Pricing Techniques for the Oil Industry

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Abstract

The field of financial mathematics is not limited to just the financial sector, but can be extended to many other industrial problems. This dissertation aims to introduce financial mathematical techniques and their application to the oil industry. We propose a methodology for pricing certain apparatus used in the oil industry, concentrating on a multi-phase flow meter produced by Schlumberger.

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Chapter 1

Introduction

In this project we propose a methodology for pricing a multi-phase flow meter produced by Schlumberger.

Multi-phase flow meters are instruments that provide information on the contents and concentration of produce from a reservoir. Sub-sea meters are used to monitor production from offshore wells, whilst terrain meters are used to monitor on-shore wells. The information from such a meter is vital in determining the value of a reservoir, and assists decisions relating to the operation of the reservoir. The main uses of such meters are:

- Allocating costs and revenues: often pipelines are shared by different companies and the cost and revenues have to be allocated between them. These can be determined by measuring the flow-rate.

- Information on reservoir structure: the data from the flow meter can assist in diagnostics on how the flow is changing with respect to time.

- Replacing other apparatus: sub-sea meters can replace deep sea pipelines. A meter installed at the gathering manifold allows a second pipeline from the manifold to the platform to be eliminated. A case study of a sub-sea "tie-back" in the Gulf of Mexico indicated the replacement of a coated pipeline of length 15 miles and diameter 6 inches, costing \$3,500,000 with installation. The cost of installing a meter would be the price of the meter, M, and the well-head hook up with cost \$500,000. This delivers a lower bound for the meter price,

$$
\begin{aligned}
M &\geq \$3,500,000 - \$500,000 \\
&= \$3,000,000
\end{aligned}
$$

Note that this is a simple lower bound because the actual metering benefits have not been considered here and must still added to the value of the meter.

- Predicting clogging of deep sea pipelines: sub-sea meters can determine when the formation of ice-like hydrates within deep sea pipelines is possible. Once a clog has occurred it is expensive to repair.

- Abandonment decisions: generally the production of a well is viewed as having an exponential decline with the possibility of noise, about which little is known. Relating the production information, obtained from the flow meter, to the market price of the asset determines the optimal decision of when to abandon a certain project.

- Production decisions from satellite reservoirs: if a platform has several wells feeding into it and there are limits on the total rates of gas and water flow that the platform can handle, then the flow meters on the incoming wells can be used to determine which of the wells to take full advantage of and how to allocate the capacity of the pipeline as to obtain optimal production.

Readings from such meters are approximately 95% accurate, accuracy greater than any other competing apparatus. Alternative hardware to multi-phase flow meters include large settling tanks, called separators. These require a seven man crew in comparison to a multi-phase flow meters two man crew. Multi-phase flow meters are generally bought for permanent installation, but periodic testing is available.

An ideal starting point would be the case of a perfect meter. We value such a meter and decide what additional risks occur if the meter were not perfectly accurate.

In chapter 2 we discuss various mathematical methods used in approaching a solution to such a problem. The case of an ideal sub-sea meter is considered in chapter 3, then extended to the non-ideal case in chapter 4. Finally, in chapter 5, we consider the use of a multi-phase flow meter in maximising the profit from two separate wells.

Chapter 2

Probabilistic and Qualitative Methods

Decision techniques for the optimal operation of a company have in recent years become more sophisticated with the assistance of probabilistic and qualitative methods. The oil industry is at the forefront of the adoption of these techniques and has an emerging interest in option theory and value pricing. This dissertation aims to use the recent developments in decision techniques to present both a general methodology for value-pricing hardware, and the particular application of that to the case of multi-phase flow meters.

Decision analysis represents a general framework for modelling and analysing decisions as to help the decision maker. There is a variety of different techniques including discounted cash flow, Monte Carlo simulation, portfolio theory, option theory and utility theory. This discussion will begin with the definitions of risk and uncertainty which are generally viewed as different concepts.

Risk is the chance or probability of some event occurring. It acts as a binary concept being the probability of an event occurring or not occurring; this probability can be estimated by considering previous experience.

Uncertainty is the range of possible values or sizes of that same event if it occurs. Each possible outcome of the event has a certain probability associated with it and these possible outcomes form a continuum.

2.1 Discounted Cash Flow

This is the most widely used tool in the industry and is based on the idea that an amount of money received at some point in the future is worth less than the same amount received at the present time. For example, an amount E received at some future time T would have a different value at the present time. If we assume that interest rates are constant and discount E using a continuously compounded interest rate r, the present value $M(t)$ at a certain time $t \leq T$, can be calculated by solving

$$\frac{\mathrm{d}M}{M} = r\mathrm{d}t.$$

The solution to this is

$$M = ce^{rt},$$

where c is a constant from integration. Using the fact that at time $t = T$ we have $M = E$, the value at a certain time t is given by

$$M = Ee^{-r(T-t)}.$$

Note that if the interest rate, r, were not constant but a known function of time, say $r(t)$ then this would be modified to

$$M = Ee^{-\int_t^T r(s)\,\mathrm{d}s}.$$

The associated concept of **Net Present Value**, **NPV**, is the sum of the discounted cash flows and represents the difference between the present discounted values of the cash outflow over the projected life of the project and the present values of the cash inflows.

Discounted cash flow analysis can be illustrated by considering the value of a $2300 net cash flow that is received annually in a period of up to five years, as a result of investing $5000 today, having different worth for each year:

Year	Revenue	Expenses	NCF	10% discounted NCF	20% discounted NCF
0			−$5000	−$5000	−$5000
1	$3000	$700	$2300	$2091	$1917
2	$3000	$700	$2300	$1900	$1597
3	$3000	$700	$2300	$1728	$1331
4	$3000	$700	$2300	$1571	$1109
5	$3000	$700	$2300	$1428	$924
Total	$15000	$3500	$6500	$1878	$3114

Note that here we have assumed that the revenue and operating expenses remain constant over the period, but could increase in value proportionally to inflation. Using a 10% discount rate, the value of a $2300 net cash flow, (NCF), that is received in year 5, as a result of investing $5000 today, is worth $1428. The sum of all the discounted net cash flows, at a discount rate of 10%, is $3718. Thus, had we invested the $5000 at an interest rate of 10% over the five year period we would earn $3718 less than had we invested in the project. This example also shows calculations when a discount rate of 20% is used.

We assume that if the **NPV** is positive, a profit would be earned and the project is a good investment; otherwise if negative. This is the **NPV Rule** which dictates to invest if and only if

$$\mathbf{NPV} \geq 0.$$

A vital element in this calculation is the discount rate applied. If the rate were viewed as the risk-free rate, i.e. the rate that a bank would offer; then a negative NPV would imply that investing the money in a bank account would yield a higher return than the proposed project.

This method has one downfall in that it is limited due to its insensitivity to changes in circumstances in the industry over long time intervals. It can be combined with sensitivity analysis which views consequences of possible changes but fails in quantifying the probability of any given outcome. The **NPV** rule ignores investing at a latter date and so is correct for *now or never* investments such as the decision to invest before the anticipated entry of a competitor firm in a market large enough for only one firm.

The value to invest at a latter date, thus the value of the *opportunity* to invest latter, for example *now or in a year*, would be the positive difference between the **NPV** in a year and the present **NPV**. For a more thorough description of this technique see Dixit and Pindyck [3].

2.2 Monte Carlo Simulation

This statistical approach determines the probability of an event by calculating distributions of possible inputs and then randomly sampling these inputs in a "trial". The result of a trial provides one possible answer. The random sampling of input and trials thereon is performed typically 1000 to $100,000$ times, so that a distribution of possible outcomes is obtained. The more likely outcomes will be sampled more than the extremes because there are more examples in that range. The end result is a distribution curve of a range of possible outcomes and the probability of any particular value occurring. The shape of these distributions can be various i.e. normal, triangular, log-normal , depending on the information available or the model used.

One problem with this method is that due to incorrect measurements, little data availability or limited sample size there is an intrinsic error within this method; see Forth [4]. Monte Carlo simulation is widely used for estimating reserves and to a far lesser extent for economic decisions.

2.3 Portfolio Theory

Portfolio theory is concerned with finding the most desirable group of assets to hold. The aim is to assemble and exploit the best possible combination of these assets to achieve a maximum rate of return and minimise risk over a single time period where each asset has a given level of risk attached to it. Typical assets used in such a portfolio are stock, bonds and traded options.

For example consider two stocks, S_1 and S_2, and a risk-less asset such as a bond, B. The existence of risk means that one can no longer associate a single payoff with investment in any asset. The payoff must be described as a set of possible outcomes with each having an associated probability of occurrence, called a return distribution. These distributions have a measure of central tendency, called the expected return, and a measure of risk or dispersion around the mean, called the standard deviation. Indicate the expected return and standard deviation of our stocks, S_1 and S_2, by μ_1, μ_2 and σ_1, σ_2 respectively; see Figure 2.1.

This marked bold line in this figure indicates the **efficient frontier**, which are all the possible combinations of these assets which are desirable to hold. If we combine
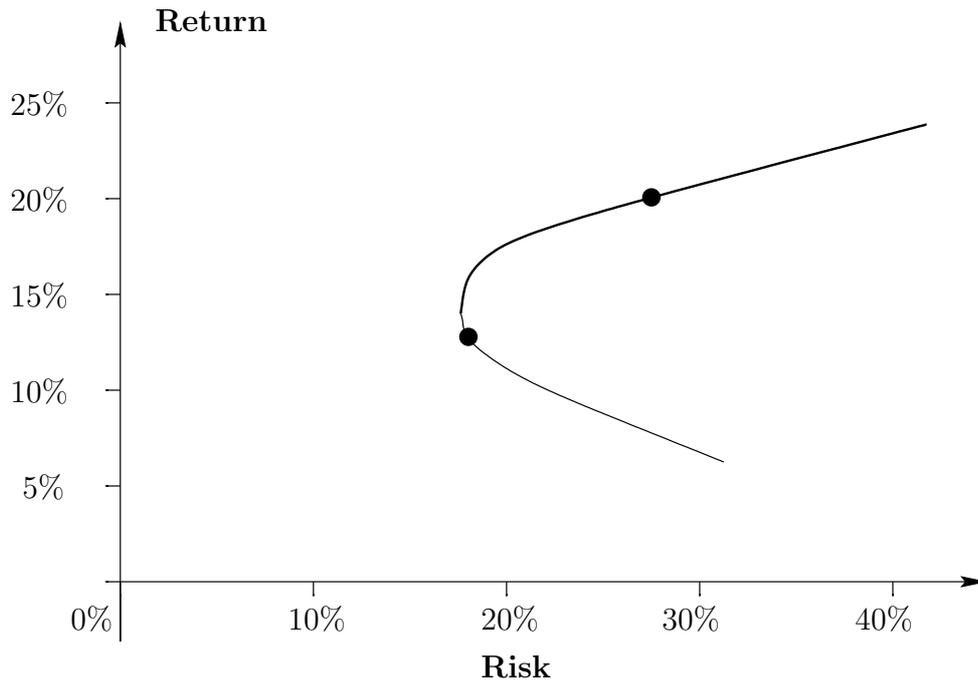
Figure 2.1: Two stocks and any combination.

the two stocks in a portfolio, Π, the expected return of our portfolio is by definition

$$\mu_\Pi = w_1\mu_1 + w_2\mu_2,$$

where w_1 and w_2 are the proportions of our wealth invested in S_1 and S_2 respectively. The variance of the return of our portfolio is

$$\sigma_\Pi{}^2 = w_1{}^2\sigma_1{}^2 + 2w_1(1 - w_1)\rho\sigma_1\sigma_2 + (1 - w_1)^2\sigma_2{}^2,$$

where the second right-hand term accommodates for correlation, ρ, between the two stocks. Including a risk-free investment, such as a bond, guarantees a rate of return, say r. The volatility of such an asset is zero, and we now get a new efficient frontier which is the straight line in Figure 2.2.

The portfolio for which the straight line touches the original efficient frontier is called the **market portfolio**, and the straight line itself is called the **capital market line**. The position taken on the efficient frontier is subjective and depends on the investors risk preference. For a further discussion see Elton & Gruber [10] and Wilmott [11].

2.4 Financial Options

The stock price or *asset price* is assumed to follow a log-normal random walk. With time, t, and price of the asset, S, on the time interval from t to $t + \mathrm{d}t$ of size $\mathrm{d}t$,
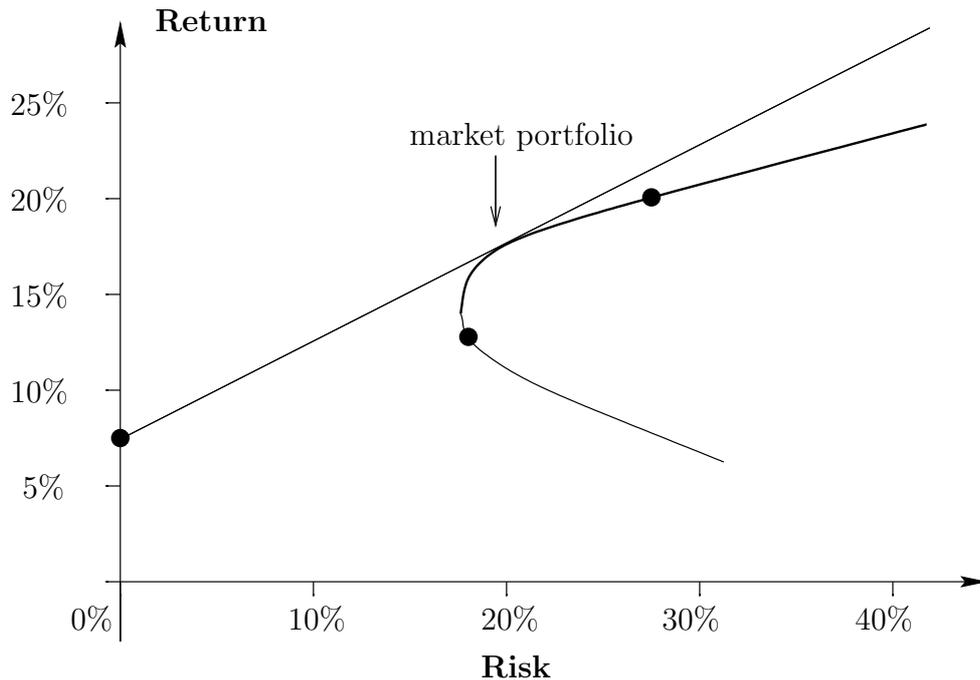
Figure 2.2: Portfolio possibilities and the efficient frontier.

a change of $\mathrm{d}S$ in the asset price occurs. The return, being change in price divided by the original value, $\mathrm{d}S/S$, can be separated into deterministic return and random changes in asset price due to external market effects [5]. The deterministic return would be similar to return obtained on money invested in a risk-free bank account, and is given by

$$\mu\,\mathrm{d}t$$

where μ is the growth rate of the asset known as drift. We would expect $\mu > r$ to include increase due to inflation (the continuous decrease or depreciation of currency value) which can be a random variable. The random change can be represented as a random sample from a normal distribution with mean zero and which contributes

$$\sigma\,\mathrm{d}X$$

where $\mathrm{d}X$ represents the random variable with mean zero and variance $\mathrm{d}t$, and σ is the volatility of the asset. The term $\mathrm{d}X$ is scaled with $\sqrt{\mathrm{d}t}$ such that

$$\mathrm{d}X^2 = \mathrm{d}t.$$

Now we can model the asset price fluctuations with the stochastic differential equation

$$\frac{\mathrm{d}S}{S} = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}X. \tag{2.1}$$

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Define $V(S, t)$ as the value of an option on the underlying asset S at time t. Then using Ito's lemma,

$$dV = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \mu S \frac{\partial V}{\partial S}\right) dt + \sigma S \frac{\partial V}{\partial S} dX.$$

We construct a portfolio, Π, consisting of one option of value, V, and a number $-\triangle$ of underlying assets with value, S. The value of the portfolio is given by

$$\Pi = V - \triangle S. \tag{2.2}$$

The change in value of the portfolio for one time-step, if holding $\triangle$ fixed for that period is

$$d\Pi = dV - \triangle dS.$$

The portfolio here would follow a random walk and choosing $\triangle$ correctly as to have a deterministic change in portfolio value avoiding arbitrage would illustrate a form of $\triangle$ hedging. For the discussed portfolio choosing

$$\triangle = \frac{\partial V}{\partial S} \tag{2.3}$$

would eliminate the random component, where $\triangle$ is the value of $\partial V/\partial S$ at the start of the time-step dt. The resulting portfolio has an increment which is wholly deterministic:

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt.$$

To avoid arbitrage opportunities a risk free portfolio must earn the risk free rate r,

$$\partial\Pi = r\Pi dt = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt. \tag{2.4}$$

Substituting (2.2) and (2.3) into equation (2.4) and dividing throughout by dt gives the **Black-Scholes partial differential equation**:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0.$$

For further details see, Wilmott *et al* [5].

2.5   Real Options

Decisions in the oil industry are to a large extent irreversible and input information can change over time, altering the outcome for decisions taken at different points in

time. Unlike discounted cash flow, option theory captures the flexibility inherent with most projects such as intervention for abandonment or expanding of a project. These "real" options are analogous to financial options; using a valuation formula to value the project significantly with greater accuracy than by discounted cash flow.

For example consider the suspension of an oil field. Assume we own an oil field which produces oil at a cost per barrel, C. We assume the market price of the oil to be S, yielding a profit of $P = S - C$ per barrel. The change in market price follows the Brownian motion

$$\mathrm{d}S = a\mathrm{d}t + b\mathrm{d}X, \tag{2.5}$$

with a and b respectively the *drift* and *volatility* of S. The variables a and b are assumed for simplicity to be constant. Once again we assume $a > 0$ because oil is a limited resource, though it does not change the form of our equation here. The value of all future profits V as a function of S and t satisfies

$$\frac{\partial V}{\partial t} + \frac{1}{2}b^2\frac{\partial^2 V}{\partial S^2} + a\frac{\partial V}{\partial S} - rV + S - C = 0,$$

the Black-Scholes equation with no dividend but continuous profit P.
The final condition

$$V(S, T) = 0$$

reflects that no more profit can be made from a *dry* oil well; while

$$V(S, t) \geq 0$$

is required for optimal behaviour.
It is also required that $\frac{\partial V}{\partial P}$ is continuous, resulting in a set of equations similar to that of an American option see Wilmott, Howison & Dewynne [5]. Solving this equation we can obtain an optimal time to suspend the production of oil.

2.6 Barrier Options

Barrier options are path dependent options. A path dependent option is an option whose payoff not only depends on the underlying asset spot price at expiry; but also of the past history of the underlying asset.

Barrier options differ from vanilla options in that part of the option contract is triggered if the underlying asset price, S, hits a certain barrier, B, at a time prior to

expiry. The right to exercise the option may be forfeited at this barrier, an out barrier; or the option only comes into existence if the asset price crosses a certain value, an in barrier. Barrier options can be either a put or a call and are classified as follows:

- **up-and-in**: the option expires worthless unless the barrier $S = B$ is reached from below before expiry.

- **down-and-in**: the option expires worthless unless the barrier $S = B$ is reached from above before expiry.

- **up-and-out**: the option expires worthless if the barrier $S = B$ is reached from below before expiry.

- **down-and-out**: the option expires worthless if the barrier $S = B$ is reached from above before expiry.

Often a rebate is allowed, whereby the holder of the option receives a specified amount if the barrier is crossed in the case of out-barriers or not crossed in the case of in-barriers.

Barrier options payoff depends on the path taken by the asset up to expiry, yet this dependency is weak. The only information on the path that is required is whether or not the barrier has been triggered. The Black-Scholes equation

$$\frac{\partial V_{d/o}}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_{d/o}}{\partial S^2} + r\frac{\partial V_{d/o}}{\partial S} - rV_{d/o} = 0, \tag{2.6}$$

is still satisfied. If the underlying asset reaches an 'out' barrier the contract becomes worthless, then we have the boundary condition

$$V_{d/o}(S_d, t) = 0 \text{ for } t < T; \tag{2.7}$$

or

$$V_{d/o}(S_d, t) = R \text{ for } t < T \tag{2.8}$$

if a rebate is included in the contract. With the out-barrier at $S = S_d$, we must solve the Black-Scholes equation for $0 \leq S \leq S_d$ with this condition on $S = S_d$ and a final condition relating to the payoff if this barrier is not triggered. The value for a call option at expiry, where the barrier is not triggered, would be

$$V_{d/o}(S, T) = \max(S - E, 0). \tag{2.9}$$

2.7 Utility Theory

This aspect of decision making recognises that different companies have different attitudes towards money. For example a small company would be less able to sustain losses than a large one.

Each company's top management has a definite trend towards decisions on losses, profits and their associated risk. By analysing previous decisions it is possible to construct a utility curve which acts as a representation of the decision making attitude of the management towards risk. Such a curve could be used to guide decision makers further down in the hierarchy as which decisions to make in coherence with that of the company as a whole.

Initially supposing that W is your total wealth we would introduce a utility function, say

$$U(W),$$

which would quantify the benefit of this wealth to you. Note that this utility function is not well defined, in the sense that in the future personal preferences may change and so too the usefulness of wealth. If x and y are two amounts such that $x < y$ then

$$U(x) < U(y)$$

as to imply more wealth is more useful than less wealth. Thus if our change in wealth, $\mathrm{d}W$ is such that

$$\mathrm{d}W > 0$$

then expanding with a Taylor series about W yields

$$U(W) < U(W + \mathrm{d}W) = U(W) + U'(W)\mathrm{d}W + O(\mathrm{d}W^2),$$

where we have assumed that $U(W)$ is differentiable such that $U'(W) > 0$.

The expected utility of an uncertain outcome, $\mathbb{E}[U(W)]$, for W being a random variable with a continuum of possible values with probability density function, $p(W)$, is

$$\mathbb{E}[U(W)] = \int_{-\infty}^{\infty} U(W)p(W)\,\mathrm{d}W.$$

Note that W could take discrete values and that $U(\mathbb{E}[W])$ was not used because of it not being well defined for certain events, $\mathbb{E}[W]$, which would not occur.

To consider the options of being **risk averse**, **risk neutral** or **risk seeking** view a *double or nothing* game i.e. playing roulette on red or black. Assuming you have

wealth, W, and bet $\mathrm{d}W$ to either win $2\mathrm{d}W$ or nothing, with equal probability of $\frac{1}{2}$. If you do not play then your expected future utility is certain at

$$\mathbb{E}[U(W)|W = W_0 \text{ before bet}] = U(W_0).$$

If you play your wealth will become either $W - \mathrm{d}W$ or $W + \mathrm{d}W$ each with the same probability, $\frac{1}{2}$, yielding an expected future utility of

$$\mathbb{E}[U(W)] = \frac{1}{2}U\left(W - \mathrm{d}W\right) + \frac{1}{2}U\left(W + \mathrm{d}W\right).$$

If you chose not to play the game you are **risk averse**, such that $U(W) > \mathbb{E}[U(W)]$ and

$$U(W) > \frac{1}{2}U\left(W - \mathrm{d}W\right) + \frac{1}{2}U\left(W + \mathrm{d}W\right).$$

Expanding using Taylor series about W yields

$$U(W) > U(W) + \frac{1}{2}U''(W)(\mathrm{d}W^2) + O\left((\mathrm{d}W)^3\right),$$

such that $U''(W) < 0$.

If you are **risk neutral** then $U(W) = \mathbb{E}[U(W)]$, such that

$$U''(W) = 0;$$

and if you are **risk seeking** then $U(W) < \mathbb{E}[U(W)]$ such that

$$U''(W) > 0.$$

An important concept illustrated here is that the benefit of wealth is not necessarily proportional to the wealth. Generalising the game such that if you posses an amount of wealth W, you can play the game for free gaining a arbitrary small random amount ϕ which has zero mean and a variance of $\sigma_\phi{}^2$;

$$\mathbb{E}[\phi] = 0, \ \mathbb{E}[\phi^2] = \sigma_\phi{}^2.$$

If you choose not to play, your expected wealth will be the same as your current wealth

$$\mathbb{E}[W + \phi] = W + \mathbb{E}[\phi] = W.$$

Consider what amount of wealth $W - \psi$, with certainty, would be equivalent to playing the game and having wealth $W + \phi$ with uncertainty thus

$$\mathbb{E}[U(W - \psi)] = U(W - \psi) = \mathbb{E}[U(W + \phi)].$$

Since ϕ is small we can Taylor expand $U(W + \phi)$ about the point W:

$$U(W + \phi) = U(W) + U'(W)\phi + \frac{1}{2}U''(W)\phi^2 + \cdots$$

such that

$$
\begin{aligned}
\mathbb{E}[U(W + \phi)] &= U(W) + U'(W)\mathbb{E}[\phi] + \frac{1}{2}U''(W)\mathbb{E}[\sigma^2] + \cdots \\
&= U(W) + \frac{1}{2}\sigma_\phi{}^2 U''(W) + \cdots
\end{aligned}
$$

Similarly expanding $U(W - \psi)$ about W:

$$U(W - \psi) = U(W) - \psi U'(W) + \cdots$$

such that

$$U(W) - \psi U'(W) = U(W) + \frac{1}{2}\sigma_\phi{}^2 U''(W),$$

giving

$$\psi = -\frac{1}{2}\frac{U''(W)}{U'(W)}\sigma_\phi{}^2.$$

The amount one would pay to avoid playing the game, ψ, is proportional to the risk, $\sigma_\phi{}^2$, with a proportionality constant

$$A(W) = -\frac{U''(W)}{U'(W)},$$

known as the absolute level of risk aversion. Certainty equivalent of wealth is what quantity of certain wealth W_c has the same utility as the utility of the uncertain wealth, W_{uc};

$$U(W_c) = \mathbb{E}[U(W_{uc})].$$

Given that $U(W)$ is continuous, differentiable and strictly increasing ($U'(W) > 0$) and that we know $\mathbb{E}[U(W_{uc})]$, then W_c is unique. We can approximate the certainty equivalent of risk of W_{uc} by

$$
\begin{aligned}
W_c &\sim \mathbb{E}[W_{uc}] + \frac{1}{2}\frac{U''(W)}{U'(W)}\sigma_\phi{}^2 \\
&= \mathbb{E}[W_{uc}] - \frac{1}{2}A(W)\sigma_\phi{}^2,
\end{aligned}
$$

where $\sigma_\phi{}^2$ is the variance of the uncertain wealth outcomes.

For more specifics on to what degree each of these methods are used in the industry see Simpson [1].

Chapter 3

An ideal sub-sea meter

Sub-sea meters are used to monitor production from offshore wells. These meters relay data on flow rate and concentration of produce from these wells. When considering a meter on an individual well, or indeed on several wells, we can isolate the important use of predicting methane hydrate formations which clog the pipeline. These are expensive to repair.

On the ocean bed at a temperature of approximately 3^oC and under elevated pressure the mixing of water and gas, such as methane, ethane and hydrocarbon species, from the well can react to form an ice-like hydrate that blocks the pipeline. Meters give data on amounts of water and methane gas within the pipe, enabling the prediction of clogs occurring and allowing prevention methods to be instigated. For simplicity an *ideal* or perfect meter will be considered on wells with no uncertainty. The meter acts as an insurance against clogging and can be priced using an **NPV** calculation.

We assume the clogging follows a Poisson process dq, so that

$$dq = \left\{ \begin{array}{ll} 0 & \text{with probability } 1 - \lambda\, dt \\ 1 & \text{with probability } \lambda\, dt, \end{array} \right.$$

with

$$P(\text{clog in } \delta t) = \lambda \delta t + O(\delta t), \qquad (3.1)$$

where λ is the intensity of the Poisson process; see Figure 3.1.

Assuming the price of repairing the pipe, R, remains constant then the present discounted cost is given by

$$\mathbf{NPV} = \int_0^\infty P\left(\text{clog at t}\right) Re^{-rt}\, dt.$$
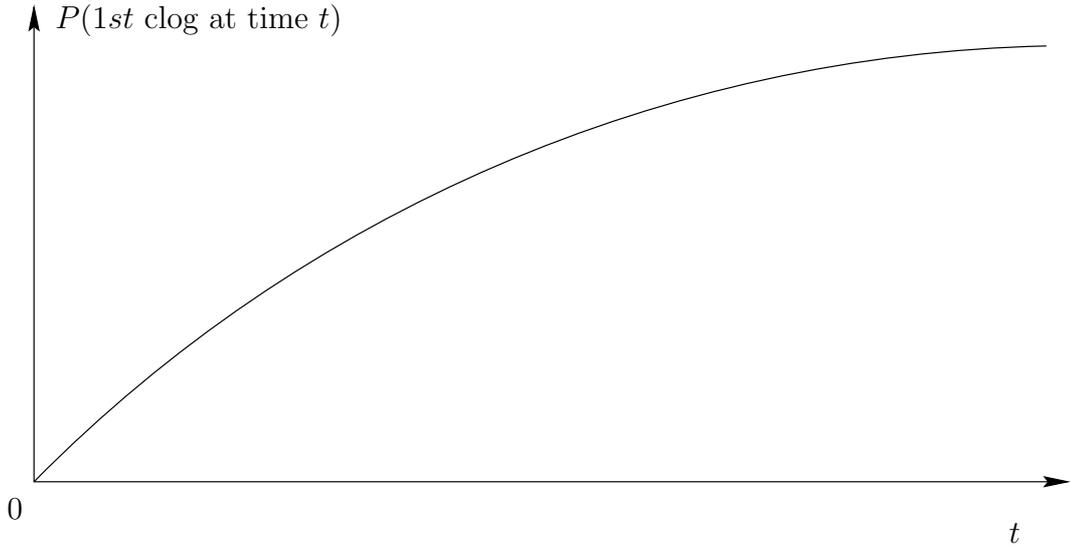
Figure 3.1: A Poisson distribution for clogging.

Thus for example if we expected only one clogging to occur at $t = 1$ then

$$
\begin{aligned}
\mathbf{NPV} &= \int_0^\infty \delta\left(t - 1\right) Re^{-rt}\,\mathrm{d}t \\
&= Re^{-r},
\end{aligned}
$$

where $\delta(.)$ is the Dirac-delta function.

For the lifetime of production from the well, T, let N_T represent the number of clogs which have occurred in the period $[0, T]$. The random process N_T is assumed to obey the following conditions:

1. N_t with $0 \leq t \leq T$ is a random variable taking non-negative integer values,

2. $N_0 = 0$,

3. $N_s \leq N_t$ if $s \leq t$,

4. the number of clogs occurring in a period $(s, t]$ where $0 \leq s < t$ is independent of clogs occurring prior to time s,

5. there exists a number $\lambda > 0$ such that for a small period of time, δt,

   - $P(N_{t+\delta t} = n + 1 | N_t = n) = \lambda \delta t + o(\delta t)$,

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- $P(N_{t+\delta t} = n | N_t = n) = 1 - \lambda \delta t + o(\delta t)$. This assumes that the probability a clog forms in the period $(t, t + \delta t]$ is a linear function of δt, and from this we see that the probability of two or more clogs forming in this interval is

$$P(N_{t+\delta t} \geq n + 2 | N_t = n) = o(\delta t).$$

  Thus it is only required to view the possible events of either one or no clogs forming in the time interval $(t, t + \delta t]$.

With the above assumptions we assume that clogging is independent of past clogs so that the expected time of first clogging is

$$\mathbb{E}[1st \text{ clogging time}] = \frac{1}{\lambda},$$

and the expected number of clogs in the interval $(0, T]$ is

$$
\begin{aligned}
\mathbb{E}[N_T] &= \frac{T}{\mathbb{E}[1st \text{ clogging time}]} \\
&= \frac{T}{\frac{1}{\lambda}} \\
&= \lambda T.
\end{aligned}
$$

The expected cost of the first clog, costing R to repair, is

$$
\begin{aligned}
\mathbb{E}\left[Re^{-rt_1}\right] &= \int_0^\infty \lambda e^{-\lambda t_1} Re^{-rt_1} \, \mathrm{d}t_1 \\
&= \frac{R\lambda}{r + \lambda},
\end{aligned}
$$

where t_1 is time to first clog and r is our discounting factor. If you assume the cost of clogging increases with time as Re^{rt} this calculation is much simpler because each clog has an **NPV** of r; but then the total cost of clogging is ∞. This would not occur because it is usual to have a discount rate $> r$. The expected cost of the second clog is given by the tower rule as

$$\mathbb{E}\left[\text{cost of 2nd clog}\right] = \mathbb{E}\left[\mathbb{E}\left[\text{cost of 2nd clog} | 1\text{st clog is at time } t_1\right]\right].$$

The inner expectation is

$$\int_0^\infty \lambda e^{-\lambda t_2} Re^{-r(t_2 + t_1)} \, \mathrm{d}t_2,$$

where t_2 is the time from t_1 to t_2 which is

$$\frac{R\lambda}{r + \lambda} e^{-rt_1}.$$

So now we need

$$\mathbb{E}\left[\frac{R\lambda}{r+\lambda}e^{-rt_1}\right] = R\left(\frac{\lambda}{r+\lambda}\right)^2$$

as above. The expected cost from clogging is

$$
\begin{aligned}
\mathbb{E}\left[\text{cost from clogging}\right] &= \sum_{j=1}^{\infty} R\left(\frac{\lambda}{r+\lambda}\right)^j \\
&= \frac{R\lambda}{r+\lambda}\cdot\frac{1}{1-\frac{\lambda}{r+\lambda}} \\
&= \frac{R\lambda}{r};
\end{aligned}
\tag{3.2}
$$

which is dimensionally correct. Suppose you have observed values $t_1, t_2, \ldots, t_N$ for the stopping times. Then you can think of $T_1 = t_1$, $T_2 = t_2 - t_1$, $T_3 = t_3 - t_2, \ldots T_{N-1} = t_N - t_{N-1}$ as a series of first exit times. The joint probability density function of $T_1, \ldots T_N$ is

$$
\begin{aligned}
L &= \lambda e^{-\lambda t_1 - \lambda t_2 \ldots - \lambda t_N} \\
&= \lambda^N e^{-\lambda \sum t_i} \text{ for i=1,2,\ldots N-1.}
\end{aligned}
$$

Noting that

$$\frac{\partial}{\partial\lambda}\left(\log L\right) = 0,$$

implies that

$$\frac{\partial}{\partial\lambda}\left(N\log\lambda - \lambda\sum T_i\right) = 0.$$

The expected first clogging time is thus given by

$$\frac{1}{\lambda} = \frac{\sum T_i}{N},$$

which can be substituted into equation (3.2).

A statistical approach delivers a similar result for the average first exit time,

$$\frac{1}{N}\sum_{n=1}^{N-1} T_n$$

which might reasonably be thought to be a good estimate for the expected first exit, and can be derived by minimising the sum of squares,

$$\frac{\partial}{\partial\lambda}\left(\sum_{n=1}^{N-1}(T_n - 1/\lambda)^2\right) = 0.$$

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Another model is where the methane and water concentrations are represented by random walks; predicting the expected time of the pipe first clogging with barrier options. Clogging occurs when there is a certain percentage of water, B_ω, and certain percentage of methane gas, B_θ, in the production. Assuming the percentage of each substance follows a random walk, we have two random walks which when both are above the mentioned concentrations or *barriers* react to from a clog see Figure 3.2.
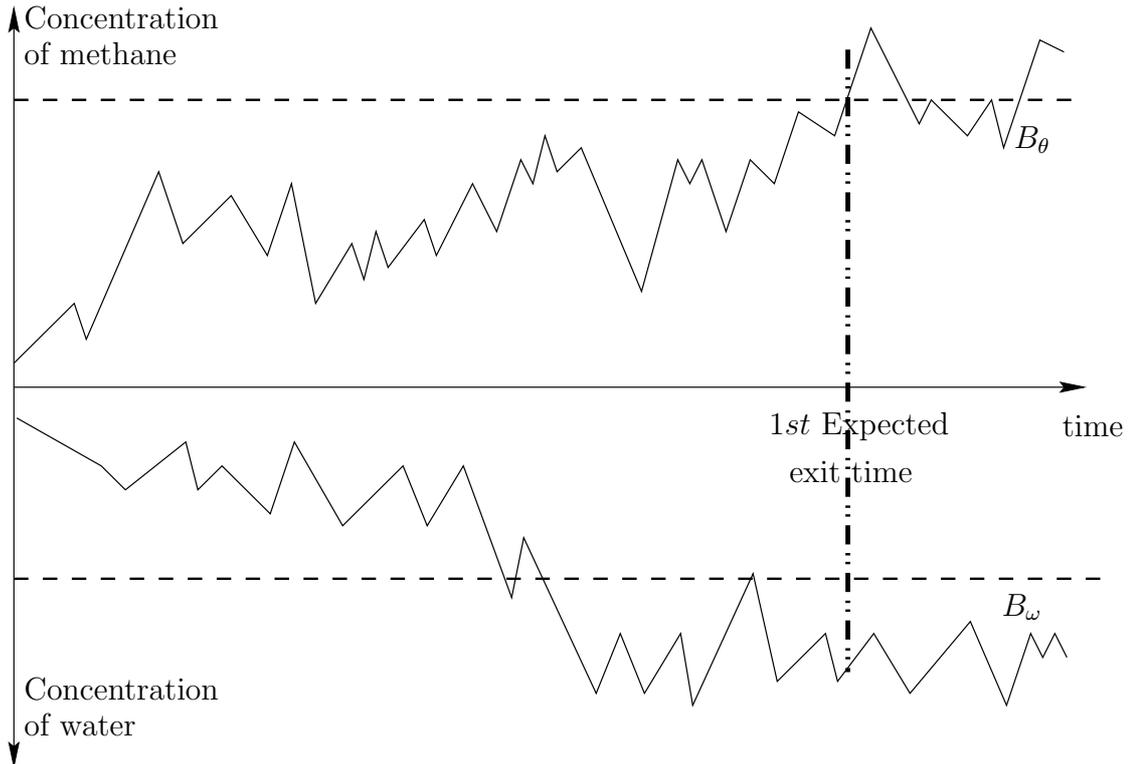

Figure 3.2: Representation of random walks for water and methane gas.

Determining the expected time at which both random walks are above the barrier gives a first expected clogging time, $\frac{1}{\lambda}$, which can be used to calculate the expected total cost of clogging, which is the **NPV** of the meter, with

$$\mathbb{E}\left[\text{cost from clogging}\right] = \frac{R\lambda}{r}. \tag{3.3}$$

To be accurate one would have to determine all the expected times of clogging for the two random walks and sum the discounted cost of repair. For simplicity we concentrate on predicting the first expected clogging time which is similar to theory on correlated financial barrier options. We assume that the concentrations of water, ω, and concentration of methane gas, θ, follows general random walks of the form

$$d\theta = a_1 dt + b_1 dX_1,$$

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$$\mathrm{d}\omega \quad = \quad a_2\mathrm{d}t + b_2\mathrm{d}X_2;$$

where

$$\mathbb{E}[\mathrm{d}X_i{}^2] \quad = \quad \mathrm{d}t \text{ for i=1,2 and}$$
$$\mathbb{E}[\mathrm{d}X_1\,\mathrm{d}X_2] \quad = \quad \rho\,\mathrm{d}t, \tag{3.4}$$

and ρ is the correlation coefficient, see Wilmott *et al* [7]. For our specific example we assume that the concentration of water and methane gas follow more specific random walks with mean reversion of the form

$$\mathrm{d}\theta \quad = \quad -k_1(\theta - \theta_\infty)\mathrm{d}t + \sigma\theta(1-\theta)\mathrm{d}X_1,$$
$$\mathrm{d}\omega \quad = \quad -k_2(\omega - \omega_\infty)\mathrm{d}t + \bar{\sigma}\omega(1-\omega)\mathrm{d}X_2.$$

In our random walks k_i is the speed of mean reversion, θ_∞ and ω_∞ the expected average concentration of methane and water respectively and $\sigma, \bar{\sigma}$ the variability of concentration. Note that these two random walks would not remain at the values of 0 and 1 if reached, due to them having a non-zero drift term. The first expected exit time, q, can be calculated from the Kolmogorov equation, see Wilmott *et al* [7],

$$\frac{1}{2}b_1{}^2\frac{\partial^2 q}{\partial\theta^2} + \rho b_1 b_2 \frac{\partial^2 q}{\partial\theta\partial\omega} + \frac{1}{2}b_2{}^2\frac{\partial^2 q}{\partial\omega^2} + a_1\frac{\partial q}{\partial\theta} + a_2\frac{\partial q}{\partial\omega} = -1 \tag{3.5}$$

with the condition that $q = 0$ on the barrier and for the given specific walks $a_1 = -k_1(\theta - \theta_\infty)$, $b_1 = \sigma\theta(1-\theta)$, $a_2 = -k_2(\omega - \omega_\infty)$ and $b_2 = \bar{\sigma}\omega(1-\omega)$. This is an elliptic partial differential equation which can be solved numerically using forward Euler to find a first expected exit time. A more common and simpler method for attaining this exit time would be using Monte Carlo simulation. Discretising the random walks of (3.4) gives

$$\theta(t+\delta t) \quad = \quad \theta(t) - k_1(\theta(t) - \theta_\infty)\delta t + \sigma\theta(t)(1-\theta(t))\sqrt{\delta t}\phi_1,$$
$$\omega(t+\delta t) \quad = \quad \omega(t) - k_2(\omega(t) - \omega_\infty)\delta t + \bar{\sigma}\omega(t)(1-\omega(t))\sqrt{\delta t}\phi_2; \tag{3.6}$$

which can be implemented in a Matlab program, see Appendix A, by continually generating two pseudo-random normally distributed variables ϕ_1 and ϕ_2 to simulate and plot the walks; see Figure 3.3. Choosing to program in C for industrial popularity, we can extend the Matlab code to return the average first expected exit time. This program runs Monte Carlo simulation on two mean reverting random walks and returns an averaged time, across the simulations, of when both concentrations θ and
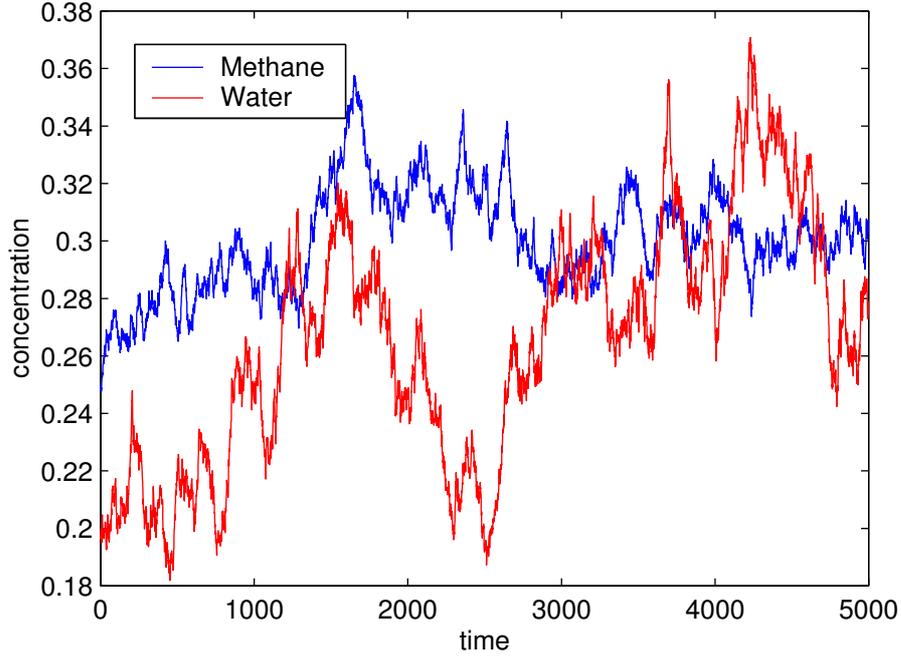
Figure 3.3: Matlab simulated random walks for water and methane gas.

ω are simultaneously across the respective thresholds for clogging, B_θ and B_ω; see Appendix B. To illustrate the program we run it using the sensible values:

$$
\begin{aligned}
k_1 \text{ and } k_2 &= 0.015 \\
\sigma \text{ and } \bar{\sigma} &= 1.0 \\
B_\theta \text{ and } B_\omega &= 0.5 \\
\theta_\infty \text{ and } \omega_\infty &= 0.5
\end{aligned}
$$

and initial concentrations on water and methane gas, $\theta_0 = 0.1$ and $\omega_0 = 0.1$ respectively. We run three different simulations of different lengths to determine and compare the first expected clogging time delivers:

- for $10,000$ simulations a result of 0.986599 hours,

- for $100,000$ simulations a result of 0.979228 hours,

- for $1,000,000$ simulations a result of 0.978882 hours.

Quality control for the C code is performed by extensively repeating our simulation for θ (ω would be identical), and calculating an empirical probability density function

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(pdf) for θ to compare this to the analytical steady-state distribution of θ, calculated from the Kolmogorov equation (3.5). The forward Kolmogorov equation is

$$P_t = \frac{\partial^2}{\partial \theta^2} \left(\frac{1}{2} b^2 p \right) - \frac{\partial}{\partial \theta} (ap),$$

if $t \to \frac{1}{\sigma^2}$ we get a steady state pdf, p_∞ given by

$$(ap_\infty)' = \frac{1}{2} \left(b^2 p_\infty \right)''.$$

This is the ordinary differential equation (ode),

$$
\begin{aligned}
0 \;=\; & \sigma^2(1-\theta)^2 p_\infty - 4\sigma^2\theta(1-\theta)p_\infty \\
& + 2\sigma^2\theta(1-\theta)^2 p_\infty{}' + \sigma^2\theta^2 p_\infty - 2\sigma^2\theta^2(1-\theta)p_\infty{}' \\
& + \frac{1}{2}\sigma^2\theta^2(1-\theta)^2 p_\infty{}'' + k_1(\theta - \theta_\infty)p_\infty{}' + k_1 p_\infty
\end{aligned}
$$

which can be solved numerically. We know that

$$\int_0^\infty p_\infty \, \mathrm{d}\theta = 1,$$

from which we can numerically calculate the constant in our ode solution. Plotting this analytical pdf against the empirical pdf; see Figure 3.4 would indicate if the the C code is adequate.

With historical well clogging data we can determine the values of $\sigma, \bar{\sigma}, k_1$ and k_2 for the program by matching the simulation to the data. Using equation (3.3) and the result from the C program, we can easily determine the value of this insurance against clogging for an ideal meter.
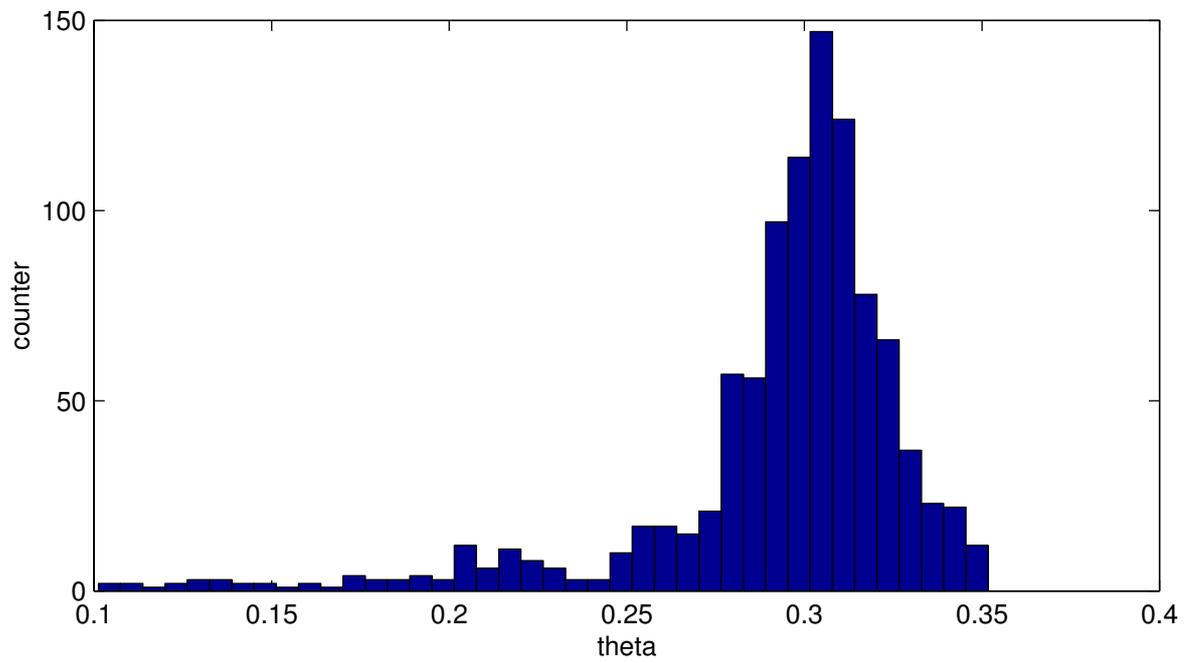
Figure 3.4: Histogram of the empirically calculated pdf for one concentration random walk.

Chapter 4

Non-ideal sub-sea meter

Multi-phase flow meters are not perfect and their readings tend to be in an error band. A meter can be tested in a laboratory to determine this error deviance, ϵ , from the true reading. In the previous chapter we considered an ideal meter whose readings exactly depict the true concentrations of water and methane gas; shown in Figure 3.3. Considering a non-ideal meter requires consideration of not only having an exact reading, as in the previous chapter, but also the possibility of attaining a reading in the error band above and below the exact concentration. This provides four cases for our meter readings:

- the meter predicts a genuine clog and is correct,

- the meter predicts no clog while there is a genuine clog,

- the meter predicts a clog while there is no genuine clog,

- the meter predicts no clog and there is no genuine clog.

We assume that the error in the meter readings are inconsistent and occur randomly within the error band. If the readings had a consistent error the meter could be calibrated within the lab and viewed as an ideal meter. Assuming that θ and ω follow mean reverting random walks as in equation (3.4) and the error is $\epsilon \leq |5\%|$; we simulate the non-ideal meter readings by generating two pseudo-random normally distributed variables between 0.95 and 1.05 at each time-step and multiplying this with the actual or ideal meter simulated readings at each time-step. Programming in Matlab, see appendix C, we can simulate a non-ideal meter's readings in comparison to the exact concentrations; see Figure 4.1.
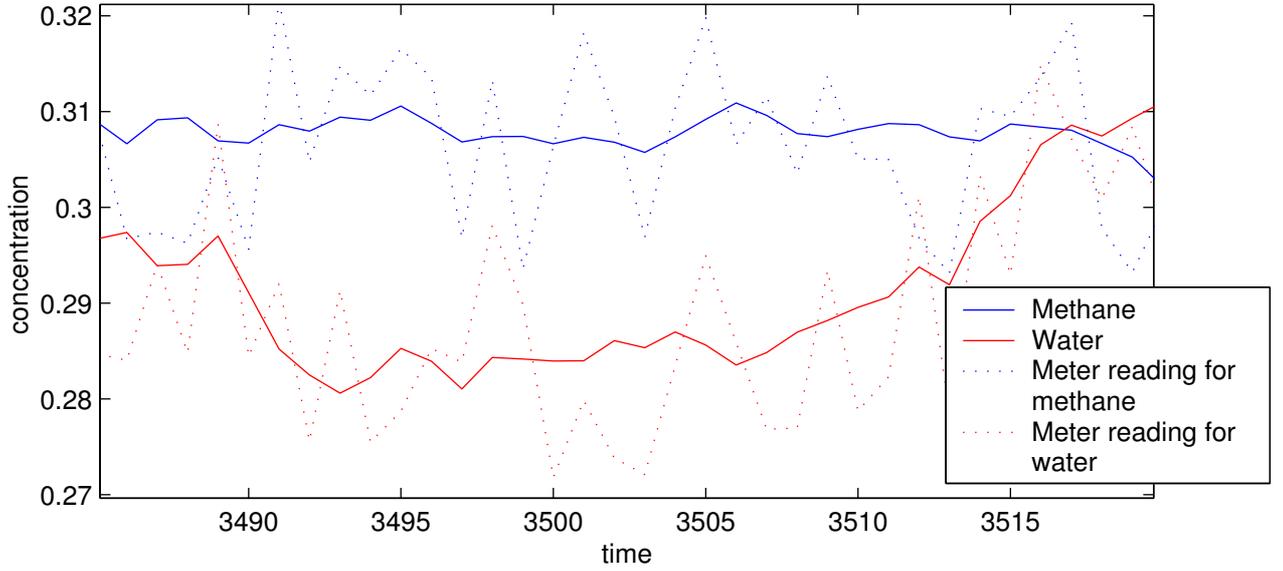
Figure 4.1: Matlab simulated random walks for concentration of water and methane gas and their readings on a non-ideal meter.

With a counter in this program for each of the four cases we can determine the possibility of each case occurring by dividing its counter by the number of time-steps for which the walk is run. For example using the parameters :

$$
\begin{aligned}
k_1 &= 0.05 \\
\sigma &= 0.15 \\
B_\theta &= 0.3 \\
\theta_\infty &= 0.3 \\
k_2 &= 0.07 \\
\bar{\sigma} &= 0.25 \\
B_\omega &= 0.3 \\
\omega_\infty &= 0.3 \\
\text{timestep} &= 0.0025
\end{aligned}
$$

with initial values for θ and ω of 0.25 and 0.2 respectively; we attain

$$
\begin{aligned}
P\,(\text{Detects a genuine clog}) &= 0.1094, \\
P\,(\text{Misses a genuine clog}) &= 0.0402, \\
P\,(\text{Predicts a false clog}) &= 0.0220, \\
P\,(\text{Predicts correctly continuing production}) &= 0.8284.
\end{aligned}
$$

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Note that the probability of a meter reading that results in incorrect action is

$$P\left(\text{Prompting incorrect action}\right) = P\left(\text{Misses a genuine clog}\right) + P\left(\text{Predicts a false clog}\right)$$
$$= 0.0622,$$

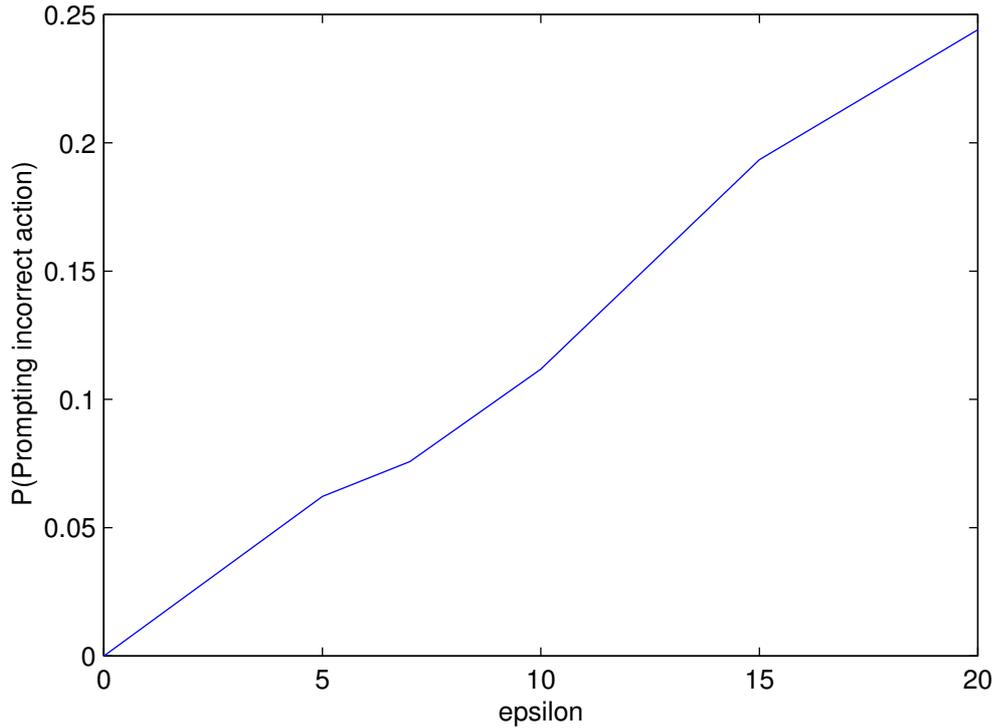which is dependent on the error band, ϵ; see Figure 4.2.

Figure 4.2: Probability of meter prompting incorrect action against error.

We adapt the Matlab code to keep a record of the time-step for each of the possible cases occurring, i.e. times of missed clogs $= t_{m1}, t_{m2} \ldots$ and times of false clogs $= t_{s1}, t_{s2} \ldots$, we could calculate

$$\mathbb{E}[\text{Price missed clogs}] = \sum_{i=1}^{\text{Number missed clogs}} R e^{-rt_{mi}},$$

$$\mathbb{E}[\text{Price false shutdowns}] = \sum_{j=1}^{\text{Number false shutdowns}} S e^{-rt_{sj}};$$

where R and S is the price of repair and shut down respectively. Note this is for a finite time period and that for pricing the prediction of a false clog we could take into consideration that if a genuine clog were to appear within a short period after shut

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down, then would also incur the price of production from shut down to the actual clog above the price of shut down. We can price the non-ideal meter with

$$
\begin{aligned}
\mathbb{E}[\text{Price non-ideal meter}] \;=\; & \mathbb{E}[\text{Price ideal meter}] - \mathbb{E}[\text{Price false shutdowns}] \\
& - \mathbb{E}[\text{Price missed clogs}].
\end{aligned}
$$

Chapter 5

An ideal meter reading of two wells

As a sub-sea meter reading of a number of wells provides production data from these wells it can be used to maximise the production. Due to a limited pipeline capacity, our aim is to attain the highest profitable produce from these wells. At a first glance one would assume that the best result would be to produce from a single well beginning with that which had the most useful produce content and then moving on to the well with the next best estimated production. Though due to changing environmental conditions we assume that the production of valuable produce per unit time, P, from the ith well follows a random walk with mean reversion such as

$$\mathrm{d}P_i = -k_i \left(P_i - \bar{P}_i \right) \mathrm{d}t + \sigma_i P_i \, \mathrm{d}X_i \text{ with } i = 1 \text{ to number of wells.} \qquad (5.1)$$

Note here that k_i is the speed of mean reversion, σ_i depicts the variability in well production and $\bar{P}_i$ is the expected average production of the ith well. Thus production from a well may not always be profitable having phases of unprofitable production, which we would like to eliminate and price. Considering production from two wells, thus $i = 1, 2$, we can view the meter as determining when to switch production between wells as to attain the maximum production, see Figure 5.1.
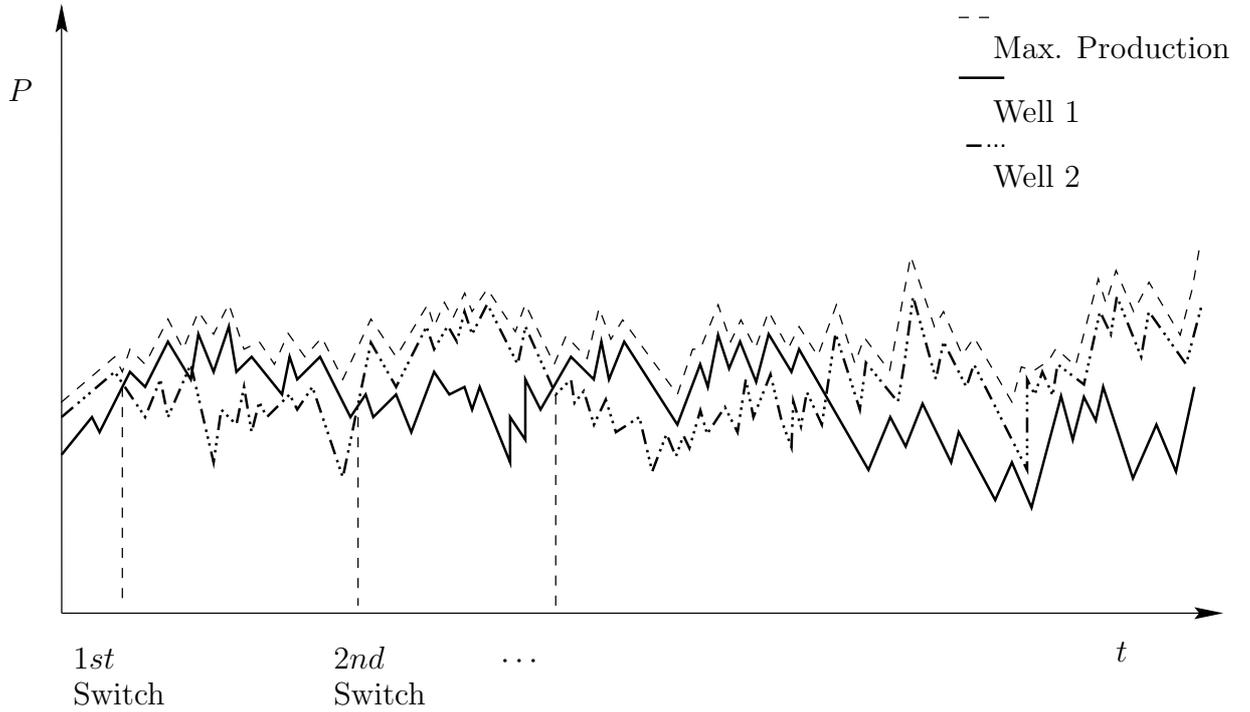
Figure 5.1: Maximising production from two wells.

We can determine total production, Q_j, in a period $(0, t]$ from the equation

$$Q_1 = \int_0^t \alpha P_1 + (1 - \alpha) P_2 \, \mathrm{d}t, \qquad (5.2)$$

with relation to the random walks of P_1 and P_2 we would like to determine the binary function $\alpha(t)$ which takes values of either 1 or 0 during the period as to maximise the production. This maximised production can be compared with producing initially from the expected most profitable well and then from the second well

$$Q_2 = \int_0^{t_1} P_1 \, \mathrm{d}t + \int_{t_1}^t P_2 \, \mathrm{d}t \text{ where } t_1 \leq t \text{ and } P_2 < P_1,$$

to show production benefits and thus pricing this aspect of an ideal meter.

Through Monte Carlo simulation in Matlab, see appendix D, we can attain a representation of the random walks in equation 5.1 and graphically determine the times of switching, see Figure 5.2, which can be used in equation 5.2. This code compares the two random walks for production at every time-step, indicating the value of α by placing an $\times$ at either 1 or 0 on the second graph in Figure 5.2.

Using historical data of the wells one can determine the values of σ_i and k_i and then using equation 5.2 determine the maximum production from the two wells in
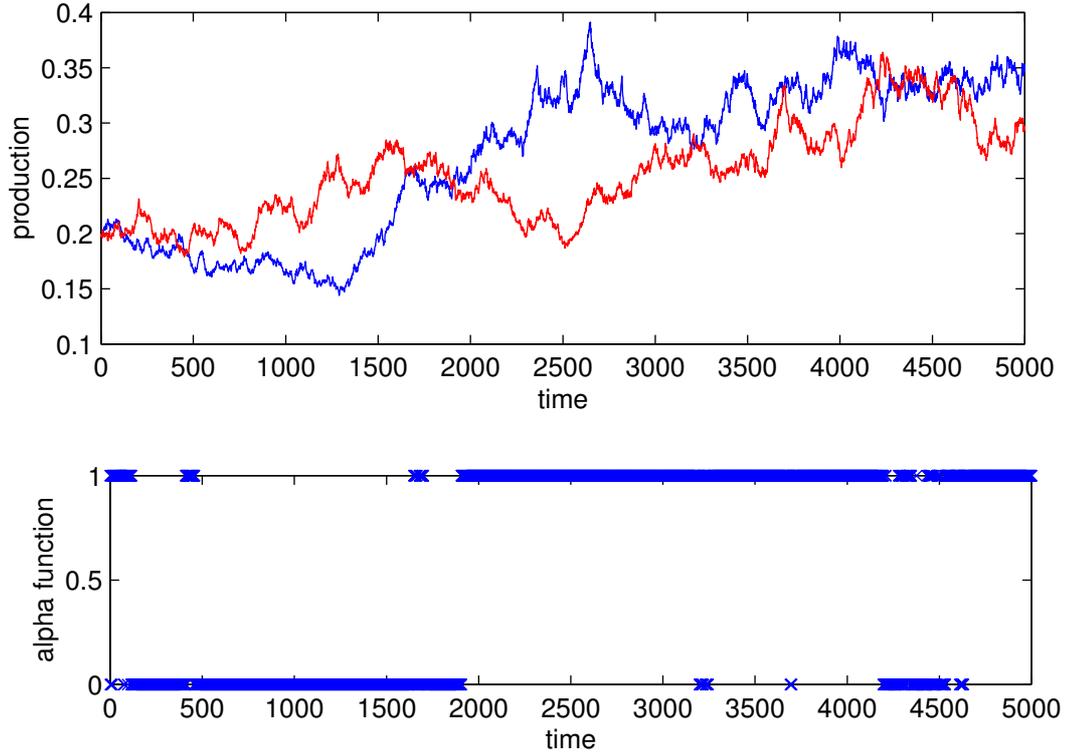
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Figure 5.2: Matlab simulated random walks for the production from two wells and associated alpha function.

the desired period which would be at largest the smaller of the two wells expected production time. With an estimated oil price, κ, we can determine the value of production Q_1 by discounting with some discounting factor, r, as

$$V_1 = \int_0^t \left(\alpha P_1 + (1 - \alpha) P_2 \right) e^{-rt} \kappa \, \mathrm{d}t,$$

and the value of production Q_2 as

$$V_2 = \int_0^{t_1} P_1 e^{-rt} \kappa \, \mathrm{d}t + \int_{t_1}^t P_2 e^{-rt} \kappa \, \mathrm{d}t.$$

The value of the meter, V_m, would be the value created by switching:

$$V_m = V_1 - V_2.$$

Note that here switching is viewed as cost-less.

Considering the possibility of production from both wells simultaneously draws parallels to stochastic optimal control for mutual funds see Björk [8]. Assuming that

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both well production follows the random walks as in equation (5.1) the combined production or yield, Y, is given by

$$
\begin{aligned}
\mathrm{d}Y &= \left(-w_1 k_1 \left(P_1 - \bar{P}_1\right) - w_2 k_2 \left(P_2 - \bar{P}_2\right)\right) \mathrm{d}t + \\
&\quad + w_1 \sigma_1 \mathrm{d}X_1 + w_2 \sigma_2 \mathrm{d}X_2,
\end{aligned} \tag{5.3}
$$

where w_1 and w_2 are the respective weights of production from well 1 and well 2 respectively and

$$
\sum_{i=1}^{2} w_i = 1.
$$

Note that as before

$$
\begin{aligned}
\mathbb{E}[\mathrm{d}X_i{}^2] &= \mathrm{d}t \text{ for i=1,2 and} \\
\mathbb{E}[\mathrm{d}X_1 \, \mathrm{d}X_2] &= \rho \, \mathrm{d}t;
\end{aligned}
$$

such that

$$
\mathrm{d}Y^2 \sim \left(w_1{}^2 \sigma_1{}^2 + w_2{}^2 \sigma_2{}^2\right) \mathrm{d}t + 2 w_1 w_2 \sigma_1 \sigma_2 \rho.
$$

We would like to maximise

$$
\mathbb{E}_0 \left[\int_0^T Y(s) \, \mathrm{d}s\right],
$$

with T some given time horizon being at most the smallest expected production lifespan of the two wells. Introduce a value function, J, defined by

$$
\begin{aligned}
J(x,t) &= \max_w \mathbb{E}_t \left[\int_t^T Y(s) \, \mathrm{d}s\right] \\
&= \mathbb{E}_t \left[\int_{t+dt}^T Y(s) \, \mathrm{d}s + Y(t)\mathrm{d}t\right] \\
&= \mathbb{E}_t \left[J(x + dx, t + dt) + Y(t)\mathrm{d}t\right]
\end{aligned}
$$

such that

$$
0 = \mathbb{E}_t \left[\mathrm{d}J + Y(t)\mathrm{d}t\right].
$$

The Hamilton-Jacobi-Bellman, (**HJB**), equation for this problem is

$$
\max_A \left[\frac{\partial J}{\partial t} - \left(w_1 k_1 \left(P_1 - \bar{P}_1\right) - w_2 k_2 \left(P_2 - \bar{P}_2\right)\right) \frac{\partial J}{\partial y} + \frac{1}{2} \left(w_1{}^2 \sigma_1{}^2 + w_2{}^2 \sigma_2{}^2\right) \frac{\partial^2 J}{\partial y^2} + Y\right] = 0,
$$

where

$$
A = \left\{ \begin{array}{l} w_1 + w_2 = 1 \\ 0 \leq w_1, w_2 \leq 1 \end{array} \right. ,
$$

and y represents the realisation of Y which may not be integrable from equation (5.3). The given **HJB** equation is highly non-linear and could be solved numerically using

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an explicit method to determine the weights evolvement for maximum production. From the unsolved **HJB** equation one is still able to get an indication of possible values for the weights which can be separated into three cases; see Figure 5.3,

1. $w_1 = 1$ and $w_2 = 0$,

2. $w_1 = 0$ and $w_2 = 1$,

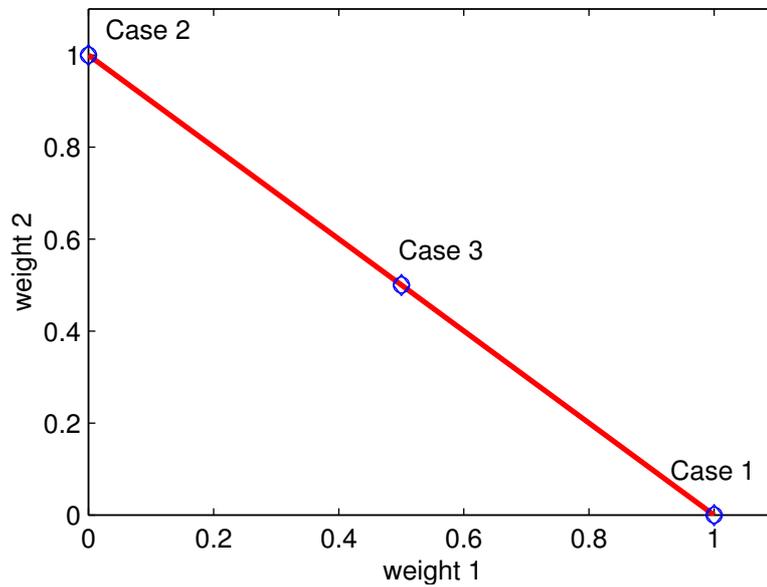3. $0 < w_1 < 1$, $0 < w_2 < 1$ and $w_1 + w_2 = 1$.

Figure 5.3: Possible combinations of weights.

Define the variables, Q, α, β and γ, such that

$$
\begin{aligned}
Q &= \frac{\partial J}{\partial t} - \left(w_1 k_1 \left(P_1 - \bar{P}_1\right) - w_2 k_2 \left(P_2 - \bar{P}_2\right)\right) \frac{\partial J}{\partial y} + \frac{1}{2} \left(w_1{}^2 \sigma_1{}^2 + w_2{}^2 \sigma_2{}^2\right) \frac{\partial^2 J}{\partial y^2} \\
&= \frac{\partial J}{\partial t} - \left(w_1 k_1 \left(P_1 - \bar{P}_1\right) - (1 - w_1) k_2 \left(P_2 - \bar{P}_2\right)\right) \frac{\partial J}{\partial y} + \frac{1}{2} \left(w_1{}^2 \sigma_1{}^2 + (1 - w_1)^2 \sigma_2{}^2\right) \frac{\partial^2 J}{\partial y^2} \\
&= \alpha w_1 + \beta w_1{}^2 + \gamma.
\end{aligned}
$$

with γ a constant. It is economically feasible to assume a convex payoff, $\frac{\partial^2 J}{\partial y^2} > 0$ and so $\beta > 0$, thus if $\alpha + \beta < 0$ then $w_1 = 0$ and if $\alpha + \beta > 0$ then $w_1 = 1$, switching between these two at $\alpha + \beta = 0$; see Figure 5.4.

This implies a *bang-bang* control such as indicated by the Monte Carlo simulation.
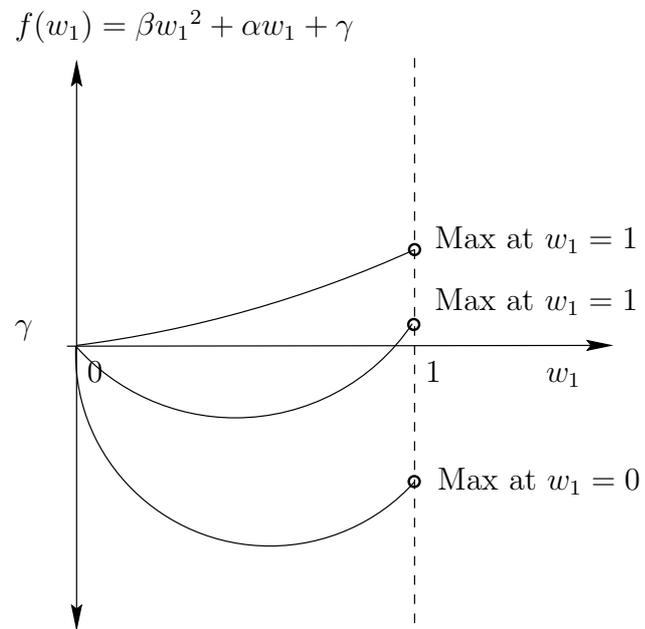
Figure 5.4: Three possible behaviours of the quadratic equation.

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Appendix A

Matlab code for two concentration random walks

Note that the function **normrnd** is part of the Matlab Statistics Toolbox.

```
iter=5000

%normal rv's
z1=normrnd(0,1,1,iter)';
z2=normrnd(0,1,1,iter)';

kx=0.05
sigx=0.15
xb=0.4
x0=0.2

ky=0.1
sigy=0.15
yb=0.4
y0=0.2

dt=0.0025
sdt=sqrt(dt)

%break
```

```
clear x
clear y

x(1)=x0;
y(1)=y0;

for i=2:length(z1)
  x(i)=x(i-1)-k1*(x(i-1)-a)+x(i-1)*(1-x(i-1))*sigx*sdt*z1(i);
  y(i)=y(i-1)-k2*(y(i-1)-b)+y(i-1)*(1-y(i-1))*sigy*sdt*z2(i);
end

figure
hold on
plot(x,'b-')
plot(y,'r-')
hold off
```

Appendix B

C code for predicting first average exit time

Note that in this program the random walks are viewed as uncorrelated, `rho`=0, but could be adjusted without complicating the simulation. This program runs $100,000$ simulations but could be increased depending on available computing time.

```c
#include <math.h>

#define MAX_RAND 2147483647

long random(void);
double zgauss(void);
double dosim(void);
void randomize(void);

int main()
{
    randomize();
    printf("\n\n\n%lf\n",dosim());
    return 12789;
}


double dosim()
{
```

```
double dt=0.01;
long sims=100000;
double time;
double runningSum=0;

double x0 = 0.1;
double y0 = 0;
double xt, yt;

//define constants
const double k1 = 0.005;
const double sigx = 0.15;
const double thresx = 0.4;      //threshold level for walk x
const double a = 0.3;           //expected mean for walk x

const double k2 = 0.001;
const double sigy = 0.15;
const double thresy=0.3;        //threshold level for walk y
const double b = 0.3;           //expected mean for walk y

long i, j;

double e1,e2,z1,z2;

const double rho = 0.0;
const double srho = sqrt(1.0-rho*rho);
const double sdt = sqrt(dt);

  for(i=0; i<sims; ++i) {
    xt = x0;
    yt= y0;
    j = 0;
    do {

      //generate normal random variables
      e1=zgauss();
```

```
        e2=zgauss();
        z1=e1;
        z2=rho*e1+srho*e2;

        //random walks
        xt += -k1*(xt-a)+xt*(1-xt)*sigx*sdt*z1;
        yt += -k2*(yt-b)+yt*(1-yt)*sigy*sdt*z2;

        ++j;
      } while ( (xt <thresx) && ( yt < thresy));
      time = j*dt;
      runningSum += time;
      /*
      if (i % 1000 == 0) {
        printf("\n%ld",i);
      }
      */
    }

  return(runningSum/sims);

}



void init_state(istate, size, seed)
long unsigned *istate;
long size;
long seed;
{
      int i;

      srand(seed);

      for(i=0; i<size; ++i)
              istate[i]=rand()+rand();
}
```

```
void init_random(size,seed)
long seed;
long size;
{
        static long unsigned istate[64];

        if (size < 8)
                size = 2;
        else if (size < 16)
                size = 8;
        else if (size < 32)
                size = 16;
        else if (size < 64)
                size = 32;
        else
                size = 64;

        init_state(istate,size,seed);

        initstate( (unsigned)seed, (char *)istate, size);
}


double zgauss()
{
        static int    set = 0;
        static double gset;
        double fac, r, v1, v2;

        if ( !set ) {
                do {
                        v1 = (double)rand()/(double)MAX_RAND;
                        v2 = (double)rand()/(double)MAX_RAND;
                        v1=2*v1-1.0;
                        v2=2*v2-1.0;
```

```
                              r = v1*v1+v2*v2;
/*
                              printf("v1=%lf v2=%lf r=%lf\n",v1,v2,r);
*/
              } while (r>1.0);
              fac = sqrt( -2.0*log(r)/r );
              gset = v1*fac;
              ++set;
              return(v2*fac);
      }
      else {
              set = 0;
              return(gset);
      }
}


void randomize()
{
      long time();
      long time1, time2;
      long dummy1, dummy2;

      time1 = time(&dummy1);

      init_random( 64L, time1 );

      time2 = time(&dummy2);

      srandom(time2);
}
```

Appendix C

Matlab code for two concentration random walks and non-ideal meter readings

```
iter=5000

%normal rv's
z1=normrnd(0,1,1,iter)';
z2=normrnd(0,1,1,iter)';

k1=0.005
sigx=0.15
x0=0.25
threshx=0.3 %concentration barrier for x
a=0.3

k2=0.005
sigy=0.25
y0=0.2
threshy=0.3 %concentration barrier for y
b=0.3

dt=0.0025
sdt=sqrt(dt)
```

```
clear x
clear y
counter1=0;
counter2=0;
counter3=0;
counter4=0;


x(1)=x0;
y(1)=y0;
x1(1)=x0;
y1(1)=y0;
z1(1)=1;
z2(1)=1;


for i=2:length(z1)
    x(i)=x(i-1)-k1*(x(i-1)-a)+x(i-1)*(1-x(i-1))*sigx*sdt*z1(i);
    y(i)=y(i-1)-k2*(y(i-1)-b)+y(i-1)*(1-y(i-1))*sigy*sdt*z2(i);
    z1(i)=normrnd(0,1)/10+0.95;
    z2(i)=normrnd(0,1)/10+0.95;
    while ((z1(i)<0.95)|(z1(i))> 1.05 ),
        z1(i)=normrnd(0,1)/10+0.95;
    end
    while ((z2(i)<0.95)|(z2(i))> 1.05 ),
        z2(i)=normrnd(0,1)/10+0.95;
    end
    x1(i)=x(i)*z1(i);
    y1(i)=y(i)*z2(i);
end
for i=2:length(z1)
    %predicts a genuine clog
 if ((x(i)>=threshx) & (y(i)>=threshy) & (x1(i)>=threshx) & (y1(i)>=threshy))
        counter1=counter1+1;
    %misses a genuine clog
 elseif ((x(i)>=threshx) & (y(i)>=threshy) & (x1(i)<threshx) & (y1(i)<threshy))
        counter2=counter2+1;
    %predicts a false clog
```

```
  elseif ((x(i)<threshx) & (y(i)<threshy) & (x1(i)>=threshx) & (y1(i)>=threshy))
      counter3=counter3+1;
    %does not predict a clog when no clog present
    else
      counter4=counter4+1;
      end
end

plot(x,'b-')
hold on
plot(y,'r-')
plot(x1,'b:')
plot(y1,'r:')
hold off
```

Appendix D

Matlab code for two production random walks

```
iter=5000

%normal rv's
z1=normrnd(0,1,1,iter)';
z2=normrnd(0,1,1,iter)';

kx=0.05
sigx=0.15
xb=0.4
x0=0.2

ky=0.1
sigy=0.15
yb=0.4
y0=0.2

dt=0.0025
sdt=sqrt(dt)

clear x
clear y
```

```
x(1)=x0;
y(1)=y0;

for i=2:length(z1)
  x(i)=x(i-1)-kx*(x(i-1)-xb)*dt+sigx*x(i-1)*sdt*z1(i);
  y(i)=y(i-1)-ky*(y(i-1)-yb)*dt+sigy*y(i-1)*sdt*z2(i);
end

figure
subplot(2,1,1)
plot(x,'b-')
hold on
plot(y,'r-')
hold off

%decide most profitable

t=[x' y']
t2=max(t,[],2)
index=(t==[t2 t2])
subplot(2,1,2)
plot(index(:,1),'x')
```