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# Ruin probabilities of the Parisian type for small claims

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## Abstract

In this paper, we extend the concept of ruin in risk theory to the Parisian type of ruin. For this to occur, the surplus process must fall below zero and stay negative for a continuous time interval of specified length. We obtain the probability of ruin in the infinite horizon for the case when the process starts from zero and the asymptotic form of the probability of ruin in the infinite horizon for the case when the process starts from the point far above zero. We see that in the small claim case an asymptotic formula similar to Cramér's formula is true.

**Keywords:** ruin, Parisian type of ruin, risk process, ruin probability, adjustment coefficient.

## 1 Introduction

We consider a classical surplus process in continuous time  $\{X_t\}_{t \geq 0}$

$$X_t = u + ct - \sum_{k=0}^{N_t} Y_k, \quad (1)$$

where  $u \geq 0$  is the initial reserve,  $c$  is a constant rate of premium payment per time unit, and  $\{N_t\}_{t \geq 0}$  is a Poisson process with parameter  $\lambda$  representing the numbers of claims up to time  $t$ . The sequence  $\{Y_k\}$ ,  $k = 1, 2, \dots$ , are claim sizes which are independent and identically distributed non-negative random variables that are also independent of the number of claims. We also assume  $c > \lambda E(Y_1)$  (the *net profit condition*). Define the stopping time

$$T = \inf \{t > 0 \mid X_t < 0\}. \quad (2)$$

The event of ruin in infinite time horizon can be expressed as  $\{T < \infty\}$ . The density of  $T$  and the probability of ruin have been widely studied. See for example [3], [4], [9], [10], [11], [12], [13], [16], [17], [14], [20] and [23].

In this paper, we extend the concept of ruin to the Parisian type of ruin. The idea comes from Parisian options, the prices of which depend on the excursions

of the underlying asset prices above or below a barrier. An example is a *Parisian down-and-out option*, the owner of which loses the option if the underlying asset price  $S$  reaches the level  $l$  and remains constantly below this level for a time interval longer than  $d$ . For details and extensions, see [2], [5], [6], [7], [8], [21] and [22].

Parisian type ruin will occur if the surplus falls below zero and stays below zero for a continuous time interval of length  $d$ . In some respects, this is a more appropriate measure of risk than classical ruin as it gives the office some time to put its finances back in order. In practice, the bankruptcy procedures in many countries allow for this "grace" period, such as the Chapter 11 bankruptcy of the United States' Bankruptcy Code. Similar bankruptcy regulations are also applied to Japan and France (see [1]).

In order to introduce the concept of Parisian type of ruin mathematically, we will first define the excursion. Set

$$g_t = \sup\{s < t \mid \text{sign}(X_s) \text{sign}(X_t) \leq 0\}, \quad (3)$$

$$d_t = \inf\{s > t \mid \text{sign}(X_s) \text{sign}(X_t) \leq 0\}, \quad (4)$$

with the usual convention,  $\sup\{\emptyset\} = 0$  and  $\inf\{\emptyset\} = \infty$ , where

$$\text{sign}(x) = \begin{cases} 1, & \text{if } x > 0 \\ -1, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \end{cases}.$$

The trajectory between  $g_t$  and  $d_t$  is the excursion of process  $X$  below or above zero which straddles time  $t$ . Assuming  $d > 0$ , we now define

$$\tau_d = \inf\{t > 0 \mid \mathbf{1}_{\{X_t < 0\}}(t - g_t) \geq d\}. \quad (5)$$

We can see that  $\tau_d$  is therefore the first time that the length of the excursion of process  $X$  below zero reaches given level  $d$ . We then define the events  $\{\tau_d < \infty\}$  to be the Parisian type of ruin in the infinite horizon. We are interested in the corresponding probabilities

$$P(\tau_d < \infty).$$

In Section 2 we calculate the Parisian type ruin probability for the case when the initial reserve is zero. In Section 3 we study the case when the initial reserve is greater than zero. The asymptotic form of the Parisian type ruin probability will be given for the small claim case. We conclude our results in Section 4 and point out some directions for the future research.

## 2 The ruin probability for the case when the initial reserve is zero

In this section, we are going to consider a simplified case with no initial reserve, i.e.

$$X_t = ct - \sum_{k=0}^{N_t} Y_k. \quad (6)$$

Set

$$G(y) = P(Y_i < y), \quad \bar{G}(y) = P(Y_i > y);$$

$$m = E(Y_i), \quad \hat{g}(v) = \int_0^\infty e^{-vy} dG(y).$$

Denote the ruin probabilities to be

$$\psi(u) = P(T < \infty \mid X_0 = u), \quad \psi_d(u) = P(\tau_d < \infty \mid X_0 = u).$$

Since  $T < \tau_d$ , it is clear that  $\psi(u) > \psi_d(u)$ .

**Theorem 1** *For the process  $X$  defined by (6), we have that*

$$\psi_d(0) = \frac{\lambda m \bar{H}(d)}{c - \lambda m H(d)}, \quad (7)$$

where

$$H(d) = \mathcal{L}_\beta^{-1} \left( \frac{cv_\beta^+ - \beta}{\lambda m \beta v_\beta^+} \right), \quad (8)$$

$$\bar{H}(d) = 1 - H(d), \quad (9)$$

and  $v_\beta^+$  is the unique positive solution of

$$-\beta + cv_\beta + \lambda(\hat{g}(v_\beta) - 1) = 0. \quad (10)$$

**Proof:** It is well-known that

$$\psi(0) = \frac{\lambda m}{c}, \quad (11)$$

and that the overshoot  $-X_T$  is a non-negative continuous random variable with density

$$\frac{\bar{G}(x)}{m}. \quad (12)$$

See for example [9], [10], [11], [12], [16], [17], [18], [19], [20] and [23]. Furthermore, define

$$T^* = \inf \{t > 0, X_t = 0 \mid X_0 = x, x < 0\}. \quad (13)$$

It has been shown in [15] that

$$E(\exp(-\beta T^*)) = \exp\left(v_\beta^+ x\right). \quad (14)$$

We use  $h(t)$  to denote the density of the first (and actually any, due to the Markov property of the process  $X$ ) excursion below zero. Its Laplace transform

can be obtained as follows:

$$\begin{aligned}
\hat{h}(\beta) &= \int_0^\infty e^{-\beta t} h(t) dt \\
&= \int_0^\infty E(\exp(-\beta T^* | X_0 = -y)) \frac{\bar{G}(y)}{m} dy \\
&= \int_0^\infty \exp(-v_\beta^+ y) \frac{\bar{G}(y)}{m} dy \\
&= \frac{1 - \hat{g}(v_\beta^+)}{m v_\beta^+} = \frac{c v_\beta^+ - \beta}{\lambda m v_\beta^+}.
\end{aligned}$$

Define then the cumulative distribution function of  $T^*$  to be

$$H(d) = P(T^* < d). \quad (15)$$

We have actually

$$H(d) = \int_0^d h(t) dt = \mathcal{L}_\beta^{-1} \left( \frac{\hat{h}(\beta)}{\beta} \right) = \mathcal{L}_\beta^{-1} \left( \frac{c v_\beta^+ - \beta}{\lambda m \beta v_\beta^+} \right). \quad (16)$$

Moreover, the number of excursions  $N$  below zero has a geometric distribution such that

$$P(N = n) = \left(1 - \frac{\lambda m}{c}\right) \left(\frac{\lambda m}{c}\right)^n, \quad n = 0, 1, 2, \dots \quad (17)$$

As a result, the largest ever excursion below zero, denoted by  $L$ , is such that

$$P(L \leq d) = \sum_{i=0}^{\infty} (H(d))^i \left(1 - \frac{\lambda m}{c}\right) \left(\frac{\lambda m}{c}\right)^i = \frac{1 - \frac{\lambda m}{c}}{1 - \frac{\lambda m}{c} H(d)}. \quad (18)$$

Hence we have

$$\psi_d(0) = 1 - P(L \leq d) = \frac{\lambda m \bar{H}(d)}{c - \lambda m H(d)}. \quad (19)$$

□

**Remark:** It is clear that  $\psi_d(0) < \psi(0)$  by simply comparing (7) and (11). Also, we can obtain  $\psi(0)$  by taking  $d \rightarrow 0$  in (7).

### 3 An asymptotic formula for the ruin probability

In this section we focus on the asymptotic form for the Parisian ruin probability as  $u \rightarrow \infty$ . We assume that we have small claims.

*Assumption:* The Laplace transform  $\hat{g}(v)$  is defined for all  $v \in (\alpha, \infty)$  for some  $\alpha < 0$ .

**Theorem 2** For the process  $X$ ,  $X_0 = u$ , when  $u \rightarrow \infty$  we have that

$$\psi_d(u) \sim C_d e^{-Ru}, \quad (20)$$

where

$$C_d = C \left\{ 1 - \frac{R}{c - \lambda m H(d)} Q(d) \right\}, \quad (21)$$

$$C = \frac{c - m\lambda}{R\lambda} \left[ \int_0^\infty y e^{Ry} \bar{G}(y) dy \right]^{-1}, \quad (22)$$

$$Q(d) = \mathcal{L}_\beta^{-1} \left( \frac{1}{v_\beta^+ (v_\beta^+ + R)} \right), \quad (23)$$

and  $R$  is the adjustment coefficient which is the unique positive root of

$$-cR + \lambda (\hat{g}(-R) - 1) = 0. \quad (24)$$

**Proof:** First of all, the Parisian ruin probability can be written as follow:

$$\begin{aligned} & \psi_d(u) \\ = & P(\tau_d < \infty \mid X_0 = u) \\ = & P(\tau_d < \infty, T < \infty, T^* < d \mid X_0 = u) + P(\tau_d < \infty, T < \infty, T^* \geq d \mid X_0 = u) \\ = & P(T < \infty, T^* < d \mid X_0 = u) P(\tau_d < \infty \mid X_0 = 0) \\ & + P(T < \infty, T^* \geq d \mid X_0 = u). \end{aligned}$$

That last equality is due to the strong Markov property of  $X$ . We have obtained  $P(\tau_d < \infty \mid X_0 = 0)$  in (7). Furthermore, we have

$$\begin{aligned} & \int_0^\infty e^{-\beta d} \lim_{u \rightarrow \infty} e^{Ru} P(T < \infty, T^* < d \mid X_0 = u) dd \\ = & \lim_{u \rightarrow \infty} \int_0^\infty e^{-\beta d} e^{Ru} P(T < \infty, T^* < d \mid X_0 = u) dd \\ = & \lim_{u \rightarrow \infty} e^{Ru} E \left( \frac{e^{-\beta T^*}}{\beta} \mathbf{1}_{\{T < \infty\}} \mid X_0 = u \right) \\ = & \lim_{u \rightarrow \infty} \int_0^\infty E \left( \frac{e^{-\beta T^*}}{\beta} \mid -X_T = z \right) e^{Ru} P(T < \infty, -X_T \in dz \mid X_0 = u) \\ = & \lim_{u \rightarrow \infty} \int_0^\infty E \left( \frac{e^{-\beta T^*}}{\beta} \mid -X_T = z \right) P(-X_T \in dz \mid T < \infty, X_0 = u) e^{Ru} \psi(u) \\ = & \int_0^\infty E \left( \frac{e^{-\beta T^*}}{\beta} \mid -X_T = z \right) \lim_{u \rightarrow \infty} P(-X_T \in dz \mid T < \infty, X_0 = u) e^{Ru} \psi(u). \end{aligned}$$

By (14) we have that

$$E \left( \frac{e^{-\beta T^*}}{\beta} \mid -X_T = z \right) = \frac{e^{-v_\beta^+ z}}{\beta}.$$

It is well-known that

$$\begin{aligned} \lim_{u \rightarrow \infty} P(-X_T \in dz \mid T < \infty, X_0 = u) &= \frac{\lambda R}{c - \lambda m} \int_0^\infty e^{Rx} \bar{G}(x+z) dx, \\ \lim_{u \rightarrow \infty} e^{Ru} \psi(u) = C &= \frac{c - m\lambda}{R\lambda} \left[ \int_0^\infty y e^{Ry} \bar{G}(y) y \right]^{-1}. \end{aligned}$$

For more details see [3], [13] and [23]. We have therefore that

$$\begin{aligned} & \int_0^\infty e^{-\beta d} \lim_{u \rightarrow \infty} e^{Ru} P(T < \infty, T^* < d \mid X_0 = u) dd \\ &= C \int_0^\infty \frac{e^{-v_\beta^+ z}}{\beta} \frac{\lambda R}{c - \lambda m} \int_0^\infty e^{Rx} \bar{G}(x+z) dx \\ &= \frac{C\lambda}{c - \lambda m} \frac{1}{\beta} \left( \frac{\hat{g}(-R) - \hat{g}(v_\beta^+)}{v_\beta^+ + R} - \frac{1 - \hat{g}(v_\beta^+)}{v_\beta^+} \right) \\ &= \frac{CR}{c - \lambda m} \left( \frac{1}{v_\beta^+ (v_\beta^+ + R)} \right). \end{aligned}$$

As a result,

$$\lim_{u \rightarrow \infty} e^{Ru} P(T < \infty, T^* < d \mid X_0 = u) = \frac{CR}{c - \lambda m} Q(d), \quad (25)$$

where

$$Q(d) = \mathcal{L}_\beta^{-1} \left( \frac{1}{v_\beta^+ (v_\beta^+ + R)} \right);$$

and hence

$$P(T < \infty, T^* < d \mid X_0 = u) \sim e^{-Ru} \frac{CR}{c - \lambda m} Q(d). \quad (26)$$

Also, we have

$$\begin{aligned} & P(T < \infty, T^* \geq d \mid X_0 = u) \\ &= \psi(u) - P(T < \infty, T^* < d \mid X_0 = u) \\ &\sim C e^{-Ru} \left( 1 - \frac{R}{c - \lambda m} Q(d) \right). \end{aligned} \quad (27)$$

We have therefore proved (20).

□

**Remark 1:** The constant  $C$  given by (22) is the well-know Cramér constant. This theorem gives the modified version of the Cramér constant,  $C_d$  for the Parisian ruin case, which is given by (21).

**Remark 2:** It is easy to see that  $C_d < C$ , and hence  $\psi_d(u) < \psi(u)$ .

## 4 Conclusion

In Section 3 we obtain the asymptotic result for the small claim case (see Assumption). Note that it is not the case for the result in Section 2, which is true for all claim distributions. When  $u > 0$ , the difficulty with the large claim case is that we do not have a nice form for the distribution of overshoot on which the length of excursions below zero depend. The investigation of the large claim case can be a topic of future research.

For the small claim case, instead of asymptotic form we obtained here, it would also be nice to get a formula for  $\psi_d(u)$  for a general  $u > 0$ . One of the difficulties is that the length of the excursions below zero depends on the length of the preceding excursion above zero since the overshoots depend on the length of the excursion above zero. However, in the case of exponential distributed claims, we do not have such problem since the overshoot is independent of the excursion and the explicit form for  $\psi_d(u)$  can be obtained (see [8]).

Furthermore, as another direction of future research, one should try to study the Parisian ruin probability in finite time horizon, i.e.  $P(\tau_d < t)$ .

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