means that ϕ_∞ will be of the form

$$\phi_\infty(x, y, t; \lambda) = 1 + i \sum_{i=1}^{2n} \frac{\phi_i(x, y, t)}{\lambda - \lambda_i}, \tag{4.47}$$

for some ϕ_i independent of λ. The line bundle is of degree $2n$ so there exists $2n+1$ global sections. Thus, given the $2n$ conditions from equation (4.46) and the boundary condition as $\lambda \to \infty$, we can generically determine ϕ uniquely.

To derive the Lax pair, we now seek operators L_j so that $L_j \phi = 0$. To do this, we need to find operators L such that

$$e^{-\mu} L \phi_0 = O(|\lambda|^{2n-1}) \quad \text{near } \lambda = \infty,$$

and such that $e^{-\mu} L \phi_0$ is smooth everywhere. Then by Liouville's theorem, $L_j \phi = 0$. Consider $\tilde{L} = i\partial_y - \partial_x^2$.

$$\begin{aligned} e^{-\mu} \tilde{L} \phi_0 &= e^{-\mu} \left(i\partial_y - \partial_x^2 \right) \left(e^{\mu} \prod_{i=1}^{2n} (\lambda - \lambda_i) \right) \left(1 + i \sum_{i=1}^{2n} \frac{\phi_i}{\lambda - \lambda_i} \right) \\ &= \prod_{i=1}^{2n} (\lambda - \lambda_i) \left(i \sum_{i=1}^{2n} \frac{\tilde{L}\phi_i}{\lambda - \lambda_i} \right) - 2i\lambda \prod_{i=1}^{2n} (\lambda - \lambda_i)\, i \sum_{i=1}^{2n} \frac{\partial_x \phi_i}{\lambda - \lambda_i} \\ &= \lambda^{2n} \left[2 \sum_{i=1}^{2n} \partial_x \phi_i \right] + O(|\lambda|^{2n-1}) \end{aligned} \tag{4.48}$$

Hence setting $u(x, y, t)$ as

$$u(x, y, t) = - \sum_{i=1}^{2n} \partial_x \phi_i,$$

and $L_2 = i\partial_y - \partial_x^2 + 2u$, we see that

$$e^{-\mu} L_2 \phi_0 \sim O(|\lambda|^{2n-1}),$$

and hence

$$L_2 \phi = 0$$

everywhere. By a similar procedure, the entire Lax system $L_i \phi = 0$ associated to the KP-I hierarchy can be derived. □

One thing to consider is the reality condition. In Section 4.3.1, we choose the points where ϕ has poles as $\{\lambda_i, \bar{\lambda}_i; i = 1, \ldots, n\}$, whereas in the geometric construction described in Theorem 4.3, we take the points to be a general set of $2n$ complex numbers. The reason for the discrepancy is simple; a solution will be generated for any set of $2n$ distinct complex numbers, but it will only be real if that set consists of n distinct *strictly* complex numbers (i.e. non-real) and their complex conjugates.

It is worth pointing out that this geometric construction is only valid for the lump solutions of the KP-I equation. The complete set of solutions to the KP-I equation consists of a nonlinear superposition of the lump solutions and the solutions arising from the nonlocal Riemann-Hilbert problem. As such, this construction will need to be somehow incorporated with one that corresponds to the nonlocal Riemann-Hilbert problem to completely describe the solutions of the KP-I equation.

Despite this, it may be possible to generalise this construction to produce new lump-like solutions of the KP-I equation. For example the condition of the existence of a connection on $2n$ fibres could be replaced with an identification between fibres. To date, however, we have had little success with this.

Chapter 5

Connections between the KP and KdV Equations

The intention of this research is to develop the correspondence for the KP equations and to make clear the analogy with the Ward construction. The majority of the work has concentrated on the KP-II equation as its semi-local inverse scattering has always seemed to hold more potential for a local theory than the nonlocal inverse scattering of the KP-I equation. However we would hope that any twistor correspondence derived for the KP-II equation would be generalisable to the KP-I equation.

In this chapter, we first review how the inverse scattering techniques for the KP-I and KP-II equations reduce to that of the KdV equation, for which there is a standard twistor correspondence. We also consider what happens to the Dirac construction for the KP-II equation from the previous chapter under this reduction.

We then review the inverse scattering for both the KP-I and KP-II equations highlighting the similarities between them. Finally we describe a formal construction (the KP-σ equation) showing how the KP-I nonlocal Riemann-Hilbert problem can be viewed, at least formally, as a singular version of the d-bar problem.

5.1 Reduction from KP to KdV

The KdV equation is closely related to the KP equation. The KdV equation is given by

$$4u_t + 12uu_x - u_{xxx} = 0. \tag{5.1}$$

Clearly, any solution of the KdV equation is also a y-independent solution of the KP equation (KP-I *or* KP-II),

$$(4u_t + 12uu_x - u_{xxx})_x - 3\sigma^2 u_{yy} = 0. \tag{5.2}$$

Thus one would hope that the inverse scattering transforms are also closely related.

Reduction of the KP-I Inverse Scattering

Recall that the inverse scattering transform for the KP-I equation is a non-local Riemann-Hilbert problem of the form

$$\phi_+(\lambda) - \phi_-(\lambda) = \int_{\kappa\in\mathbb{R}} \alpha(\lambda,\kappa) e^{i(\lambda-\kappa)x - i(\lambda^2-\kappa^2)y} \phi_-(\kappa) d\kappa, \tag{5.3}$$

for $\lambda \in \mathbb{R}$, and ϕ^+ (ϕ_-) possesses an analytic continuation into the upper (lower) half-plane in $\lambda \in \mathbb{C}$. For ϕ and hence u to be independent of y, we choose α of the form

$$\alpha(\lambda,\kappa) = \alpha^+(\lambda)\delta(\lambda-\kappa) + \alpha^-(\lambda)\delta(\lambda+\kappa), \quad \lambda,\kappa \in \mathbb{R}. \tag{5.4}$$

This picks out the points in the integrand where $\lambda^2 = \kappa^2$, hence removing any dependence on y. This gives the expression

$$\phi_+(\lambda) - \phi_-(\lambda) = \alpha^+(\lambda)\phi_-(\lambda) + \alpha^-(\lambda)e^{2i\lambda x}\phi_-(-\lambda), \quad \lambda \in \mathbb{R}. \tag{5.5}$$

In effect, the spectral parameter for the KdV equation ζ is related to the parameter for the KP equation λ by $\zeta = \lambda^2$. We will refrain from introducing ζ in our description, but will instead ensure that the functions in the final inverse scattering transform are unchanged under the symmetry $\lambda \to -\lambda$. Bearing this in mind, we define the vector-valued function $\mathbf{\Phi}_\pm(\lambda)$ by

$$\mathbf{\Phi}_\pm(\lambda) = \begin{pmatrix} \phi_\pm(\lambda) + \phi_\pm(-\lambda) \\ \lambda^{-1}\left[\phi_\pm(\lambda) - \phi_\pm(-\lambda)\right] \end{pmatrix}, \tag{5.6}$$

which is invariant under the symmetry. We obtain the *local* Riemann-Hilbert problem

$$\mathbf{\Phi}_+(\lambda) - \mathbf{\Phi}_-(\lambda) = \frac{1}{2}\left(A(\lambda) + B(\lambda)\right)\mathbf{\Phi}_-(\lambda), \quad \lambda \in \mathbb{R}, \tag{5.7}$$

where

$$A = \begin{pmatrix} \alpha^+(\lambda) + \alpha^+(-\lambda) & \lambda(\alpha^+(\lambda) - \alpha^+(-\lambda)) \\ \lambda^{-1}(\alpha^+(\lambda) - \alpha^+(-\lambda)) & \alpha^+(\lambda) + \alpha^+(-\lambda) \end{pmatrix}, \tag{5.8}$$

and the matrix B is given by

$$\begin{pmatrix} \alpha^-(\lambda)e^{2i\lambda x} + \alpha^-(-\lambda)e^{-2i\lambda x} & -\lambda(\alpha^-(\lambda)e^{2i\lambda x} - \alpha^-(-\lambda)e^{-2i\lambda x}) \\ \lambda^{-1}(\alpha^-(\lambda)e^{2i\lambda x} - \alpha^-(-\lambda)e^{-2i\lambda x}) & -(\alpha^-(\lambda)e^{2i\lambda x} + \alpha^-(-\lambda)e^{-2i\lambda x}) \end{pmatrix}. \tag{5.9}$$

This is the Riemann-Hilbert problem associated with the KdV equation. It is further discussed, although in a slightly different formulation by Ablowitz and Clarkson, [2], although both formulations have equation (5.5) as a common early step and are equivalent.

Reduction of the KP-II Inverse Scattering

We adopt a similar approach to the KP-II case. We take the d-bar equation

$$\partial_{\bar{\lambda}}\phi(\lambda) = e^{(\bar{\lambda}-\lambda)x + (\bar{\lambda}^2-\lambda^2)y}\alpha(\lambda,\bar{\lambda})\phi(\bar{\lambda}), \tag{5.10}$$

and, in a similar manner to the KP-I case, choose

$$\alpha(\lambda,\bar{\lambda}) = \alpha^+(\lambda,\bar{\lambda})\delta(\lambda - \bar{\lambda}) + \alpha^-(\lambda,\bar{\lambda})\delta(\lambda + \bar{\lambda}). \tag{5.11}$$

Hence ϕ is holomorphic, except on the real and imaginary λ-axes. Recall from the previous section that we are in fact dealing with a spectral parameter of the form $\zeta = \lambda^2$, and so considering ϕ as a function of ζ, it will only be the real ζ-axis on which the function is not holomorphic. Hence ϕ will only have a jump on the real ζ-axis and we can proceed to derive the local Riemann-Hilbert problem as for the KP-I case.

Note that as we would expect, the inverse scattering for both the KP-I and KP-II equations reduces to equivalent expressions for the inverse scattering of the KdV, as those inverse scattering expressions are derived primarily from the first Lax operators for the respective equations. These operators only differ in their dependence on ∂_y, which is the symmetry imposed to make this reduction, hence the separate types of inverse scattering for the KP equations reduce to the same expression for KdV.

5.2 The Inverse Scattering for the Linearised KP Equations

The aim of this and the following section is to show that the inverse scattering for the KP-I equation and the inverse scattering for the KP-II equation can be viewed as analytic continuations of each other. However, before we consider the inverse scattering transform in its entirety, let us first look at the simpler case of the linearised KP equations. The linearisation is achieved by letting

$$\phi = \phi_0 + \epsilon\phi_1 + O(\epsilon^2),$$

$$\alpha = \epsilon\alpha_1 + O(\epsilon^2),$$

and

$$u = \epsilon u_1 + O(\epsilon^2), \tag{5.12}$$

where $0 < \epsilon << 1$, in both the inverse scattering equations.

The linearised KP-II equation

We consider the KP-II equation first, as the expressions are more straightforward. The boundary conditions on ϕ become $\phi_0 \to 1$ and $\phi_1 \to 0$ as $\lambda \to \infty$.

The d-bar equation for ϕ gives

$$\partial_{\bar{\lambda}} \phi_0 = 0,$$

and

$$\partial_{\bar{\lambda}} \phi_1 = e^{\bar{\mu}-\mu} \alpha_1 \bar{\phi_0}.$$

The first equation tells us that ϕ_0 is holomorphic everywhere and is 1 at $\lambda = \infty$. Hence $\phi_0 \equiv 1$. Using the extended form of Cauchy's formula, that is

$$f(\lambda, \bar{\lambda}) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{f(\xi)}{\xi - \lambda} d\xi + \frac{1}{2\pi i} \int_{\mathcal{R}} \frac{\partial f}{\partial \bar{\xi}} \frac{1}{\xi - \lambda} d\xi \wedge d\bar{\xi}, \tag{5.13}$$

where $\mathcal{C}$ is a closed contour in $\mathbb{C}$ and $\mathcal{R}$ is the enclosed region in the complex plane, we take $\mathcal{C}$ as a circular contour at infinity and $\mathcal{R}$ as the complex plane. This gives

$$\begin{aligned} \phi_1(\lambda, \bar{\lambda}) &= \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial \phi_1}{\partial \bar{\xi}} \frac{1}{\xi - \lambda} d\xi \wedge d\bar{\xi} \\ &= \frac{1}{2\pi i} \int_{\mathbb{C}} e^{\bar{\mu}_\xi - \mu_\xi} \alpha_1(\xi, \bar{\xi}) \frac{1}{\xi - \lambda} d\xi \wedge d\bar{\xi} \end{aligned} \tag{5.14}$$

where $\mu_\xi = \xi x + \xi^2 y + \xi^3 t$. Note that the first term of equation (5.13) vanishes because $\phi_1 = 0$ at $\lambda = \infty$. Hence, from the equation $L_2 \phi = 0$, it follows that $u_1(x, y, t)$ is given by

$$\begin{aligned} u_1(x, y, t) &= -\frac{1}{2} \left(\partial_y - \partial_x^2 - 2\lambda \partial_x \right) \phi_1 \\ &= \frac{1}{2\pi i} \int_{\mathbb{C}} \left(\xi - \bar{\xi} \right) e^{\bar{\mu}_\xi - \mu_\xi} \alpha_1(\xi, \bar{\xi}) d\xi \wedge d\bar{\xi} \\ &= \partial_x \left(-\frac{1}{2\pi i} \int_{\mathbb{C}} e^{\bar{\mu}_\xi - \mu_\xi} \alpha_1(\xi, \bar{\xi}) d\xi \wedge d\bar{\xi} \right). \end{aligned} \tag{5.15}$$

The linearised KP-I equation

We now consider the KP-I case. We start with the nonlocal Riemann-Hilbert problem for the KP-I equation,

$$\left(\phi^{+}-\phi^{-}\right)(\lambda)=\int_{\mathbb{R}} e^{\mu_\kappa-\mu_\lambda}\alpha(\lambda,\kappa)\phi^{-}(\kappa)d\kappa,$$

where $\phi^{+}(\lambda)\,(\phi^{-}(\lambda))$ is a solution of $L\phi=0$ that is holomorphic in λ in the upper (lower) half-plane.

On linearisation and application of the boundary condition, we obtain the results

$$\phi_0^{\pm}=1$$

and

$$\left(\phi_1^{+}-\phi_1^{-}\right)(\lambda)=\int_{\mathbb{R}} e^{\mu_\kappa-\mu_\lambda}\alpha(\lambda,\kappa)d\kappa.$$

This second equation is a straightforward local Riemann-Hilbert problem. Hence $\phi_1^{\pm}$ are evaluated by the splitting formula

$$\phi_1^{\pm}(\lambda)=\pm\frac{1}{2}f(\lambda)+\frac{1}{2\pi i}\int_{\mathbb{R}}\frac{f(\tau)}{\tau-\lambda}d\tau, \tag{5.16}$$

where

$$f(\lambda)=\int_{\mathbb{R}} e^{\mu_\kappa-\mu_\lambda}\alpha(\lambda,\kappa)d\kappa. \tag{5.17}$$

Define $\Gamma^{+}_{R,\delta}$ $(\Gamma^{-}_{R,\delta})$ to be a semicircular contour along the real axis and in the upper (lower) half-plane of radius R $(R>\lambda)$, with a small semicircular indentation around $\tau=\lambda$ of radius δ, $(0<\delta<<1)$, extending into the lower (upper) half plane. Then equation (5.16) is equivalent to

$$\begin{aligned}\phi_1^{\pm}(\lambda) &= \frac{1}{2\pi i}\int_{\Gamma^{\pm}}\frac{f(\tau)}{\tau-\lambda}d\tau\\ &= \frac{1}{2\pi i}\lim_{\substack{R\to\infty\\ \delta\to 0}}\int_{\Gamma^{\pm}_{R,\delta}}\frac{f(\tau)}{\tau-\lambda}d\tau. \end{aligned}\tag{5.18}$$

Inserting the formula for $f(\lambda)$ from equation (5.17) gives the expression

$$\phi_1^{\pm}(\lambda)=\frac{1}{2\pi i}\int_{\tau\in\Gamma^{\pm}}\int_{\kappa\in\mathbb{R}} e^{\mu_\kappa-\mu_\tau}\alpha(\tau,\kappa)\frac{1}{\tau-\lambda}d\kappa d\tau. \tag{5.19}$$

Following a similar calculation to the KP-II case, this gives that

$$u_1(x,y,t) = i\partial_x \left(-\frac{1}{2\pi i} \int_{\tau\in\Gamma^\pm} \int_{\kappa\in\mathbb{R}} e^{\mu_\kappa - \mu_\tau} \alpha(\tau,\kappa) d\kappa d\tau \right).$$

Comparison of linearised KP-I and KP-II

We are now able to prove the following theorem.

Theorem 5.1 *Suppose $\alpha(\xi,\eta)$ defined on $\mathbb{CP}^1 \times \mathbb{CP}^1$ has an analytic continuation from the contour given by $\eta = \bar{\xi}$ to the contour given by $\xi \in \mathbb{R}$, $\eta \in \Gamma^\pm$, where $\Gamma^\pm$ is as defined above. Then there is an analytic continuation of the inverse scattering transform of the linearised KP-I equation to the inverse scattering transform of the KP-II equation, and vice versa.*

Proof: Compare equation (5.14) with equation (5.19). Consider the integrand in equation (5.14) as a function on the contour of $\mathbb{CP}^1 \times \mathbb{CP}^1$ given by $(\xi, \bar{\xi})$ for $\xi \in \mathbb{CP}^1$. Hence, given that the integrand has an analytic continuation to the second contour, by performing this continuation the inverse scattering for the linearised KP-I equation is obtained. □

5.3 Comparison of the Inverse Scattering for the Full KP-I and KP-II Equations

One would expect this analytic continuation to extend to the full KP equations, and indeed, it does. The relations become, for the KP-I equation

$$\phi^\pm(\lambda) = 1 + \frac{1}{2\pi i} \int_{\tau\in\Gamma^\pm} \int_{\kappa\in\mathbb{R}} e^{\mu_\kappa - \mu_\tau} \alpha(\tau,\kappa) \frac{\phi^\pm(\tau)}{\tau - \lambda} d\kappa d\tau; \tag{5.20}$$

this is the same expression as for the linearised case, equation (5.19), except for an additional $\phi^\pm$ term in the integrand. The KP-II relation becomes

$$\phi(\lambda) = 1 + \frac{1}{2\pi i} \int_{\mathbb{C}} e^{\bar{\mu}_\xi - \mu_\xi} \alpha_1(\xi,\bar{\xi}) \frac{\phi(\bar{\xi})}{\xi - \lambda} d\xi \wedge d\bar{\xi}. \tag{5.21}$$

Again, this is comparable to the linearised inverse scattering relation, equation (5.14), except for an extra $\phi(\bar{\lambda})$ in the integrand. Thus, at least at a formal level, the analytic continuation described for the linear case remains. However there is an additional problem in the case of the full KP-I equation, which is the existence of lump solutions, as described in Section 4.3.

5.3.1 Analytic Continuation and Lump Solutions

In our work on the KP-II equation, we have assumed to date that there are no lump solutions to that equation. This is true, as proved rigorously by Fokas and Sung in [10]. The details of this proof are beyond the scope of this thesis and will not be presented. However the authors prove that, subject to suitable conditions on u and the boundary conditions already imposed, there is a unique solution to the equation (3.16) and hence there will be no such lump solutions for the KP-II equation. In fact, they prove a stronger result - there will be a unique solution of the more general equation

$$\sigma\phi_y - \phi_{xx} - 2\lambda\phi_x = -2u\phi, \tag{5.22}$$

where σ is a constant complex number, provided that the real part of σ is non-zero. It does not explicitly prove that uniqueness fails, and hence homogeneous solutions exist, for σ imaginary, but such solutions can be explicitly shown to exist, for example in Section 4.3.

The obvious question that this raises is, what happens to the lump solution terms in the KP-I inverse scattering? What causes them to apparently disappear? The answer, quite simply, is that they don't. However the lump solutions for the KP-I equation, under the transformation $y \to iy$ become singular. Hence as the solution is not defined for all x, y, t, it does not belong to the class of solutions under consideration.

This adds a new complication to a theoretical construction covering both the KP-I and KP-II equations. Should such singular solutions for KP-II be included in the correspondence, and, if there should not be, how would the lump solutions be incorporated without incorporating the singular solutions?

5.3.2 The Čech-Dolbeault Isomorphism

Previously, we have mentioned the equivalent Čech and Dolbeault descriptions for holomorphic vector bundles and hence for the Ward construction, and the similarities between the Čech description and the nonlocal Riemann-Hilbert problem and between the Dolbeault description and the Dirac operator.

In Section 5.2, we have shown that the inverse scattering for the linearised KP equations is given by, respectively, a local Riemann-Hilbert problem (Čech) or a d-bar operator (Dolbeault), which are then shown to be analytic continuations of each other. This would seem to indicate that the KP theories and the Ward construction are closely related. However, they is a slight complication.

The Dirac operator does not correspond to the d-bar operator for the Ward correspondence, as the d-bar operator has gauge freedom. The Dirac operator has none. Hence, what we must presume is that the Dirac operator corresponds to a *gauge-fixed* version of the Ward construction (and similarly the nonlocal Riemann-Hilbert problem), rather than the general standard Ward construction.

5.4 The KP-σ equation

When looking for a geometric structure that covers both KP-I and KP-II, one might expect that there would be a structure where each case would correspond only to the choice of the constant σ in the first Lax operator, rather than have each case as the reduction of a larger system. In this section, we consider equation (5.22) where we assume that σ is a constant of the form $\sigma = e^{i\theta}$. We shall use this expression to show how the inverse scattering for KP-II tends to the inverse scattering for KP-I as $\theta \to \frac{\pi}{2}$.

Theorem 5.2 *Suppose ϕ satisfies the equation $L\phi = 0$, where*

$$L = \sigma\partial_y - \partial_x^2 + 2u,$$

where $\sigma = e^{i\theta}$, and $\phi \to 1$ as $x^2 + y^2, |\lambda| \to \infty$ for $x, y \in \mathbb{R}$ and $\lambda \in \mathbb{C}$. Then there ϕ satisfies a d-bar relation of the form

$$\frac{\partial\phi}{\partial\tilde{\lambda}} = \exp\{\tilde{\mu} - \mu\} f(\lambda, \tilde{\lambda})\phi(\tilde{\lambda}), \tag{5.23}$$

where $\tilde{\lambda} = \bar{\lambda} + i(\lambda - \bar{\lambda})\tan\theta$. Moreover, the d-bar relationship does not hold in the limit $\theta \to \pi/2$.

Proof: Consider equation (5.22) with $\sigma = e^{i\theta}$ where $0 \leq \theta \leq \pi/2$. We take μ to have the form $\mu = \sigma\lambda x + \sigma\lambda^2 y$. The first Lax operator (for ϕ) is given by

$$L = \sigma\partial_y - \partial_x^2 - 2\sigma\lambda\partial_x + 2u. \tag{5.24}$$

The differential part of L has the Green's function

$$\begin{aligned} G &= -\frac{1}{4\pi^2}\int\int \frac{\exp\{i(\xi x + \eta y)\}}{\xi^2 - 2i\xi\sigma\lambda + i\sigma\eta} d\xi d\eta & (5.25)\\ &= -\frac{\bar{\sigma}}{2\pi}\text{sgn}y \int_{\mathbb{R}} e^{i\xi x + 2i\lambda\xi y - \bar{\sigma}\xi^2 y}\theta\left(\left(2\lambda_I\xi + \xi^2\cos\theta\right) y\right) d\xi.. & (5.26)\end{aligned}$$

The denominator of the first integrand has discrete zeroes (provided $\theta \neq \frac{\pi}{2}$) at the points $(\xi, \eta) = (0, 0)$ and $(\xi, \eta) = (\xi_0, \eta_0)$, where

$$(\xi_0, \eta_0) = \left(-\frac{2\lambda_I}{\sigma_R}, -\frac{4\lambda_I(\sigma\lambda)_R}{\sigma_R^2}\right). \tag{5.27}$$

In the KP-II case ($\sigma = 1$) the d-bar relation relates λ and $\bar{\lambda}$. We need to consider in this more general case the variables λ and $\tilde{\lambda}$ where $\tilde{\lambda} = \bar{\lambda} + i(\lambda -$

$\bar{\lambda})\tan\theta$. This gives that

$$\begin{aligned}\frac{\partial}{\partial\tilde{\lambda}} &= \frac{\partial\bar{\lambda}}{\partial\tilde{\lambda}}\frac{\partial}{\partial\bar{\lambda}}\\ &= \frac{1}{1-i\tan\theta}\frac{\partial}{\partial\bar{\lambda}}\\ &= \frac{\sigma_R}{\bar{\sigma}}\frac{\partial}{\partial\bar{\lambda}}.\end{aligned} \tag{5.28}$$

It is also worth noting that $i\xi_0 = \sigma(\tilde{\lambda}-\lambda)$ and $i\eta_0 = \sigma(\tilde{\lambda}^2-\lambda^2)$. There are two results crucial to the final result. The first is that

$$G(\tilde{\lambda}) = \exp(\mu-\tilde{\mu})G(\lambda) \tag{5.29}$$

and the second that

$$\frac{\partial G}{\partial\tilde{\lambda}} = \frac{i}{2\pi}\mathrm{sgn}\lambda_I \exp(\tilde{\mu}-\mu). \tag{5.30}$$

Using these results, and following the derivation of the KP-II d-bar equation, this gives the relation

$$\frac{\partial\phi}{\partial\tilde{\lambda}} = \exp\{\tilde{\mu}-\mu\}f(\lambda,\tilde{\lambda})\phi(\tilde{\lambda}), \tag{5.31}$$

where

$$f(\lambda,\tilde{\lambda}) = \frac{i}{2\pi}\mathrm{sgn}\lambda_I \int\int \exp\{\mu'-\tilde{\mu}'\}2u(x',y')\phi(x',y';\lambda)dx'dy'. \tag{5.32}$$

It should be noted that this is identical to the corresponding function in the KP-II case, with $\bar{\lambda}$ and $\tilde{\lambda}$ interchanged.

Thus we have a d-bar relation for $0\le\theta<\pi/2$, but this range does not include the case for KP-I ($\theta=\pi/2$). The majority of the above calculation is invalid for $\theta=\pi/2$, so we need to go back to the expressions for the Green's function. Consider the expression in equation (5.26), with $\theta=\pi/2$ and $\sigma=i$. This gives

$$G(x,y;\lambda) = \frac{i}{2\pi}\mathrm{sgn}y \int_{\mathbb{R}} e^{i\xi x+2i\lambda\xi y+i\xi^2 y}\theta(2\lambda_I\xi y)d\xi. \tag{5.33}$$

This is defined for $\lambda_I \neq 0$ and is holomorphic in λ (the λ_I term in the Heaviside function ensures that the integral exists). So for $\theta = \pi/2$ and $\lambda_I \neq 0$, we have a holomorphic Greens function and a d-bar equation of sorts, namely $\partial_{\bar{\lambda}}\phi = 0$. When $\lambda_I = 0$, the equation (5.26) makes no sense, and, on examination of the integral in equation (5.25), we find that the integral does not exist - the zeros of the denominator of the integrand are no longer isolated in the ξ, η plane (and hence integrable), but dense. Hence there is a jump in the Green's function across the real axis, which in turn gives rise to a nonlocal Riemann-Hilbert problem on the real axis. $\square$

So is this KP$-\sigma$ construction a suitable choice for a general construction? The answer is no. This construction does not actually generate the nonlocal Riemann-Hilbert problem for KP-I, it merely breaks down when $\sigma = i$. Also, the lump solutions are not dealt with. Fokas and Sung prove in [10] that equation (5.22) with suitable boundary condition has a unique solution for all λ provided that $\operatorname{Re}\sigma \neq 0$, meaning that lump solutions will not exist. So when the limit $\theta = \pi/2$ is attained, the terms corresponding to the lump solutions seemingly appear from nowhere. This construction is clearly too limited to deal with the case $\sigma = i$ and therefore inappropriate as a construction for the KP-equations.

Part III

OTHER INTEGRABLE SYSTEMS

Chapter 6

The Davey-Stewartson Equations

The other well-known $(2+1)$-dimensional integrable system that does not seem to possess a standard twistor correspondence is the Davey-Stewartson (DS) equations. The Lax pair for the DS equations are matrix operators as opposed to scalar operators as for the KP equations, but the operators have a notably different form; the first Lax operator is hyperbolic (DS-I) or elliptic (DS-II), whereas the first KP operator is parabolic. Regardless of this difference, the inverse scattering for the equations is similar.

In this section, we present a reformulation of the DS-II equations and their Lax pair, to express them in terms of complex spatial variables z and $\bar{z}$. Following this, we demonstrate the link between DS-I and DS-II by considering the linearised case of the inverse scattering transform. (Recall that for the KP equations, we considered the linearised Lax pair.) We then outline the derivation of the Dirac operator for DS-II from the Lax pair, based on a proof first given in [12].

Following this, we then investigate, for the DS-II equations, the possibility of a localised Dirac operator, by considering the Lax operator with a boundary condition for $|z| = 1$, rather than $|z| = \infty$.

6.1 Derivation of the Lax Operators for the DS-II Equations

The standard formulation of the Davey-Stewartson equation is in terms of real variables x, y and t, with some basic symmetry in the x and y variables. It would seem sensible then for the Lax pair to have the same symmetry. In this section we use the freedom that exists in the formulation of the Lax pair to derive a version that depends on $z = x + iy$ and $\bar{z}$, rather than x and y and, moreover, is symmetric in z and $\bar{z}$.

Recall the general form of Dirac operator as given in Definition 3.1. Namely that the Dirac operator D_α is given by

$$D_\alpha = \begin{pmatrix} \partial_{\bar{\lambda}} & \alpha \\ \bar{\alpha} & \partial_\lambda \end{pmatrix}, \tag{6.1}$$

where $\alpha(\mu, \bar{\mu}; \lambda, \bar{\lambda})$ is taken to be a smooth function with the form

$$\alpha(\lambda, \bar{\lambda}, \mu, \bar{\mu}) = \exp \bar{\mu}\, \alpha_0(\lambda, \bar{\lambda}) \exp -\mu. \tag{6.2}$$

Definition 6.1 *A Dirac operator for the DS-II hierarchy is as given in Definition 3.1 where $\alpha_0(\lambda, \bar{\lambda})$ is taken to be a smooth $\mathrm{GL}(2, \mathbb{C})$-valued function of the form*

$$\alpha_0(\lambda, \bar{\lambda}) = \begin{pmatrix} 0 & \alpha_1(\lambda, \bar{\lambda}) \\ \alpha_2(\lambda, \bar{\lambda}) & 0 \end{pmatrix}, \tag{6.3}$$

such that $\alpha_i \sim O(|\lambda|^{-3})$ as $\lambda \to \infty$ and μ is given by

$$\mu_\lambda(z, \bar{z}, t) = \begin{pmatrix} \lambda z + \lambda^2 t & 0 \\ 0 & \lambda \bar{z} + \lambda^2 t \end{pmatrix}. \tag{6.4}$$

Note that α_0 is taken as a multiplicative operator acting on the right.

We introduce the notation $\Sigma_\lambda = e^{\mu_\lambda}$ and take Ψ to be a 2×2 matrix-valued function, such that $\Psi \Sigma_\lambda^{-1} \to 1$ as $\lambda \to \infty$. As for the KP-II case, we define Φ by $\Psi = \Phi \Sigma_\lambda$, giving that $\Phi \to I$ as $\lambda \to \infty$.

Proposition 6.2 *Given a Dirac operator as in Definition 6.1, there exists an associated sequence of Lax operators for the DS-II hierarchy and hence there exists an associated solution to the DS-II equations with boundary conditions* $\beta, \gamma, \rho \to 0$ *as* $|z| \to \infty$.

Proof: Following the method outlined for the KP-II equation, we derive the Lax pair related to this operator. As Φ is holomorphic near $\lambda = \infty$, it has the expansion

$$\Phi = I + \lambda^{-1} \begin{pmatrix} \phi^{11} & \phi^{12} \\ \phi^{21} & \phi^{22} \end{pmatrix} + \lambda^{-2} \begin{pmatrix} \hat{\phi}^{11} & \hat{\phi}^{12} \\ \hat{\phi}^{21} & \hat{\phi}^{22} \end{pmatrix} + O(\lambda^{-3}).$$

We look for operators L_i such that

$$L_i \left(\Phi \Sigma_\lambda\right) \Sigma_\lambda^{-1} = O(\lambda^{-1}).$$

The first such operator is given by

$$L_1 = \begin{pmatrix} \partial_{\bar{z}} & 0 \\ 0 & \partial_z \end{pmatrix} - \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}, \tag{6.5}$$

where $\beta(z, \bar{z}) = \phi^{12}$ and $\gamma(z, \bar{z}) = \phi^{21}$. There is some freedom in the form of the second operator; we choose a form that has z and $\bar{z}$ on an equal footing. This is given (up to the addition of a multiple of L_1) by

$$L_2 = \partial_t - \begin{pmatrix} \partial_z^2 & 0 \\ 0 & \partial_{\bar{z}}^2 \end{pmatrix} - \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} \begin{pmatrix} \partial_z & 0 \\ 0 & \partial_{\bar{z}} \end{pmatrix} + A, \tag{6.6}$$

where $A = (A^{ij})$ is a 2×2 matrix whose entries satisfy

$$\begin{aligned} A^{11} &= 2\phi_z^{11}, \\ A^{12} &= \phi^{12}\phi^{22} - \hat{\phi}^{12}, \\ A^{21} &= \phi^{11}\phi^{21} - \hat{\phi}^{21}, \\ A^{22} &= 2\phi_{\bar{z}}^{22}. \end{aligned} \tag{6.7}$$

On consideration of the compatibility condition below, we derive an alternative expression for A, namely that the entries of A must satisfy

$$\begin{aligned} \partial_{\bar{z}} A^{11} &= 2(\beta\gamma)_z, \\ A^{12} &= \beta_{\bar{z}}, \\ A^{21} &= \gamma_z, \\ \partial_z A^{22} &= 2(\beta\gamma)_{\bar{z}}. \end{aligned} \tag{6.8}$$

The compatibility condition for the operators is given by

$$[L_1, L_2] + NL_1 = 0, \tag{6.9}$$

where N is given by the operator

$$N = \begin{pmatrix} 0 & \beta \\ -\gamma & 0 \end{pmatrix} (\partial_z - \partial_{\bar{z}}) + \begin{pmatrix} 0 & \beta_{\bar{z}} + 2\beta_z \\ \gamma_z + 2\gamma_{\bar{z}} & 0 \end{pmatrix}. \tag{6.10}$$

This gives rise to the Davey-Stewartson-II equations

$$\begin{aligned} \beta_t + \left(\partial_{\bar{z}}^2 - \partial_z^2 + \rho\right)\beta = 0 \\ \gamma_t - \left(\partial_{\bar{z}}^2 - \partial_z^2 + \rho\right)\gamma = 0 \\ \rho_{z\bar{z}} = 2(\partial_z^2 - \partial_{\bar{z}}^2)(\beta\gamma). \end{aligned} \tag{6.11}$$

where $\rho = A^{11} - A^{22}$. □

Note that this is not the standard expression used for the DS-II equations in the literature - the standard expression is equation (2.7). However, if we seek a geometric construction for the equations, it seems that treating the space variables as one complex coordinate on $\mathbb{CP}^1$ would seem to be more natural. Note that the expression derived here is equivalent to the standard expressions used. (See, for example, Ablowitz and Clarkson,[2] or Konopelchenko,[12]).

6.2 Derivation of the Dirac Operator for the Davey-Stewartson Equations

In this section we will derive the Dirac operator for the DS-II equations from the Lax operator. In this case, it is much more straightforward to see where the non-holomorphy of the Green's function originates than for the KP-II case (although in both cases it is due to the boundary conditions imposed on ϕ near $\lambda = \infty$). However for the DS-II case, we have to take into account the fact that lump solutions may exist, corresponding to solutions of the Lax pair that tend to zero as z goes to infinity. For the time being, we will assume that these solutions do not exist, and briefly discuss the lump solutions in Section 6.3.2.

Proposition (Konopelchenko) 6.3 *Suppose β and γ are such that there are no solutions to*

$$L_1(\Phi\Sigma_\lambda) = 0 \quad \text{and} \quad L_2(\Phi\Sigma_\lambda) = 0,$$

such that $\Phi \to 0$ as $z \to \infty$. Then if $\Phi\Sigma_\lambda$ satisfies the Lax pair and $\Phi \to I$ as $z \to \infty$, then Φ also satisfies a Dirac equation of the type defined in Definition 6.1.

Proof: Denote the components of Φ by

$$\Phi = \begin{pmatrix} \Phi^{11} & \Phi^{12} \\ \Phi^{21} & \Phi^{22} \end{pmatrix}$$

The equation $L_1(\Phi\Sigma_\lambda) = 0$ implies that

$$\left[\begin{pmatrix} \partial_{\bar{z}} & 0 \\ 0 & \partial_z \end{pmatrix} - \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} \right] \Phi + \lambda \begin{pmatrix} 0 & \Phi^{12} \\ \Phi^{21} & 0 \end{pmatrix} = 0. \tag{6.12}$$

The solution to this equation that tends to I as $\lambda \to \infty$ and $|z| \to \infty$ can be written as

$$\Phi(z, \bar{z}; \lambda) = I + \hat{G}_\lambda \left[B(.)\Phi(.) \right] (z, \bar{z}), \tag{6.13}$$

where

$$B = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix},$$

and the operator $\hat{G}_\lambda$ is defined on a matrix-valued function C of z and $\bar{z}$ by

$$(\hat{G}_\lambda C)(z,\bar{z}) = \frac{1}{2\pi i}\int_{\mathcal{D}} \begin{pmatrix} \frac{C^{11}(\zeta,\bar{\zeta})}{\zeta - z} & e^{\lambda(\bar{\zeta}-\bar{z})-\bar{\lambda}(\zeta - z)}\frac{C^{12}(\zeta,\bar{\zeta})}{\zeta - z} \\ e^{\lambda(\zeta - z)-\bar{\lambda}(\bar{\zeta}-\bar{z})}\frac{C^{21}(\zeta,\bar{\zeta})}{\bar{\zeta}-\bar{z}} & \frac{C^{22}(\zeta,\bar{\zeta})}{\bar{\zeta}-\bar{z}} \end{pmatrix} d\zeta d\bar{\zeta}. \tag{6.14}$$

This operator is simply the application of the Green's function of the homogeneous part of equation (6.12). Note the addition of terms in $\bar{\lambda}$ in the exponentials in the integrand. Without the boundary condition on Φ, these can be taken as arbitrary functions of $\bar{\lambda}$. With the boundary condition, they are necessarily of this form to make the Green's function bounded. It is these terms that cause the non-holomorphy of Φ for the d-bar problem.

Differentiating equation (6.13) with respect to $\bar{\lambda}$ gives

$$\partial_{\bar{\lambda}}\Phi(z,\bar{z};\lambda) = (\partial_{\bar{\lambda}}\hat{G}_\lambda)(B\Phi) + \hat{G}_\lambda(B\partial_{\bar{\lambda}}\Phi). \tag{6.15}$$

On differentiation of equation (6.14), it can be shown that the first term on the right hand side can be written in the form

$$(\partial_{\bar{\lambda}}\hat{G}_\lambda)(B\Phi) = \tilde{\Sigma}_{\bar{\lambda}} \begin{pmatrix} 0 & \alpha_1 \\ \alpha_2 & 0 \end{pmatrix} \tilde{\Sigma}_{\lambda}^{-1}, \tag{6.16}$$

where

$$\begin{aligned} \alpha_1(\lambda,\bar{\lambda}) &= -\frac{1}{2\pi i}\int_{\mathcal{D}} e^{\lambda\bar{\zeta}-\bar{\lambda}\zeta}\beta(\zeta,\bar{\zeta})\Phi^{22}(\zeta,\bar{\zeta};\lambda)d\zeta\wedge d\bar{\zeta}, \\ \alpha_2(\lambda,\bar{\lambda}) &= -\frac{1}{2\pi i}\int_{\mathcal{D}} e^{\lambda\zeta-\bar{\lambda}\bar{\zeta}}\gamma(\zeta,\bar{\zeta})\Phi^{11}(\zeta,\bar{\zeta};\lambda)d\zeta\wedge d\bar{\zeta}, \end{aligned} \tag{6.17}$$

and $\tilde{\Sigma}_\lambda$ is given by

$$\tilde{\Sigma}_\lambda(z,\bar{z}) = \begin{pmatrix} e^{\lambda z} & 0 \\ 0 & e^{\lambda\bar{z}} \end{pmatrix}. \tag{6.18}$$

Now let $N(z,\bar{z};\lambda)$ be a solution of the integral equation

$$N(z,\bar{z};\lambda) = \begin{pmatrix} 0 & e^{\bar{\lambda}z-\lambda\bar{z}} \\ e^{\bar{\lambda}\bar{z}-\lambda z} & 0 \end{pmatrix} + \hat{G}_\lambda(BN)(z,\bar{z};\lambda), \tag{6.19}$$

that is the solution to equation (6.12) that tends to $\begin{pmatrix} 0 & e^{\bar{\lambda}z-\lambda\bar{z}} \\ e^{\bar{\lambda}\bar{z}-\lambda z} & 0 \end{pmatrix}$ as λ tends to infinity. Hence it follows from equations (6.15), (6.16) and (6.19) that the expressions

$$\Lambda = \partial_{\bar{\lambda}}\Phi(z,\bar{z};\lambda) \quad \text{and} \quad \Lambda = N(z,\bar{z};\lambda)\begin{pmatrix} \alpha_2 & 0 \\ 0 & \alpha_1 \end{pmatrix}$$

are both solutions to the integral equation

$$\Lambda(z,\bar{z};\lambda) = \begin{pmatrix} 0 & e^{\bar{\lambda}z-\lambda\bar{z}}\alpha_1 \\ e^{\bar{\lambda}\bar{z}-\lambda z}\alpha_2 & 0 \end{pmatrix} + \hat{G}_\lambda(B\Lambda)(z,\bar{z};\lambda). \tag{6.20}$$

This means that both expressions satisfy $L_1(\Lambda\Sigma_\lambda) = 0$ with the same boundary condition as $\lambda \to \infty$. Hence, by the uniqueness of the solution of equation (6.12) it follows that

$$\partial_{\bar{\lambda}}\Phi(z,\bar{z};\lambda) = N(z,\bar{z};\lambda)\begin{pmatrix} \alpha_2 & 0 \\ 0 & \alpha_1 \end{pmatrix}. \tag{6.21}$$

Now let the matrix $\Gamma(z,\bar{z};\lambda,\bar{\lambda})$ be defined by

$$\Gamma(z,\bar{z};\lambda,\bar{\lambda}) = \Phi(\bar{\lambda})\begin{pmatrix} 0 & e^{\bar{\lambda}z-\lambda\bar{z}} \\ e^{\bar{\lambda}\bar{z}-\lambda z} & 0 \end{pmatrix}. \tag{6.22}$$

On exchanging λ with $\bar{\lambda}$ in equations (6.13) and (6.14) and multiplying on the right by the matrix $\begin{pmatrix} 0 & e^{\bar{\lambda}z-\lambda\bar{z}} \\ e^{\bar{\lambda}\bar{z}-\lambda z} & 0 \end{pmatrix}$, one can see that Γ satisfies the equation

$$\Gamma(z,\bar{z};\lambda) = \begin{pmatrix} 0 & e^{\bar{\lambda}z-\lambda\bar{z}} \\ e^{\bar{\lambda}\bar{z}-\lambda z} & 0 \end{pmatrix} + \hat{G}_\lambda(B\Gamma)(z,\bar{z};\lambda), \tag{6.23}$$

that is the same equation as N satisfies and has the same behaviour as $\lambda \to \infty$. Hence by uniqueness of solution,

$$N(z, \bar{z}; \lambda) = \Gamma(z, \bar{z}; \lambda) = \begin{pmatrix} 0 & e^{\bar{\lambda}z - \lambda\bar{z}} \\ e^{\bar{\lambda}\bar{z} - \lambda z} & 0 \end{pmatrix} + \hat{G}_\lambda(B\Gamma)(z, \bar{z}; \lambda). \quad (6.24)$$

On substituting into equation (6.21), this gives the d-bar relation

$$\begin{aligned} \partial_{\bar{\lambda}}\Phi(\lambda) &= \Gamma \begin{pmatrix} \alpha_2 & 0 \\ 0 & \alpha_1 \end{pmatrix} \\ &= \Phi(\bar{\lambda})\tilde{\Sigma}_{\bar{\lambda}} \begin{pmatrix} 0 & \alpha_1 \\ \alpha_2 & 0 \end{pmatrix} \tilde{\Sigma}_{\lambda}^{-1}. \end{aligned} \quad (6.25)$$

Now let us determine the dependence on t of the α_i terms. Multiplying equation (6.25) on the right by Σ_λ (as defined in equation (6.4), and writing $\Psi(\lambda) = \Phi(\lambda)\Sigma_\lambda$, we have

$$\partial_{\bar{\lambda}}\Psi(\lambda) = \Psi(\bar{\lambda}) \left[\begin{pmatrix} 0 & \alpha_1(\lambda, \bar{\lambda}; t) \\ \alpha_2(\lambda, \bar{\lambda}; t) & 0 \end{pmatrix} e^{(\lambda^2 - \bar{\lambda}^2)t} \right]. \quad (6.26)$$

Now $\partial_{\bar{\lambda}}$ commutes with the Lax operator L_2, so as $L_2\Psi = 0$, then $L_2(\partial_{\bar{\lambda}}\Psi) = 0$. Hence applying L_2 to equation (6.26) gives

$$0 = \partial_t \left[\begin{pmatrix} 0 & \alpha_1(\lambda, \bar{\lambda}; t) \\ \alpha_2(\lambda, \bar{\lambda}; t) & 0 \end{pmatrix} e^{(\lambda^2 - \bar{\lambda}^2)t} \right]. \quad (6.27)$$

Thus

$$\alpha_i(\lambda, \bar{\lambda}; t) = \alpha_i(\lambda, \bar{\lambda}) e^{(\bar{\lambda}^2 - \lambda^2)t}, \quad (6.28)$$

where $\alpha_i(\lambda, \bar{\lambda})$ are as defined in equations (6.17). Thus the d-bar equation can be written

$$\partial_{\bar{\lambda}}\Phi(\lambda) = \Phi(\bar{\lambda})\Sigma_{\bar{\lambda}} \begin{pmatrix} 0 & \alpha_1 \\ \alpha_2 & 0 \end{pmatrix} \Sigma_{\lambda}^{-1}. \quad (6.29)$$

Given that β and γ are both real-valued functions of z, it follows that $\Phi(\bar{\lambda}, \lambda) = \overline{\Phi(\lambda, \bar{\lambda})}$. Hence the semi-local d-bar equation becomes the local equation

$$\partial_{\bar{\lambda}}\Phi = \overline{\Phi}\Sigma_{\bar{\lambda}} \begin{pmatrix} 0 & \alpha_1 \\ \alpha_2 & 0 \end{pmatrix} \Sigma_{\lambda}^{-1}, \tag{6.30}$$

giving rise to the Dirac operator equation as given in Definition 6.1. □

6.3 The DS-I and DS-II Equations

Now that we have derived the Lax pair for the DS-II equations, we will take a brief aside to consider the DS-I equation as well. As in the case for the KP-I equation, the inverse scattering is of a different form for that of the DS-II equation, that of a nonlocal Riemann-Hilbert problem. We again check to see how closely the inverse scattering transforms for the two cases are linked. It is possible that the difference could be more severe than in the KP case, as in that case, both of the first Lax operators were parabolic, whilst in the DS case, one (DS-I) is hyperbolic and the other (DS-II) is elliptic. To check whether there is a problem, we again consider the linearised case. We can then extend the result to the case of the full equations.

6.3.1 The Linearised DS Equations

We first consider the linearised form of the Dirac equation for the DS-II equations.

Linearised DS-II

The d-bar relation for the DS-II equation is given by

$$\partial_{\bar{\lambda}}\Phi = \bar{\Phi} \begin{pmatrix} 0 & \alpha_1 e^{\bar{\lambda}z - \lambda\bar{z} + (\bar{\lambda}^2 - \lambda^2)t} \\ \alpha_2 e^{\bar{\lambda}\bar{z} - \lambda z + (\bar{\lambda}^2 - \lambda^2)t} & 0 \end{pmatrix}.$$

We linearise by putting

$$\Phi = I + \epsilon \begin{pmatrix} \phi^{11} & \phi^{12} \\ \phi^{21} & \phi^{22} \end{pmatrix}, \quad \alpha_i = \epsilon\tilde{\alpha}_i,$$

in this equation. Hence, the linearised equations are given by

$$\begin{aligned} \partial_{\bar{\lambda}}\phi^{11} &= 0; \\ \partial_{\bar{\lambda}}\phi^{12} &= \tilde{\alpha}_1(\lambda, \bar{\lambda})e^{\bar{\lambda}z - \lambda\bar{z} + (\bar{\lambda}^2 - \lambda^2)t}; \\ \partial_{\bar{\lambda}}\phi^{21} &= \tilde{\alpha}_2(\lambda, \bar{\lambda})e^{\bar{\lambda}\bar{z} - \lambda z + (\bar{\lambda}^2 - \lambda^2)t}; \\ \partial_{\bar{\lambda}}\phi^{22} &= 0. \end{aligned}$$

It follows from this that

$$\begin{aligned} \phi^{11} &= 0; \\ \phi^{12} &= \frac{1}{2\pi i}\int_{\mathbb{CP}^1} \frac{\tilde{\alpha}_1(\xi, \bar{\xi})e^{\bar{\xi}z - \xi\bar{z} + (\bar{\xi}^2 - \xi^2)t}}{\xi - \lambda} d\xi d\bar{\xi}; \\ \phi^{21} &= \frac{1}{2\pi i}\int_{\mathbb{CP}^1} \frac{\tilde{\alpha}_2(\xi, \bar{\xi})e^{\bar{\xi}\bar{z} - \xi z + (\bar{\xi}^2 - \xi^2)t}}{\xi - \lambda} d\xi d\bar{\xi}; \\ \phi^{22} &= 0. \end{aligned}$$

From the first Lax equation, we see that β and γ must both be of order ϵ. Putting $\beta = \epsilon\tilde{\beta}$ and $\gamma = \epsilon\tilde{\gamma}$, we can see that $\tilde{\beta}$ is given by

$$\begin{aligned} \tilde{\beta}(z, \bar{z}) &= \lambda\phi^{12} + \phi^{12}_{\bar{z}} \\ &= -\frac{1}{2\pi i}\int_{\mathbb{CP}^1} \tilde{\alpha}_1(\xi, \bar{\xi})e^{\bar{\xi}z - \xi\bar{z} + (\bar{\xi}^2 - \xi^2)t} d\xi d\bar{\xi}. \end{aligned}$$

Similarly, $\tilde{\gamma}$ is given by

$$\tilde{\gamma}(z, \bar{z}) = -\frac{1}{2\pi i}\int_{\mathbb{CP}^1} \tilde{\alpha}_2(\xi, \bar{\xi})e^{\bar{\xi}\bar{z} - \xi z + (\bar{\xi}^2 - \xi^2)t} d\xi d\bar{\xi}. \tag{6.31}$$

Linearised DS-I

The first Lax operator for the DS-I equations is given by

$$\left[\begin{pmatrix} \partial_\eta & 0 \\ 0 & \partial_\xi \end{pmatrix} - \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}\right] \Phi \begin{pmatrix} e^{i\lambda\xi} & 0 \\ 0 & e^{i\lambda\eta} \end{pmatrix} = 0, \tag{6.32}$$

where ξ and η are real variables. We will not present the derivation of the inverse scattering transform for the DS-I equation for reasons of space, but the transform is a nonlocal Riemann-Hilbert problem of the form

$$(\Phi_+ - \Phi_-)(\lambda) = \int_{\mathbb{R}} \Phi_-(\kappa) \begin{pmatrix} 0 & \alpha_1(\lambda,\kappa)e^{i(\kappa\xi-\lambda\eta)} \\ \alpha_2(\lambda,\kappa)e^{i(\kappa\eta-\lambda\xi)} & 0 \end{pmatrix} e^{i(\kappa^2-\lambda^2)t} d\kappa.$$

Linearising as before, we put

$$\Phi_\pm = I + \epsilon \begin{pmatrix} \phi_\pm^{11} & \phi_\pm^{12} \\ \phi_\pm^{21} & \phi_\pm^{22} \end{pmatrix}, \quad \alpha_i = \epsilon\tilde{\alpha}_i,$$

which reduces the problem to four local Riemann-Hilbert problems, namely

$$\begin{aligned} (\phi_+^{11} - \phi_-^{11})(\xi,\eta,\lambda) &= 0; \\ (\phi_+^{12} - \phi_-^{12})(\xi,\eta,\lambda) &= \int_{\mathbb{R}} \tilde{\alpha}_1(\lambda,\kappa)e^{i(\kappa\xi-\lambda\eta+(\kappa^2-\lambda^2)t)}d\kappa; \\ (\phi_+^{21} - \phi_-^{21})(\xi,\eta,\lambda) &= \int_{\mathbb{R}} \tilde{\alpha}_2(\lambda,\kappa)e^{i(\kappa\eta-\lambda\xi+(\kappa^2-\lambda^2)t)}d\kappa; \\ (\phi_+^{22} - \phi_-^{22})(\xi,\eta,\lambda) &= 0. \end{aligned}$$

It follows that $\phi_\pm^{11} = 0 = \phi_\pm^{22}$, but the expressions for ϕ^{12} and ϕ^{21} are a little more complicated. Following the procedure used for the linearised KP-I from Section 5.2, it can be shown that

$$\phi_\pm^{12}(\lambda) = \frac{1}{2\pi i}\int_{\tau\in\Gamma^\pm}\int_{\kappa\in\mathbb{R}} e^{i(\kappa\xi-\tau\eta+(\kappa^2-\tau^2)t)}\tilde{\alpha}_1(\tau,\kappa)\frac{1}{\tau-\lambda}d\kappa d\tau, \qquad (6.33)$$

where the contours $\Gamma^\pm$ are as defined in Section 5.2. Similarly,

$$\phi_\pm^{21}(\lambda) = \frac{1}{2\pi i}\int_{\tau\in\Gamma^\pm}\int_{\kappa\in\mathbb{R}} e^{i(\kappa\eta-\tau\xi+(\kappa^2-\tau^2)t)}\tilde{\alpha}_2(\tau,\kappa)\frac{1}{\tau-\lambda}d\kappa d\tau. \qquad (6.34)$$

Hence it follows from the first Lax equation that, putting $\beta = \epsilon\tilde{\beta}$ and $\gamma = \epsilon\tilde{\gamma}$,

$$\begin{aligned} \tilde{\beta}(\xi,\eta,t) &= -\frac{1}{2\pi}\int_{\tau\in\Gamma^\pm}\int_{\kappa\in\mathbb{R}} e^{i(\kappa\xi-\tau\eta+(\kappa^2-\tau^2)t)}\tilde{\alpha}_1(\tau,\kappa)d\kappa d\tau; \\ \tilde{\gamma}(\xi,\eta,t) &= -\frac{1}{2\pi}\int_{\tau\in\Gamma^\pm}\int_{\kappa\in\mathbb{R}} e^{i(\kappa\eta-\tau\xi+(\kappa^2-\tau^2)t)}\tilde{\alpha}_2(\tau,\kappa)d\kappa d\tau. \qquad (6.35) \end{aligned}$$

Analytic Continuation

With these results, we can prove the following theorem.

Theorem 6.4 *Suppose $\tilde{\alpha}_1(\xi,\eta)$ and $\tilde{\alpha}_1(\xi,\eta)$, defined on $\mathbb{CP}^1 \times \mathbb{CP}^1$ have an analytic continuation from the contour given by $\eta = \bar{\xi}$ to the contour given by $\xi \in \mathbb{R}$, $\eta \in \Gamma^{\pm}$, where $\Gamma^{\pm}$ is as defined in Section 5.2. Then there is an analytic continuation of the inverse scattering transform of the linearised DS-I equation to the inverse scattering transform of the DS-II equation, and vice versa.*

Proof: The proof is similar to the proof of Theorem 5.1 for the KP equations. By comparing equations (6.31) and (6.35), the result follows. □

6.3.2 Extension to the Full DS Equations

As for the KP equations, this result can be shown to hold for the full Davey-Stewartson equations, but again, the existence of lump solutions arises. In the DS case, though, the lump solutions exist for the DS-II equations, rather than the equations associated to the nonlocal Riemann-Hilbert problem, as in the KP case.

We have made the assumption in the derivation of the Dirac operator that the lump solutions do not exist. In fact, these will modify the Dirac operator so that it becomes

$$\partial_{\bar{\lambda}}\Phi(\lambda) = \Phi(\bar{\lambda})\Sigma_{\bar{\lambda}} \begin{pmatrix} 0 & \alpha_1 \\ \alpha_2 & 0 \end{pmatrix} \Sigma_{\lambda}^{-1} + S, \tag{6.36}$$

where S is a matrix-valued function depending on λ in the form of the sum of delta functions at a finite number of poles. This complicates matters, and for the time being, for simplicity, we have chosen to deal with the Dirac operator without lump solutions. For the rest of this chapter, we shall concentrate on the DS-II equations, with the assumption that lump solutions do not exist.

6.4 The Dirac Operator: Generic Boundary Conditions

Thus far, we have only considered Φ defined for all z, with a boundary condition at $z = \infty$. As for the KP equations, we would like to be able to consider Φ with a finite boundary condition on z, to draw a parallel to the Ward construction. We had no success with such a construction for the KP-II equation (see Section 3.3). This is in part due to the fact that the first Lax operator for the KP-II equation is parabolic; as with the standard heat operator, one cannot impose a generic boundary condition on the equation $L_2\phi = 0$ if the boundary is a closed curve. The first Lax operator for the DS-II equation, however, is an elliptic operator - these have a rather different behaviour to parabolic operators, notably admitting solutions that are specified on a closed boundary. Hence we will attempt to produce a localised version of the inverse scattering transform for the DS-II equations. For ease of calculation, we will consider z taking values on the unit disc.

A Basic Result

Let us first consider the problem

$$\partial_{\bar{z}}\phi(z,\bar{z};w;\lambda) = f(z,\bar{z})\phi(z,\bar{z};w;\lambda), \tag{6.37}$$

with boundary condition

$$\phi(z,\bar{z};w;\lambda) = \frac{1}{z-w}, \quad |w| > 1, \quad |z| = 1. \tag{6.38}$$

Then ϕ satisfies the integral equation

$$\phi(z,\bar{z};w;\lambda) = \frac{1}{z-w} + \frac{1}{2\pi i}\int_{\mathcal{D}} \frac{f(\xi,\bar{\xi})}{\xi - z}\phi(\xi,\bar{\xi};w;\lambda)d\xi \wedge d\bar{\xi}. \tag{6.39}$$

We can use ϕ to generate solutions of the equation (6.37), but with different boundary conditions. Suppose that ϕ_g satisfies equation (6.37) and

$\phi_g = g(z)$ for $|z| = 1$, where $g(z)$ is holomorphic on the unit disc. We have that

$$\phi_g(z, \bar{z}; \lambda) = g(z) + \frac{1}{2\pi i} \int_{\mathcal{D}} \frac{f(\xi, \bar{\xi})}{\xi - z} \phi_g(\xi, \bar{\xi}; \lambda) d\xi \wedge d\bar{\xi}. \tag{6.40}$$

On multiplying equation (6.39) by $(2\pi i)^{-1} g(w)$ and integrating around $\partial\mathcal{D}$, where $\partial\mathcal{D} = \{z : |z| = 1\}$, we obtain that

$$\frac{1}{2\pi i} \int_{\partial\mathcal{D}} g(w) \phi(z, \bar{z}; w; \lambda) dw \tag{6.41}$$

also satisfies equation (6.40). By the uniqueness of solution of (6.37), this gives that

$$\phi_g(z, \bar{z}; \lambda) = \frac{1}{2\pi i} \int_{\partial\mathcal{D}} g(w) \phi(z, \bar{z}; w; \lambda) dw. \tag{6.42}$$

The function $\phi(z, \bar{z}; w; \lambda)$ can effectively be used as a generating function for solutions with any boundary condition that has a holomorphic extension over the unit disc.

We will use this idea of a generating function for ϕ to derive as far as possible a localised inverse scattering theory for the DS-II equations.

We should note that for space considerations, when we say that a function depends on λ, we mean that it depends on both λ and $\bar{\lambda}$. However we should make clear that when we say that a function depends on the variable w (or $\bar{w}$), we mean that the function is holomorphic in that variable.

The DS-II case

Theorem 6.5 *Suppose that Φ_{-1} satisfies $L_1(\Phi_{-1}\Sigma_\lambda) = 0$ and*

$$\Phi_{-1} = \frac{1}{2\pi i} \begin{pmatrix} (w - z)^{-1} & 0 \\ 0 & (\bar{w} - \bar{z})^{-1} \end{pmatrix} \quad \text{for} \quad |z| = 1, t = 0.$$

Then Φ_{-1} satisfies the equations

$$\partial_{\bar{\lambda}} \Phi^{i1}_{-1}(z, \bar{z}; w; \lambda) = \alpha_1(w; \lambda, \bar{\lambda}) e^{\bar{\lambda}\bar{z} - \lambda z} \int_{\partial\mathcal{D}} \Phi^{i2}_{-1}(z, \bar{z}; \bar{w}; \bar{\lambda}) d\bar{w} \tag{6.43}$$

$$\partial_{\bar{\lambda}} \Phi^{i2}_{-1}(z, \bar{z}; \bar{w}; \lambda) = \alpha_2(\bar{w}; \lambda, \bar{\lambda}) e^{\bar{\lambda} z - \lambda\bar{z}} \int_{\partial\mathcal{D}} \Phi^{i1}_{-1}(z, \bar{z}; w; \bar{\lambda}) dw, \tag{6.44}$$

where

$$\alpha_1(w;\lambda,\bar{\lambda}) = -\frac{1}{2\pi i}\int_{\mathcal{D}} e^{\lambda\xi-\bar{\lambda}\bar{\xi}}\gamma(\xi,\bar{\xi})\Phi^{11}_{-1}(\xi,\bar{\xi};w;\lambda,\bar{\lambda})d\xi d\bar{\xi}, \tag{6.45}$$

$$\alpha_2(\bar{w};\lambda,\bar{\lambda}) = -\frac{1}{2\pi i}\int_{\mathcal{D}} e^{\lambda\bar{\xi}-\bar{\lambda}\xi}\beta(\xi,\bar{\xi})\Phi^{22}_{-1}(\xi,\bar{\xi};\bar{w};\lambda,\bar{\lambda})d\xi d\bar{\xi}. \tag{6.46}$$

Proof: Consider the first Lax operator of the Davey-Stewartson equations, acting on $\Phi = (\Phi^{ij})$. This is given by

$$\begin{aligned}
\partial_{\bar{z}}\Phi^{11}(z,\bar{z};w;\lambda) &= \beta(z,\bar{z})\Phi^{21}(z,\bar{z};w;\lambda),\\
(\partial_z+\lambda)\Phi^{21}(z,\bar{z};w;\lambda) &= \gamma(z,\bar{z})\Phi^{11}(z,\bar{z};w;\lambda),\\
(\partial_{\bar{z}}+\lambda)\Phi^{12}(z,\bar{z};\bar{w};\lambda) &= \beta(z,\bar{z})\Phi^{22}(z,\bar{z};\bar{w};\lambda),\\
\partial_z\Phi^{22}(z,\bar{z};\bar{w};\lambda) &= \gamma(z,\bar{z})\Phi^{12}(z,\bar{z};\bar{w};\lambda).
\end{aligned} \tag{6.47}$$

Note that the variable w has been introduced. This is because the boundary condition for this "generating" Φ will depend on w, where w is assumed to be outside the unit disc. We define the function Φ_{-1} as the matrix-valued function satisfying the above equations, having the boundary condition

$$\Phi_{-1}(z,\bar{z};w;\lambda) = \frac{1}{2\pi i}\begin{pmatrix} (w-z)^{-1} & 0 \\ 0 & (\bar{w}-\bar{z})^{-1}\end{pmatrix} \tag{6.48}$$

for $z\in\partial\mathcal{D}$. There is a lot of freedom in the most general solution to this problem. This corresponds to the multiplication of the four equations above by, respectively, $e^{h_{11}}, e^{h_{21}}, e^{h_{12}}, e^{h_{22}}$, where $h_{1i} = h_{1i}(z;\lambda,\bar{\lambda})$ and $h_{2i} = h_{2i}(\bar{z};\lambda,\bar{\lambda})$. This gives the general solution

$$\Phi^{11}_{-1}(z,\bar{z};w;\lambda) = \frac{1}{2\pi i}\left(\frac{1}{w-z}+\int_{\mathcal{D}}\frac{e^{h_{11}(\xi)-h_{11}(z)}}{\xi-z}\beta(\xi,\bar{\xi})\Phi^{21}_{-1}(\xi,\bar{\xi};w;\lambda)d\xi d\bar{\xi},\right) \tag{6.49}$$

$$\Phi^{21}_{-1}(z,\bar{z};w;\lambda) = \frac{1}{2\pi i}\int_{\mathcal{D}}\frac{e^{h_{21}(\bar{\xi})-h_{21}(\bar{z})}e^{\lambda(\xi-z)}}{\bar{\xi}-\bar{z}}\gamma(\xi,\bar{\xi})\Phi^{11}_{-1}(\xi,\bar{\xi};w;\lambda)d\xi d\bar{\xi}, \tag{6.50}$$

$$\Phi^{12}_{-1}(z,\bar{z};\bar{w};\lambda) = \frac{1}{2\pi i}\int_{\mathcal{D}}\frac{e^{h_{12}(\xi)-h_{12}(z)+\lambda(\bar{\xi}-\bar{z})}}{\xi-z}\beta(\xi,\bar{\xi})\Phi^{22}_{-1}(\xi,\bar{\xi};\bar{w};\lambda)d\xi d\bar{\xi}, \tag{6.51}$$

$$\Phi^{22}_{-1}(z,\bar{z};\bar{w};\lambda) = \frac{1}{2\pi i}\left(\frac{1}{\bar{w}-\bar{z}} + \int_{\mathcal{D}} \frac{e^{h_{22}(\bar{\xi})-h_{22}(\bar{z})}}{\bar{\xi}-\bar{z}}\gamma(\xi,\bar{\xi})\Phi^{12}_{-1}(\xi,\bar{\xi};\bar{w};\lambda)d\xi d\bar{\xi}.\right) \tag{6.52}$$

We require certain conditions on these formulae, primarily that they be manifestly bounded for all of $\lambda \in \mathbb{CP}^1$. This is sufficient to ensure that $h_{11} = 0$, $h_{21}(\bar{z}) = -\bar{\lambda}\bar{z}$, $h_{12}(z) = -\bar{\lambda}z$ and $h_{22} = 0$, all up to an additive constant which can be taken to be zero without loss of generality. This gives the results

$$\Phi^{11}_{-1}(z,\bar{z};w;\lambda) = \frac{1}{2\pi i}\left(\frac{1}{w-z} + \int_{\mathcal{D}} \frac{1}{\xi - z}\beta(\xi,\bar{\xi})\Phi^{21}_{-1}(\xi,\bar{\xi};w;\lambda)d\xi d\bar{\xi}\right), \tag{6.53}$$

$$\Phi^{21}_{-1}(z,\bar{z};w;\lambda) = \frac{1}{2\pi i}\int_{\mathcal{D}} \frac{e^{\lambda(\xi-z)-\bar{\lambda}(\bar{\xi}-\bar{z})}}{\bar{\xi}-\bar{z}}\gamma(\xi,\bar{\xi})\Phi^{11}_{-1}(\xi,\bar{\xi};w;\lambda)d\xi d\bar{\xi}, \tag{6.54}$$

$$\Phi^{12}_{-1}(z,\bar{z};\bar{w};\lambda) = \frac{1}{2\pi i}\int_{\mathcal{D}} \frac{e^{\lambda(\bar{\xi}-\bar{z})-\bar{\lambda}(\xi-z)}}{\xi - z}\beta(\xi,\bar{\xi})\Phi^{22}_{-1}(\xi,\bar{\xi};\bar{w};\lambda)d\xi d\bar{\xi}, \tag{6.55}$$

$$\Phi^{22}_{-1}(z,\bar{z};\bar{w};\lambda) = \frac{1}{2\pi i}\left(\frac{1}{\bar{w}-\bar{z}} + \int_{\mathcal{D}} \frac{1}{\bar{\xi}-\bar{z}}\gamma(\xi,\bar{\xi})\Phi^{12}_{-1}(\xi,\bar{\xi};\bar{w};\lambda)d\xi d\bar{\xi}\right). \tag{6.56}$$

The solutions Φ^{i1}_{-1} will be holomorphic on w for $w \in \{\mathbb{CP}^1 - \mathcal{D}\}$ and Φ^{i2}_{-1} will be anti-holomorphic on w for $w \in \{\mathbb{CP}^1 - \mathcal{D}\}$. We now establish the d-bar relation for these equations. Differentiating these equations with respect to $\bar{\lambda}$ gives

$$\partial_{\bar{\lambda}}\Phi^{11}_{-1}(z,\bar{z};w;\lambda) = \frac{1}{2\pi i}\int_{\mathcal{D}} \frac{1}{\xi - z}\beta(\xi,\bar{\xi})\partial_{\bar{\lambda}}\Phi^{21}_{-1}(\xi,\bar{\xi};w;\lambda)d\xi d\bar{\xi}, \tag{6.57}$$

$$\partial_{\bar{\lambda}}\Phi^{21}_{-1}(z,\bar{z};w;\lambda) = \tilde{\alpha}_1 + \frac{1}{2\pi i}\int_{\mathcal{D}} \frac{e^{\lambda(\xi-z)-\bar{\lambda}(\bar{\xi}-\bar{z})}}{\bar{\xi}-\bar{z}}\gamma(\xi,\bar{\xi})\partial_{\bar{\lambda}}\Phi^{11}_{-1}(\xi,\bar{\xi};w;\lambda)d\xi d\bar{\xi}, \tag{6.58}$$

$$\partial_{\bar{\lambda}}\Phi^{12}_{-1}(z,\bar{z};\bar{w};\lambda) = \tilde{\alpha}_2 + \frac{1}{2\pi i}\int_{\mathcal{D}} \frac{e^{\lambda(\bar{\xi}-\bar{z})-\bar{\lambda}(\xi-z)}}{\xi - z}\beta(\xi,\bar{\xi})\partial_{\bar{\lambda}}\Phi^{22}_{-1}(\xi,\bar{\xi};\bar{w};\lambda)d\xi d\bar{\xi}, \tag{6.59}$$

$$\partial_{\bar{\lambda}}\Phi^{22}_{-1}(z,\bar{z};\bar{w};\lambda) = \frac{1}{2\pi i}\int_{\mathcal{D}} \frac{1}{\bar{\xi}-\bar{z}}\gamma(\xi,\bar{\xi})\partial_{\bar{\lambda}}\Phi^{12}_{-1}(\xi,\bar{\xi};\bar{w};\lambda)d\xi d\bar{\xi}, \tag{6.60}$$

where

$$\tilde{\alpha}_1(z,\bar{z};w;\lambda;\bar{\lambda}) = e^{-\lambda z+\bar{\lambda}\bar{z}}\alpha_1(w;\lambda,\bar{\lambda}), \tag{6.61}$$

$$\tilde{\alpha}_2(z,\bar{z};\bar{w};\lambda;\bar{\lambda}) = e^{\bar{\lambda}z-\lambda\bar{z}}\alpha_1(w;\lambda,\bar{\lambda}), \tag{6.62}$$

and

$$\alpha_1(w;\lambda,\bar{\lambda}) = -\frac{1}{2\pi i}\int_{\mathcal{D}} e^{\lambda\xi-\bar{\lambda}\bar{\xi}}\gamma(\xi,\bar{\xi})\Phi^{11}_{-1}(\xi,\bar{\xi};w;\lambda,\bar{\lambda})d\xi d\bar{\xi}, \tag{6.63}$$

$$\alpha_2(\bar{w};\lambda,\bar{\lambda}) = -\frac{1}{2\pi i}\int_{\mathcal{D}} e^{\lambda\bar{\xi}-\bar{\lambda}\xi}\beta(\xi,\bar{\xi})\Phi^{22}_{-1}(\xi,\bar{\xi};\bar{w};\lambda,\bar{\lambda})d\xi d\bar{\xi}. \tag{6.64}$$

Let us now consider the function defined by

$$\hat{\Phi}(z,\bar{z};\lambda) = \partial_{\bar{\lambda}}\Phi_{-1}(z,\bar{z};\lambda)\begin{pmatrix} e^{\lambda z-\bar{\lambda}\bar{z}} & 0 \\ 0 & e^{\lambda\bar{z}-\bar{\lambda}z}\end{pmatrix}. \tag{6.65}$$

$\hat{\Phi}$ satisfies

$$\hat{\Phi}^{11}(z,\bar{z};w;\lambda) = \frac{1}{2\pi i}\int_{\mathcal{D}}\frac{e^{\lambda(z-\xi)-\bar{\lambda}(\bar{z}-\bar{\xi})}}{\xi-z}\beta(\xi,\bar{\xi})\hat{\Phi}^{21}(\xi,\bar{\xi};w;\lambda)d\xi d\bar{\xi}, \tag{6.66}$$

$$\hat{\Phi}^{21}(z,\bar{z};w;\lambda) = \alpha_1 + \frac{1}{2\pi i}\int_{\mathcal{D}}\frac{1}{\bar{\xi}-\bar{z}}\gamma(\xi,\bar{\xi})\hat{\Phi}^{11}(\xi,\bar{\xi};w;\lambda)d\xi d\bar{\xi}, \tag{6.67}$$

$$\hat{\Phi}^{12}(z,\bar{z};\bar{w};\lambda) = \alpha_2 + \frac{1}{2\pi i}\int_{\mathcal{D}}\frac{1}{\xi-z}\beta(\xi,\bar{\xi})\hat{\Phi}^{22}(\xi,\bar{\xi};\bar{w};\lambda)d\xi d\bar{\xi}, \tag{6.68}$$

$$\hat{\Phi}^{22}(z,\bar{z};\bar{w};\lambda) = \frac{1}{2\pi i}\int_{\mathcal{D}}\frac{e^{\lambda(\bar{z}-\bar{\xi})-\bar{\lambda}(z-\xi)}}{\bar{\xi}-\bar{z}}\gamma(\xi,\bar{\xi})\partial_{\bar{\lambda}}\Phi^{12}(\xi,\bar{\xi};\bar{w};\lambda)d\xi d\bar{\xi}, \tag{6.69}$$

Now define a second function $\tilde{\Phi}$ by

$$\tilde{\Phi}^{i1}(z,\bar{z};\lambda) = \int_{\partial\mathcal{D}}\Phi^{i1}_{-1}(z,\bar{z};w;\bar{\lambda})dw, \tag{6.70}$$

$$\tilde{\Phi}^{i2}(z,\bar{z};\lambda) = \int_{\partial\mathcal{D}}\Phi^{i2}_{-1}(z,\bar{z};\bar{w};\bar{\lambda})d\bar{w}. \tag{6.71}$$

This is the function that satisfies equations (6.47) with the boundary condition that $\Phi = I$ for $|z| = 1$.

On integrating as above the equations (6.49) to (6.52), and comparing them with equations (6.66) to (6.69), we see that

$$\alpha_2\tilde{\Phi}^{i1} = \hat{\Phi}^{i2}, \quad \alpha_1\tilde{\Phi}^{i2} = \hat{\Phi}^{i1}, \tag{6.72}$$

that is

$$\partial_{\bar{\lambda}}\Phi^{i1}_{-1}(z,\bar{z};w;\lambda) = \alpha_1(w;\lambda,\bar{\lambda})e^{\bar{\lambda}\bar{z}-\lambda z}\int_{\partial\mathcal{D}}\Phi^{i2}_{-1}(z,\bar{z};\bar{w};\bar{\lambda})d\bar{w} \tag{6.73}$$

$$\partial_{\bar{\lambda}}\Phi^{i2}_{-1}(z,\bar{z};\bar{w};\lambda) = \alpha_2(\bar{w};\lambda,\bar{\lambda})e^{\bar{\lambda}z-\lambda\bar{z}}\int_{\partial\mathcal{D}}\Phi^{i1}_{-1}(z,\bar{z};w;\bar{\lambda})dw. \tag{6.74}$$

□

Thus we have an inverse scattering transform of a sort for the problem as described. Unfortunately, a problem now arises. In the global case, the boundary condition on Φ is that $\Phi \to 1$ as $z \to \infty$ *for all t*. We assume that β and γ initially have compact support on some bounded set containing $z = 0$ and the nature of the problem, with this initial condition ensures that the boundary condition is viable. However, for the case of the finite boundary, β and γ will not vanish on the boundary for $t > 0$ and we will need to be able to determine how the boundary value for Φ propagates from its initial value I using the second Lax operator. This calculation is currently beyond our capabilities, but it is worth noting that we have come up against the same problem as we did with the localised KP-II equation - a parabolic operator, in this case the second Lax operator rather than the first. Although β and γ may have compact support initially, for any $t > 0$ this will not be the case (as in the heat equation), although they will still tend to zero at $z = \infty$, removing the problem in the global case.

However we do have a d-bar-like relation for Φ_{-1} at $t = 0$. This is of the form

$$\partial_{\bar{\lambda}}\Phi_{-1}(\lambda) = \left[\oint_{w\in\partial\mathcal{D}} dw\Phi_{-1}(\bar{\lambda})\right]\Sigma_{\bar{\lambda}}\begin{pmatrix} 0 & \alpha_2(\bar{w},\lambda,\bar{\lambda}) \\ \alpha_1(w,\lambda,\bar{\lambda}) & 0 \end{pmatrix}\Sigma^{-1}_{\lambda}, \tag{6.75}$$

where we define $\oint dwA(w,\bar{w})$ by

$$\oint dwA(w,\bar{w}) = \begin{pmatrix} \oint A^{11}(w,\bar{w})dw & \oint A^{12}(w,\bar{w})d\bar{w} \\ \oint A^{21}(w,\bar{w})dw & \oint A^{22}(w,\bar{w})d\bar{w} \end{pmatrix},$$

and Σ_λ is given by

$$\Sigma_\lambda = \begin{pmatrix} e^{\lambda z} & 0 \\ 0 & e^{\lambda \bar{z}} \end{pmatrix}.$$

Thus even though this inverse scattering relation is only valid for $t = 0$, it is much more complicated than the standard d-bar relation due to the contour integral.

Corollary 6.6 *Suppose Φ_g satisfies $L_1(\Phi_g \Sigma_\lambda) = 0$ and*

$$\Phi_g = \begin{pmatrix} g_1(z) & 0 \\ 0 & g_2(\bar{z}) \end{pmatrix} \quad \text{for} \quad |z| = 1, t = 0,$$

for any function g_1 holomorphic on the unit disc and any function g_2 anti-holomorphic on the unit disc. Then equation (6.75) can be used as a generator for the inverse scattering of Φ_g at $t = 0$, obtaining the d-bar-like relation

$$\partial_{\bar{\lambda}} \Phi_g(\lambda) = \left[\oint_{w \in \partial \mathcal{D}} dw \Phi_{-1}(\bar{\lambda}) \right] \Sigma_{\bar{\lambda}} \begin{pmatrix} 0 & \tilde{\alpha}_2(\lambda, \bar{\lambda}) \\ \tilde{\alpha}_1(\lambda, \bar{\lambda}) & 0 \end{pmatrix} \Sigma_{\bar{\lambda}}^{-1}, \qquad (6.76)$$

where

$$\begin{aligned} \tilde{\alpha}_1(\lambda, \bar{\lambda}) &= \frac{1}{2\pi i} \oint_{|w|=1} g_1(w) \alpha_1(w, \lambda, \bar{\lambda}) dw \\ \tilde{\alpha}_2(\lambda, \bar{\lambda}) &= \frac{1}{2\pi i} \oint_{|w|=1} g_2(\bar{w}) \alpha_2(\bar{w}, \lambda, \bar{\lambda}) d\bar{w} \end{aligned} \qquad (6.77)$$

Proof: This result is obtained as indicated previously, by multiplying equation (6.75) by the boundary value of Φ and integrating around $|w| = 1$. $\square$

Thus we have produced an inverse scattering result for a general boundary condition on the unit disc. Apart from the fact that this only holds for $t = 0$, we also have a second problem in that the above result relates $\partial_{\bar{\lambda}} \Phi_g$ with $\Phi_{-1}(\bar{\lambda})$ rather than $\Phi_g(\bar{\lambda})$. Thus this result has fewer similarities to the Ward construction than previous results. Note we would also have to deal with how the boundary condition evolves through time and how that would effect the relation.

Taking into account these additional complications, it would seem that a general result for all t, if it does exist, would be considerably more complicated than the Dirac operator for the global case.

6.5 Inversion of the D-Bar Problem and the First Lax Operator

We have chosen the Dirac operator as the basis for the geometrical construction due to the similarities between the d-bar equation

$$\begin{pmatrix} \partial_{\bar{\lambda}} & 0 \\ 0 & \partial_{\lambda} \end{pmatrix} \Phi - \begin{pmatrix} 0 & \alpha \\ \tilde{\alpha} & 0 \end{pmatrix} \bar{\Phi} = 0 \tag{6.78}$$

and the Sparling equation $(\partial_{\bar{\lambda}} + \alpha)\psi = 0$ defining holomorphic vector bundles. In this section, we give a reformulation of the above Dirac equation, and demonstrate a duality of sorts between the first Lax operator for the DS-II equations and the reformulated Dirac operator.

Reformulation of the D-bar equation

The first Lax operator L_1,

$$L_1 = \begin{pmatrix} \partial_{\bar{z}} & 0 \\ 0 & \partial_{z} \end{pmatrix} - \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}$$

acting on Ψ does not give four linked equations but two pairs of coupled equations. Thus when we introduce the spectral parameter, there is no need for it to be the same parameter for each pair - we chose the parameter to be λ in both cases to get the d-bar relation in the form we desired. For this case though, we make a different choice of parameter by replacing λ in the second set of equations with $\bar{\lambda}$, i.e. taking $L_1\Psi = 0$, with Ψ of the form

$$\Psi = \Phi \begin{pmatrix} e^{\lambda z+\lambda^2 t} & 0 \\ 0 & e^{\bar{\lambda}\bar{z}+\bar{\lambda}^2 t} \end{pmatrix}, \tag{6.79}$$

where, as before, $\Phi \to 1$ as $\lambda, |z| \to \infty$. The four equations on the components of Φ are

$$\begin{aligned} \partial_{\bar{z}}\Phi^{11}(z,\bar{z};\lambda) &= \beta(z,\bar{z})\Phi^{21}(z,\bar{z};\lambda), \\ (\partial_z+\lambda)\Phi^{21}(z,\bar{z};\lambda) &= \gamma(z,\bar{z})\Phi^{11}(z,\bar{z};\lambda), \\ (\partial_{\bar{z}}+\bar{\lambda})\Phi^{12}(z,\bar{z};\lambda) &= \beta(z,\bar{z})\Phi^{22}(z,\bar{z};\lambda), \\ \partial_z\Phi^{22}(z,\bar{z};\lambda) &= \gamma(z,\bar{z})\Phi^{12}(z,\bar{z};\lambda). \end{aligned} \tag{6.80}$$

Let us work for the time being with Φ defined for all $z \in \mathbb{C}$ and $\Phi \to 1$ as $|z|, |\lambda| \to \infty$. This problem can be rewritten as the integral equations

$$\Phi^{11}(z,\bar{z};\lambda) = 1 + \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{1}{\xi - z}\beta(\xi,\bar{\xi})\Phi^{21}(\xi,\bar{\xi};\lambda)d\xi d\bar{\xi}, \tag{6.81}$$

$$\Phi^{21}(z,\bar{z};\lambda) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{e^{\lambda(\xi-z)-\bar{\lambda}(\bar{\xi}-\bar{z})}}{\bar{\xi}-\bar{z}}\gamma(\xi,\bar{\xi})\Phi^{11}(\xi,\bar{\xi};\lambda)d\xi d\bar{\xi}, \tag{6.82}$$

$$\Phi^{12}(z,\bar{z};\lambda) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{e^{\bar{\lambda}(\bar{\xi}-\bar{z})-\lambda(\xi-z)}}{\xi-z}\beta(\xi,\bar{\xi})\Phi^{22}(\xi,\bar{\xi};\lambda)d\xi d\bar{\xi}, \tag{6.83}$$

$$\Phi^{22}(z,\bar{z};\lambda) = 1 + \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{1}{\bar{\xi}-\bar{z}}\gamma(\xi,\bar{\xi})\Phi^{12}(\xi,\bar{\xi};\lambda)d\xi d\bar{\xi}. \tag{6.84}$$

Now utilising the similar methods to the derivation of the Dirac operator for the Davey Stewartson equations, one sees that

$$\partial_{\bar{\lambda}}\Phi^{i1}(z,\bar{z};\lambda) = e^{-\lambda z+\bar{\lambda}\bar{z}}\alpha(\lambda,\bar{\lambda})\Phi^{i2}(z,\bar{z};\lambda) \tag{6.85}$$

and

$$\partial_{\lambda}\Phi^{i2}(z,\bar{z};\lambda) = e^{+\lambda z-\bar{\lambda}\bar{z}}\tilde{\alpha}(\lambda,\bar{\lambda})\Phi^{i1}(z,\bar{z};\lambda), \tag{6.86}$$

where

$$\alpha(\lambda,\bar{\lambda}) = -\frac{1}{2\pi i}\int_{\mathbb{C}} e^{\lambda\xi-\bar{\lambda}\bar{\xi}}\gamma(\xi,\bar{\xi})\Phi^{11}(\xi,\bar{\xi};\lambda)d\xi d\bar{\xi} \tag{6.87}$$

and

$$\tilde{\alpha}(\lambda, \bar{\lambda}) = -\frac{1}{2\pi i} \int_{\mathbb{C}} e^{\bar{\lambda}\bar{\xi} - \lambda\xi} \beta(\xi, \bar{\xi}) \Phi^{22}(\xi, \bar{\xi}; \lambda) d\xi d\bar{\xi}. \tag{6.88}$$

In matrix form, this is equivalent to the equation

$$\left[\begin{pmatrix} \partial_{\bar{\lambda}} & 0 \\ 0 & \partial_{\lambda} \end{pmatrix} - \begin{pmatrix} 0 & \hat{\alpha} e^{+\lambda z - \bar{\lambda}\bar{z}} \\ \alpha e^{-\lambda z + \bar{\lambda}\bar{z}} & \end{pmatrix} \right] \Phi^T = 0 \tag{6.89}$$

Substituting Ψ into the equation, we have the relation

$$\tilde{D}\Psi := \left[\begin{pmatrix} \partial_{\bar{\lambda}} & 0 \\ 0 & \partial_{\lambda} \end{pmatrix} - \begin{pmatrix} 0 & \alpha(\lambda, \bar{\lambda}) \\ \hat{\alpha}(\lambda, \bar{\lambda}) & 0 \end{pmatrix} \right] \Psi^T = 0. \tag{6.90}$$

Thus we have an alternative to the d-bar equation given previously. This equation also resembles the Sparling equation, although with the differential operator of a different form, rather than the multiplicative operator acting on a conjugate.

An Inversion of Operators

The Dirac operator and the first Lax operator for the DS-II equation are strikingly similar. An obvious question to ask is whether any new relations can be generated by "swapping" them over. In other words, what happens if we use the methods utilised to derive the d-bar relation from the Lax operator on the Lax operator itself, and similarly, what happens if we use the methods that derive the Lax operator from the d-bar relation on the d-bar relation? It seems possible that a series of similar operators could be generated thus; in this section, using the reformulation of the Dirac operator, we see this is not the case.

Proposition 6.7 *Suppose we treat the reformulated Dirac operator as a Lax operator acting on Φ. Then the associated inverse scattering operator is the standard first Lax operator for the DS-II equations. Also, if we consider the first Lax operator as a Dirac operator, then the associated first Lax operator is the reformulated Dirac operator.*

Proof: We start with equations (6.85) and (6.86) and integrate them directly. Φ still has the boundary condition that $\Phi \to 1$ as $|\lambda|$, $|z| \to \infty$, hence we have that

$$\begin{aligned}
\Phi^{11}(z,\lambda) &= 1 + \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{e^{-\eta z+\bar{\eta}\bar{z}}}{\eta-\lambda}\alpha(\eta)\Phi^{12}(z,\eta)d\eta d\bar{\eta} \\
\Phi^{21}(z,\lambda) &= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{e^{-\eta z+\bar{\eta}\bar{z}}}{\eta-\lambda}\alpha(\eta)\Phi^{22}(z,\eta)d\eta d\bar{\eta} \\
\Phi^{12}(z,\lambda) &= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{e^{+\eta z-\bar{\eta}\bar{z}}}{\bar{\eta}-\bar{\lambda}}\tilde{\alpha}(\eta)\Phi^{11}(z,\eta)d\eta d\bar{\eta} \\
\Phi^{22}(z,\lambda) &= 1 + \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{e^{+\eta z-\bar{\eta}\bar{z}}}{\bar{\eta}-\bar{\lambda}}\tilde{\alpha}(\eta)\Phi^{21}(z,\eta)d\eta d\bar{\eta} \qquad (6.91)
\end{aligned}$$

We now consider the expressions Φ_i, $i = 1, 2$, where

$$\Phi_1 = \begin{pmatrix} \partial_{\bar{z}}\Phi^{11} & e^{-\bar{\lambda}\bar{z}}\partial_{\bar{z}}(e^{\bar{\lambda}\bar{z}}\Phi^{12}) \\ e^{-\lambda z}\partial_z(e^{\lambda z}\Phi^{21}) & \partial_z\Phi^{22} \end{pmatrix},$$

and

$$\Phi_2 = \begin{pmatrix} \kappa_1\Phi^{21} & \kappa_2\Phi^{22} \\ \kappa_1\Phi^{11} & \kappa_2\Phi^{12} \end{pmatrix},$$

where κ_i are given by

$$\begin{aligned}
\kappa_1(z) &= -\frac{1}{2\pi i}\int_{\mathbb{C}} e^{+\eta z-\bar{\eta}\bar{z}}\tilde{\alpha}(\eta)\Phi^{11}(\eta)d\eta d\bar{\eta}, \\
\kappa_2(z) &= -\frac{1}{2\pi i}\int_{\mathbb{C}} e^{-\eta z+\bar{\eta}\bar{z}}\alpha(\eta)\Phi^{22}(\eta)d\eta d\bar{\eta}.
\end{aligned}$$

It can be shown, by differentiation of equations (6.91) with respect to z or $\bar{z}$, that Φ_i satisfies

$$\begin{aligned}
\Phi_i^{11}(z,\lambda) &= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{e^{-\eta z+\bar{\eta}\bar{z}}}{\eta-\lambda}\alpha(\eta)\Phi_i^{12}(z,\eta)d\eta d\bar{\eta}, \\
\Phi_i^{12}(z,\lambda) &= \kappa_1(z) + \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{e^{+\eta z-\bar{\eta}\bar{z}}}{\bar{\eta}-\bar{\lambda}}\tilde{\alpha}(\eta)\Phi_i^{11}(z,\eta)d\eta d\bar{\eta}, \\
\Phi_i^{21}(z,\lambda) &= \kappa_2(z) + \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{e^{-\eta z+\bar{\eta}\bar{z}}}{\eta-\lambda}\alpha(\eta)\Phi_i^{22}(z,\eta)d\eta d\bar{\eta}, \\
\Phi_i^{22}(z,\lambda) &= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{e^{+\eta z-\bar{\eta}\bar{z}}}{\bar{\eta}-\bar{\lambda}}\tilde{\alpha}(\eta)\Phi_i^{21}(z,\eta)d\eta d\bar{\eta},
\end{aligned}$$

for each $i = 1, 2$. Hence Φ_1 and Φ_2 both satisfy the same integral equations which, by assumption, have a unique solution, and hence $\Phi_1 = \Phi_2$. In other words, we have derived the first Lax operator from the (reformulated) Dirac operator.

It is straightforward to show that a similar result holds if we treat the Lax operator as a Dirac operator and derive an operator using the methods detailed in the proof of Proposition 3.4; the operator derived is simply the reformulated Dirac operator. □

As shown above, we cannot derive new integrable systems related to the Davey-Stewartson equations by this method. In the next chapter, however, we demonstrate a number of generalisations of the Dirac operator that give rise to new systems of equations.

Chapter 7

Generalisations of the Dirac operator

The Dirac operator can be used to generate more integrable systems than just the KP hierarchy and the DS equations. There are several changes that can be made to the set-up that give different results. Notably these are

- The form of μ. The most interesting results arise when negative powers of the spectral parameter λ are introduced, by taking a different inhomogeneous form of the variable μ. This procedure can be adopted for both the KP and the DS case, giving rise to a number of known and unknown equations. This procedure can be used to effectively extend the KP and DS hierarchies.

- The boundary condition of ϕ as λ tends to ∞.

- The form of ϕ. In the KP case, it is taken to be scalar valued, but much more freedom is produced by making it take values in $GL(2, \mathbb{C})$. More results may follow by changing this Lie group, either by restricting it to a subgroup of $GL(2, \mathbb{C})$ or by increasing the size of the matrix to $GL(N, \mathbb{C})$. Note that if ϕ is matrix-valued, then so will be α.

- An alteration of the form of the Dirac operator itself. One can take the standard Dirac operator as

$$\not{D}_A = \begin{pmatrix} \partial_{\bar{\lambda}} & 0 \\ 0 & \partial_\lambda \end{pmatrix} + A,$$

  and replace the matrix A with a more general operator. For example, if A is an integral operator, one essentially obtains the nonlocal d-bar problem, which is a more general inverse scattering transform than the semi-local d-bar problem used for KP-II and DS-II.

Below we detail some examples of generalisations of the Dirac operator, using a variety of the above choices. The majority of the results arise from the first generalisation, that of altering the form of μ to include negative powers of λ.

Note that one further modification that could be considered is the form of α. However, for the Dirac operator picture to be effective, α must take the form

$$\alpha(\mu, \bar{\mu}; \lambda, \bar{\lambda}) = \exp -\mu\, \alpha_0(\lambda, \bar{\lambda})\, \exp \bar{\mu}.$$

with any modifications being in the form of μ or the form of ϕ.

7.1 The "Modified" Hierarchies

The first alternative form of μ that we shall consider is that obtained by replacing λ with λ^{-1}, in both the KP and the DS d-bar equations. Applying this generalisation to the KP form of the Dirac operator will give the appropriate Dirac operator for the modified KP equation (mKP) and applying it to the DS form will give the operator for the Ishimori equation, amongst others.

Note that in this chapter, all of the equations mentioned are in fact the type-II equations, i.e. by the KP equation, we actually mean the KP-II equation.

Negative Powers of the Spectral Parameter

What the first generalisation of the system involves is to keep the boundary condition on ϕ to be $\phi \to 1$ as $\lambda \to \infty$ whilst changing the form of μ. Recall in the case of the KP equation $\mu = \lambda x + \lambda^2 y + \lambda^3 t$, which is a coordinate on the fibre of the line bundle $\mathcal{O}(3)$ over $\mathbb{CP}^1$. In spinor notation, let π_A be homogeneous global coordinates on $\mathbb{CP}^1$. On a region containing the origin, $(\pi_0, \pi_1) = (1, \lambda)$. For the KP case, we chose

$$\mu = \frac{\tilde{\mu}}{(\pi_0)^4}, \tag{7.1}$$

where $\tilde{\mu}$ is the homogeneous form of μ given by

$$\tilde{\mu} = \sum_{i=0}^{3} t_i \pi_0^{3-i} \pi_1^i. \tag{7.2}$$

Note that as the t_i are real, the w variable makes no contribution to the Dirac operator. We shall consider as an alternative

$$\hat{\mu} = \frac{\tilde{\mu}}{(\pi_1)^4} = w + \lambda^{-1} x + \lambda^{-2} y + \lambda^{-3} t, \tag{7.3}$$

where the t_i have been labelled to agree with the standard notation. Now the potential singularities have been moved to the origin, but we will retain the normalisation that $\phi = 1$ at $\lambda = \infty$.

The mKP Hierarchy

Definition 7.1 *A Dirac operator for the mKP-II hierarchy is as defined in Definition 3.3 where $\alpha(\mu, \bar{\mu}; \lambda, \bar{\lambda})$ is taken to be a smooth scalar-valued function on $\mathcal{O}(n)$, restricted to a section $\mu = \sum t_i \lambda^{-i}$, where α_0 vanishes near $\lambda = 0$ and $\lambda = \infty$ and has the reality condition $\alpha_0(\lambda, \bar{\lambda}) = \overline{\alpha_0(\bar{\lambda}, \lambda)}$.*

Theorem 7.2 *Given a Dirac operator as in Definition 7.1, there exists an associated sequence of Lax operators for the mKP-II hierarchy and hence there exists an associated solution u to the mKP-II hierarchy with the boundary condition $u(x, y, t) \to 0$ as $x^2 + y^2 \to \infty$.*

Proof: We wish to find candidates L_i for Lax operators such that

$$\begin{aligned} L_i\phi &= O(\lambda^{-1}) \quad \text{near} \quad \lambda = \infty \\ L_i\phi &= O(1) \qquad \text{near} \quad \lambda = 0. \end{aligned} \tag{7.4}$$

We do not require $L_i\phi$ to vanish at both $\lambda = 0$ and $\lambda = \infty$ as we are effectively treating ϕ as a function of the variable $\lambda' = \lambda^{-1}$, defined on $\mathbb{CP}^1 - \{0\}$. To consider the necessary behaviour on $\mathbb{CP}^1 - \{\infty\}$, we multiply by the patching function λ. Thus we only require ϕ as given to have order zero near $\lambda = 0$ and not order one.

As we need to consider the behaviour near the origin, we will simplify things slightly by constraining α to vanish not only at $\lambda = \infty$ but also at $\lambda = 0$.

Now, as we did for the derivation of the Dirac operator for the KP-II equation, we consider operators of the form

$$D_{t_i}^n = e^{-\mu}\partial_{t_i}^n e^{\mu}, \tag{7.5}$$

as the components of the operator L_i. We will take, again as per the KP-II equations, the leading order terms of L_i to be given by

$$\tilde{L}_i = D_{t_i} - D_x^i. \tag{7.6}$$

As α vanishes near both $\lambda = \infty$ and $\lambda = 0$, then ϕ is holomorphic at these points and thus has the expansion

$$\phi \sim 1 + \lambda^{-1}\phi^1 + \ldots, \tag{7.7}$$

near $\lambda = \infty$ and

$$\phi \sim \hat{\phi}^0 + \lambda\hat{\phi}^1 + \ldots \tag{7.8}$$

near $\lambda = 0$.

It follows that any expression of the form $D_{t_i}^n\phi$ near $\lambda = \infty$ is automatically of order λ, so it remains to consider the expressions near $\lambda = 0$.

As an example, the leading order part of L_2 has the representation

$$(D_y - D_x^2)\phi = -2\lambda^{-1}\hat{\phi}_x^0 + O(1). \tag{7.9}$$

The full expression for this operator to have the correct order is given by

$$L_2 = D_y - D_x^2 + 2uD_x, \tag{7.10}$$

where $u = \hat{\phi}_x^0/\hat{\phi}^0$. The second operator is given by

$$L_3 = D_t - D_x^3 + 3uD_x^2 + vD_x, \tag{7.11}$$

where v is an expression in $\hat{\phi}_0$ and $\hat{\phi}_1$. Thus the Lax pair acting on ψ is given by

$$\begin{aligned} \hat{L}_2 &= \partial_y - \partial_x^2 + 2u\partial_x, \\ \hat{L}_3 &= \partial_t - \partial_x^3 + 3u\partial_x^2 + v\partial_x, \end{aligned} \tag{7.12}$$

the Lax pair for the mKP-II equation. □

The procedure can be continued to find operators of the form

$$\hat{L}_n = \partial_{t_n} - \partial_x^n + \sum_{r=1}^{n-1} u^{(r)}\partial_x^r, \tag{7.13}$$

for all positive integer n. This is the *mKP hierarchy.*

Another Generalisation leading to the mKP Equation

There is another equally valid generalisation of the d-bar relation that gives rise to the mKP hierarchy. Let μ be as in the KP case, but now change the boundary condition of ϕ so that $\phi \to 0$ as $\lambda \to \infty$, rather than tending to unity. Following through the calculation as per the derivation of the KP hierarchy in Section (3.1) we arrive at the same Lax pair as above. The calculation is very similar to the one above, as it is basically the same case with λ replaced directly with λ^{-1}.

The "mDS" Equations

It seems a sensible approach to perform the same procedure on the Dirac operator for the Davey-Stewartson equations. In this case, we obtain something rather interesting, notably a number of different equations.

We will first outline the most basic of these. Suppose we put $\Psi = \Phi\Sigma_\lambda$, where

$$\begin{aligned} \Sigma_\lambda &= \exp\begin{pmatrix} \lambda^{-1}z + \lambda^{-2}t & 0 \\ 0 & \lambda^{-1}\bar{z} + \lambda^{-2}t \end{pmatrix} \\ &= \exp\left[\lambda^{-1}Ix + i\lambda^{-1}\sigma_3 y + \lambda^{-2}It\right], \end{aligned} \tag{7.14}$$

where σ_3 is given by

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We will work in the real variables x and y for the time being (rather than the complex z) as the calculation is rather more straightforward. Suppose near $\lambda = 0$, Φ has the expansion

$$\Phi = g + \lambda h + \dots.$$

Then the relevant expressions we need are

$$\begin{aligned} D_x\Phi &= \lambda^{-1}g + O(1), \\ D_y\Phi &= i\lambda^{-1}g\sigma_3 + O(1), \\ D_x^2\Phi &= \lambda^{-2}g + \lambda^{-1}(2g_x + h) + O(1), \\ D_t\Phi &= \lambda^{-2}g + \lambda^{-1}h + O(1). \end{aligned} \tag{7.15}$$

We again look for operators such that $L\Phi = O(1)$; the simplest of these are

$$\begin{aligned} L &= \partial_y - i(g\sigma_3 g^{-1})\partial_x, \\ M &= \partial_t - \partial_x^2 + 2g_x g^{-1}\partial_x. \end{aligned} \tag{7.16}$$

The compatibility condition $[L, M] = 0$ gives the system of equations

$$A_x = [B, A]\,, \tag{7.17}$$

and

$$2B_y + i\,(A_t - A_{xx}) - 2i\,(AB_x - BA_x) = 0, \tag{7.18}$$

where

$$A = g\sigma_3 g^{-1} \quad \text{and} \quad B = g_x g^{-1}. \tag{7.19}$$

Substituting in for g shows that the first equation is automatically satisfied, and the second equation is given by

$$\begin{aligned} & 2g_{xy} - 2g_x g^{-1} g_y + ig_t\sigma_3 - ig\sigma_3 g^{-1} g_t \\ & -ig\sigma_3 g^{-1} g_{xx} - ig_{xx}\sigma_3 + 2ig_x g^{-1} g_x \sigma_3 \;=\; 0 \end{aligned} \tag{7.20}$$

There is, however, some freedom in the second operator M above. The other choices for the operator include

$$\begin{aligned} M_2 &= \partial_t + \partial_y^2 - 2g_y g^{-1}\partial_y \\ M_3 &= \partial_t + i\partial_x\partial_y + ig_x g^{-1}\partial_y + ig_y g^{-1}\partial_x, \end{aligned} \tag{7.21}$$

which will give an equivalent integrable system, as these operators can be obtained by adding AL to M_1, where A is the appropriate differential operator. Another choice that can be made is in the multiplication of the operators by a function independent of λ. One such variation is described by Konopelchenko and Matkarimov in [13], where the choice of Lax operators are given by

$$\begin{aligned} L' &= ih\partial_y - (\sigma_3 h)\partial_x, \\ M' &= h\partial_t - h\partial_x\partial_y - 2h_x\partial_x - 2h_y\partial_y, \end{aligned} \tag{7.22}$$

where $h = g^{-1}$. This choice requires an involved compatibility condition of the form $[L', M'] + AL' + BM' = 0$ to give the integrable system, but has the advantage that with a certain choice of variables, one can reduce this complicated system to either the Davey-Stewartson-II equations, or the Ishimori-II equations.

Note that as in the case of the mKP equation, the mDS equations and the corresponding hierarchy can also be derived by changing the boundary condition on ϕ as $\lambda \to \infty$ to $\phi \to 0$. The calculations follow the same lines as those above.

7.2 Positive and Negative Powers of λ in μ

We now consider μ in the form

$$\mu = \frac{\tilde{\mu}}{(\pi_0)^{n-1}\pi_1}, \tag{7.23}$$

that is $\mu = \lambda^{-1}u + w + \lambda x + \cdots \lambda^{n-1}t_{n-1}$.

We now need to consider the behaviour of ϕ at both $\lambda = 0$ and $\lambda = \infty$. We will do this by instead of considering a scalar value of ϕ, we look at a vector-valued ϕ.

We suppose ϕ is homogeneous of degree one. Then by the index theorem, there exists a two-dimensional solution space, so we can normalise at two points on the Riemann sphere instead of one, as in the KP case. Because of the $GL(2, \mathbb{C})$ freedom in the construction, we can define ϕ^0 as the solution to

$$\frac{\partial \phi}{\partial \bar{\lambda}}(\lambda) = e^{\bar{\mu}-\mu}\alpha(\lambda, \bar{\lambda})\phi(\bar{\lambda}) \tag{7.24}$$

such that

$$\phi^0(\lambda = 0) = 0 \quad \text{and} \quad \phi^0(\lambda = \infty) = 1,$$

and ϕ^1 as the solution such that

$$\phi^1(\lambda = 0) = 1 \quad \text{and} \quad \phi^1(\lambda = \infty) = 0.$$

We replace the space-time coordinates t_{-1}, t_1 and t_2 with, respectively u, x and y. Recall that, as detailed for the mKP equation above, for an expression to vanish near the origin, it needs to be of order zero, rather than of order one due to the transition function.

$\alpha(\lambda, \bar{\lambda})$ is taken to vanish in the neighbourhood of $\lambda = 0$ and $\lambda = \infty$ and to be smooth elsewhere, implying that ϕ^i are holomorphic in these regions and hence admit expansions of the form

$$\begin{aligned} \phi^0 &= \quad\quad\ \ \beta^{(1)} + \beta^{(2)}\lambda + \cdots \\ \phi^1 &= \lambda^{-1} + \ \gamma^{(1)} + \gamma^{(2)}\lambda + \cdots, \end{aligned} \tag{7.25}$$

near $\lambda = 0$ and

$$\begin{aligned} \phi^0 &= 1 + \ \delta^{(1)}\lambda^{-1} + \delta^{(2)}\lambda^{-2} + \cdots \\ \phi^1 &= \quad\quad \epsilon^{(1)}\lambda^{-1} + \epsilon^{(2)}\lambda^{-2} + \cdots \end{aligned} \tag{7.26}$$

near $\lambda = \infty$.

7.2.1 Matrix Lax Operators

Proposition 7.3 *Suppose that $\Phi = (\phi^0 \quad \phi^1)^T$ is a two-component column vector where ϕ^0 and ϕ^1 have the behaviour near $\lambda = 0$ and $\lambda = \infty$ defined above. Then if Φ satisfies a Dirac equation of the form*

$$\begin{pmatrix} \partial_{\bar{\lambda}} & \alpha_0 e^{\bar{\mu}-\mu} \\ \overline{\alpha_0} e^{\mu-\bar{\mu}} & \partial_{\lambda} \end{pmatrix} \begin{pmatrix} \Phi \\ \overline{\Phi} \end{pmatrix} = 0,$$

where $\mu = \lambda^{-1}u + w + \lambda x + \lambda^2 y$. Then this generates the Lax pair and hence a solution of a new $(2+1)$-dimensional integrable system given by

$$\begin{aligned} \Delta_u - (\beta\epsilon)_x &= 0, \\ \kappa_u - (\beta_x\epsilon)_x &= 0, \\ \beta_t - \beta_{xxx} + 3\beta_x\Delta - 3\beta\kappa &= 0, \\ \epsilon_t - \epsilon_{xxx} + 3\kappa\epsilon + 3(\epsilon\Delta)_x &= 0. \end{aligned}$$

Proof: We wish to construct a series of differential operators M_k such that $e^{-\mu}M_k e^{\mu}\Phi$ has a Taylor expansion containing only negative powers of λ at $\lambda = \infty$ and only non-negative powers at $\lambda = 0$. Then, as $e^{-\mu}M_k e^{\mu}\Phi$ is smooth on $\mathbb{CP}^1$ and vanishes at $\lambda = 0$ and $\lambda = \infty$, then it must vanish everywhere. Hence the operators M_k must commute and their commutation relations give rise to the system of differential equations.

We first consider the region near $\lambda = 0$. We define the operator D_x by

$$D_x = e^{-\mu}\partial_x e^{\mu},$$

and similarly D_u and D_y. We see that, near $\lambda = 0$,

$$\phi^1 = \lambda^{-1} + O(1),$$

$$D_u\phi^0 = \beta^{(1)}\lambda^{-1} + O(1).$$

Hence the expression

$$D_u\phi^0 - \beta^{(1)}\phi^1$$

is smooth on $\mathbb{CP}^1$ and vanishes at both $\lambda = 0$ and $\lambda = \infty$. Hence the expression must be zero everywhere. In a similar manner, we consider the neighbourhood of $\lambda = \infty$. There, we have that

$$D_x\phi^1 - \epsilon^{(1)}\phi^0 = O(\lambda^{-1}),$$

and hence by the same argument as above, this must also vanish everywhere. Hence we have the first Lax operator

$$M_1 = \begin{pmatrix} \partial_u & 0 \\ 0 & \partial_x \end{pmatrix} - \begin{pmatrix} 0 & \beta^{(1)} \\ \epsilon^{(1)} & 0 \end{pmatrix}.$$

To obtain the second Lax operator, we consider again the neighbourhood of $\lambda = \infty$, and look at the expressions $D_y - D_x^2\phi^i$. It follows that

$$(D_y - D_x^2)\phi^0 + 2\delta_x^{(1)}\phi^0 = O(\lambda^{-1}),$$

$$(D_y - D_x^2)\phi^1 + 2\epsilon_x^{(1)}\phi^0 = O(\lambda^{-1}).$$

By the earlier argument, both of these expression must vanish, giving the second Lax operator

$$M_2 = \partial_y - \partial_x^2 + \begin{pmatrix} 2\delta_x^{(1)} & 0 \\ 2\epsilon_x^{(1)} & 0 \end{pmatrix}.$$

The compatibility condition

$$[M_1, M_2] + AM_1 = 0$$

is attained by taking

$$A = \begin{pmatrix} 0 & 2\beta_x^{(1)} \\ 2\epsilon_x^{(1)} & 0 \end{pmatrix}.$$

This condition gives the equations (with upper indices suppressed)

$$\beta_y - \beta_{xx} + 2\beta\delta = 0,$$

$$\epsilon_y + \epsilon_{xx} - 2\epsilon\delta = 0,$$

$$\delta_u - (\beta\epsilon)_x = 0. \tag{7.27}$$

This equation is a reduction of the Davey-Stewartson equation. It is discussed by Konopelchenko in [12] but in a different context to this. This is not the only pair of Lax operators that can be derived. An alternative choice of second operator is

$$M_2 = \begin{pmatrix} \partial_y - \partial_x^2 & 0 \\ -\epsilon^{(1)}\partial_x & \partial_y \end{pmatrix} + \begin{pmatrix} 2\delta_x^{(1)} & 0 \\ \epsilon_x^{(1)} & 0 \end{pmatrix}, \tag{7.28}$$

but this simply gives rise to the same system of equations, i.e. equations (7.27). To obtain a new system, we take the first operator from the above Lax pair, but take a second operator with leading term $\partial_t - \partial_x^3$. This is given by

$$M_3 = \partial_t - \partial_x^3 + 3\begin{pmatrix} \delta_x & 0 \\ \epsilon_x & 0 \end{pmatrix}\partial_x - 3\begin{pmatrix} \kappa & 0 \\ -\epsilon\delta_x & 0 \end{pmatrix}. \tag{7.29}$$

The commutation relation

$$[M_1, M_3] + 3 \begin{pmatrix} 0 & \beta_x \partial_x + \beta_{xx} - \beta \delta_x \\ \epsilon_x \partial_x + \epsilon \delta_x & 0 \end{pmatrix} M_3 = 0, \tag{7.30}$$

gives rise to the system of equations

$$\begin{aligned} \Delta_u - (\beta\epsilon)_x &= 0, \\ \kappa_u - (\beta_x \epsilon)_x &= 0, \\ \beta_t - \beta_{xxx} + 3\beta_x \Delta - 3\beta\kappa &= 0, \\ \epsilon_t - \epsilon_{xxx} + 3\kappa\epsilon + 3(\epsilon\Delta)_x &= 0, \end{aligned} \tag{7.31}$$

where $\Delta = \delta_x$.

Note that an obvious hierarchy exists, consisting of operators M_n of the form

$$M_n = \partial_{t_n} - \partial_x^n + \sum_{r=0}^{n-2} \begin{pmatrix} \kappa^{(r)} & 0 \\ \rho^{(r)} & 0 \end{pmatrix} \partial_x^r,$$

for some functions $\kappa^{(r)}$ and $\rho^{(r)}$. □

7.2.2 Scalar Lax Operators

Theorem 7.4 *The KP hierarchy of Lax operators L_n, $n = 2, 3, \ldots$ can be extended in the negative direction to form new integrable systems. The first such operator is given by*

$$L_1 = \partial_x \partial_u - \beta \partial_u - \rho_u,$$

and the compatibility condition $[L_1, L_2] + 2\beta_x L_1 = 0$ gives the integrable system

$$\begin{aligned} \beta_y - \beta_{xx} - 2\beta\beta_x + 2\rho_{xx} &= 0, \\ (\rho_y + \rho_{xx})_u - 2\beta\rho_{xu} + \beta_x \rho_u &= 0. \end{aligned} \tag{7.32}$$

Proof: We again combine the negative and positive powers of λ in μ although in this case with respect to scalar-valued functions. By considering operators acting on ϕ_0 as defined in the previous section, we can show that the expressions

$$\begin{aligned} \hat{L}_1\phi^0 &= (D_x D_u - \beta D_u - \rho_u)\,\phi^0, \\ \hat{L}_2\phi^0 &= \left(D_y - D_x^2 + 2\rho_x\right)\phi^0, \end{aligned} \tag{7.33}$$

where $\rho = u + \delta^{(1)}$ and $\beta = \partial_x\left(\log\beta^{(1)}\right)$, vanish at both $\lambda = 0$ and $\lambda = \infty$, as, in fact, do all the operators from the KP hierarchy. Hence putting

$$\begin{aligned} L_1 &= \partial_x\partial_u - \beta\partial_u - \rho_u, \\ L_2 &= \partial_y - \partial_x^2 + 2\rho_x, \end{aligned} \tag{7.34}$$

and considering the compatibility condition

$$[L_1, L_2] + 2\beta_x L_1 = 0, \tag{7.35}$$

we obtain the corresponding equation. One can extend this hierarchy by introducing variables that are the coefficients of λ^{-n}, $n = 2, 3, \ldots$ in μ. □

There is a similar result for the mKP hierarchy.

Theorem 7.5 *The mKP hierarchy of Lax operators can be extended in the negative direction to form new integrable systems. The first such operator is given by*

$$M_1 = \partial_x\partial_u - \zeta_u\partial_x - \eta,\,,$$

and the compatibility condition $[M_1, M_2] + 2\zeta_{xx}M_1 = 0$ *gives the integrable system*

$$\begin{aligned} \eta_y - \eta_{xx} + 2\left(\zeta_x\eta\right)_x &= 0 \\ \left(\zeta_y + \zeta_{xx} + \zeta_x^2\right)_u &= 2\eta_x. \end{aligned} \tag{7.36}$$

Proof: This result is proved as for the KP extension, although operators are considered acting on ϕ^1 rather than ϕ^0. □

Construction of Hierarchies

For each of these systems, there is an immediate hierarchy of equations that can be obtained. Note that the second Lax operator in each system is that of the KP-II equation and the mKP-II equation respectively. It follows that the operator M_1 can be appended to the KP hierarchy to form a new hierarchy and similarly $\tilde{M}_1$ can be added to the mKP hierarchy. Note however that these hierarchies can be extended further by considering the variables given by the coefficients of $\lambda^{-2}, \lambda^{-3}, \ldots$ in μ.

7.3 A Simple Generalisation of the DS Equations

Generalisations need not be so involved. This is a straightforward generalisation of the Davey-Stewartson operator.

Proposition 7.6 *By taking the Davey-Stewartson form of the Dirac operator, with* $\Sigma_\lambda = \lambda(Ix + gy) + \lambda^2 t$ *for some constant invertible matrix* g*, then there is a corresponding integrable system given by the equation*

$$2A_{xy} - 2gA_{xx} + [A_t - A_{xx}, g] = 0.$$

Proof: By following the method of derivation of the Lax operators of the DS-II equation, where α is taken to vanish in the neighbourhood of $\lambda = \infty$ and Φ has the expansion

$$\Phi = 1 + \lambda^{-1}A + \cdots,$$

we obtain a Lax pair given by

$$L_1 = \partial_y - g\partial_x - [A, g]$$

$$L_2 = \partial_t - \partial_x^2 + 2A_x.$$

The commutation condition $[L_1, L_2] = 0$ gives the stated differential equation.□

7.4 Further Generalisations of the Dirac Operator

In this section, we discuss the generalisations of the d-bar relation (and hence the Dirac operator) that other integrable systems have as their inverse scattering transforms. Whatever construction is developed to describe solutions of the KP equations should be potentially generalisable to include these systems as well.

The Nonlocal D-bar Problem

This form of inverse scattering transform is more general than the d-bar problem discussed earlier and also incorporates the nonlocal Riemann-Hilbert problem, i.e., in the case of the KP equation, its nonlocal d-bar problem covers both KP-I and KP-II. The nonlocal d-bar problem is given by

$$\frac{\partial \phi}{\partial \bar{\lambda}}(\lambda, \bar{\lambda}) = \int_{\mathbb{CP}^1} \phi(\lambda', \bar{\lambda}') R(\lambda', \bar{\lambda}'; \lambda, \bar{\lambda}) d\lambda' d\bar{\lambda}'. \tag{7.37}$$

One obtains the d-bar problem by choosing $R = \delta(\lambda' - \bar{\lambda})\tilde{R}(\lambda, \bar{\lambda})$ and the nonlocal Riemann-Hilbert problem by $R = \delta(\lambda - \bar{\lambda})\delta(\lambda' - \bar{\lambda}')\tilde{R}(\lambda, \bar{\lambda})$. The potential R will also depend on the variables x_i, in the form of exponentials as per KP-II, DS-II, i.e.,

$$R(\lambda, \bar{\lambda}; \lambda', \bar{\lambda}'; x, y, t) = e^{-\mu_{\lambda'}} R_0(\lambda, \bar{\lambda}; \lambda' \bar{\lambda}') e^{+\mu_\lambda}.$$

The Lax hierarchies are chosen by finding operators of the form

$$L = \sum u_{lmn} D_x^l D_y^m D_t^n, \tag{7.38}$$

(where $D_x = e^{-\mu}\partial_x e^{\mu}$) such that

$$\left[\frac{\partial}{\partial \bar{\lambda}}, L\right] \phi = 0. \tag{7.39}$$

Assuming that the nonlocal d-bar problem has a unique solution (up to scale), this gives that $L\phi = 0$.This is effectively the same procedure as for the local d-bar problem, except for the fact that no boundary conditions are needed to be imposed on the solution, giving rise to a larger class of solutions of the equations that can be generated.

Equations generated by the nonlocal d-bar problem include the (2+1)-dimensional Harry Dym equation (see Dubrovsky and Konopelchenko,[7]) and the (2+1)-dimensional equations of Loewner type (see Konopelchenko and Rogers,[14], [15]).

The Veselov-Novikov (NVN-I) Equation

This is given by

$$u_t + u_{zzz} + u_{\bar{z}\bar{z}\bar{z}} - 3(u\partial_z^{-1}u_{\bar{z}})_{\bar{z}} - 3(u\partial_{\bar{z}}^{-1}u_z)_z + 3\epsilon(\partial_z^{-1}u_{\bar{z}\bar{z}} + \partial_{\bar{z}}^{-1}u_{zz}) = 0. \quad (7.40)$$

This derives from the expression

$$[L_1, L_2] = BL_1,$$

where the Lax pair L_1 and L_2 and the operator B are given by

$$\begin{aligned} L_1 &= -\partial_z\partial_{\bar{z}} + u(z,\bar{z},t) - \epsilon \\ L_2 &= \partial_t + \partial_z^3 + \partial_{\bar{z}}^3 - 3(\partial_z^{-1}u_{\bar{z}})\partial_{\bar{z}} - 3(\partial_{\bar{z}}^{-1}u_z)\partial_z, \\ B &= -3(\partial_z^{-1}u_{\bar{z}\bar{z}} + \partial_{\bar{z}}^{-1}u_{zz}) \end{aligned} \quad (7.41)$$

The form of the inverse scattering depends on the sign on ϵ. The case for $\epsilon = 0$ is somewhat complicated; for information on this, see Boiti *et al*, [3]. The case for $\epsilon < 0$ (the NVN-I_- equation) has a reasonably straightforward d-bar relation. The spectral parameter is introduced by putting

$$\psi = \phi e^{-\tau(\lambda z + \lambda^{-1}\bar{z})}, \quad \text{where} \quad \tau^2 = -\epsilon. \quad (7.42)$$

The inverse scattering for this equation is a straightforward d-bar relation of the form

$$\frac{\partial\phi}{\partial\bar{\lambda}}(\lambda,\bar{\lambda}) = F(\lambda,\bar{\lambda})e^{\tau\left[(\lambda-\bar{\lambda}^{-1})z-(\bar{\lambda}-\lambda^{-1})\bar{z}\right]}\phi(\bar{\lambda}^{-1},\lambda^{-1}). \quad (7.43)$$

This relation can be used to create a Dirac operator, as per the KP-II equation. A little care is needed to deal with the singularity at $\lambda = 0$, but the methods used for the mKP equation should cover this.

A more interesting situation occurs for the case $\epsilon > 0$, (the NVN-I_+ equation). The Green's function for L_1 is non-holomorphic everywhere, giving rise to a d-bar problem, but also is discontinuous at $|\lambda| = 1$. As such, if we put $\psi = \phi e^{\left[-ik(\lambda z+\lambda^{-1}\bar{z})\right]}$, $(k^2 = \epsilon)$ then ϕ not only satisfies a d-bar relation of the form

$$\frac{\partial \phi}{\partial \bar{\lambda}}(\lambda, \bar{\lambda}) = F(\lambda, \bar{\lambda}) e^{ik\left[(\lambda+\bar{\lambda}^{-1})z+(\bar{\lambda}+\lambda^{-1})\bar{z}\right]} \phi(-\bar{\lambda}^{-1}, -\lambda^{-1}), \quad |\lambda| = 1 \tag{7.44}$$

but also satisfies a nonlocal Riemann-Hilbert on the unit circle

$$\left[\phi^+ - \phi^-\right](\lambda, \bar{\lambda}) = \oint R(\lambda, \lambda') e^{ik\left[(\lambda-\lambda')z+(\bar{\lambda}-\bar{\lambda}')\bar{z}\right]} \phi^-(\lambda', \bar{\lambda}') d\lambda', \tag{7.45}$$

where ϕ^+ (ϕ^-) is the boundary value of the function for $|\lambda| > 1$ ($|\lambda| < 1$). This equation seems to be the simplest example of this combination of d-bar and nonlocal Riemann-Hilbert inverse scattering. We would need to use a nonlocal d-bar relation with $\mu_\lambda = -ik(\lambda z + \lambda^{-1}\bar{z})$ and with R given by

$$R(\lambda, \bar{\lambda}; \lambda', \bar{\lambda}') = e^{\mu_\lambda - \mu_{\lambda'}} \left[F(\lambda)\delta(\lambda - \bar{\lambda}') + \tilde{F}(\lambda)\delta(|\lambda| - 1)(|\lambda'| - 1)\right].$$

This appears to be the most complicated reduction of the nonlocal d-bar relation that gives rise to a known integrable system. It should be noted that any results concerning these equations are uniformly for each individual case, rather than for the nonlocal d-bar equation in its general form. This is due to the fact that every reduction of the nonlocal d-bar equation that gives rise to an integrable system uses a delta function, which will have different effects on the analysis involved, depending on the form of the delta function. The behaviour of the KP equations is a good example of this.

Part IV

CONCLUSION

Chapter 8

Conclusion

The primary goal of this thesis was to find and construct a Ward-like correspondence for the KP equations. The Dirac operator certainly fulfils this to an extent, but also falls short of what would be desirable in some areas. In this final chapter, we consider where the Dirac operator succeeds in being a Ward-like construction, where it fails, and what uses can be made of it in its current form. We propose a possible alternative correspondence, and present an open problem.

Is the Dirac Operator a Suitable Correspondence?

There is no conclusive answer to this question, unfortunately. It depends on how close to the Ward construction one wants the structure to be. For example, the Dirac operator (and the nonlocal Riemann-Hilbert problem) are both constructions on minitwistor space and provide a link between the minitwistor variables and space-time. However the operators contain rather more structure than the Ward correspondence and seem to be nonlocalisable. It is this property (that the Ward construction possesses) that the Dirac operator is currently missing.

The Localisation Issue

Is the Dirac operator a localisable theory? The results of our work toward this goal are ultimately inconclusive, but we believe that it is not localisable in the standard form given for the quasi-local d-bar operator, for a number of reasons.

First, as described in Section 3.3, the nonholomorphy of the function ϕ depends explicitly on the boundary condition at $x^2 + y^2$ and $\lambda = \infty$ and the fact that the Green's function must be bounded for all x, y and λ. Localising in any of the spatial variables will introduce a quite different problem (the Green's function for $\mathbb{R}^2$ is an exponential, whilst that for a semi-infinite strip is a Fourier series) which may well not lead to a Dirac operator.

Second, there is the question of boundary conditions. As demonstrated for the DS-II equation in Section 6.4, imposing a certain simple local boundary condition complicates the form of the d-bar equation - it becomes

$$\partial_{\bar{\lambda}}\phi(\lambda) + A\phi(\bar{\lambda}) = 0,$$

where A is an integral operator. What is more, should the boundary condition be generalised to any holomorphic function, the equation becomes

$$\partial_{\bar{\lambda}}\phi(\lambda) + \tilde{A}\tilde{\phi}(\bar{\lambda}) = 0,$$

where $\tilde{A}$ is an integral operator and $\tilde{\phi}$ is a solution of the Lax pair with different boundary behaviour to ϕ. The corresponding Dirac operator no longer acts on a single function and hence becomes unsuitable for the structure proposed. Finally, regarding boundary conditions, there is the question of the evolution of the boundary condition. For example, in the DS-II case, we cannot tell how the boundary condition chosen for $t = 0$ evolves in time, but this would introduce yet more complications to the construction. A similar problem occurs for the semi-infinite strip for KP-II in determining the evolution of the boundary data in the y-direction.

The idea of localised solutions for certain integrable systems is an ongoing research interest of Fokas. He has worked to date on localisation issues for the KdV and linearised KdV equations - see [9]. This has also produced results involving complicated contour integrals. It seems reasonable to assume that any extension to KP will be just as involved, if not more so.

The question now becomes, if we cannot localise the structure in a suitable way, do we actually want and/or need to?

The ideal construction that we would hope for as an end product of this work would be one that not only contains the KP equations as a reduction, but also the ASDYM equations and all of the other $(2+1)$-dimensional integrable systems that are not obtainable as reductions of the ASDYM equations. One might speculate that this general construction will not be a local one, but that the reduction that produces the ASDYM equations is such that local solutions become possible.

Inclusion of KP-I

Is the Dirac construction a suitable alternative for the Ward correspondence? It would appear not to be the case. Putting aside this issue of localisation, the construction perhaps should not be considered until it incorporates the KP-I equation (and its lump solutions) as well. Also, unlike the Ward correspondence, there is no gauge freedom in the Dirac operator.

We have proposed a geometric structure that corresponds to the lump solutions in Chapter 4, however this seems to be a quite separate entity from the Dirac operator and does not incorporate the solutions arising from the nonlocal Riemann-Hilbert problem. Until we obtain a full understanding of the underlying geometry of the nonlocal Riemann-Hilbert problem (and, to an extent, the Dirac operator), a suitable construction incorporating both equations will prove elusive.

An Alternative Construction

There is, in fact, a construction that incorporates both the KP-I and KP-II equations and many others - the nonlocal d-bar problem (as described in Section 7.4). This construction, given a suitable reduction, will generate either the d-bar problem or the nonlocal Riemann-Hilbert problem, including the lump solutions. In fact, if the integral kernel in equation (7.37) is taken to be of the form

$$R(\lambda', \bar{\lambda}'; \lambda, \bar{\lambda}) = -\alpha(\lambda, \bar{\lambda})\delta(\lambda - \lambda'),$$

one obtains the Sparling equation $(\partial_{\bar{\lambda}} + \alpha)\phi = 0$, which can be used to generate the ASDYM equations.

The problem with this construction is that it is manifestly nonlocal and is basically a formal construction. For example, it seems that to consider any properties of the nonlocal d-bar construction, one must first make a reduction of it and then consider the behaviour of the reduction. This is in part due to the nature of the reductions - they invariably involve the use of delta function. However, if one is willing to sacrifice the condition that the general Ward-like construction needs to be local, then the nonlocal d-bar problem would seem to be a place to start.

The Dirac Operator

Putting aside the question of whether the Dirac operator is suitable as a Ward-like construction, what else can we use this construction for?

- As detailed in Section 3.2, the Dirac operator encompasses the theory of the KP equations as proposed by Segal and Wilson in [23]. In fact, it covers more solutions than that theory does, as Segal-Wilson theory requires that the functions be holomorphic in λ, whereas the Dirac operator merely requires them to be smooth.

- The Dirac operator can be used to generate new hierarchies of integrable systems. For example, the equations presented in Sections 7.2.1 and 7.2.2 are new integrable systems and there is the potential for many more systems to be generated.

Integrability of the KP-II Equation

One question that perhaps should be asked is, is the KP-II equation actually integrable? A survey of the literature on the subject produces no exact solutions to the equation. There are no examples of solutions and the corresponding functions satisfying the d-bar relation and it does not possess a standard Ward correspondence. So should we even be looking for one?

Clearly, the answer is yes. In Section 3.2 we present a solution in power series form. Whilst this does not have a closed form, it satisfies the d-bar problem as required. The lump solutions for the KP-I equations can be analytically continued to provide singular solutions to KP-II. Also there are solutions to the KP-II equation that are not related to the d-bar problem - they do not vanish at $x^2+y^2 \to \infty$ and hence do not satisfy the conditions for the inverse scattering procedures. Also, there exist action-angle variables for the equation (as detailed by Lipovsky in [16]). Indeed, the existence of the Lax pair for the equation is enough to show that solutions to the equations exist.

Outstanding Problems

Let us assume that the Dirac operator is the best "Ward-like" correspondence that we can construct for the KP-II equation. What other results could be proved about it?

One of the most important aspects of the work of Segal and Wilson, [23], on the KdV and KP equations is the theory of tau functions. The tau function τ is a form of potential for the equations, as

$$u = \partial_x^2 (\log \tau)$$

is a solution of the relevant equation. Briefly, the theory concerns the Grassmannian Gr of the Hilbert space $L^2(S^1; \mathbb{C})$. The tau function measures the action of the group Γ_+ on the canonical section of the determinant line bundle over Gr, where Γ_+ is the group of non-vanishing holomorphic functions on the unit disc, acting multiplicatively. In [19], Mason, Woodhouse and Singer prove a similar result, but related to twistor theory for the KdV equation, namely that the tau function is related to the determinant of the d-bar operator on a holomorphic vector bundle over twistor space corresponding to the KdV equation.

The Segal and Wilson tau function exists for both the KdV and the KP equations, so the obvious conjecture to make is that if we replace the $\bar{\partial}$-operator in the KdV argument above with the Dirac operator, do we obtain the tau function for the KP equation?

This question has yet to be answered, although one might well expect it to be true. There are a number of problems with calculating the determinant of the Dirac operator that have yet to be overcome, unfortunately.

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