Twisted Strings, Vertex Operators and Algebras.

by

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You stay at home, important work to do.
Unroll the ancient parchments of your theories,
And range the elements of life in series,
Relating them together, as you ponder.
Reflect on what, still more on how and why;
And while in sundry foreign parts I wander
I may unearth the dot upon the "I".
Then is fulfilled the mighty aim:
Renown for such high toil will not desert you:
You have your due reward, your gold, your fame,
Your wealth of knowledge and—perhaps—of virtue.

Johann Wolfgang von Goethe, Faust Part II.

To Nessie.

PREFACE

The work presented in this thesis was carried out between October 1985 and July 1988 in the Department of Mathematical Sciences at the University of Durham, under the supervision of Dr. E.F. Corrigan.

The material in this thesis has not been submitted previously for any degree in this or any other university.

No claim of originality is made for the review in chapter 2, except for the construction of the algebras which generalize the Lie and cross-bracket, which are believed to be original. The material in chapter 3 was inspired by [1-3]. The treatment of the zero-modes and, in particular, the twisted operator cocycles is original, being based on work done in collaboration with Richard Myhill. A more complete treatment appears in [4]. The construction of the twisted string emission vertex in chapter 4 is based on original work with Ed Corrigan [5]. The discussions of the reflection twist in chapters 5, 6 and 7 and the third order twist of E_6 in chapter 8 are original; most of it was done in collaboration with Ed Corrigan, and is written up in [6], although the extension of the algebra enhancement mechanism to the Lie algebras of rank 8 and the cross-bracket algebra, the discussion of the characters and the shifted picture, are due solely to the author.

I should like to express my gratitude to Ed Corrigan for motivating and inspiring much of this work, and for fruitful collaboration on this and other topics in string theory. I also have pleasure in thanking Richard Myhill, for many discussions on algebras and automorphisms, and also Graeme Robertson, Patrick Dorey and Harry Braden for discussions. Finally I acknowledge and thank the Science and Engineering Research Council for supporting this work.

ABSTRACT

This work is principally concerned with the operator approach to the orbifold compactification of the bosonic string. Of particular importance to an operator formalism is the conformal structure and the operator product expansion. These are introduced and discussed in detail. The Frenkel–Kac–Segal mechanism is then examined and is shown to be a consequence of the duality of dimension one operators of an analytic bosonic string compactified on a certain torus. Possible generalizations to higher dimension operators are discussed, this includes the cross–bracket algebra which plays a central rôle in the vertex operator representation of Griess's algebra, and hence the Fischer–Griess Monster Group.

The mechanism of compactification is then extended to orbifolds. The exposition includes a detailed account of the twisted sectors, especially of the zero–modes and the twisted operator cocycles. The conformal structure, vertex operators and correlation functions for twisted strings are then introduced. This leads to a discussion of the vertex operators which represent the emission of untwisted states. It is shown how these operators generate Kac–Moody algebras in the twisted sectors. The vertex operators which insert twisted states are then constructed, and their rôle as intertwining operators is explained. Of particular importance in this discussion is the rôle of the operator cocycles, which are seen to be crucial for the correct working of the twisted string emission vertices.

The previously established formalism is then applied in detail to the reflection twist. This includes an explicit representation of the twisted operator cocycles by elements of an appropriate Clifford algebra and the elucidation of the operator algebra of the twisted emission vertices, for the ground and first excited states in the twisted sector. This motivates the 'enhancement mechanism', a generalization of the Frenkel–Kac–Segal mechanism, involving twisted string emission vertices, in dimensions 8, 16 and 24, associated with rank 8 Lie algebras, rank 16 Lie algebras and the cross–bracket algebra for the Leech lattice, respectively. Some of the relevant characters of the 'enhanced' modules are determined, and the connection of the cross–bracket algebra to the phenomenon of 'Monstrous Moonshine' and the Monster Group is explained. Algebra enhancement is then discussed from the greatly simplified shifted picture and extensions to higher order twists are considered.

Finally, a comparison of this work with other recent research is given. In particular, the connection with the path integral formalism and the extension to general asymmetric orbifolds is discussed. The possibility of reformulating the moonshine module in a 'covariant' twenty–six dimensional setting is also considered.

CONTENTS

1. INTRODUCTION.

String theory is still, more than 20 years after its conception, rather a nebulous subject. However, to its adherents it offers the realistic hope of being a consistent quantum theory of gravity which incorporates gauge symmetries in a natural way. If one can point to a unifying principle underlying string theory then it would be the conformal structure, at least for the first quantized formalism. Viewing the first quantized string as a two dimensional conformal field theory yields considerable rewards of insight and systematization. Such theories are highly restrictive in nature, almost all the relevant information being encoded in the conformal dimensions of the fields and their operator product expansion [7–9]. Most of the 'new formalism', based on conformal field theory, is a re-couching of results obtained in the dual model, the previous incarnation of string theory. However, one can recognize a number of important advances. For example the BRST quantization has been elucidated for both the bosonic and superstring [9]. Introducing Fadeev–Popov ghosts has led to considerable simplifications, especially with regard to the superstring. Considerable advances have also been accomplished with the technique of bosonization. Indeed in the recent four dimensional string models, based on the covariant lattice approach, all the internal fermionic degrees of freedom are bosonized [10]. The first quantized theory in a flat background is as a result of these advances well understood, although one short coming is the absence of a proof of finiteness. This work proposes to tackle another problem posed by the string, that of dimensional reduction. Rather than be an encumbrance the extra dimensions, over and above the four dimensions of space–time, allow the string to have non–abelian gauge symmetries by compactification on tori via the Frenkel–Kac–Segal mechanism [11–14]. We will consider generalizations of this mechanism to orbifold backgrounds.

Even if string theory fails to achieve the lofty goal of being the ultimate theory of the universe, from an abstract point of view it is rich in structure and one sees many nascent connections with previously unrelated branches of pure mathematics. Perhaps the most important area of overlap is the theory of affine Lie (or Kac–Moody) algebras. After the rebirth of the string in the early 1980's, and particularly after the invention of the heterotic string, there has been an explosion of interest in such algebras. In fact the string seems to be particularly adept at representing many symmetry structures including affine Lie algebras and their supersymmetric generalizations, general coordinate invariance (supergravity) and finite groups. There is even recent evidence that Jordan algebras can be included in this list [15–18]. This work proposes to investigate the connection between certain (twisted) string models and representations of Kac–Moody algebras. Before we describe the program in detail it is worthwhile reviewing in a historical context the appearance of twisted strings and the corresponding developments in the theory of Kac–Moody algebras.

Half–integrally moded string fields first appeared in the literature as early as 1971 in attempts to generalize the dual model [19,20]. Later these half–integrally moded fields appeared in one attempt to describe off–shell amplitudes in the dual model from an operator point of view [21]. This old application of twisted strings has recently been revived in refs.[22,23], where evidence was found that a careful treatment of the BRST ghosts remedies the major difficulties of the old off–shell model: namely its critical dimension of sixteen which is in conflict with the on–shell critical dimension of twenty–six.

While string theory underwent something of a hibernation, mathematicians were beginning to discover that affine Lie algebras could be represented by operators which were basically vertex operators of the dual model built from non–integrally moded string fields [24]. These constructions actually predated the Frenkel–Kac–Segal mechanism [13,14], which was a construction of untwisted algebras of type $A^{(1)}$, $D^{(1)}$ and $E^{(1)}$, in terms of conventional untwisted vertex operators. The connection between the untwisted and twisted approaches was elucidated to a certain extent in ref.[25], where the rôle of the underlying automorphism was emphasized. To bring events full circle, refs.[26–28] discussed a model based on a certain reflection automorphism whose vertex operators were identical to the operators appearing in the off–shell model, except for the zero–mode operator cocycle pieces which are absent in the off–shell formalism. Subsequently all the vertex operator constructions have been unified and elucidated by Lepowsky [3]. Almost simultaneously, string theory was being revived as a possible unified theory of all the known interactions. The main achievement at this time was the construction of the heterotic string [11,12], which is formed from the left–moving degrees of freedom of the closed bosonic string in twenty–six dimensions and the right–moving degrees of freedom of the closed superstring in ten dimensions. The sixteen internal mismatched dimensions are used to generate a rank 16 gauge group via the Frenkel–Kac–Segal mechanism. In this picture the internal degrees of freedom are viewed as resulting from the compactification of a closed purely left–moving bosonic string on the torus formed by quotienting flat space by a lattice. To avoid global diffeomorphism anomalies the lattice

lattice has to be the root lattice of $E_8 + E_8$ or $spin(32)/\mathbb{Z}_2$ †, the resulting theory having a gauge group of $E_8 \times E_8$ or $SO(32)$.

Although the heterotic string has many desirable features (*e.g.* supersymmetry and a non–abelian gauge group), it is nevertheless a ten dimensional theory. The obvious way to reconcile this with four dimensional space–time is to invoke compactification. In the first attempt to compactify the heterotic string [29], the vacuum configuration of the ten dimensional space–time was taken to be $M_4 \times K$, where M_4 is four dimensional Minkowski space and K is some compact six dimensional manifold. Requiring a realistic gauge group and an unbroken $N = 1$ supersymmetry, consistent with string propagation, pinned K down as a Calaubi–Yau manifold. Although these early attempts were encouraging they were not inherently 'stringy'. Due to the complicated nature of Calaubi–Yau manifolds only the effective field theory of the massless sector could be considered. The introduction of orbifolds allowed the construction of 'realistic' models in four dimensions from an inherently stringy compactification [1,2,30]. The theory of a string on an orbifold naturally leads to non–integrally moded fields, and some of the vertex operators which appear are precisely the objects discovered by mathematicians in the theory of affine Lie algebras.

The heterotic construction has since been generalized following the seminal work of Narain [31], who found that by mixing the internal sixteen dimensions and the other six dimensions one could construct four dimensional models in a unified way. Further generalizations of this idea have been attempted based on the bosonization of the superstring [10]. In fact the number of consistent four dimensional models seems to be embarrassingly enormous. The quintessential feature of these models is an underlying lattice which defines a torus in a sort of enlarged Frenkel–Kac–Segal mechanism. The nice feature about these models is that modular invariance only constrains the lattice to have certain generic features. Twisted generalizations of these models involving asymmetric orbifolds have also been considered [32].

From a conformal field theory point of view orbifold models now represent a whole class of theories which are completely solvable, at least in principle. Twisted string correlation functions have been considered in refs.[33,34]. These functions exhibit non–perturbative instanton effects from the two dimensional point of view. One of the conclusions of this work is the recognition that these non–perturbative contributions

† The $spin(32)/\mathbb{Z}_2$ lattice is the root lattice of D_{16} appended with one of the spinor weight cosets.

come from the operator cocycles in the operator formalism. We now outline the content of this work in more detail.

Chapter 2 is mainly a review of bosonic string theory, which is not intended to be didactic but rather its aim is to highlight some important concepts which underlie the rest of the work. Of particular importance is the rôle of conformal invariance. Following [7–9] we discuss the bosonic string as a paradigm of two dimensional conformal field theory. Special emphasis is placed on the operator product expansion of the fields; it is shown how this relates to the properties of correlation functions, in particular duality. We show how the requirements of duality and locality place restrictions on the structure constants of the operator algebra. The operator algebra can be written in an equivalent commutator form, and so the (analytic) vertex operators generate an infinite Lie algebra. The problem of compactifying the string on a flat torus is considered. By taking the dimension one operators of the analytic sector in isolation one is led to the Frenkel–Kac–Segal construction. We then consider how the mechanism can be generalized to higher dimension operators, and find an infinite series of algebras. The algebra defined on operators of dimension ≤ 2 is just the cross–bracket algebra of refs.[26–28], which is a key ingredient in the construction of the 'Moonshine Module', and hence the Fischer–Griess Monster group , as we explain in a later chapter.

In chapter 3 we introduce the concept of an orbifold, as a way of generalizing toroidal compactifications. A detailed discussion of the boundary conditions and classical motion of a closed bosonic string moving on such a space is developed. We place particular emphasis on the possibility of 'twisted' boundary conditions. The theory is then quantized following [2], and the Hilbert space structure is elucidated. We then consider the conformal structure of twisted string models, showing how the ground states of the twisted sectors have a non–zero conformal weight. Generic features of vertex operators and correlation functions are then considered. The importance of the twist projection, as a way to restore locality, is highlighted. The structure of the zero–mode space in the twisted sectors is alluded to. following the more detailed analysis presented in ref.[4]. In order to make contact with the vertex operator representations of Kac–Moody algebras the vertices which emit untwisted states are constructed and their operator algebra is worked out in detail.

Building on the formalism established in chapter 3, in chapter 4 we construct the vertices which emit twisted states from untwisted strings [5]. The oscillator contributions to these vertices turn out to be a fairly simple generalization of part of the old off–shell vertex [21]. However, one of the most important conclusions from this section is the realization that the zero–mode operator cocycle contribution to the vertex is crucial for its correct working. We also highlight the intertwining property of these vertices, originally studied in refs.[35,36], in fact this property is actually used as a guide for constructing the vertices. There follows an evaluation of the operator product expansion of a twisted string emission vertex with an untwisted string emission vertex. We conclude the chapter with some comments on the conformal properties of the vertex.

Chapters 5 and 6 are an investigation into the $\mathbb{Z}_2$ reflection twist. In chapter 5 the twisted operator cocycles are evaluated in terms of an appropriate Clifford algebra for all the simply–laced Lie algebras as well as for the $spin(32)/\mathbb{Z}_2$ lattice and the Leech lattice. Chapter 6 is devoted to a detailed discussion of the operator algebra of the twisted string emission vertices. It is found that two twisted vertices factorize onto untwisted vertices, furthermore the results are consistent with the conformal structure of the theory in the sense that the structure constants are nothing but the 3–point functions. The calculation in the untwisted sector is relatively straightforward, whereas the calculation in the twisted sector, which amounts to an evaluation of the four twisted string correlation function, is quite lengthy. One important conclusion emerges, that is, the operator algebra is only well defined if the dual of the underlying lattice is 'integral'† . These calculations suggest that the Frenkel–Kac–Segal mechanism could be generalized with twisted vertices. We investigate this hypothesis in chapter 7. When the operator algebra is well defined and there exist twisted emission vertices with conformal dimension one, we show how the algebra generated in the untwisted and twisted sectors separately is enhanced by the twisted vertices which mix the two sectors. Such level matching occurs for rank 8 and rank 16 algebras. For certain algebras in these dimensions enhancement occurs. This mechanism allows us to discuss the relationship between the two original heterotic string theories, previously established in the fermionic picture [2,37,38]. We also show that enhancement occurs for the cross–bracket algebra based defined on the Leech lattice. This is essentially the construction of the 'Moonshine Module' discovered by Frenkel Lepowsky and Meurman [26–28]. We now have an explicit operator representation of Griess's algebra, whose automorphism group is the Fischer–Griess Monster group.

We then consider the case when the twist is an inner automorphism of the underlying

† By 'integral' we mean that all the vectors have an integral squared length, in contrast to the usual terminology.

algebra. In these cases it is possible to reconstruct the theory in terms of shifted vertex operators. This has the advantage of considerably simplifying the operator algebra of the twisted emission vertices. The algebra enhancement mechanism in the shifted picture is then elucidated, giving a more simplified perspective on the phenomenon. We conclude the chapter with some comments concerning generalizations of algebra enhancement to higher order twists. In particular we consider a third order twist in E_6 and E_8 as examples [6].

In the last chapter the extension of the formalism to asymmetric orbifolds is examined. For a symmetric orbifold we show how the twisted operator cocycles have a very simple geometrical interpretation. The four twisted string interaction, which was calculated in chapter 6 for the reflection twist, is then considered. We compare our expression with other work on twisted string interactions [33,34]. In particular we discover that from an operator point of view the classical instanton contribution of the path integral formalism comes from the twisted operator cocycles.

To conclude the last chapter we consider shifting and twisting theories in twenty-four dimensions with the Niemeier lattices. The question as to whether the Moonshine Module can be imbedded in a covariant 26 dimensional framework, in analogy with the bosonic string, is briefly addressed.

2. THE STRING AND CONFORMAL FIELD THEORY.

In this chapter we give a brief introduction to the bosonic string from the two dimensional conformal field theory perspective. The operator algebra, which is a consequence of the operator product expansion, is highlighted and discussed in some detail because it forms the central theme for the rest of the material. We place particular emphasis on a discussion of the factorization and duality of closed strings because these properties lie at the heart of the appearance of the Frenkel–Kac–Segal (FKS) mechanism in string theory. After this, we consider a closed string compactified on a flat torus, which leads to the concept of an analytic string and the FKS mechanism. The cross–bracket algebra is then introduced, since it seems to be the central object underlying the 'Moonshine Module', an infinite $\mathbb{Z}$–graded representation of the Fischer–Griess Monster group.

2.1 FIRST QUANTIZED STRING THEORY.

The first quantized theory of the bosonic string is a two–dimensional field theory of the map $X(\xi)$, from the world–sheet of the string $\mathcal{M}$, $\xi \in \mathcal{M}$, into space–time $\mathcal{S}$. The geometry of $\mathcal{S}$ is specified a priori, although hopefully in a consistent second quantized theory it would be determined dynamically. We will only consider cases where $\mathcal{S}$ is locally flat†. The dynamics of the first quantized theory are determined by the action:

$$S = \frac{1}{2}\int d^2\xi\sqrt{-g}g^{ab}\partial_a X.\partial_b X, \tag{2.1}$$

where $\xi^a = (\tau, \sigma)$ and g^{ab}, the *intrinsic* metric of $\mathcal{M}$, is treated as an auxiliary field. In evaluating the path integral over surfaces

$$\mathcal{Z} = \int \mathcal{D}g_{ab}\mathcal{D}X\, e^{iS[X,g]}.$$

one must be careful to account for the gauge invariances of S, which arise from the invariance of $\mathcal{M}$ under diffeomorphisms (reparameterizations), and Weyl rescaling of the metric g. The most suitable way to deal with the gauge invariance is to set $g_{ab} = e^{\varsigma}\hat{g}_{ab}$, where $\hat{g}_{ab}$ is some fixed reference metric. For tree–level contributions to the partition function $\hat{g}_{ab}$ is most conveniently chosen to be the flat metric diag$(-1, 1)$. The scaling field ς can be successfully decoupled from the theory if the dimension of the background space–time is twenty–six [39], the **critical dimension** of the bosonic string.

† Except, perhaps, at a finite number of points when Orbifolds are considered.

If we continue to Euclidean space $(\tau, \sigma) \rightarrow (\overline{\tau} = i\tau, \sigma)$ then the world-sheet, which at the tree-level has the topology S^2, can be parameterized by the complex coordinates $z(\overline{z}) = \exp(\overline{\tau} \pm i\sigma)$, in which case

$$S = \int d^2z\, \partial X . \overline{\partial} X, \tag{2.2}$$

where the integral ranges over the complex plane for closed strings, and the upper half plane for open strings.

The gauge fixing procedure leaves as a residual invariance analytic and anti-analytic conformal transformations. $z \rightarrow f(z)$ ($\overline{z} \rightarrow \overline{f}(\overline{z})$). This residual conformal invariance plays a crucial rôle in the theory, since it ensures that ghost states decouple from physical amplitudes. The conformal transformations are generated by the stress-energy tensor, which being automatically traceless has just two components, one analytic the other anti-analytic [7–9]:

$$\begin{aligned} T(z) &= -\frac{1}{2}\partial X . \partial X, \\ \overline{T}(\overline{z}) &= -\frac{1}{2}\overline{\partial} X . \overline{\partial} X. \end{aligned} \tag{2.3}$$

The classical equation of motion is just Laplace's equation in two dimensions,

$$\partial \overline{\partial} X = 0,$$

which is readily solved by

$$X(z, \overline{z}) = X(z) + \overline{X}(\overline{z}).$$

Before we go on to set up the first quantized theory we pause to discuss **two dimensional Conformal field theories** (2d–CFT) in general; of which the bosonic string is but one example.

2.2 Two Dimensional Conformal Field Theories.

The local fields $\{\phi_i\}$ of a 2–d CFT are characterized by their **conformal scaling dimension** [7–9]; under $z \rightarrow \lambda z$

$$\phi_i(z) \longmapsto \lambda^{\Delta_i} \phi_i(\lambda z).$$

For the moment we will concentrate on the analytic sector of the theory, a parallel treatment can be given for the anti-analytic sector.

The quintessential property of 2–d CFTs is that virtually all the structure of the theory is encoded in the spectrum of fields and their **Operator Product Expansions** (OPEs). The OPE describes the short distance behaviour of two fields on the complex plane:

$$\phi_i(z)\phi_j(\omega) = \sum_k (z-\omega)^{\Delta_k-\Delta_i-\Delta_j} c_{ij}{}^k \phi_k(\omega). \tag{2.4}$$

The **structure constants** of the operator algebra, $c_{ij}{}^k$, satisfy a set of non–linear constraints (the bootstrap constraints) which result from the associativity of the OPE [7].

The operators can be represented in an appropriate Hilbert space and each is in one–to–one correspondence with a state such that

$$\mid i\rangle = \phi_i(0)\mid 0\rangle, \tag{2.5}$$

where $\mid 0\rangle$ is the **vacuum state**, to be defined below. Recall that conformal transformations are generated by the stress–energy tensor $T(z)$. The field $T(z)$, along with the unit operator, generate the **Conformal** or **Virasoro algebra**,

$$T(z)T(\omega) = \frac{c/2}{(z-\omega)^4} + \frac{2T(\omega)}{(z-\omega)^2} + \frac{\partial T(\omega)}{z-\omega} + Reg. \tag{2.6}$$

Here, '*Reg*' indicates the existence of terms which are regular as $z \to \omega$. c is the **central charge** which characterizes the particular theory. In common with most OPEs, as we shall show later, the algebra can be written in an equivalent commutator form [7,8]† . We define the moments of a generic field $\phi_i(z)$ by the Laurent expansion:

$$\phi_i(z) = \sum_{n\in\mathbf{Z}} \phi^i_n z^{-n-\Delta_i}. \tag{2.7}$$

$T(z)$ has scaling dimension 2 and so its moments, L_n, are defined via

$$T(z) = \sum_{n\in\mathbf{Z}} L_n z^{-n-2}.$$

The equivalent commmutator form of the Virasoro algebra is

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}. \tag{2.8}$$

This is precisely the Lie algebra of conformal transformations, with an anomaly term parameterized by the central charge. The subalgebra $\{L_{\pm 1}, L_0\}$ generates the Lie al-

† This is appropriate for the bosonic theories we are considering.

gebra $sl(2,\mathbb{C})$ without anomaly, and it can therefore be *exponentiated* to the group of **projective transformations**, $SL(2,\mathbb{C})$:

$$z \longrightarrow \frac{az+b}{cz+d}, \qquad ad - bc = 1.$$

This group plays an important rôle in the construction of scattering amplitudes.

The vacuum state can now be defined as the state annihilated by the semi–infinite subalgebra $\{L_n \mid n \geq -1\}$.

$$L_n \mid 0\rangle = 0 \qquad n \geq -1. \tag{2.9}$$

An important class of fields are the **primary**, or **highest weight**, fields $\{\psi_a\}$, which transform in a simple way under the conformal algebra:

$$T(z)\psi_a(\omega) = \frac{\Delta_a \psi_a(\omega)}{(z-\omega)^2} + \frac{\partial\psi_a(\omega)}{z-\omega} + Reg, \tag{2.10}$$

or equivalently

$$[L_n, \psi_a(z)] = z^n(z\partial + \Delta_a(n+1))\psi_a(z). \tag{2.11}$$

These fields correspond to **highest weight states** of a **Verma module**, which satisfy

$$L_n \mid a\rangle = \begin{cases} 0 & n > 0 \\ \Delta_a \mid a\rangle & n = 0. \end{cases} \tag{2.12}$$

All other fields in the theory, which fill out the Verma modules, are *descended* from the primary fields in the sense that the states to which they correspond are of the form [7]

$$L_{-k_1} \ldots\ldots L_{-k_m} \mid a\rangle.$$

2.3 THE STRING AS A CONFORMAL FIELD THEORY.

The string field $X(z)$ itself does not have a well defined scaling dimension and hence is *not* a conformal field. However, the basic fields of the theory are all built out of the fields $\exp(i\alpha.X(z))$, of dimension $\alpha^2/2$, and $\partial^n X(z)$ ($\partial^n X(z) \equiv d^n/dz^n X(z)$) with $n > 0$, of dimension n.

The conformal algebra in the analytic sector is generated by the stress–energy tensor

$$T(z) = -\frac{1}{2} : \partial X(z).\partial X(z) :, \tag{2.13}$$

where ': :' implies **normal ordering**; that is, the singular part of the OPE is subtracted. In fact

$$\partial X^{\mu}(z)\partial X^{\nu}(\omega) = -\frac{\delta^{\mu\nu}}{(z-\omega)^2} + Reg.$$

therefore

$$T(z) = -\frac{1}{2}\lim_{\omega\to z}\left\{\partial X(z).\partial X(\omega) + \frac{d}{(z-\omega)^2}\right\}.$$

The solution to the classical equation of motion on flat space–time is

$$X_{open}(z,\overline{z}) = \frac{1}{2}(X(z) + X(\overline{z})), \tag{2.14}$$

for open strings, and for closed strings

$$X_{closed}(z,\overline{z}) = \frac{1}{2}(X(z) + \overline{X}(\overline{z})), \tag{2.15}$$

where $X(z)$ and $\overline{X}(\overline{z})$ are the **Fubini–Veneziano fields**:

$$\begin{aligned} X(z) &= q - ip\ln z + i\sum_{\substack{n\in\mathbf{Z}\\ n\neq 0}}\frac{\alpha_n}{n}z^{-n}, \\ \overline{X}(z) &= q - ip\ln\overline{z} + i\sum_{\substack{n\in\mathbf{Z}\\ n\neq 0}}\frac{\overline{\alpha}_n}{n}\overline{z}^{-n}. \end{aligned} \tag{2.16}$$

Canonical quantization leads to the following non–zero commutation relations

$$\begin{aligned} [q,p] &= i, & q^\dagger &= q,\ p^\dagger = p. \\ [\alpha_n,\alpha_m] &= n\delta_{n+m,0}, & \alpha_n^\dagger &= \alpha_{-n}, \\ [\overline{\alpha}_n,\overline{\alpha}_m] &= n\delta_{n+m,0}, & \overline{\alpha}_n^\dagger &= \overline{\alpha}_{-n}. \end{aligned} \tag{2.17}$$

For closed strings, with which we are exclusively interested, the analytic and anti–analytic sectors are, apart from the zero–modes, effectively decoupled and we can treat them separately. Later. we shall find that for compactified closed strings, *even* the zero–modes are decoupled.

The Hilbert space has a decomposition

$$\mathcal{H} = \mathcal{F} \otimes \overline{\mathcal{F}} \otimes \mathcal{P}, \tag{2.18}$$

where $\mathcal{F}(\overline{\mathcal{F}})$ is the Fock–space spanned by the identity and the creation operators $\alpha_{-n}(\overline{\alpha}_{-n})$, $n > 0$, and $\mathcal{P}$ is a complex span of momentum states, *i.e* states of the form

$$\mid \alpha\rangle = e^{i\alpha.q} \mid 0\rangle, \quad p \mid \alpha\rangle = \alpha \mid \alpha\rangle.$$

The field corresponding to the state

$$\mid \phi\rangle = \prod_{i=1}^{N} \epsilon_i.\alpha_{-n_i} \mid \alpha\rangle,$$

in the analytic sector. is

$$V(\phi, z) = : \prod_{i=1}^{N} \frac{i\epsilon_i.\partial^{n_i} X(z)}{(n_i - 1)!} e^{i\alpha.X(z)} :, \tag{2.19}$$

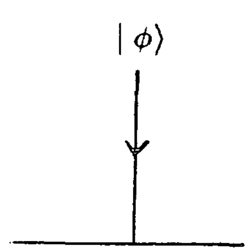
where $\Delta_\phi = \alpha^2/2 + \sum_{i=1}^{N} n_i$. At the level of the oscillators normal ordering is defined as

$$: \alpha_n \alpha_m : = \begin{cases} \alpha_n \alpha_m & m > n \\ \alpha_m \alpha_n & m < n \end{cases}$$

$$: pq : = qp.$$

In string theory the conformal field $V(\phi, z)$ is frequently called a **Vertex Operator**, since it represents the insertion of the state $\mid \phi\rangle$ at the point z on the world–sheet. We shall sometimes represent the vertex operators diagrammatically:

$V(\phi, z)$:



The vertex operator which represents the emission of the state $\mid \phi\rangle$ is

$$V(\overline{\phi}, z) = z^{-2\Delta_\phi} \Big(V(\phi, 1/\overline{z}) \Big)^\dagger .$$

L_0 grades the Hilbert space $\mathcal{H}$:

$$\mathcal{H} = \bigoplus_n \mathcal{H}[n], \tag{2.20}$$

where the grade of a state is its **conformal weight** (eigenvalue of L_0), or equivalently the dimension of the vertex operator to which it corresponds. This allows us to introduce the **partition function** or **character** for the analytic sector:

$$\begin{aligned}\chi(q) &= \mathrm{Tr}\left(q^{L_0}\right) \\ &= \sum_{|\phi\rangle} \langle \phi \mid q^{L_0} \mid \phi \rangle.\end{aligned} \tag{2.21}$$

This function plays an important rôle in the construction of one–loop amplitudes and in the discussion of global diffeomorphism anomalies and modular invariance, in string theory [40].

2.4 Correlation Functions, Factorization and Duality.

In order to construct scattering amplitudes for strings we need to evaluate auxiliary objects in the 2–d CFT, known as **correlation functions**:

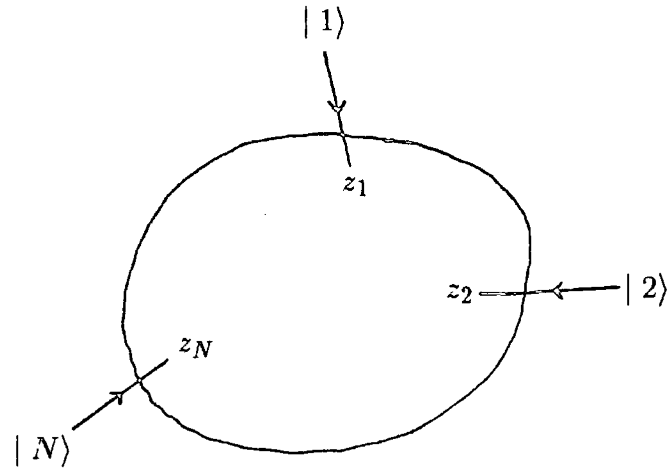
$$\mathcal{F}_N = \int \mathcal{D}X\mathcal{D}g \; e^{-S} \prod_{i=1}^{N} \phi_i(z_i), \tag{2.22}$$

where the $\{\phi_i\}$ are arbitrary fields. Alternatively, we may evaluate a correlation function as a time ordered expectation value in the Hilbert space:

$$\mathcal{F}_N = \langle 0 \mid T \left\{ \prod_{i=1}^{N} \phi_i(z_i) \right\} \mid 0 \rangle. \tag{2.23}$$

Since $z = \exp(\overline{\tau} + i\sigma)$, time ordering is equivalent to *radial* ordering on the complex plane. Conventionally, the insertion $\phi_i(z)$ ($\phi_{\overline{i}}(z)$) is interpreted as absorbing (emitting) the state $\mid i\rangle$ on the world-sheet:

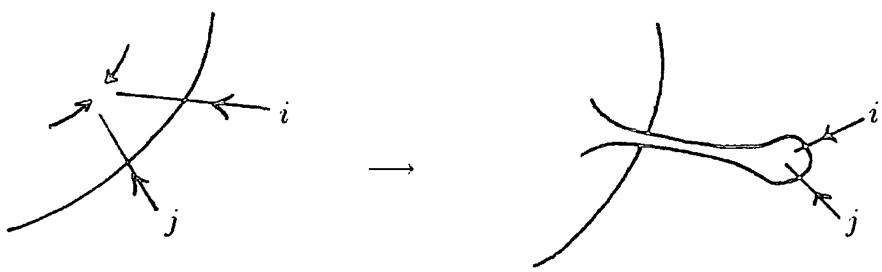
$\mathcal{F}_N$:



The correlation function $\mathcal{F}_N$ is a meromorphic function of the z_i's, with poles as $z_i \to z_j$; this is the property **factorization**. At an operator level it is simply a consequence of the OPE

$$\phi_i(z_i)\phi_j(z_j) \sim (z_i - z_j)^{\Delta_k - \Delta_i - \Delta_j} c_{ij}{}^k \phi_k(z_j),$$

and corresponds (conformally) to the configuration where there is a narrow tube separating strings i and j from the rest of the world-sheet:

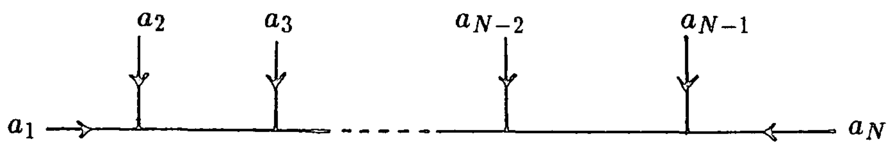


The narrow tube is indicative of on-shell intermediate string states which give rise to the poles in the correlation function. It is apparent that the structure constants of the operator algebra are related to the 3-point functions, we will make this correspondence more precise below.

Consider factorization in more detail. We can associate a particular ordering of the correlation function, say

$$\mathcal{F}_N(a_i, \ldots, a_N) = \langle 0 \mid \phi_{a_1}(z_{a_1}) \ldots \phi_{a_N}(z_{a_N}) \mid 0\rangle, \quad \mid z_{a_{i-1}} \mid > \mid z_{a_i} \mid,$$

with the **dual diagram**:



These functions exhibit poles in *all* possible **planar channels**. A planar channel corresponds to a partition of $b_1, ..., b_N$ into two sets $[b_1, ..., b_j][b_{j+1}, ..., b_N]$, where $b_1, ..., b_N$ is *any* cyclic permutation of $a_1, ..., a_N$. Planar factorization implies that if we analytically continue $\mathcal{F}_N \to \tilde{\mathcal{F}}_N$ to cover the whole complex plane then

$$\tilde{\mathcal{F}}_N(a_1, ..., a_N) = \tilde{\mathcal{F}}_N(b_1, ..., b_N).$$

We call this **planar duality**, since for the four point function we have

$$\tilde{\mathcal{F}}_4(1,2,3,4) = \tilde{\mathcal{F}}_4(2,3,4,1),$$

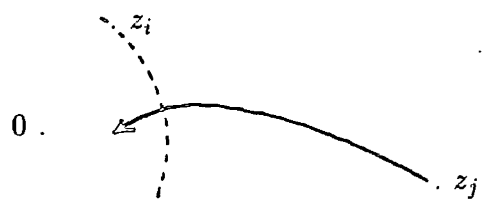
which exhibits the usual 'duality' between the s and t channels. At the level of the operator algebra this implies that the structure constants satisfy the cyclic identity

$$c_{ij}{}^{k} = c_{\bar{k}i}{}^{\bar{j}} = c_{j\bar{k}}{}^{\bar{i}}. \tag{2.24}$$

Planar duality is characteristic of open string correlation functions, where the inserted states have a given order along the string boundary, invariant under cyclic permutation. For closed strings we demand a much stronger form of 'duality', which we call **exchange duality**. It arises from the fact that for closed strings, where the insertions are made anywhere on the world–sheet, it would be unacceptable for $\mathcal{F}_N$ to have discontinuities at places where time–ordering caused a re–ordering of operators† . This occurs when

† For fermionic operators, on the contrary, we would expect a change of sign on re–ordering.

any z_i moves from $|z_i| > |z_j|$ to $|z_i| < |z_j|$;



For closed strings, where we have both analytic *and* anti-analytic degrees of freedom, exchange duality implies at the level of the operator algebra

$$c_{ij}{}^{k} = (-)^{\Delta_i+\Delta_j-\Delta_k-\overline{\Delta}_i-\overline{\Delta}_j+\overline{\Delta}_k} c_{ji}{}^{k}. \tag{2.25}$$

where Δ_i and $\overline{\Delta}_i$ are the analytic and anti-analytic scaling dimensions of the operator $\phi_i(z,\overline{z})$, respectively. If we consider just the analytic sector in isolation, which we do when we introduce **analytic strings**‡ , then exchange duality implies

$$c_{ij}{}^{k} = (-)^{\Delta_i+\Delta_j-\Delta_k} c_{ji}{}^{k}. \tag{2.26}$$

This constraint on the structure constants of the operator algebra is important for a correct understanding of the FKS mechanism, as we shall see. We will sometimes express exchange duality by the relation

$$\phi_i(z)\phi_j(\omega) \overset{\text{dual}}{\longleftrightarrow} \phi_j(\omega)\phi_i(z).$$

What is meant by this is that there exists a *meromorphic* operator function $S(z,\omega)$, such that

$$S(z,\omega) = \begin{cases} \phi_i(z)\phi_j(\omega) & |z| > |\omega| \\ \phi_j(\omega)\phi_i(z) & |\omega| > |z| \, . \end{cases}$$

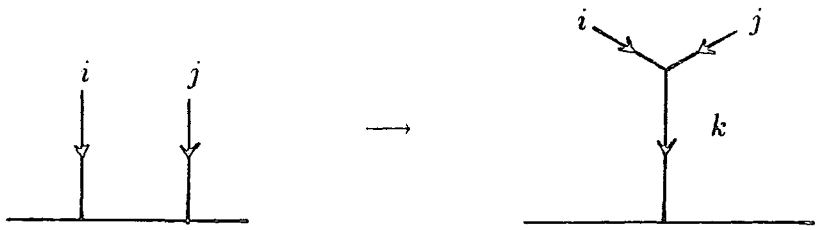
The connection between the structure constants and the 3-point functions can be

‡ Such constructions are used for the internal space of the heterotic string, for example.

made explicit by considering the factorization of $\mathcal{F}_N$ in the $[ij]$ channel:

$$\mathcal{F}_N(...i,j...) = \sum_k \mathcal{F}_{N-1}(...k...)c_{ij}{}^k(z_i - z_j)^{\Delta_k - \Delta_i - \Delta_j}.$$

In terms of dual diagrams we have



The structure constants are nothing but the $SL(2,\mathbb{C})$ invariant 3–point functions

$$c_{ij}{}^k = \mathcal{F}_3(j,\overline{k},i)(z_j - z_i)^{[ij,k]}(z_k - z_j)^{[jk,i]}(z_i - z_k)^{[ik,j]},$$

where $[ij,k] = \Delta_i + \Delta_j - \Delta_k$. $SL(2,\mathbb{C})$ invariance can now be used to send $(z_j, z_k, z_i) \to (\infty, 1, 0)$:

$$c_{ij}{}^k = \langle j \mid \phi_{\overline{k}}(1) \mid i \rangle. \tag{2.27}$$

A knowledge of the 3–point functions is therefore enough to determine the operator algebra, and by successive application of the OPE *any* correlation function.

2.5 THE OPERATOR ALGEBRA.

The operator algebra can be written in an equivalent commutator form, as we have already remarked, when exchange duality holds. Exchange duality, between two operators, implies that there exists a meromorphic function $\overline{\phi_i(z)\phi_j(\omega)}$, such that

$$\overline{\phi_i(z)\phi_j(\omega)} = \begin{cases} \phi_i(z)\phi_j(\omega) & \mid z \mid > \mid \omega \mid \\ \phi_j(\omega)\phi_i(z) & \mid \omega \mid > \mid z \mid, \end{cases}$$

i.e. $\overline{\phi_i(z)\phi_j(\omega)} = T\{\phi_i(z)\phi_j(\omega)\}$. We now define the '$\circ$' product [41]

$$\Phi \circ \Psi(\omega) = \oint_\omega \frac{dz}{2\pi i} \overline{\Phi(z)\Psi(\omega)}, \tag{2.28}$$

where Φ and Ψ are arbitrary products of fields and meromorphic functions, and the contour surrounds the point ω, but no other possible poles. We also define the residue

$$\mathrm{Res}(\Phi) = \oint_0 \frac{dz}{2\pi i} \Phi(z),$$

where the contour surrounds $z = 0$, but no other possible poles. Consider the ordered product

$$\mathrm{Res}(\Phi)\mathrm{Res}(\Psi) = \oint_0 \frac{d\omega}{2\pi i} \oint_C \frac{dz}{2\pi i} \overline{\Phi(z)\Psi(\omega)},$$

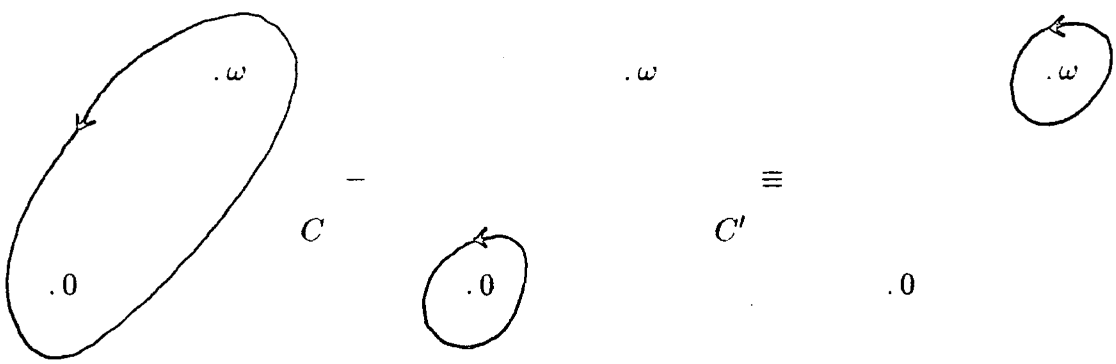
where on C $\mid z \mid > \mid \omega \mid$. The product with opposite ordering is

$$\mathrm{Res}(\Psi)\mathrm{Res}(\Phi) = \oint_0 \frac{d\omega}{2\pi i} \oint_{C'} \frac{dz}{2\pi i} \overline{\Phi(z)\Psi(\omega)},$$

where on C' $\mid z \mid < \mid \omega \mid$. Therefore

$$[\mathrm{Res}(\Phi), \mathrm{Res}(\Psi)] = \oint_0 \frac{d\omega}{2\pi i} \left\{ \oint_C \frac{dz}{2\pi i} - \oint_{C'} \frac{dz}{2\pi i} \right\} \overline{\Phi(z)\Psi(\omega)}.$$

Now the effective difference between the two integrals in curly brackets is an integration around the point $z = \omega$ only; this is the standard 'contour deformation argument' [41,42]:



Therefore the operators have a natural Lie algebra structure:

$$[\mathrm{Res}(\Phi), \mathrm{Res}(\Psi)] = \mathrm{Res}(\Phi \circ \Psi). \tag{2.29}$$

Notice that the antisymmetric nature of the commutator implies

$$\mathrm{Res}(\Phi \circ \Psi + \Psi \circ \Phi) = 0. \tag{2.30}$$

In general we can take the field $\Phi(z)$ to be a linear combination of conformal fields with coefficients in the space of meromorphic functions:

$$\Phi(z) = \sum a_i(z)\phi_i(z).$$

By suitably choosing the $a_i(z)$ we can write the commutator in terms of the moments of $\phi_i(z)$, which were defined in equation (2.7),

$$\phi^i_n = \mathrm{Res}\left(z^{n+\Delta_i-1}\phi_i(z)\right).$$

To calculate $f\phi_i \circ g\phi_j$, we use the OPE

$$f\phi_i \circ g\phi_j(\omega) = \sum_k c_{ij}{}^k g(\omega)\phi_k(\omega) \oint_\omega \frac{dz}{2\pi i} f(z)(z-\omega)^{-[ij,k]}.$$

From this we deduce

$$[\phi^i_n, \phi^j_m] = \sum_k \frac{1}{2}\left\{\binom{n+\Delta_i-1}{[ij,k]-1} - \binom{m+\Delta_j-1}{[ij,k]-1}\right\} c_{ij}{}^k \phi^k_{n+m}, \qquad (2.31)$$

which is equivalent to the operator algebra in the short distance expansion form.

There is a neat way to calculate the operator algebra of a given set of operators in closed form, based on the work of ref.[43], and developed in ref.[44]. The idea is to extend the definition of a field, $\phi_i(z) = \phi(\mid i\rangle, z)$ corresponding to the state $\mid i\rangle$, by linearity

$$\phi(a \mid i\rangle + b \mid j\rangle, z) = a\phi_i(z) + b\phi_j(z).$$

In particular, we can consider operators of the form

$$\phi\left(\phi_i(z) \mid j\rangle, \omega\right).$$

The other fact we need is that

$$e^{zL_{-1}}\phi(\mid i\rangle, 0)e^{-zL_{-1}} = \phi(\mid i\rangle, z),$$

so that

$$\phi(\mid i\rangle, z) \mid 0\rangle = e^{zL_{-1}} \mid i\rangle.$$

Considering the time–ordered product, $|z| > |\omega|$,

$$\begin{aligned}\phi_i(z)\phi_j(\omega) \mid 0\rangle &= \phi_i(z)e^{\omega L_{-1}} \mid j\rangle \\ &= e^{\omega L_{-1}}\phi_i(z-\omega) \mid j\rangle \\ &= \phi\big(\phi_i(z-\omega) \mid j\rangle, \omega\big) \mid 0\rangle,\end{aligned}$$

by linearity. But $\phi_i(z-\omega) = \sum \phi^i_n (z-\omega)^{-n-\Delta_i}$, so the OPE can be expressed as

$$\phi_i(z)\phi_j(\omega) = \sum_n (z-\omega)^{-n-\Delta_i}\phi\big(\phi^i_n \mid j\rangle, \omega\big). \tag{2.32}$$

The sum is well defined because in general there exists an N such that $\phi^i_n \mid j\rangle = 0$ for $n > N$, and the expansion only has a finite number of poles.

2.6 Compactified Closed Strings.

In this section we will consider in detail the motion of a closed string on a torus [45–47]. This exercise will form an important preliminary to orbifold compactification, which we discuss in later chapters.

For the moment we consider compactifications on tori formed by quotienting flat space by some d dimensional lattice Λ:

$$T = \frac{\mathbf{R}^d}{2\pi\Lambda}.$$

Later we will generalize the discussion to **asymmetric tori**. The solution to the classical equations of motion for closed strings is

$$X_{closed}(\sigma,\tau) = X + \frac{1}{2}P\tau + L\sigma + \frac{i}{2}\sum_{\substack{n\in\mathbf{Z}\\ \neq 0}}\left\{\frac{\alpha_n}{n}e^{in(\tau-\sigma)} + \frac{\overline{\alpha}_n}{n}e^{in(\tau+\sigma)}\right\}, \tag{2.33}$$

where the winding vector $L \in \Lambda$, to ensure that the string is *indeed* closed on T. The zero–modes present us with a subtlety; we must introduce a variable Q canonically conjugate to L, which does not appear in the mode expansion. After quantization the zero–modes satisfy the following non–zero commutation relations

$$[X, P] = i, \quad [Q, L] = i.$$

When constructing the Hilbert space we must ensure that it is invariant under the lift of the abelian group Λ, generated by

$$e^{2\pi i\lambda.P}, \quad \lambda \in \Lambda.$$

This constrains the eigenvalues of P to lie on the dual lattice Λ^*, *i.e.*

$$P \in \Lambda^*, \quad L \in \Lambda.$$

In this situation the analytic and anti–analytic, or left and right moving, sectors completely decouple, including the zero–modes:

$$X_{closed}(z, \overline{z}) = \frac{1}{2}(X(z) + \overline{X}(\overline{z})), \tag{2.34}$$

where

$$\begin{aligned} X(z) &= q - ip\ln z + i\sum \frac{\alpha_n}{n} z^{-n}, \\ \overline{X}(\overline{z}) &= \overline{q} - i\overline{p}\ln \overline{z} + i\sum \frac{\overline{\alpha}_n}{n}\overline{z}^{-n}. \end{aligned} \tag{2.35}$$

In the above $q = X + \frac{1}{2}Q$, $\overline{q} = X - \frac{1}{2}Q$, $p = \frac{1}{2}P + L$ and $\overline{p} = \frac{1}{2}P - L$, so that *all* barred operators commute with all unbarred operators. The vertex operators are also completely decoupled [45]. For example, the operator wich represents the absorption of a state with momentum k and winding number l is

$$V(\alpha, \overline{\alpha}, z, \overline{z}) = : e^{i\alpha.X(z)} :: e^{\overline{\alpha}.\overline{X}(\overline{z})} : C(\alpha, \overline{\alpha}),$$

where $\alpha = \frac{1}{2}k + l$ and $\overline{\alpha} = \frac{1}{2}k - l$. $\hat{C}(\alpha, \overline{\alpha}) = e^{i\alpha.q + i\overline{\alpha}.\overline{q}}C(\alpha, \overline{\alpha})$ is an operator on the zero–mode space which must be included to ensure exchange duality. These operators are variously known as **operator cocycles** or **generalized Klein factors** in the literature.

We will adopt the former terminology. By interchanging two vertex operators with momenta and winding numbers (k, l) and $(k'. l')$ (or $(\alpha, \overline{\alpha})$ and $(\beta, \overline{\beta})$) we deduce that

$$\hat{C}(\alpha, \overline{\alpha})\hat{C}(\beta, \overline{\beta}) = (-)^{\alpha.\beta - \overline{\alpha}.\overline{\beta}}\hat{C}(\beta, \overline{\beta})\hat{C}(\alpha, \overline{\alpha}). \tag{2.36}$$

With the hindsight of Narain [31] we can see that it is advantageous to introduce the Lorentzian lattice Γ:

$$\Gamma \equiv \left\{ A = \left(\frac{1}{2}k + l, \frac{1}{2}k - l\right) \mid k \in \Lambda^*,\ l \in \Lambda \right\},$$

of dimension $2d$ and metric $\mathrm{diag}((+)^d, (-)^d)$. Γ is in fact an even lattice, since for $A = (\alpha, \overline{\alpha}) \in \Gamma$

$$A.A = \alpha^2 - \overline{\alpha}^2 = 2k.l \in 2\mathbb{Z}.$$

It is also integral

$$A.B = \alpha.\beta - \overline{\alpha}.\overline{\beta} = k.l' + k'.l \in \mathbb{Z},$$

futhermore it can be shown to be self–dual. The operator cocycles can now be written in terms of the vectors in Γ. Equation (2.36) becomes

$$\hat{C}(A)\hat{C}(B) = (-)^{A.B}\hat{C}(B)\hat{C}(A).$$

At this point we notice that since the left and right movers are completely decoupled there is no a priori reason why they should not move on different tori, with not even necessarily the same dimension, *i.e.* we can take $\Gamma = (\Lambda_L, \Lambda_R)$. This is essentially the generalization considered in ref.[31]. A stringy interpretation is in some senses lost, however. we do not need to consider these tori as being 'real', we view them as simply being the internal degrees of freedom of the lower dimensional uncompactified theory. We could go to the extreme and consider just the left–movers or analytic sector in isolation. This case is discussed in the next section because it forms an integral part of the heterotic string and leads to the Frenkel–Kac–Segal mechanism.

Narain showed that modular invariance requires Γ to be an even self–dual lattice with respect to the Lorentzian metric. It is important to realize that the Lorentzian character of Γ has nothing to do with the Euclidean space underlying the construction, it is purely a convenience. These toroidal models can also be coupled to a background antisymmetric tensor field, which can be interpreted as rotations of the lattice [46.47].

2.7 The Frenkel–Kac–Segal mechanism.

For a purely analytic model the Hilbert space has a decomposition

$$\mathcal{H} = \mathcal{F} \otimes \mathcal{P}(\Lambda), \tag{2.37}$$

where $\mathcal{F}$ is the Fock space spanned by the identity and the operators α_{-n}, $n > 0$. and $\mathcal{P}(\Lambda)$ is the complex span of momentum eigenstates on the lattice Λ. If the operator algebra of the model is to be local, that is having no branch point singularities, then Λ must be an **even lattice**, this implies that $\Lambda_2 \subset \Lambda$, the vectors of length squared two, is the root system of a simply–laced Lie algebra, g.

L_0 grades $\mathcal{H}$:

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}[n], \tag{2.38}$$

where the grade of a state is

$$\Delta = \frac{1}{2}\alpha^2 + N.$$

Here α is the momentum of the state and N is its occupation number. The even condition on the lattice Λ ensures that states have integral conformal weight. which implies that the operator algebra is local.

The vertex operator corresponding to the state

$$\mid \phi\rangle = \sum_{i=1}^{N} \epsilon_i.\alpha_{-n_i} \mid \alpha\rangle,$$

is

$$V(\phi, z) =: \prod_{i=1}^{N} \frac{i\epsilon_i \partial^{n_i} X(z)}{(n_i - 1)!} e^{i\alpha.X(z)} : C(\alpha). \tag{2.39}$$

Notice that this differs from (2.19) by the addition of the **operator cocycle** factor $C(\alpha)$. If we define $\hat{C}(\alpha) = e^{i\alpha.q}C(\alpha)$, then to ensure exchange duality we require

$$\hat{C}(\alpha)\hat{C}(\beta) = \Omega(\alpha, \beta)\hat{C}(\beta)\hat{C}(\alpha), \tag{2.40}$$

where the *symmetry factor* is

$$\Omega(\alpha, \beta) = (-)^{\alpha.\beta}. \tag{2.41}$$

In order that the operator algebra closes, the operators $\hat{C}(\alpha)$ must also generate a projective representation of Λ:

$$\hat{C}(\alpha)\hat{C}(\beta) = \varepsilon(\alpha,\beta)\hat{C}(\alpha+\beta), \tag{2.42}$$

where the factor set ε satisfies

$$\frac{\varepsilon(\alpha,\beta)}{\varepsilon(\beta,\alpha)} = \Omega(\alpha,\beta). \tag{2.43}$$

The associativity of the operator algebra implies that ε forms a 2–cocycle map ε : $\Lambda \times \Lambda \rightarrow \{\pm 1\}$:

$$\varepsilon(\alpha,\beta+\gamma)\varepsilon(\beta,\gamma) = \varepsilon(\alpha,\beta)\varepsilon(\alpha+\beta,\gamma),$$

which is the cocycle corresponding to the central extension of Λ by $\mathbb{Z}_2 \simeq \{\pm 1\}$:

$$1 \longrightarrow \{\pm 1\} \longrightarrow \hat{\Lambda} \longrightarrow \Lambda \longrightarrow 1,$$

where an element $\hat{\alpha} \in \hat{\Lambda}$ is (ζ,α), $\zeta \in \{\pm 1\}$, and $\hat{\Lambda}$ has a product

$$\hat{\alpha} \circ \hat{\alpha}' = (\zeta\zeta'\varepsilon(\alpha,\alpha'),\alpha+\alpha').$$

$c_{\alpha\beta}{}^{\gamma} = \varepsilon(\alpha,\beta)\delta_{\alpha+\beta,\gamma}$ is effectively the structure constant of the operator algebra of the momentum fields.

It is not necessary to increase the size of the Hilbert space to accommodate the operators $\hat{C}(\alpha)$, they have a natural action on $\mathcal{P}(\Lambda)$ [13,14,48]:

$$\hat{C}(\alpha) : \ |\beta\rangle \longmapsto \varepsilon(\alpha,\beta)\,|\alpha+\beta\rangle\,,$$

realized by

$$\hat{C}(\alpha) = \sum_{\beta\in\Lambda} \varepsilon(\alpha,\beta)\mid \alpha+\beta\rangle\langle\beta\mid. \tag{2.44}$$

The operator algebra is a map

$$\mathrm{W}_n \times \mathrm{W}_m \longrightarrow \mathrm{W}_{n+m-1},$$

where W_n is the space of conformal fields of dimension *at most* n. Notice immediately that a closed subalgebra is generated by W_1, the operators of dimension 0 and 1. For the analytic string these are

$$\begin{cases} V(\alpha, z) & \alpha \in \Lambda_2 \\ i\epsilon.\partial X(z) & \\ 1, & \text{(the unit operator).} \end{cases}$$

Here, Λ_p is set of vectors defined by

$$\Lambda_p \equiv \{\alpha \in \Lambda \mid \alpha^2 = p\}. \tag{2.45}$$

The operator algebra of these operators is easily deduced,

$$\begin{aligned} V(\alpha, z)V(\beta, \omega) &= \begin{cases} \frac{\varepsilon(\alpha,\beta)}{z-\omega} V(\alpha+\beta, \omega) & \alpha.\beta = -1 \\ \frac{1}{(z-\omega)^2}(1 + (z-\omega)i\alpha.\partial X(\omega)) & \alpha + \beta = 0 \\ 0 & \text{otherwise} \end{cases} \\ i\epsilon.\partial X(z)V(\alpha, \omega) &= \frac{\epsilon.\alpha}{z-\omega} V(\alpha, \omega) \\ i\epsilon.\partial X(z) i\eta.\partial X(\omega) &= \frac{\epsilon.\eta}{(z-\omega)^2}, \end{aligned} \tag{2.46}$$

modulo regular terms. We have taken $\varepsilon(\alpha, -\alpha) = 1$. This algebra is in fact the **affinization**, $\hat{g}$, of the finite Lie algebra g, the algebra whose root system is Λ_2. The affinization of a finite Lie algebra is the central extension of the **loop algebra** of g. In the algebra above the unit operator plays the rôle of the extension. $\hat{g}$ is in fact the **untwisted Kac–Moody algebra** associated with g, which is usually denoted $g^{(1)}$ to distinguish it from the other **twisted algebras**, $g^{(\tau)}$ $\tau = 2, 3$, which can exist for certain g [49]. $\hat{g}$ is more recognizable in its commutator form:

$$\begin{aligned} [V_n(\alpha), V_m(\beta)] &= \begin{cases} \varepsilon(\alpha, \beta)V_{n+m}(\alpha+\beta) & \alpha.\beta = -1 \\ (\alpha.\alpha_{n+m} + n\delta_{n+m,0}) & \alpha + \beta = 0 \\ 0 & \text{otherwise} \end{cases} \\ [\epsilon.\alpha_n, V_m(\alpha)] &= \epsilon.\alpha V_{n+m}(\alpha) \\ [\epsilon.\alpha_n, \eta.\alpha_m] &= (\epsilon.\eta)n\delta_{n+m,0}. \end{aligned} \tag{2.47}$$

With L_0 playing the rôle of the **derivation** this corresponds to the **homogeneous gradation** of $g^{(1)}$ [49]. In this case the **horizontal algebra**, the subalgebra commuting with the derivation, is simply g itself. The structure constants of the operator algebra, restricted to $\Lambda_2 \times \Lambda_2$, are identified with the structure constants of g.

$\mathcal{H} = \mathcal{F} \otimes \mathcal{P}(\Lambda)$ is the level-one basic $g^{(1)}$-module. We would have obtained the other possible level-one $g^{(1)}$-modules by taking the momenta of the zero-mode space to lie in a different coset of Λ^*/Λ, *i.e.*

$$\mathcal{H}^\lambda = \mathcal{F} \otimes \mathcal{P}(\Lambda + \lambda), \tag{2.48}$$

where λ is a minimal weight of g. The number of inequivalent level-one modules is equal to the order of the centre of g [48,49]. The character of the module $\mathcal{H}^\lambda$ is:

$$\chi^\lambda(q) = q^{-d/24} \frac{\sum\limits_{\alpha \in \Lambda} q^{\frac{1}{2}(\alpha+\lambda)^2}}{\prod\limits_{n=1}^{\infty}(1 - q^n)^d}, \tag{2.49}$$

where the factor $q^{-d/24}$ is conventional.

$\mathcal{H}[1]$ is an adjoint representation of g in the sense that there exists a map $g \rightarrow \mathcal{H}[1]$, given by

$$x \in g \longmapsto | x\rangle = V_1(x) \mid 0\rangle,$$

and the following action of g on $\mathcal{H}[1]$:

$$V_0(y) \mid x\rangle = | [x, y]\rangle.$$

Vertex operator representations of Kac–Moody algebras have been extensively studied in the literature, for example they are discussed in refs.[42,48–50]. Their appearance in string theory was precipitated by the discovery of the heterotic string [11,12], where the FKS mechanism is responsible for the appearance of non-abelian gauge symmetries. The FKS mechanism has now also been extended to non-simply laced Lie algebras, using fermionic fields [51].

2.8 The Cross–Bracket Algebra and its Generalizations.

It is pertinent to investigate whether there exist other algebras which close on higher dimension operators. The problem is that the operator algebra as it stands maps

$$\mathrm{W}_n \times \mathrm{W}_m \longrightarrow \mathrm{W}_{n+m-1},$$

and so when $n, m > 1$ the operator algebra does not close. Consider for the moment, the case when $n = m = 2$. In order to cancel the operators of dimension 3 Frenkel, Lepowsky and Meurman (FLM) [26–28,52], were led to consider the **cross–bracket algebra**, which we define via the **cross–product**:

$$(A \times B)(\omega) = \frac{1}{2}\left\{(zA)\circ(\omega^{-1}B) - (z^{-1}A)\circ(\omega B)\right\}, \tag{2.50}$$

for A and $B \in \mathrm{W}_2$, *i.e.* for operators of the form

$$\sum_{\Delta_\phi \leq 2} a_i(z)\phi_i(z),$$

where the $a_i(z)$ are meromorphic functions. This product closes on W_2. To see this, take $A = f(z)\phi(z)$ and $B = g(z)\psi(z)$, where ϕ and ψ are conformal fields of dimension 2. We compute

$$\begin{aligned} A \times B(\omega) =& \frac{1}{12}\left(g\omega^{-1}\partial^3(f\omega) - g\omega\partial^3(f\omega^{-1})\right)\Theta^0(\omega) \\ &+ \frac{1}{4}\left(g\omega^{-1}\partial^2(f\omega) - g\omega\partial^2(f\omega^{-1})\right)\Theta^1(\omega) \\ &+ \frac{1}{2}\left(g\omega^{-1}\partial(f\omega) - g\omega\partial(f\omega^{-1})\right)\Theta^2(\omega) \\ &+ \frac{1}{2}(gf - fg)\Theta^3(\omega), \end{aligned}$$

for generic fields Θ^0, Θ^1, Θ^2 and Θ^3, of dimension 0, 1, 2 and 3, respectively. Notice that the dimension three piece is cancelled in this product. We can now define the commutative, non–associative, **cross-bracket algebra**:

$$[\mathrm{Res}(A) \times \mathrm{Res}(B)] = \mathrm{Res}(A \times B). \tag{2.51}$$

Commutativity follows from

$$\mathrm{Res}(A \times B - B \times A) = 0,$$

due to equation (2.30).

A further generalization of this is possible. For A and B in W_n, we define the n**–cross–product** by expressing it in terms of the $(n-1)$–product

$$A \times_n B(\omega) = \frac{1}{2}\left\{(zA)\times_{n-1}(\omega^{-1}B) - (z^{-1}A)\times_{n-1}(\omega B)\right\}. \tag{2.52}$$

This product maps $\mathrm{W}_n \times \mathrm{W}_n \to \mathrm{W}_n$. From this we construct the following algebras,

H_n:

$$[\mathrm{Res}(A) \times_n \mathrm{Res}(B)] = \mathrm{Res}(A \times_n B), \qquad A,\ B \in \mathrm{W}_n. \tag{2.53}$$

H_1 is the familiar Lie algebra, while H_2 is the cross–bracket algebra. For general n, H_n is antisymmetric for n odd and symmetric for n even.

2.9 The Leech Lattice and Monstrous Moonshine.

The cross–bracket algebra seems to be particularly relevant to the Leech lattice, which is the unique Euclidean even self–dual lattice in $d = 24$ for which Λ_2 is empty. The significance of this is that the cross–bracket algebra, which we call $\hat{S}$, is spanned by the dimension 2 operators and the unit operator

$$\begin{cases} V(\alpha, z), & \alpha \in \Lambda_4 \\ V(\epsilon, \eta, z) = -:\epsilon.\partial X(z)\eta.\partial X(z): & \\ i\mu.\partial^2 X(z) & \\ 1. & \end{cases}$$

As it stands we must also include the dimension one operators $i\mu.\partial X(z)$ in order that the algebra closes. If we ignore these dimension one operators then $\hat{S}$ can be viewed as the *affinization* of the finite algebra S; in analogy with the Lie algebra case. Later we shall show how these awkward dimension one operators are removed. The $\hat{S}$–module $\mathcal{H} = \mathcal{F} \otimes \mathcal{P}(\Lambda^L)$ has character

$$\chi(q) = q^{-1} \frac{\sum\limits_{\alpha \in \Lambda^L} q^{\frac{1}{2}\alpha^2}}{\prod\limits_{n=1}^{\infty} (1 - q^n)^{24}}.$$

This character is related to the famous normalized modular function $J(q)$:

$$\chi(q) = J(q) + 24.$$

Notice that the existence of dimension one operators is directly related to the mismatch between $\chi(q)$ and $J(q)$. It was pointed out by McKay that the coefficients of $J(q)$ are in fact simple linear combinations of dimensions of the irreducible representations of the Fischer–Griess Monster group, F_1, the largest **sporadic group** in the classification of finite groups. This led to the speculation that there exists an infinite dimensional

$\mathbb{Z}$–graded F_1–module with character $J(q)$. This conjecture was given even more weight by Conway and Norton [53], in their paper 'Monstrous Moonshine', and was elucidated when FLM actually constructed the natural module for F_1. Their construction is based on the cross–bracket algeba $\hat{S}$, in a way we will explain more fully in chapter 7.

3. STRINGS ON ORBIFOLDS.

For our purposes, an orbifold is constructed by taking a manifold and quotienting it by some discrete symmetry group which does not act freely, *i.e.* the orbifold has a set of points at which its curvature is ill-defined.

Orbifolds were first introduced into string theory as an alternative to Calabi–Yau manifolds, for the compactification of the heterotic string [1,2]. Now they stand in their own right as consistent background spaces for string propagation. More generally, they give rise to a whole class of solvable conformal field theories. The question which we pose in this and the following chapters is: what is the structure of the conformal field theory and, in particular, what are the vertex operators for strings propagating on orbifolds? These questions have been partially answered in refs.[33,34], where a path integral approach was followed. On the contrary, we will take an operator approach (also considered in refs.[55,56]), because this picture is particularly appropriate for discussing the symmetries and algebras which result from orbifold models. One of the crucial features of string propagation on orbifolds is the existence of the twisted sectors, *i.e.* strings which wind around the curvature singularities. These sectors ensure the consistency of theory and they couple in a non-trivial way to the normal (untwisted) sector; we will discuss these special string states in detail. At the fundamental level, the consistency of such theories is questionable, based as they are on such singular objects. However, perturbatively at least, the string seems to be able to propagate quite consistently on orbifolds.

From a mathematical point of view, twisted strings have turned up in generalizations of vertex operator representations of Kac–Moody algebras [3,24]. We will show how such algebras arise from considering the operator algebra of vertex operators for strings on orbifolds. This will involve the vertex which inserts untwisted states. In later chapters we will also construct the vertices that insert twisted states and consider their algebra.

In this chapter we shall be concerned with just the analytic degrees of freedom of the string. We shall briefly consider orbifold models which have both analytic and anti-analytic degrees of freedom in chapter 9.

3.1 ORBIFOLDS.

The orbifolds that we will consider have the form

$$\Omega = \frac{\mathbb{R}^d}{G}, \tag{3.1}$$

where G is some **space group** with the semi–direct product form

$$G = (2\pi\Lambda) \times W.$$

In the above, Λ is a lattice and W, the 'twist', is a point group which must have a well defined action on Λ; *i.e.* W is a subgroup of the automorphism group of Λ which leaves the origin fixed.

An element $g = (\alpha, u) \in G$, $\alpha \in \Lambda$ $u \in W$, has the following action on a vector $x \in \mathbb{R}^d$:

$$g(x) = u(x) + 2\pi\alpha.$$

To form the orbifold we now *identify* all points in $\mathbb{R}^d$ that lie on an orbit of G. Equivalently, and more suitably for our exposition, we may construct the orbifold by identifying all points on the torus

$$T = \frac{\mathbb{R}^d}{2\pi\Lambda}, \tag{3.2}$$

that lie in an orbit of W,

$$\Omega = \frac{T}{W}. \tag{3.3}$$

In general W does not act **freely** on T, in other words, there will be points fixed by the action of an element of W. At these points the curvature of Ω is ill–defined. Defining the set

$$M_u = \{x \in \mathbb{R}^d \mid (1-u)x \in 2\pi\Lambda\},$$

allows us to define the set of fixed point singularities as

$$\overline{M}_u = \frac{M_u}{2\pi\Lambda}. \tag{3.4}$$

Let us consider this in more detail. If x corresponds to a fixed point then

$$(1 - u)x = 2\pi\alpha, \qquad \alpha \in \Lambda.$$

Next we define the set

$$K_u \equiv (1 - P_u)\mathbb{R}^d \cap \Lambda, \tag{3.5}$$

where P_u is the projection operator onto the u–invariant subspace:

$$P_u = \frac{1}{N_u} \sum_{q=0}^{N_u - 1} u^q, \tag{3.6}$$

(N_u is the order of u). We see that since

$$(1 - P_u)(1 - u)x = (1 - u)x,$$

implies $\alpha \in K_u$ and

$$M_u = 2\pi(1 - u_*)^{-1}K_u + P_u(\mathbb{R}^d).$$

Here, u_* means the restriction of u to K_u, that is $u_* = (1 - P_u)u$. Therefore, in general the fixed point singularities are $\dim(P_u(\mathbb{R}^d))$ dimensional and there are

$$F_u = \left| \frac{(1 - u_*)^{-1}K_u}{\Lambda} \right|$$

of them. This is in fact equal to [4,32]

$$F_u = \left| \frac{K_u}{(1 - u)\Lambda} \right| = \det(1 - u_*). \tag{3.7}$$

In what follows we will sometimes be concerned with the situation where each $u \in W$ ($u \neq e$, the identity) leaves no directions fixed, *i.e.* $\det(1 - u) \neq 0$. We shall adopt the convention of [4], and call these cases **No Fixed Point Automorphisms** (NFPAs). These fixed points should *not* be confused with the fixed point singularities of the orbifold itself.

3.2 The Classical String on an Orbifold.

The possible boundary conditions of a closed string moving on the covering space T are

$$X_u(ze^{2\pi i}) = uX_u(z) \mod \Lambda, \quad u \in W. \tag{3.8}$$

To discuss the solutions to the classical equations of motion it is expedient to introduce the complexification V of $\mathbb{R}^d$, since we can diagonalize u in V:

$$u = \mathrm{diag}\left(e^{2\pi i\lambda_1}, \ldots.., e^{2\pi i\lambda_d}\right).$$

In general $0 \leq \lambda_i < 1$ and if $\lambda \neq 0$ or $\frac{1}{2}$, then the $e^{2\pi i\lambda_i}$'s come in conjugate pairs. If u has order N_u then we can decompose V into the following eigenspaces

$$V = \bigoplus_{q=0}^{N_u-1} V_{q/N_u},$$

where

$$V_\lambda \equiv \left\{x \in V \mid u(x) = e^{2\pi i\lambda}x\right\}.$$

To go along with each eigenspace V_λ we introduce a set of basis vectors $\{e^a_\lambda\}$, $a = 1, \ldots., \dim(V_\lambda)$, such that

$$e^a_\lambda . e^b_\mu = \begin{cases} \delta^{ab} & \lambda + \mu \in \mathbb{Z} \\ 0 & \text{otherwise}. \end{cases}$$

In the above basis we can easily solve the classical equations of motion. Splitting the string field $X_u(z)$ into its zero-mode and oscillator pieces

$$X_u(z) = X^0_u(z) + X^\pm_u(z),$$

we have

$$\begin{aligned} X^0_u(z) &= q^u - ip^u \ln z \\ e^a_\lambda . X^\pm_u &= i \sum_{\substack{r \in \mathbb{Z}+\lambda \\ \neq 0}} \frac{(\alpha^u_r)^a}{r} z^{-r}. \end{aligned} \tag{3.9}$$

The boundary condition (3.8) requires

$$\begin{aligned} (1-u)q^u &= 2\pi(p^u + \alpha), \quad \alpha \in \Lambda \\ (1-u)p^u &= 0. \end{aligned} \tag{3.10}$$

First of all consider the allowed momenta. Recall for toroidal compactifications that we had the constraint $p \in \Lambda^*$, which was required so that the states were invariant under $\exp(2\pi i \alpha . p)$, $\alpha \in \Lambda$. Define the subspace of vectors invariant under u:

$$I_u \equiv \{\alpha \in \Lambda | (1-u)\alpha = 0\}. \tag{3.11}$$

This subspace is not affected by the 'twisting' so we demand that the states are invariant under $\exp(2\pi i \alpha . p^u)$, $\alpha \in I_u$. This implies that the eigenvalues of p^u lie in $I_u^* = P_u(\Lambda^*)$. We now choose a particular coset, generalizing what we did for untwisted analytic strings:

$$p^u \in \Lambda^u, \tag{3.12}$$

where Λ^u is the projected lattice:

$$\Lambda^u \equiv P_u(\Lambda) = \left\{ \gamma \mid \gamma = \frac{1}{N_u} \sum_{q=0}^{N_u - 1} u^q \alpha, \quad \alpha \in \Lambda \right\}. \tag{3.13}$$

For a NFPA $p^u = 0$. Now consider the other boundary condition. Let $q^u = q_0^u + q_1^u$, where $q_0^u = P_u(q^u)$. Clearly q_0^u is not constrained, *i.e.* $q_0^u \in P_u(\mathbb{R}^d)$. q_1^u satisfies

$$(1-u)q_1^u = 2\pi(\alpha + p^u).$$

Therefore

$$\alpha + p^u \in (1 - P_u)\mathbb{R}^d \cap (\Lambda + p^u),$$

and we deduce that

$$q^u \in \frac{2\pi(1-u_*)^{-1}\left((1-P_u)\mathbb{R}^d \cap (\Lambda + p^u)\right)}{\Lambda} + P_u\left(\mathbb{R}^d\right). \tag{3.14}$$

Comparing this with our discussion of the fixed point singularities of the orbifold shows that there is an inequivalent q^u for each singularity.

3.3 The Quantum String on an Orbifold.

The first task in setting up the quantum theory is to construct the Hilbert space $\mathcal{H}_\Omega$. This involves a number of steps [2].

(1). The Hilbert space, $\mathcal{H}_T$, for the theory on the covering space T is constructed. In constructing $\mathcal{H}_T$ we must allow the string to have the non-trivial boundary conditions of equation (3.8). because such **twisted strings** are closed on Ω. This gives rise to the **twisted sectors** and we write

$$\mathcal{H}_T = \bigoplus_{u \in W} \mathcal{H}_u. \tag{3.15}$$

$\mathcal{H}_e$ is conventionally known as the untwisted sector: we call $\mathcal{H}_u$ the u–twisted sector.

(2). Construct a lift of W into the Hilbert space $\mathcal{H}_T$; we call it $\hat{W}$.

(3). Finally we identify $\mathcal{H}_\Omega$ as the $\hat{W}$ invariant subspace of $\mathcal{H}_T$:

$$\mathcal{H}_\Omega = \overline{\mathcal{H}}_T = \frac{1}{N} \sum_{\hat{u} \in \hat{W}} \hat{u}\,(\mathcal{H}_T). \tag{3.16}$$

Here N is the order of $\hat{W}$ †.

If W is non–abelian then the action of $\hat{W}$ will in general mix states in sectors corresponding to conjugate elements [1,2]. For example, under the action of $v \in W$ a u–twisted string becomes a vuv^{-1}–twisted string:

$$\begin{aligned} vX_u(ze^{2\pi i}) &= vuX_u(z) \\ &= (vuv^{-1})vX_u(z). \end{aligned}$$

This implies

$$\hat{v} : \ \mathcal{H}_u \longrightarrow \mathcal{H}_{vuv^{-1}}. \tag{3.17}$$

Notice that if v lies in the centralizer (or little group) of u, that is the subgroup

$$G_u \equiv \{v \in W \mid [u, v] = 0\},$$

then $\hat{v} : \mathcal{H}_u \to \mathcal{H}_u$.

† When elements of W leave subspaces of T fixed, it is possible that the order of $\hat{W}$ is twice the order of W [3.54].

Before we go on to describe the $\hat{W}$ projection let us make the following definitions:

1. Let $\{u_i\}$, $i = 1, ..., M$, be the set of conjugacy classes of W, each represented by an element u_i.
2. Consider the right cosets W/G_{u_i}, and label representatives of these cosets v_a^i, $a = 1, ..., m_i$.

To realize the $\hat{W}$ projection we must first project each $\mathcal{H}_{u_i}$ onto its $\hat{G}_{u_i}$ invariant subspace $\widetilde{\mathcal{H}}_{u_i}$, and then sum over all states in different $\widetilde{\mathcal{H}}_v$, with v conjugate to u_i. By this we mean form

$$\overline{\mathcal{H}}_{u_i} = \bigoplus_{a=1}^{m_i} \hat{v}_a^i \left(\widetilde{\mathcal{H}}_{u_i}\right). \tag{3.18}$$

Finally we have

$$\mathcal{H}_\Omega = \bigoplus_{i=1}^{M} \overline{\mathcal{H}}_{u_i}. \tag{3.19}$$

So in this sense there is a sector for each conjugacy class of W.

The identity element of W, e, is in a conjugacy class of its own, therefore

$$\overline{\mathcal{H}}_e = \frac{1}{N} \sum_{\hat{u} \in \hat{W}} \hat{u}\,(\mathcal{H}_e).$$

If W is abelian then every element is in a conjugacy class of its own, in this case

$$\mathcal{H}_\Omega = \bigoplus_{u \in W} \overline{\mathcal{H}}_u.$$

After quantization, the string field $X_u(z)$ becomes an operator on the Hilbert space $\mathcal{H}_u$, and canonical quantization gives the following non-zero commutation relations for the oscillators,

$$[(\alpha_r^u)^a, (\alpha_s^u)^b] = r\delta^{ab}\delta_{r+s,0}, \tag{3.20}$$

where $r, s \in \mathbb{Z} + \lambda, \mathbb{Z} - \lambda$, respectively, and $a, b = 1, ..., \dim(V_\lambda)$. The hermiticity of the oscillators is $(\alpha_r^u)^\dagger = \alpha_{-r}^u$.

The Hilbert subspaces have a decomposition

$$\mathcal{H}_u = \mathcal{F}_u \otimes \mathcal{V}_u, \tag{3.21}$$

where $\mathcal{F}_u$ is the Fock space spanned by the unit operator and the creation operators α_r^u, $r < 0$, and $\mathcal{V}_u$ is the zero–mode space. $\mathcal{V}_u$ has a further decomposition

$$\mathcal{V}_u = \mathcal{P}(\Lambda^u) \otimes \mathcal{V}_u^0, \tag{3.22}$$

where $\mathcal{P}(\Lambda^u)$ is a complex span of momentum states which take values in the projected lattice Λ^u. and $\mathcal{V}_u^0$ is a finite dimensional vector space. One would guess that $\mathcal{V}_u^0$ is a space of position eigenstates corresponding to the fixed point singularities of Ω. In fact we shall discover that this guess is naive and $\mathcal{V}_u^0$ should properly be understood as the representation space for the twisted operator cocycles; in this sense the space of fixed points is reducible. However, for models which include both left and right (analytic and anti–analytic) degrees of freedom the zero–mode space can indeed be identified with the fixed point singularities. We expand upon this point in chapter 9.

To make the connection with the path integral formulation we calculate the propagator in the u–twisted sector:

$$\begin{aligned}
&{}_u\langle 0 \mid e^a_{1-\lambda}.\partial X_u(z) e^b_\lambda.\partial X_u(\omega)] \mid 0\rangle_u \\
&\qquad = -\, z^{-(1-\lambda)}\omega^{-\lambda}\frac{[(1-\lambda)z - \lambda\omega]}{(z-\omega)^2}\delta^{ab} \\
&\qquad = -\left\{\frac{1}{(z-\omega)^2} + \frac{1}{2}\lambda(1-\lambda)z^{-2} + O(z-\omega)\right\}\delta^{ab}.
\end{aligned} \tag{3.23}$$

Notice that the twisted propagator is *not* the same as the untwisted propagator: they differ by a term which is finite as $z \to \omega$. This fact will have important consequences for the conformal structure of twisted strings, as we shall see in the next section.

3.4 Conformal Properties of Twisted Strings.

To determine the physical states and to form vertex operators, and hence scattering amplitudes, we must elucidate the conformal structure of twisted strings. As before we shall focus on the analytic sector of the string.

Initially it is convenient to work in the covering space T. In view of this it is advantageous to introduce some notation to handle operators on $\mathcal{H}_T$. Let $\{E_u\}$, $u \in W$, be a basis for $\mathcal{H}_T$, with inner product $(E_u, E_v) = \delta_{u,v}$, so that any vector, $\mid \Phi\rangle \in \mathcal{H}_T$, can be written

$$\mid \Phi\rangle = \sum_u \mid \Phi\rangle_u E_u. \qquad \mid \Phi\rangle_u \in \mathcal{H}_u.$$

It is also convenient to introduce the matrices I_{uv}, with a 'one' in position uv and zeros elsewhere, so that

$$E_u = I_{uv} E_v \qquad \text{(no summation)}.$$

The stress–energy tensor $\mathbb{T}(z)$ is diagonal

$$\mathbb{T}(z) = \sum_u T^u(z) I_{uu},$$

where the components have the canonical form

$$T^u(z) = -\frac{1}{2} \lim_{\omega \to z} \left\{ \partial X_u(z).\partial X_u(\omega) + \frac{d}{(z-\omega)^2} \right\}. \tag{3.24}$$

It is important to realize that the normal ordering in the above definition is *not* the same as normal ordering with respect to the twisted oscillators α_r^u. The difference is a finite term which showed up earlier in the propagator, equation (3.23). This has the crucial consequence that

$$_u\langle 0 \mid T^u(z) \mid 0\rangle_u = \Theta^u z^{-2}, \tag{3.25}$$

where

$$\Theta^u = \frac{1}{4} \sum_{i=1}^{d} \lambda_i (1 - \lambda_i).$$

This means that the ground state in the u–twisted sector has conformal weight Θ^u [57], *i.e.* it is *not* an $SL(2,\mathbb{C})$ singlet.

$\mathbb{T}(z)$ generates a canonical copy of the Virasoro algebra, with central charge $c = d$. In OPE language:

$$\mathbb{T}(z)\mathbb{T}(\omega) = \frac{c/2}{(z-\omega)^4} + \frac{2\mathbb{T}(\omega)}{(z-\omega)^2} + \frac{\partial \mathbb{T}(\omega)}{z-\omega} + Reg.$$

Many of the results concerning conformal field theories are valid here. However, we shall discover that for certain models the conformal fields have certain pathological problems. This will become clear in the examples that are discussed in later chapters. For the moment we will ignore such subtleties and consider only general features.

L_0^u introduces a natural gradation on the Hilbert space $\mathcal{H}_u$:

$$\mathcal{H}_u = \bigoplus_{n \in \mathbb{Z}/N_u + \Theta^u} \mathcal{H}_u[n]. \tag{3.26}$$

The character, or partition function, defined in equation (2.21), can easily be computed [58]:

$$\chi_u(q) = q^{\Theta^u - d/24} \frac{\sum_{\gamma \in \Lambda^u} q^{\gamma^2/2}}{\prod_{n=1}^{\infty} (1 - q^{\frac{n}{N_u}})^{d_n}}, \tag{3.27}$$

where $d_n = \dim\left(V_{(n/N_u \bmod N_u)}\right)$.

The canonical $SL(2, \mathbb{C})$ invariant vacuum state (the state annihilated by $\mathbb{L}_n$, $n \geq -1$) is

$$\mid \mathbb{0}\rangle = \mid 0\rangle_e E_e. \tag{3.28}$$

For each state $\mid \Phi\rangle \in \mathcal{H}_T$ we introduce a conformal field, or vertex operator, such that

$$\mid \Phi\rangle = \mathbb{V}(\Phi, 0) \mid \mathbb{0}\rangle. \tag{3.29}$$

The vertex operator $\mathbb{V}(\Phi, z)$ is interpreted as the insertion of the state $\mid \Phi\rangle$ at the point z on the world-sheet, in the usual way. The absorption of a u-twisted state, ϕ_u, can be viewed as the emission of its **charge conjugate** state, ϕ_u^c. This state must be in $\mathcal{H}_{u^{-1}}$; charge conjugation must therefore map $\mathcal{H}_u \to \mathcal{H}_{u^{-1}}$.

In terms of its components

$$\begin{aligned} \mathbb{V}(\Phi, z) &= \sum_{u,v} V^{(v,u)}(\Phi, z) I_{vu} \\ \mid \Phi\rangle &= \sum_{u} V^{(u,e)}(\Phi, 0) \mid 0\rangle_e E_e. \end{aligned}$$

The component vertex operators of the form $V^{(u,e)}(\Phi, z)$, and their conjugates, play an important rôle in what follows so it is worthwhile introducing some notation:

$$\begin{aligned} V(\Phi, z) &= V^{(c,c)}(\Phi, z) \\ W(\Phi, z) &= V^{(u,c)}(\Phi, z) \\ \overline{W}(\Phi, z) &= V^{(c,u)}(\Phi, z). \end{aligned} \tag{3.30}$$

The conformal dimensions are defined as before; each of the components is a **generalized conformal field** in the sense that for a highest weight state $\mid \Phi\rangle$, *i.e.* one for which

$$\begin{aligned} \mathbb{L}_n \mid \Phi\rangle &= 0, \quad n \geq 1 \\ \mathbb{L}_0 \mid \Phi\rangle &= \Delta \mid \Phi\rangle, \end{aligned} \tag{3.31}$$

we have

$$L_n^v V^{(v,u)}(\Phi, z) - V^{(v,u)}(\Phi, z) L_n^u = z^n(z\partial + \Delta(n+1)) V^{(v,u)}(\Phi, z). \tag{3.32}$$

The moments of $\mathbb{V}(\Phi, z)$ are defined by the expansion

$$\mathbb{V}(\Phi, z) = \sum_{n \in \mathbb{Z}/N} \mathbb{V}_n(\Phi) z^{-n-\Delta}. \tag{3.33}$$

It is important to realize that in general $\mathbb{V}(\Phi, z)$ is non–integrally moded and has non–integer dimension. This means that the theory on T is **non–local**, *i.e.* the operator algebra of the fields has branch point singularities as well as poles. We shall discover that locality is restored by the projection $T \to \Omega$, in much the same way that locality is restored in the NSR Spinning string by the Gliozzi–Olive–Scherk (GSO) projection [59]. The operator algebra (assuming that it is well–defined) is expected to have the generic form:

$$\mathbb{V}(\Phi, z)\mathbb{V}(\Phi', \omega) \sim (z - \omega)^{\Delta'' - \Delta' - \Delta} c(\Phi, \Phi', \Phi'')\mathbb{V}(\Phi'', \omega),$$

where, due to the fact that the Δ's are in general $(1/N)\times$integer, there are branch point singularities.

The fact that the conformal fields have $1/N$–branch cuts at $z = 0$ is a natural feature of vertex operators in twisted string models. The fields are multi–valued on the world–sheet $\mathcal{M}$. One way to approach this feature is to work on the covering space $\overline{\mathcal{M}}$, on which fields are single valued. For an arbitrary amplitude and for general W the covering space will have a complicated topology, however, it is still a Riemann surface [34]. W acts on $\overline{\mathcal{M}}$ as a group of automorphisms (which preserve the complex structure) and

$$\mathcal{M} = \frac{\overline{\mathcal{M}}}{W}.$$

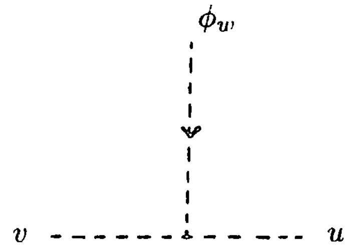
$\mathcal{M}$ is actually an orbifold itself, where the singularities correspond to the branch points where the insertions are made. It is a unique feature of these two–dimensional surfaces that these singularities do not need to be 'blown up' in order to obtain a Riemann surface. It has been conjectured that this is ultimately the reason why the string can propagate consistently on an orbifold [34]. If one could find a representation for $\mathcal{M}$ then in principle it would be possible to determine all the functions of interest for $\mathcal{M}$. The problem is that representations for $\overline{\mathcal{M}}$ are only known for simple cases, for example when $W \simeq \mathbb{Z}_N$ [34]. For the most part, we shall concentrate on the properties of the fields themselves and leave aside questions about the global properties of the Riemann surface on which they live.

Correlation functions are constructed by taking a time–ordered expectation value in the Hilbert space in the usual way:

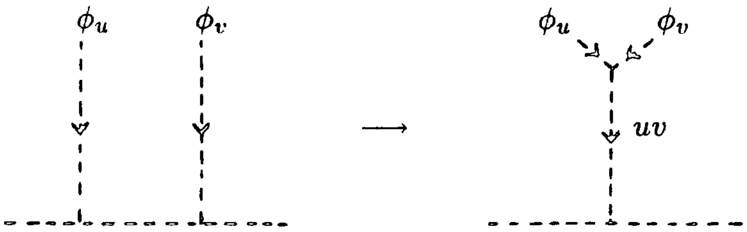
$$\langle \mathbb{0} \mid T\left\{\prod_{i=1}^{N} \mathbb{V}(\Phi_i, z_i)\right\} \mid \mathbb{0}\rangle. \tag{3.34}$$

It turns out that many of the contributions to this correlation function are in fact zero. This is due to the **twist selection rule**, which states that a correlation function is only non–vanishing if the product of group elements corresponding to the inserted states, taken in a time ordered fashion, is the identity. This is indicative of the fact that the component vertex operator $V^{(v,u)}(\phi_w, z)$, which we can associate with the diagram

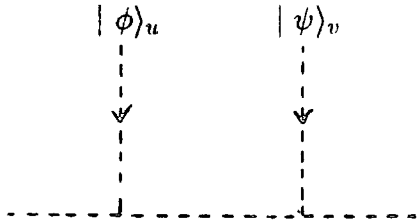
$V^{(v,u)}(\phi_w, z)$:



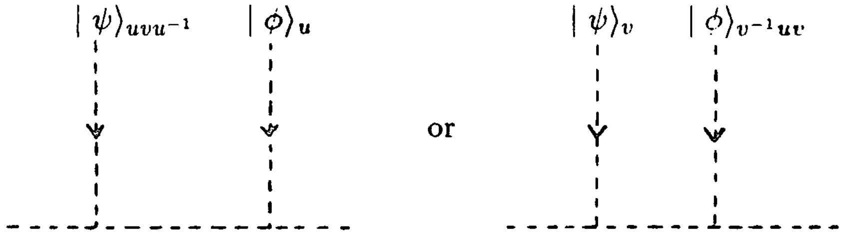
is identically zero unless $v = wu$. In particular, this implies that the structure constants of the operator algebra $c(\phi_u, \phi_v, \phi_w)$ are only non-zero when $w = uv$. The operator algebra is then a statement of the factorization:



To ensure that the resulting theory is local, we must now inquire about exchange duality. The problem here is that when W is non-abelian the interchange of two vertex operators representing the insertion of states $\mid \phi\rangle_u$ and $\mid \psi\rangle_v$ generally results in a vanishing correlation function by the twist selection rule, unless $[u,v] = 0$. So it seems that the correlation functions would in general have disastrous and unacceptable discontinuities. However, this unwelcome feature is not apparent once we have performed the projection $T \rightarrow \Omega$. To see this consider the two vertices



this can be 'dual' to either

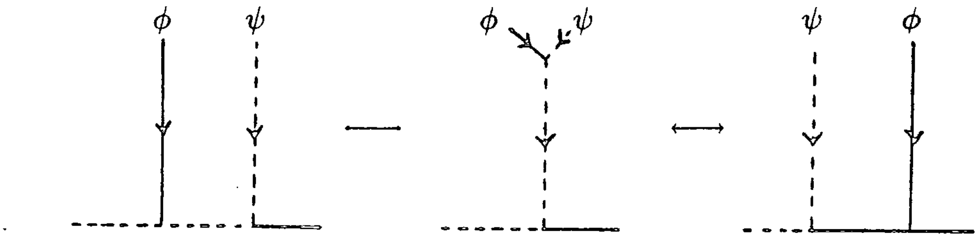


where $\mid \psi\rangle_{uvu^{-1}} = \hat{u} \mid \psi\rangle_v$ and $\mid \phi\rangle_{v^{-1}uv} = \hat{v}^{-1} \mid \phi\rangle_u$. The point is when we perform the $\hat{W}$ projection we sum over all images of a state in the same conjugacy class, and so for the projected vertex operators exchange duality would indeed hold. In this sense the theory 'knows' that it should be projected in order that the correlation functions represent a local theory.

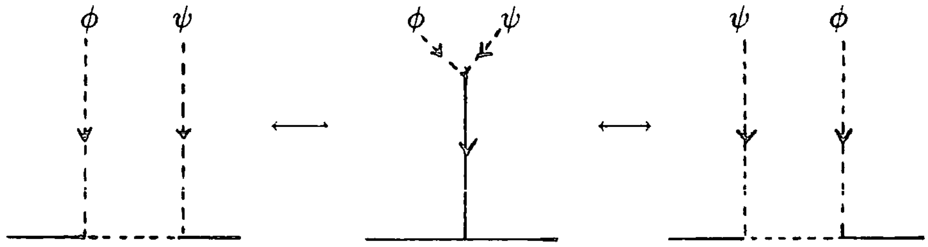
Exchange duality has some immediate consequences for the operators we defined in (3.30). Firstly,

$$V^u(\phi,z)W^{-u}(\psi,\omega) \overset{\text{dual}}{\longleftrightarrow} W^u(\psi,\omega)V(\phi,z). \tag{3.35}$$

This intertwining property will play a central rôle when we come to construct the W operators explicitly. The intertwining property is a consequence of the factorization



where we have indicated a twisted state by a broken line to distinguish it from an untwisted state. In addition from considering



we conjecture

$$\overline{W}^u(\phi,z)W^{-u}(\psi,\omega) \overset{\text{dual}}{\longleftrightarrow} \overline{W}^u(\psi,\omega)W^u(\phi,z), \tag{3.36}$$

and similarly for $W\overline{W}$. Implicit in these arguments is the fact that the fields close under OPE, generically

$$\begin{aligned} V^uW^{-u} &\longrightarrow W^u, \quad W^uV \longrightarrow W^u \\ \overline{W}^uW^{-u} &\longrightarrow V, \quad W^u\overline{W}^u \longrightarrow V^u. \end{aligned} \tag{3.37}$$

In a later chapter we will present a detailed discussion of this algebra for the **reflection twist**.

Of particular importance in our discussion is the vertex operator for the insertion of an untwisted state:

$$\mathbb{V}(\phi, z) = V(\phi, z) \oplus V^u(\phi, z) \oplus \ldots\ldots$$

These operators have been found to play an important rôle in the representation theory of Kac–Moody algebras [3]. generalizing the homogeneous FKS construction that we reviewed in chapter 2.

3.5 THE UNTWISTED VERTICES IN THE TWISTED PICTURE.

In this section we construct the component vertex operators $V^u(\phi, z)$, and discuss the locality restoring projection. In the following section we go on to consider their operator algebra, showing how the FKS mechanism can be generalized.

When $u = e$, the identity, the vertex operator is simply the expression given in equation (2.39). The expression for $V^u(\phi, z)$, $u \neq e$, has the same form as $V(\phi, z)$, with $X(z)$ replaced by the twisted field $X_u(z)$, and an overall factor of $z^{-(\alpha_1^u)^2/2}$ which ensures that $V^u(\phi, z)$ behaves in the same way with respect to the conformal algebra as $V(\phi, z)$:

$$V^u(\phi, z) = z^{-\frac{1}{2}(\alpha_1^u)^2} : \prod_{i=1}^{N} \frac{i\epsilon_i.\partial^{n_i} X_u(z)}{(n_i - 1)!} e^{i\alpha.X_u(z)} : C_u(\alpha). \tag{3.38}$$

Here, α_1^u is the projection of α perpendicular to the directions fixed by u, that is

$$\alpha_1^u = \alpha - \alpha_0^u, \quad \alpha_0^u = P_u(\alpha).$$

The **twisted operator cocycle**, $\hat{C}_u(\alpha) = e^{i\alpha.q^u} C_u(\alpha)$, is included to ensure exchange duality which is guaranteed if the operators generate a projective representation of Λ, *i.e.*

$$\hat{C}_u(\alpha)\hat{C}_u(\beta) = \varepsilon_u(\alpha, \beta)\hat{C}_u(\alpha + \beta), \tag{3.39}$$

with a 2–cocycle factor set satisfying

$$\Omega_u(\alpha, \beta) = \frac{\varepsilon_u(\alpha, \beta)}{\varepsilon_u(\beta, \alpha)} = \prod_{q=0}^{N_u - 1} \left(-e^{-\frac{2\pi i q}{N_u}}\right)^{\alpha.u^q\beta}. \tag{3.40}$$

This phase exactly cancels the phase resulting from the interchange the order of two vertex operators. Notice that $\Omega_e(\alpha, \beta) = (-)^{\alpha.\beta}$, which we recall is the untwisted expression. It is important for what follows that the two factor sets ε ($= \varepsilon_c$) and ε_u can be related by a bilinear function η_u [3]:

$$\varepsilon(\alpha,\beta) = \eta_u(\alpha,\beta)\varepsilon_u(\alpha,\beta), \tag{3.41}$$

where

$$\eta_u(\alpha.\beta) = \prod_{q=1}^{N_u-1} (1 - e^{\frac{2\pi i q}{N_u}})^{\alpha.(u^q-1)\beta}. \tag{3.42}$$

Proof : We need to show that

$$\frac{\varepsilon_u(\alpha,\beta)\eta_u(\alpha,\beta)}{\varepsilon_u(\beta,\alpha)\eta_u(\beta,\alpha)} = (-)^{\alpha.\beta}$$

$$\begin{aligned} LHS &= \Omega_u(\alpha,\beta) \prod_{q=1}^{N_u-1} \left(1 - e^{\frac{2\pi i q}{N_u}}\right)^{\alpha.u^q\beta-\beta.u^q\alpha} \\ &= \Omega_u(\alpha,\beta) \prod_{q=1}^{N_u-1} \left(-e^{\frac{2\pi i q}{N_u}}\right)^{\alpha.u^q\beta} \\ &= \Omega_u(\alpha,\beta)\Omega_u(-\alpha,\beta)(-)^{\alpha.\beta} \\ &= (-)^{\alpha.\beta} = RHS. \end{aligned}$$

3.5.1. The zero–mode space.

We have already indicated that the zero–mode space is to be understood as the representation space for the twisted operator cocycles, here we shall expand and discuss this in more detail. Recall that $\mathcal{V}_u$ has a decomposition

$$\mathcal{V}_u = \mathcal{P}(\Lambda^u) \otimes \mathcal{V}_u^0, \tag{3.43}$$

where $\mathcal{P}(\Lambda^u)$ is a complex span of momentum states on the lattice Λ^u, the projection of Λ onto the subspace fixed by u. $\mathcal{V}_u^0$ is a finite dimensional vector space, which for an irreducible representation of the operator cocycles has dimension [3,32,58]

$$c_u = \left|\frac{K_u}{(1-u)\Lambda^* \cap \Lambda}\right|^{\frac{1}{2}}. \tag{3.44}$$

For certain types of twist, where u is an *inner automorphism* of the algebra g (whose root system is Λ_2), the degeneracy can also be written [58]:

$$c_u = |\det(1-u_*)|^{1/2}\frac{\mathrm{vol}(\Lambda^u)}{\mathrm{vol}(\Lambda)}.$$

The fact that the degeneracy c_u is an integer, which is not obvious from the above formulae, is discussed in ref.[32]. The naive guess for $\mathcal{V}_u^0$, a complex span of position eigenstates corresponding to the singularities of the orbifold, is in fact reducible into a number of copies of the irreducible representation with dimension c_u [32]. Representations of the operator cocycles have been found explicitly in ref.[4], based on the original work of Lepowsky [3], here we will limit our discussion to a few points which will be relevant for what follows.

Let us define the set

$$A_u \equiv \{\alpha \mid \Omega_u(\alpha,\beta)=1, \ \ \forall\beta \in A_u\}, \tag{3.45}$$

i.e. A_u is the maximal set of commuting operator cocycles. A_u is not in fact unique, however, all choices are equivalent. It follows from its definition that the operator cocycles on A_u are diagonal. For an irreducible representation $\mathcal{V}_u^0$ is in fact isomorphic to the coset space

$$\mathcal{V}_u^0 \simeq \frac{K_u}{A_u}, \quad c_u = \left|\frac{K_u}{A_u}\right|. \tag{3.46}$$

The fact that c_u is equal to the alternative definitions, in equation (3.44), is discussed in ref.[4]. If we label a coset by a representative n_i, $i = 1,.....c_u$, then the operator cocycles have the following action on a state in $\mathcal{V}_u$:

$$\begin{aligned}&\hat{C}_u(\alpha)\mid\beta_0^u\rangle\otimes\mid n_i\rangle\\ &=\frac{\varepsilon_u(a-\alpha-\beta-n_i,\alpha+\beta+n_i)\varepsilon_u(\alpha,\beta+n_i)}{\varepsilon_u(-a+\alpha+\beta+n_i,a-\alpha-\beta-n_i)}g_u(a)\mid\alpha_0^u+\beta_0^u\rangle\otimes\mid n_j\rangle.\end{aligned} \tag{3.47}$$

where $n_j = n_i+\alpha+a \bmod \Lambda/K_u$, $a\in A_u$ and $\beta\in\Lambda/K_u$ such that $\beta_0^u = P_u(\beta)$. In the above $g_u(\alpha)$ is a one dimensional projective representation on A_u:

$$g_u(\alpha)g_u(\beta) = \varepsilon_u(\alpha,\beta)g_u(\alpha+\beta), \quad \forall\alpha,\beta\in A_u.$$

The fact that such a one dimensional representation exists is because on A_u the operator cocycles commute, *i.e.* $\varepsilon_u(\alpha,\beta)=\varepsilon_u(\beta,\alpha)$.

Notice that $\hat{C}_u(\alpha)$, $\alpha \in A_u$, is diagonal on $\mathcal{V}_u^0$ as claimed

$$\hat{C}_u(\alpha) \mid \beta_0^u\rangle \otimes \mid n_i\rangle \;=\; \Omega_u(\alpha, \beta + n_i) g_u(\alpha) \mid \alpha_0^u + \beta_0^u\rangle \otimes \mid n_i\rangle. \tag{3.48}$$

Later when we consider the transformation properties of the $\hat{C}_u(\alpha)$s under $\hat{u}$ we will need to know their representation on a subset of A_u, defined by

$$B_u \;\equiv\; \{\alpha \mid \alpha \;=\; (1-u)\beta,\; \beta \;\in\; \Lambda\}. \tag{3.49}$$

For $\alpha \in B_u$

$$\begin{aligned}\Omega_u(\alpha,\beta) &= e^{-2\pi i(\alpha')_0^u.\beta}, \qquad \beta \in \Lambda\\ \Omega_u(\alpha, n_i) &= 1, \qquad \text{since } \; n_i \in K_u,\end{aligned}$$

where $\alpha \;=\; (u\;-\;1)\alpha'$. Notice that $(\alpha')_0^u.\beta$ is unique, for a given α, up to an integer. From the above we deduce that the operator cocycles on B_u are

$$\hat{C}_u(\alpha) \;=\; g_u(\alpha) e^{-2\pi i(\alpha')_0^u.p^u}\mathbb{1}, \tag{3.50}$$

where $\mathbb{1}$ is the unit operator on $\mathcal{V}_u^0$.

3.5.2. The lift of W.

To be able to implement the projection $\mathcal{H}_T \rightarrow \mathcal{H}_\Omega$, we must find the lift of W onto $\mathcal{H}_T$.

The untwisted sector: To specify the action of $\hat{u} \in \hat{W}$ we must consider how the operators $X(z)$ and $\hat{C}(\alpha)$, which are used to build up the vertex operators, transform:

$$\begin{aligned}\hat{u}:&\quad \epsilon.X(z) \longmapsto u\epsilon.X(z)\\ \hat{u}:&\quad \hat{C}(\alpha) \longmapsto u_\alpha \hat{C}(u\alpha).\end{aligned} \tag{3.51}$$

Notice that, due to the presence of the phases u_α, the automorphism group is realized projectively on the Hilbert space. For consistency $\hat{W}$ must be an automorphism group of the operator algebra, this constrains the phases u_α to satisfy

$$\begin{aligned}u_\alpha u_\beta &= \frac{\varepsilon(\alpha,\beta)}{\varepsilon(u\alpha,u\beta)} u_{\alpha+\beta}\\ u_\alpha u_{-\alpha} &= 1\\ u_\alpha^* &= u_{-\alpha},\end{aligned} \tag{3.52}$$

where the last two constraints follows from our conventions for hermiticity $\hat{C}^\dagger(\alpha) = \hat{C}(-\alpha)$, $\varepsilon(\alpha,-\alpha) = 1$ and $\hat{C}(0) = \mathbb{1}$. For the cases where u is not a NFPA, it is possible to choose $u_\alpha = 1$ whenever $u\alpha = \alpha$, due to the freedom to make trivial automorphisms. This point is discussed in [54].

<u>*The twisted sectors*</u>: Recall that an element $\hat{u} \in \hat{W}$ in general mapped $\mathcal{H}_v \to \mathcal{H}_{uvu^{-1}}$. Let us consider just the action of the centralizer $\hat{G}_v$ on $\mathcal{H}_v$. Since $\hat{u} : V^v(\phi, z) \to V^v(\hat{u}\phi, z)$, and we have determined above how $| \phi\rangle \in \mathcal{H}_0$ transforms, for consistency we require the twisted operators to transform in the *same* way:

$$\begin{aligned} \hat{u} : \quad & \epsilon.X_v(z) \longmapsto u\epsilon.X_v(z) \\ \hat{u} : \quad & \hat{C}_v(\alpha) \longmapsto u_\alpha \hat{C}_v(u\alpha), \end{aligned} \tag{3.53}$$

where u_α is the same phase as the one that appears in (3.51). Due to the fact that

$$\frac{\varepsilon_v(\alpha,\beta)}{\varepsilon_v(u\alpha,u\beta)} = \frac{\varepsilon(\alpha,\beta)}{\varepsilon(u\alpha,u\beta)}, \quad \text{for} \quad u \in G_v,$$

$\hat{u} \in \hat{G}_v$ is also an automorphism of the operator algebra of the twisted fields $\{V^u(\phi, z)\}$.

Our main requirement of the projected theory is that it is local. This is achieved by demanding

$$\hat{u} : \quad V^u(\phi, z) \longmapsto V^u(\phi, ze^{-2\pi i}). \tag{3.54}$$

Locality now follows because $\langle \hat{u} \rangle$ (the cyclic group generated by $\hat{u}$) is a subgroup of $\hat{G}_u$ and so the projection of $\mathcal{H}_u$ onto its $\hat{G}_u$ invariant subspace, $\widetilde{\mathcal{H}}_u$, projects also onto the $\langle \hat{u} \rangle$ invariant subspace and hence integer graded oscillators. Equation (3.54) follows partly as a consequence of the twisted boundary condition for the oscillator parts of $X_u(z)$:

$$\begin{aligned} u(\epsilon).X_u^{\pm}(z) &= \epsilon.u^{-1}X_u^{\pm}(z) \\ &= \epsilon.X_u^{\pm}(ze^{-2\pi i}). \end{aligned}$$

The zero–mode piece is more subtle and requires that the twisted operator cocycle must satisfy

$$u_\alpha \hat{C}_u(u\alpha) = \hat{C}_u(\alpha) e^{\pi i (\alpha_1'')^2 - 2\pi i \alpha_0''.p''}, \tag{3.55}$$

so that

$$\hat{u} : \quad \hat{C}_u(\alpha) z^{-\frac{1}{2}(\alpha_1'')^2 + \alpha_0''.p''} \longmapsto \hat{C}_u(\alpha)(ze^{-2\pi i})^{-\frac{1}{2}(\alpha_1'')^2 + \alpha_0''.p''}.$$

We give the rather uninspiring proof of this property of the twisted operator cocycles in appendix B.

3.6 Twisted Strings and Kac–Moody Algebras.

In the last chapter we found that when we considered an analytic string moving on the torus $\mathbf{R}^d/2\pi\Lambda$, where Λ was the root lattice of a simply–laced Lie algebra g, then the operator algebra generated by

$$V(\alpha,z) \quad \alpha\in\Lambda_2,\quad i\epsilon.\partial X(z),\quad \mathbf{1},$$

was the untwisted Kac–moody algebra $\hat{g}\simeq g^{(1)}$. The question to which we address ourselves in this section is: what are the algebras, $\hat{g}^u$, generated by the corresponding operators

$$V^u(\alpha,z) \quad \alpha\in\Lambda_2,\quad i\epsilon.\partial X_u(z),\quad \mathbf{1},$$

in the u–twisted sector? This question has been answered in a mathematical context by Lepowsky in ref.[3], in what follows we shall relate these results to our string model vertex operators.

Let us define the projections

$$X_{[a]} = \frac{1}{N_u}\sum_{q=0}^{N_u-1} e^{-2\pi i q a/N_u}\hat{u}^q(X), \qquad 0\le a<N_u,$$

for a generic operator or vector X. In this basis the untwisted operator algebra, $\hat{g}$, becomes

$$\begin{aligned}
&V(\alpha,z)_{[a]}V(\beta,\omega)_{[b]}\\
&\quad= \frac{1}{N_u}\sum_{\alpha.u^q\beta=-1} e^{-2\pi i q/N_u}(u^q)_\beta\frac{\varepsilon(\alpha.u^q\beta)}{z-\omega}V(\alpha+u^q\beta,\omega)_{[a+b]}\\
&\quad+\frac{1}{N_u}\sum_{\alpha+u^q\beta=0} e^{-2\pi i q/N_u}(u^q)_\beta\left\{\frac{\delta_{a+b,0\,\mathrm{mod}\mathbf{Z}}}{(z-\omega)^2}+\frac{i\alpha_{[a+b]}.\partial X(\omega)}{z-\omega}\right\}
\end{aligned}\tag{3.56}$$

$$i\epsilon_{[a]}.\partial X(z)V(\alpha,\omega)_{[b]} = \frac{\epsilon_{[a]}.\alpha}{z-\omega}V(\alpha,\omega)_{[a+b]}$$

$$i\epsilon_{[a]}.\partial X(z)i\eta_{[b]}.\partial X(z) = \frac{\epsilon_{[a]}.\eta}{(z-\omega)^2}\delta_{a+b,0\,\mathrm{mod}\mathbf{Z}},$$

modulo regular terms. Under $\hat{u}$, $\hat{g}$ decomposes:

$$\hat{g} = \bigoplus_{a=0}^{N_u-1}\hat{g}_a[u].\tag{3.57}$$

The $\hat{u}$–invariant component generates a subalgebra, $\hat{g}_0[u]\subset\hat{g}$. From the point of view of the finite Lie algebra g, $\hat{u}$ restricted to Λ_2 (the root system), is an **automorphism**, that is, a one–to–one mapping of g onto itself that preserves the commutation relations, *i.e* $\hat{u} : g \rightarrow g$ such that

$$\hat{u}\left([x, y]\right) = [\hat{u}(x), \hat{u}(y)], \quad \forall x, y \in g.$$

Automorphisms of finite Lie algebras fall into two distinct classes. *Inner* automorphisms can be generated by the action of an element of the compact group G associated with g, *i.e* if $\hat{u}$ is inner then its action can be written as

$$\hat{u}(x) = UxU^{-1}, \quad U \in G.$$

All other automorphisms are *outer*. The only simple simply–laced Lie algebras which admit outer automorphisms are A_n, D_n and E_6. Outer automorphisms can be understood in terms of Dynkin diagram symmetries [48,49].

The distinction between inner and outer automorphisms plays an important rôle in determining the algebras $\hat{g}^u$. In fact, it is well known [48,49] that

$$\hat{g}^u \simeq g^{(\tau)},$$

where $\tau = 1$ for an inner automorphism, and $\tau = 2, 3$ for an outer automorphism; *i.e.* if $\hat{u}$ is an inner automorphism then the algebra generated in the twisted sector is actually *isomorphic* to the algebra generated in the untwisted sector. This is manifest in the **shifted picture**, where $\hat{u}$ is represented as a 'shift' in the Cartan subalgebra; we shall discuss this alternative way to formulate twisted models for inner automorphisms in chapter 8. On the contrary, if $\hat{u}$ is an outer automorphism then the algebra generated in the twisted sector is one of the **twisted Kac–Moody** algebras associated with g.

The calculation of the algebra $\hat{g}^u$ proceeds in a similar way to the untwisted case. First we 'normal order' the product

$$V^u(\alpha, z)_{[a]} V^u(\beta, \omega)_{[b]} = \frac{1}{N_u^2} \omega^{-\frac{1}{2}(\beta_1^u)^2} z^{-\frac{1}{2}(\alpha_1^u)^2} \times$$
$$\left\{ \sum_{p,q=0}^{N_u - 1} e^{-\frac{2\pi i}{N_u}(ap+bq)} (z - \omega)^{u^p\alpha . u^q\beta} \prod_{r=0}^{N_u - 1} \left(z^{\frac{1}{N_u}} - e^{\frac{2\pi i r}{N_u}} \omega^{\frac{1}{N_u}} \right)^{u^p\alpha . (u^r - 1) u^q\beta} \right.$$
$$\left. \times \ : e^{iu^p\alpha . X_u(z) + iu^q\beta . X_u(\omega)} : (u^p)_\alpha (u^q)_\beta \varepsilon_u(u^p\alpha, u^q\beta) C_u(u^p\alpha + u^q\beta) \right\}.$$

We now shift $q \rightarrow q + p$ and then use the fact that the phases $(u^p)_\alpha$ satisfy equation

(3.52), to show

$$(u^q)_\alpha(u^{p+q})_\beta\varepsilon_u(u^p\alpha, u^{p+q}\beta) \;=\; (u^p)_{\alpha+u^p\beta}(u^q)_\beta\varepsilon_u(\alpha, u^q\beta).$$

The product has a simple pole when $\alpha.u^q\beta = -1$, *i.e.* $\alpha + u^q\beta \in \Lambda_2$, and a double pole when $\alpha + u^q\beta = 0$. To aid the evaluation of the residues we now prove that

$$\begin{aligned} z^{-\frac{1}{2}(\alpha_1^u)^2}\omega^{-\frac{1}{2}(\beta_1^u)^2}\prod_{r=1}^{N_u-1}\left(z^{\frac{1}{N_u}} - e^{\frac{2\pi i r}{N_u}}\omega^{\frac{1}{N_u}}\right)^{\alpha.(u^r-1)u^q\beta} \\ = \begin{cases} \omega^{-\frac{1}{2}(\alpha_1^u+\beta_1^u)^2}\eta_u(\alpha, u^q\beta) + O(z-\omega) & \alpha.u^q\beta = -1 \\ \omega^{-\frac{1}{2}(\alpha_1^u+\beta_1^u)^2}\eta_u(\alpha, u^q\beta) + O(z-\omega)^2 & \alpha + u^q\beta = 0. \end{cases} \end{aligned}$$

Proof : Expanding the above expression to $O(z-\omega)$ we find that it equals

$$\begin{aligned} &\omega^{-\frac{1}{2}(\alpha_1^u)^2-\frac{1}{2}(\beta_1^u)^2+\alpha.\beta_0^u-\alpha.\beta}\prod_{r=1}^{N_u-1}\left(1 - e^{\frac{2\pi i r}{N_u}}\right)^{\alpha.(u^r-1)u^q\beta} \\ &\times\left\{1 + (z-\omega)\omega^{-1}\left(-\frac{1}{2}(\alpha_1^u)^2 + \frac{1}{N_u}\sum_{r=0}^{N_u-1}\frac{\alpha.(u^r-1)u^q\beta}{1-e^{2\pi i r/N_u}}\right) + O(z-\omega)^2\right\}. \end{aligned} \tag{3.58}$$

Recall that

$$\prod_{r=1}^{N_u-1}\left(1 - e^{\frac{2\pi i r}{N_u}}\right)^{\alpha.(u^r-1)u^q\beta} \;=\; \eta_u(\alpha, u^q\beta),$$

and

$$\alpha.\beta_0^u - \alpha.\beta \;=\; -\alpha_1^u.\beta_1^u.$$

When $\alpha + u^q\beta = 0$ the term $O(z-\omega)$ in round brackets in expression (3.58) is in fact zero. To see this consider

$$\frac{1}{N_u}\sum_{r=0}^{N_u-1}\frac{\alpha.(u^r-1)\alpha}{1-e^{2\pi i r/N_u}} = \begin{cases} \frac{1}{N_u}\sum\limits_{r=1}^{\frac{N_u-1}{2}}\alpha.(u^r-1)\alpha & N_u \in 2\mathbb{Z}+1 \\ \frac{1}{N_u}\sum\limits_{r=1}^{\frac{N_u-2}{2}}\alpha.(u^r-1)\alpha - \frac{\alpha^2}{N_u} & N_u \in 2\mathbb{Z}, \end{cases}$$

where we used the fact that

$$\frac{1}{1-e^{2\pi i r/N_u}} + \frac{1}{1-e^{-2\pi i r/N_u}} \;=\; 1.$$

If N_u is odd

$$\begin{aligned}\sum_{r=1}^{\frac{N_u-1}{2}} \alpha.(u^r-1)\alpha &= \frac{1}{2}\sum_{r=1}^{N_u-1} \alpha.u^r\alpha - \frac{1}{2}(N_u-1)\alpha^2 \\ &= \frac{N_u}{2}(\alpha.\alpha_0^u - \alpha^2) \\ &= -\frac{N_u}{2}(\alpha_1^u)^2,\end{aligned}$$

while if N_u is even

$$\begin{aligned}\sum_{r=1}^{\frac{N_u-2}{2}} \alpha.(u^r-1)\alpha - \alpha^2 &= \frac{1}{2}\sum_{r=1}^{N_u-1} \alpha.u^r\alpha - \frac{1}{2}(N_u-1)\alpha^2 \\ &= \frac{N_u}{2}(\alpha.\alpha_0^u - \alpha^2) \\ &= -\frac{N_u}{2}(\alpha_1^u)^2,\end{aligned}$$

Substituting these results into (3.58) when $\alpha + u^q\beta = 0$ completes the proof.

It is now a straightforward exercise to extract the singular terms and evaluate the other relevant OPEs:

$$\begin{aligned}&V^u(\alpha,z)_{[a]}V^u(\beta,\omega)_{[b]} \\ &= \frac{1}{N_u}\sum_{\alpha.u^q\beta=-1} e^{-2\pi i q/N_u}(u^q)_\beta \frac{\varepsilon(\alpha.u^q\beta)}{z-\omega}V^u(\alpha+u^q\beta,\omega)_{[a+b]} \\ &+ \frac{1}{N_u}\sum_{\alpha+u^q\beta=0} e^{-2\pi i q/N_u}(u^q)_\beta\left\{\frac{\delta_{a+b,0\,\mathrm{mod}\mathbf{Z}}}{(z-\omega)^2}+\frac{i\alpha_{[a+b]}.\partial X_u(\omega)}{z-\omega}\right\} \qquad (3.59)\\ &i\epsilon_{[a]}.\partial X_u(z)V^u(\alpha,\omega)_{[b]} = \frac{\epsilon_{[a]}.\alpha}{z-\omega}V^u(\alpha,\omega)_{[a+b]} \\ &i\epsilon_{[a]}.\partial X_u(z)i\eta_{[b]}.\partial X_u(z) = \frac{\epsilon_{[a]}.\eta}{(z-\omega)^2}\delta_{a+b,0\,\mathrm{mod}\mathbf{Z}}.\end{aligned}$$

Compare $\hat{g}$. in the $\hat{u}$ diagonal basis, with $\hat{g}^u$ above. It is immediately apparent that locally the OPEs are isomorphic. Globally, however, $V^u(\alpha,z)_{[a]}$ is $a/N_u+\mathbf{Z}$ graded, while $V(\alpha,z)_{[a]}$ is, of course, integrally graded.

The invariant generators are integer graded in both sectors, and so we have the following isomorphism:

$$\hat{g}_0[u] \simeq \hat{g}_0^u. \qquad (3.60)$$

In particular, the horizontal algebra, in the twisted sector (the subalgebra commuting with L_0^u), is isomorphic to $g_0[u]$.

Consider the action of the horizontal algebra on the ground state in the u–twisted sector

$$\tilde{V}_0^u(\alpha) \mid 0\rangle \otimes \phi = \begin{cases} \mid 0\rangle \otimes \hat{C}_u(\alpha)\phi & \alpha \in K_u \\ 0 & \text{otherwise.} \end{cases}$$

where $\mid 0\rangle \otimes \phi \in \mathcal{V}_u$. Therefore the subalgebra defined on $\alpha \in K_u$ ($\alpha^2 = 2$), must be generated by the operator cocycles themselves. This generalizes statements made in ref.[60]. $\mathcal{V}_u^0$ is an irreducible representation of this subalgebra of dimension c_u. We can determine the weights of the representation because the Cartan subalgebra, spanned by the maximal set of commuting operator cocycles, is identified with

$$\frac{1}{g_u(\alpha)}\hat{C}_u(\alpha), \quad \alpha \in \frac{\Lambda_2 \cap A_u}{\langle u\rangle}. \tag{3.61}$$

The weights follow from equation (3.48)

$$\underline{\lambda}^i = (\Omega_u(\alpha_1, n_i), \Omega_u(\alpha_2, n_i), \ldots\ldots..), \tag{3.62}$$

where $\alpha_i \in \Lambda_2 \cap A_u/\langle u\rangle$.

4. THE TWISTED STRING EMISSION VERTEX.

In this chapter we will construct and examine the **twist fields** that emit, or absorb, twisted states from an untwisted string; we shall call them **twisted string emission vertices** (TSEVs) [5].

The effect of a TSEV on a correlation function is to 'open up' a cut at the point on the world–sheet where the twisted state is inserted. The resulting cut must terminate at some other point, or points, where other twisted states are emitted or absorbed. This is just another manifestation of the twist selection rule. From an operator point of view the TSEV, $W(\phi, z)$, *intertwines* the two Hilbert spaces $\mathcal{H}_c$ and $\mathcal{H}_u$. In fact, fields of this sort have been known for a long time. The TSEV, appropriate to the reflection twist ($\alpha \longrightarrow -\alpha$, $W \simeq \mathbf{Z}_2$), without the zero–mode pieces, was first constructed in the context of off–shell amplitudes for the bosonic string in ref.[21]. These vertices were further studied in refs.[35,36], where their intertwining nature was highlighted. In addition, it was realized that the Neveu–Schwarz–Ramond (NSR) spinning string model, could also be viewed as a $\mathbf{Z}_2$ twisted fermionic string; in this picture the fermion emission vertex plays the rôle of the TSEV [61–65]. The correspondence between the fermion emission vertex and part of the off–shell vertex, has also been exploited in ref.[66]. Against this background, we will construct the TSEV for an arbitrary twist u. The oscillator pieces turn out to be a simple generalization of the off–shell vertex; the zero–mode piece, on the contrary, turns out to be rather subtle. One of the main conclusions from this chapter is the recognition that the operator cocycles are crucial for the correct working of the TSEV. They are not optional. This may be contrasted with similar recent work [67,68]. The authors of these references do not discuss the zero–mode contribution and their vertices cannot intertwine vertex operators corresponding to winding states. We will have more to say about this in the last chapter.

4.1 CONSTRUCTION OF THE TSEV.

We have already established two crucial properties of the TSEV:

$$| \phi\rangle_u = W^u(\phi, 0) \mid 0\rangle, \tag{4.1}$$

where $| \, 0\rangle$ is the $SL(2, \mathbf{C})$ invariant vacuum state, and

$$V^u(\psi, \omega) W^u(\phi, z) \xrightarrow{\text{dual}} W^u(\phi, z) V(\psi, \omega). \tag{4.2}$$

It is clear that the right hand side of equation (4.2) cannot be equal to the left, since

$V^u(\psi,\omega)$ is non-integer moded. Recall that the correspondence 'dual' means that there exists an operator function $S(\omega, z)$ such that

$$S(\omega,z) = \begin{cases} V^u(\psi,\omega)W^u(\phi,z) & I\ :\ |\,\omega\,|>|\,z\,| \\ W^u(\phi,z)V(\psi,\omega) & II:\ |\,z\,|>|\,\omega\,|\,. \end{cases}$$

We can extract what we call the 'Intertwining Relation' by analytically continuing $V^u(\psi,\omega)W^u(\phi,z)$ into region III, $|\,\omega - z\,|>|\,z\,|$, by factoring out by the $SL(2,\mathbf{C})$ group element $\exp(zL^u_{-1})$:

$$V^u(\psi,\omega)W^u(\phi,z) \longrightarrow e^{zL^u_{-1}}V^u(\psi,\omega-z)e^{-zL^u_{-1}}W^u(\phi,z).$$

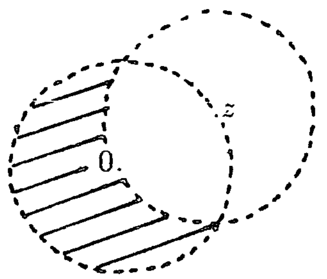
The right hand side is now well defined in region III, so in the overlap $II \cap III$ we deduce

$$V^u(\psi,\omega-z)I^u(\phi,z) = I^u(\phi,z)V(\psi,\omega), \tag{4.3}$$

where we have defined

$$I^u(\phi,z) = e^{-zL^u_{-1}}W^u(\phi,z).$$

We illustrate the overlap region, where the intertwining relation is valid, below.



The consistency of equation (4.3) is ensured in the overlap region because when $|\,z\,|>|\,\omega\,|$, $V^u(\psi,\omega-z)$ has an integer expansion in ω.

Below we shall conjecture an expression for $W^u(\phi,z)$, and then show that the intertwining relation is satisfied for the momentum vertex, $V(\alpha,z)$. Generalizations to an arbitrary vertex then follow in an obvious way. It is worth pointing out that much of the discussion mirrors very closely the analogous discussion of the fermion emission vertex, which intertwines the Neveu–Schwarz and Ramond sectors of the spinning string, we refer in particular to refs.[64,65].

Our conjectured expression is

$$W^u(\phi, z) = e^{zL^u_{-1}} \langle v \mid : \exp\left(A_u(z) + B_u(z)\right) : \mid \phi\rangle_u, \tag{4.4}$$

where

$$\begin{aligned}
A_u(z) &= -\frac{1}{2}\oint_{C_1} \frac{dx\, dy}{(2\pi i)^2}\, i\partial X(x).A_u(x,y,z)i\partial X(y) \\
B_u(z) &= B_u^-(z) + B_u^+(z) \\
B_u^-(z) &= -\oint_{C_2} \frac{dx\, dy}{(2\pi i)^2}\, i\partial X(x).B_u^-(x,y,z)i\partial X_u(y) \\
B_u^+(z) &= -\oint_{C_3} \frac{dx\, dy}{(2\pi i)^2}\, i\partial X(x).B_u^+(x,y,z)i\partial X_u(y).
\end{aligned} \tag{4.5}$$

$\langle v \mid$ is a special sum of momentum states:

$$\langle v \mid = \sum_{\beta\in\Lambda} \langle \beta \mid f_u(\beta)\hat{C}_u(\beta) \tag{4.6}$$

where $f_u(\alpha)$ is the function

$$f_u(\alpha) = \prod_{q=0}^{N_u-1} \left(1 - e^{\frac{2\pi i q}{N_u}}\right)^{\frac{1}{2}\alpha.(1-u^q)\alpha}. \tag{4.7}$$

The generating functions A_u and B_u are defined to be

$$\begin{aligned}
A_u(x,y,z) &= \sum_{q=o}^{N_u-1} (1-u^q)\ln\left\{(y-z)^{\frac{1}{N_u}} - e^{\frac{2\pi i q}{N_u}}(x-z)^{\frac{1}{N_u}}\right\} \\
B_u^-(x,y,z) &= \sum_{q=o}^{N_u-1} u^q \ln\left\{1 - e^{\frac{2\pi i q}{N_u}}\left(\frac{x-z}{y}\right)^{\frac{1}{N_u}}\right\} \\
B_u^+(x,y,z) &= -\sum_{q=o}^{N_u-1} u^q \ln\left\{(x-z)^{\frac{1}{N_u}} - e^{-\frac{2\pi i q}{N_u}} y^{\frac{1}{N_u}}\right\}
\end{aligned} \tag{4.8}$$

and the contours C_1, C_2 and C_3 are

$$\begin{aligned}
C_1 &: \mid x \mid \text{and} \mid y \mid < \mid z \mid \\
C_2 &: \mid x \mid < \mid z \mid, \mid y \mid > \mid x - z \mid \\
C_3 &: \mid x \mid < \mid z \mid, \mid y \mid < \mid x - z \mid.
\end{aligned} \tag{4.9}$$

It is relatively straightforward to evaluate the integrals in equations (4.8), and hence get an explicit expression for the TSEV in terms of the oscillators. The function $A_u(x,y,z)$

has also appeared in ref.[43], where the authors use it in the evaluation of commutators of twisted vertex operators (the $V^u(\phi, z)$'s).

Now that we have established our conjecture we can begin to examine its properties. Notice immediately that $W^u(\phi, z)$ contains only annihilation operators in the untwisted Fock space, so the first requirement, equation (4.1), follows trivially.

As promised we now prove equation (4.3) for the momentum vertex

$$V(\alpha, z) \; =: \exp(i\alpha.X(z)) : C(\alpha).$$

The general strategy we follow, is to commute this vertex through the TSEV, making judicious use of the Cambell–Baker–Hausdorff formula and keeping careful track of the residual factors which are left over. The calculation proceeds through the following steps.

1. Consider the product $I^u(\phi, z)V(\alpha, \omega)$. The creation pieces of $V(\alpha, \omega)$,

$$\exp\left(i\alpha.(q + X^-(\omega))\right),$$

does not commute with $B_u(z)$ or $A_u(z)$. When we move this term through the vertex from left to right, we pick up exponentials of the following commutators

$$\begin{aligned} [B_u^+(z), i\alpha.(q + X^-(\omega))] &= \alpha_0^u.p^u \ln(\omega - z) + i\alpha.X_u^+(\omega - z) \\ [B_u^-(z), i\alpha.(q + X^-(\omega))] &= i\alpha.X_u^-(\omega - z) \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} [A_u(z), i\alpha.(q + X^-(\omega))] = &-\frac{1}{2}\oint_{C_1} \frac{dx}{2\pi i}(\alpha.A_u(\omega, x, z)i\partial X(x) \\ &+ i\partial X(x).A_u(x, \omega, z)\alpha). \end{aligned} \tag{4.11}$$

In addition, we pick up the double commutator from the $\exp A_u(z)$ term, since it is quadratic in the untwisted field. This gives another term which is the exponential of

$$-\frac{1}{2}\alpha.A_u(\omega, \omega, z)\alpha.$$

The factor $\exp(i\alpha.X_u^-(\omega))$ gives unity on the state $\langle v \mid$. These manipulations are only valid in region II, where $\mid z \mid > \mid \omega \mid$.

2. We can rewrite equation (4.11), by using the fact that

$$\begin{aligned} \alpha.A_u(\omega,x,z)i\partial X(x) =& i\partial X(x).A_u(x,\omega,z)\alpha \\ &+ i\partial X(x).\sum_{q=0}^{N_u-1}(1-u^q)\alpha\ln\left(-e^{\frac{2\pi iq}{N_u}}\right) \end{aligned} \tag{4.12}$$

so that

$$\begin{aligned} [A_u(z), i\alpha.(q+X^+(\omega))] =& -\oint_{C_1}\frac{dx}{2\pi i}\, i\partial X(x).A_u(x,\omega,z)\alpha \\ &-\frac{1}{2}p.\sum_{q=0}^{N_u-1}(1-u^q)\alpha\ln\left(-e^{\frac{2\pi iq}{N_u}}\right). \end{aligned} \tag{4.13}$$

3. The oscillator pieces of $V^u(\alpha,\omega-z)$,

$$\exp\left(i\alpha.X_u^-(\omega-z)\right)\exp\left(i\alpha.X_u^+(\omega-z)\right),$$

can now be pulled out to the left of the TSEV, if we commute $\exp(i\alpha.X_u^+(\omega-z))$ past the term $\exp B_u^-(z)$, which produces the exponential of

$$[B_u^-(z), i\alpha.X_u^+(\omega-z)] = -\oint_{C_2}\frac{dx\ dy}{(2\pi i)^2}\, i\partial X(x).B_u(x,\omega-z,z)\alpha. \tag{4.14}$$

4. The remaining dependence on the untwisted field is contained within the exponential of

$$-\oint_{C_1}\frac{dx}{2\pi i}\, i\partial X(x).\left(A_u(x,\omega,z)+B_u(x,\omega-z,z)\right)\alpha + i\alpha.X^+(\omega).$$

We can now simplify this using the fact that in region III, $\mid\omega-z\mid>\mid z\mid$

$$A_u(x,\omega,z)+B_u(x,\omega-z,z) = \ln(\omega-x)-\ln(\omega-z)\frac{1}{N_u}\sum_{q=0}^{N_u-1}u^q. \tag{4.15}$$

In addition

$$\oint_{C_1}\frac{dx}{2\pi i}i\alpha.\partial X(x)\ln(\omega-x) = i\alpha.X^+(\omega),$$

so that the exponential of the above factor yields the term

$$(\omega - z)^{\alpha_0^u \cdot p}.$$

5. The diagonal element of A_u is given by

$$A_u(\omega,\omega,z) = (1-P_u)\ln(\omega - z) + \sum_{q=0}^{N_u-1}(1-u^q)\ln\left(1 - e^{\frac{2\pi i q}{N_u}}\right).$$

Therefore the contribution from this term is

$$(\omega - z)^{-\frac{1}{2}(\alpha_1^u)^2} f_u(\alpha)^{-1}.$$

The first term is exactly the factor needed for $V^u(\alpha, \omega - z)$.

6. Collecting all the remaining zero–mode terms together we have the messy looking expression

$$\langle v \mid \hat{C}(\alpha)(\omega - z)^{\alpha_0^u\cdot(p^u+p)} f_u(\alpha)^{-1} \prod_{q=o}^{N_u-1}\left(-e^{\frac{2\pi i q}{N_u}}\right)^{\frac{1}{2}p.(u^q-1)\alpha}, \tag{4.16}$$

which has to be simplified. However, noting that

$$\begin{aligned}\sum_{\beta\in\Lambda}\langle\beta \mid f_u(\beta)\hat{C}_u(\beta)\hat{C}(\alpha) &= \sum_{\beta\in\Lambda}\langle\beta-\alpha \mid \varepsilon(\alpha,\beta-\alpha)f_u(\beta)\hat{C}_u(\beta)\\ &= \sum_{\beta\in\Lambda}\langle\beta \mid \varepsilon(\alpha,\beta)f_u(\alpha+\beta)\hat{C}_u(\alpha+\beta)\end{aligned} \tag{4.17}$$

and

$$f_u(\alpha+\beta)\prod_{q=o}^{N_u-1}\left(-e^{\frac{2\pi i q}{N_u}}\right)^{\frac{1}{2}\beta.(u^q-1)\alpha} = \frac{f_u(\alpha)f_u(\beta)}{\eta_u(\alpha,\beta)},$$

where η_u is the function which relates ε and ε_u, we find that expression (4.16) simplifies to

$$\begin{aligned}\sum_{\beta\in\Lambda}\langle\beta \mid f_u(\beta)\varepsilon_u(\alpha,\beta)\hat{C}_u(\alpha+\beta)(\omega - z)^{\alpha_0^u\cdot(p^u+p)}&\\ = \hat{C}_u(\alpha)(\omega - z)^{\alpha_0^u\cdot p^u}\langle v \mid .&\end{aligned}$$

Where, in addition, we used the fact that

$$\hat{C}_u(\beta)(\omega - z)^{\alpha_0^u\cdot p^u} = (\omega - z)^{\alpha_0^u\cdot(p^u-\beta)}\hat{C}_u(\beta).$$

So, carefully putting all the pieces together we obtain precisely the intertwining relation:

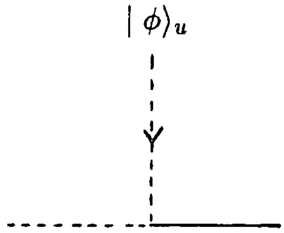
$$V^u(\alpha, \omega - z) I^u(\phi, z) = I^u(\phi, z) V(\alpha, \omega),$$

with the proviso that it is only valid in the overlap $II \cap III$.

The statement that $W^u(\phi, z)$ is an intertwining operator is now clarified. It is straightforward to go on and show that $W^u(\phi, z)$ intertwines the fields $\epsilon.\partial^n X_u(z)$ and $\epsilon.\partial^n X(z)$; in fact for these operators the cocycle contribution to the vertex plays no rôle, which explains why they were not encountered in ref.[67]. By putting together an arbitrary vertex from the building blocks $\epsilon.\partial^n X(z)$ and $V(\alpha, z)$, one can show that the intertwining relation is valid for any $V(\phi, z)$.

In our notation, $W^u(\phi, z)$ causes the absorption of the state $\mid \phi\rangle_u$ by an untwisted string:

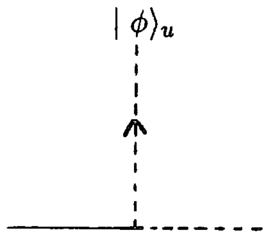
$W^u(\phi, z)$:



The conjugate vertex

$$z^{-2\Delta} W^u(\phi, 1/\bar{z})^\dagger ,$$

where Δ is the conformal weight of the state $\mid \phi\rangle_u$, represents the emission of the state $\mid \phi\rangle_u$:

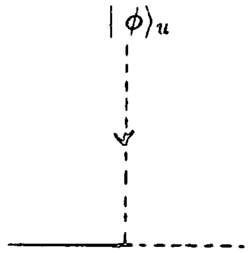


From the conjugate vertex we define the vertex

$$\overline{W}^{u^{-1}}(\phi, z) = z^{-2\Delta} W^{u^{-1}}(\phi^c, 1/\bar{z})^\dagger ,$$

which represents the absorption of $|\ \phi\rangle_u$ by a u^{-1}-twisted string:

$\overline{W}^{-u}(\phi, z)$:



4.2 The Twisted–Untwisted Operator Product.

In this section we evaluate the OPE of a TSEV with an untwisted string vertex.

In the last section we showed (at least for the momentum vertex) that in the overlap of region II and region III

$$V^u(\psi, \omega - z) I^u(\phi, z) = I^u(\phi, z) V(\psi, \omega).$$

We now show that there exists a function S', which satisfies

$$S'(\omega, z) = \begin{cases} V^u(\psi, \omega - z) I^u(\phi, z) & III \\ I^u(\phi, z) V(\psi, \omega) & II. \end{cases} \tag{4.18}$$

but is, in addition, well defined in the neighbourhood of $\omega = z$, where

$$S'(\omega, z) = I^u\big(V^u(\psi, \omega - z) \mid \phi\rangle_u, z\big). \tag{4.19}$$

The right hand side is the vertex operator corresponding to a generalized coherent state, which is defined in the spirit of our discussion at the end of section 2.5.

As in the last section, for simplicity of explanation, we shall only prove the above statement for the momentum vertex. Starting with $V^u(\alpha, \omega - z) I^u(\phi, z)$, in the overlap region $II \cap III$, there are four stages to the proof.

1. Commute $\exp(i\alpha.X_u^+(\omega - z))$ past $\exp(B_u^-(z))$, this produces a term

$$\exp\left\{\oint_0 \frac{dx}{2\pi i} i\partial X(x).B_u^-(x,\omega - z, z)\alpha\right\}.$$

2. Commute $\exp(i\alpha.X_u^-(\omega - z))$ past $\exp(B_u^+(z))$, producing a term

$$\exp\left\{\oint_0 \frac{dx}{2\pi i} i\partial X(x).B_u^-(x,\omega - z, z)\alpha\right\}.$$

3. Now consider the zero–modes. We have to move $\hat{C}_u(\alpha)(\omega - z)^{\alpha_0^u \cdot p^u}$ through $\langle v \mid$. Consider the term in the sum with momentum β:

$$\hat{C}_u(\alpha)(\omega - z)^{\alpha_0^u \cdot p^u} \hat{C}_u(\beta) = \hat{C}_u(\beta)\hat{C}_u(\alpha)\Omega_u(\alpha,\beta)(\omega - z)^{\alpha_0^u \cdot (p^u + \beta)}.$$

4. The first two contributions can be summed in the overlap region, where $\mid z \mid > \mid \omega \mid$,

$$B_u^-(x,\omega - z, z) + B_u^+(x,\omega - z, z) = -P_u \ln(\omega - z) - \sum_{q=0}^{N_u - 1} u^q \ln\left(-e^{-\frac{2\pi i q}{N_u}}\right).$$

Therefore the residual factor left over from 1 and 2, is simply

$$(\omega - z)^{-\alpha_0^u \cdot p}\Omega_u(\alpha,\beta)^{-1}.$$

This expression cancels the residual term left over from 3, and so we have succeeded in proving

$$S'(\omega, z) = I^u\big(V^u(\alpha, \omega - z) \mid \phi\rangle_u, z\big). \tag{4.20}$$

It is not difficult to extend this proof to an arbitrary untwisted state, in much the same way as the intertwining relation itself generalizes.

Effectively what we have evaluated is the OPE, in the form we introduced in section 2.5, that is

$$V^u(\psi, \omega)W^u(\phi, z) = \sum_n (\omega - z)^{-n-\Delta_\psi} W^u\big(V_n^u(\psi) \mid \phi\rangle_u, z\big). \tag{4.21}$$

As it stands, this OPE has branch cut singularities because $n \in \mathbf{Z}/N_u$. This, however, can be remedied by considering the $\hat{u}$–invariant vertex operators $\widetilde{V}(\phi, z)$ which are integer graded. Having done this we can calculate the generalized commutators. For example when $\alpha^2 = 2$

$$\widetilde{V}_0^u(\alpha) W^u(\phi, z) - W^u(\phi, z)\widetilde{V}_0(\alpha) = W^u(\widetilde{V}_0^u(\alpha) \mid \phi\rangle_u, z). \tag{4.22}$$

This encodes how the TSEV transforms under the invariant algebra $g_0[u]$.

4.3 Conformal Properties of the TSEV.

From the OPE we evaluated in the last section, we can easily deduce the fact that $W^u(\phi, z)$ is a conformal field of dimension

$$\Delta = \frac{1}{2}(\alpha_0^u)^2 + N + \Theta^u,$$

where α_0^u is the momentum and N is the occupation number of the state $\mid \phi\rangle_u$. For example, if $\mid \phi\rangle_u$ is a highest weight state,

$$\begin{aligned} L_n^u \mid \phi\rangle_u &= 0 \qquad n \geq 1 \\ L_0^u \mid \phi\rangle_u &= \Delta \mid \phi\rangle_u, \end{aligned}$$

then the OPE of $T^u(z)$, the stress–energy tensor, with $W^u(\phi, z)$ is

$$\begin{aligned} T^u(z) W^u(\phi, \omega) &= \sum_n (z-\omega)^{-n-2} W^u(L_n^u \mid \phi\rangle_u, \omega) \\ &= \frac{\Delta}{(z-\omega)^2} W^u(\phi, \omega) + \frac{1}{z-\omega} W^u(L_{-1}^u \mid \phi\rangle_u, \omega) + Reg. \end{aligned}$$

But L_{-1}^u is the generator of translations, in fact one can easily show

$$L_{-1}^u \mid \phi\rangle_u = \lim_{\omega \to 0} \partial W^u(\phi, \omega) \mid 0\rangle,$$

and so the OPE assumes the canonical form for a highest weight field:

$$T^u(z) W^u(\phi, \omega) = \frac{\Delta W^u(\phi, \omega)}{(z-\omega)^2} + \frac{\partial W^u(\phi, \omega)}{z-\omega} + Reg. \tag{4.23}$$

5. THE REFLECTION TWIST I.

In this chapter we shall consider in detail the vertex operators and algebras for the **reflection twist**. This is the twist for which

$$W = (e, \sigma) \simeq \mathbb{Z}_2, \tag{5.1}$$

where $\sigma(\alpha) = -\alpha$. Historically this was the first example where the concept of twisting was applied to the bosonic string [19–21]. It was discussed essentially because of its simple form, and also because it generalized to the bosonic string some of the more obvious features of the NSR spinning string model; that is, two sectors corresponding to the mixed boundary conditions

$$\psi(ze^{2\pi i}) = \pm\psi(z).$$

In fact, this resemblance has already been exploited in a rather elegant group theoretical way [66] for calculating some of the more complicated expressions involved in multi-fermion scattering amplitudes [69,70].

It has already been mentioned that the reflection twist appears to play a fundamental rôle in the off-shell formalism of the string. The authors of ref.[21] discovered that a Green's function representing the scattering of one off-shell string with many on-shell strings could be written as an expectation value in a twisted Fock space. This work turned out to coincide an earlier proposal of Schwarz who adopted a quite different starting point based on the dual model gauge conditions [71,72]. The twisted expectation value was then factorized in an appropriate channel, in order to isolate the bilocal off-shell vertex operator:

$$Z(z, \omega, k) = \overline{W}(z)e^{ik.q}W(\omega).$$

The fields W and $\overline{W}$ turn out to be *identical* to the twist fields we have already constructed, appropriate to the reflection twist, without the zero-mode pieces. In retrospect, it is not too surprising that the off-shell vertex operator involves twist fields. In orbifold models a cut in a correlation function, between two TSEVs on the world-sheet, accommodates a twisted boundary condition, whereas in the off-shell model it represents the boundary of a string in space-time which terminates at a finite time. In order to decouple non-physical states, the operators $\overline{W}$ and W have to be highest-weight conformal fields of unit dimension. For the reflection twist these operators have dimension $\Theta = d/16$. This exposes the unsatisfactory feature of the old off-shell model; it has critical dimension 16! Recently this apparent obstacle has been removed by including the BRST ghost contribution to the vertex [22,23]. The on-shell critical dimension of 26 is then restored. In the modern arena we are not confined to $d = 26$. We can consider the orbifolds as being ingredients in a larger framework.

In this chapter, as a preliminary to an investigation of the operator algebra generated by the fields of a reflection twisted string model, we consider the invariant algebras and the twisted operator cocycles. In the following chapters, we elucidate the operator algebra generated by the fields and then, drawing on the results that we establish, we shall be led to a generalization of the FKS mechanism involving TSEVs, as well as an explicit representation of Griess's algebra, along the lines developed in refs.[27,28].

5.1 The Twist Invariant Subalgebras.

The twist invariant subalgebras g_0 for the reflection twist have been evaluated before (see refs.[49,73] for example). By counting the invariant elements of g, the dimension of g_0 is easily determined to be

$$\dim(g_0) = \frac{1}{2}(\dim(g) - d) = \frac{1}{2}\dim(\Lambda_2),$$

where $d = \text{rank}(g)$. Later, it will be shown that the rank of g_0 equals the dimension of the maximal set of orthogonal positive roots of g. In addition if σ is inner then $\text{rank}(g_0) = \text{rank}(g)$. Below we list the simply-laced simple Lie algebras and their twist invariant subalgebras; we also indicate whether the twist is inner or outer.

g	g_0	
A_{2n}	B_n	Outer
A_{2n+1}	D_{n+1}	Outer
D_{2n}	$D_n + D_n$	Inner
D_{2n+1}	$B_n + B_n$	Outer
E_6	C_4	Outer
E_7	A_7	Inner
E_8	D_8	Inner

When we consider the Leech lattice ($d = 24$), the appropriate algebra is the cross-bracket algebra S with its twist invariant algebra S_0.

The affinizations of the invariant algebras are generated by the combinations[†]

$$\begin{aligned}
\widetilde{V}(\alpha, z) &= \frac{1}{2}\left(V(\alpha, z) + \sigma_\alpha V(-\alpha, z)\right), \\
\widetilde{V}^T(\alpha, z) &= \frac{1}{2}\left(V^T(\alpha, z) + \sigma_\alpha V^T(-\alpha, z)\right) \\
&= \frac{1}{2}\left(V^T(\alpha, z) + V^T(\alpha, z e^{2\pi i})\right),
\end{aligned} \tag{5.2}$$

in each sector, respectively, where $\alpha \in \Lambda_2$ for the Lie algebra cases and $\alpha \in \Lambda_4$ for the cross–bracket algebra case. In addition for the latter model we have to include the invariant generators

$$V(\epsilon, \eta, z) = : i\epsilon.\partial X(z).i\eta.\partial X(z) :, \tag{5.3}$$

in both sectors, *i.e.*

$$\dim(S_0) = \frac{24 \times 25}{2} + \frac{1}{2}\dim(\Lambda_4).$$

Each level in the twisted Hilbert space $\mathcal{H}_T[n + d/16]$ transforms as a g_0 representation, which we call $\mathcal{R}_n$.

5.2 The TSEV.

It is convenient to introduce the following twisted oscillators:

$$\begin{aligned}
c_r &= \frac{1}{\sqrt{r}}\alpha^\sigma_r, \quad r \in \mathbf{Z} + \frac{1}{2} > 0, \\
c_{-r} &= c_r^\dagger = \frac{1}{\sqrt{r}}\alpha^\sigma_{-r}.
\end{aligned}$$

so that

$$[c_r, c_s^\dagger] = \delta_{r,s}.$$

In terms of these oscillators the TSEV is

$$W(\phi, z) = e^{zL^T_{-1}}\langle v \,|: \exp(A + B) :|\, \phi\rangle, \tag{5.4}$$

† In what follows the sub/super-script 'T' is used to label objects associated with the twisted sector.

with

$$A(z) = \frac{1}{2} \sum_{n,m \geq 0} a_n A_{nm}(z) a_m, \tag{5.5}$$

and

$$B^{\pm}(z) = \sum_{\substack{n \geq 0 \\ r \geq \frac{1}{2}}} a_n B^{\pm}_{nr}(z) c_{\pm r}. \tag{5.6}$$

Here, $a_n = \alpha_n/\sqrt{n}$, $a_{-n} = \alpha_{-n}/\sqrt{n}$ and $a_0 = \alpha_0 = p$. The matrices A and $B^{\pm}$ are defined by the generating functions in equation (4.8). Explicitly

$$A_{nm}(z) = \begin{cases} \frac{\sqrt{nm}}{n+m} \binom{-\frac{1}{2}}{n} \binom{-\frac{1}{2}}{m} (-z)^{-n-m}, & n, m \neq 0 \\ \frac{1}{\sqrt{n}} \binom{-\frac{1}{2}}{n} (-z)^{-n}, & n \neq 0, \ m = 0 \\ -\ln(-4z), & n = m = 0, \end{cases} \tag{5.7}$$

and

$$B^{\pm}_{nr}(z) = \begin{cases} \mp \sqrt{\frac{n}{r}} \binom{\mp r}{n} (-z)^{\mp r - n}, & n \neq 0 \\ \mp \frac{1}{\sqrt{r}} (-z)^{\mp r}, & n = 0. \end{cases} \tag{5.8}$$

$W(\phi, z)$ forms one of the off–diagonal components of the conformal field of dimension $d/16 + N$, where N is the occupation number of the state. The full vertex is

$$\mathbb{V}(\phi, z) = W(\phi, z) I_{\sigma\epsilon} + \overline{W}(\phi, z) I_{\epsilon\sigma},$$

which we can write more succinctly in matrix form, as

$$\mathbb{V}(\phi, z) = \begin{pmatrix} 0 & \overline{W}(\phi, z) \\ W(\phi, z) & 0 \end{pmatrix}. \tag{5.9}$$

Here, $\overline{W}(\phi, z)$ is the conjugate field:

$$\overline{W}(\phi, z) = z^{-2\Delta_\phi} (W(\phi^c, 1/\overline{z}))^{\dagger}.$$

We shall consider charge conjugation in detail in the next section.

5.3 The Twisted Operator Cocycles.

It is convenient to define

$$C_T(\alpha) = 2^{\alpha^2}\hat{C}_\sigma(\alpha).$$

For the reflection twist $\eta(\alpha,\beta) = 4^{-\alpha.\beta}$ and $\Omega(\alpha,\beta) = (-)^{\alpha.\beta}$, and so with the above choice the twisted operator cocycles can be chosen to generate the same projective representation as the untwisted operators:

$$C_T(\alpha)C_T(\beta) = \varepsilon(\alpha,\beta)C_T(\alpha+\beta). \tag{5.10}$$

We will choose both $\hat{C}(\alpha)$ and $C_T(\alpha)$ to have the following hermiticity

$$C^\dagger(\alpha) = C(-\alpha). \tag{5.11}$$

This implies that

$$\varepsilon(-\alpha,-\beta) = \varepsilon(\beta,\alpha). \tag{5.12}$$

The phases in the transformations

$$\begin{aligned} \hat{\sigma} : \; C_T(\alpha) &\longmapsto \sigma_\alpha C_T(-\alpha), \\ \hat{\sigma} : \; \hat{C}(\alpha) &\longmapsto \sigma_\alpha \hat{C}(-\alpha), \end{aligned}$$

must satisfy equations (3.52), *i.e*

$$\begin{aligned} \sigma_\alpha\sigma_\beta\varepsilon(-\alpha,-\beta) &= \sigma_{\alpha+\beta}\varepsilon(\alpha,\beta) \\ \sigma_\alpha\sigma_{-\alpha} &= 1 \\ \sigma_\alpha^* &= \sigma_\alpha. \end{aligned}$$

Using (5.12) the first equation above becomes

$$\sigma_\alpha\sigma_\beta = \frac{\varepsilon(\alpha,\beta)}{\varepsilon(\beta,\alpha)}\sigma_{\alpha+\beta} = (-)^{\alpha.\beta}\sigma_{\alpha+\beta},$$

so with our conventions

$$\sigma_\alpha = (-)^{\frac{1}{2}\alpha^2}.$$

Furthermore, the twisted operator cocycles are invariant under σ. Summarizing, we have

$$C_T^\dagger(\alpha) \;=\; C_T(-\alpha) \;=\; (-)^{\frac{1}{2}\alpha^2} C_T(\alpha). \tag{5.13}$$

In the twisted sector the operator cocycles are just finite matrices of dimension

$$c_T \;=\; \left|\frac{\Lambda}{A}\right| \;=\; \left|\frac{\Lambda}{(2\Lambda^*)\cap\Lambda}\right|^{\frac{1}{2}}. \tag{5.14}$$

For the reflection twist, A is the set of vectors in Λ for which $(-)^{\alpha.\beta} = 1$, for all α and β in A. Below, we shall calculate the cosets Λ/A explicitly for each of the simple simply–laced Lie algebras, and hence evaluate the ground state degeneracy (or defect [58]) c_T. It is worth repeating that the choice for A is not unique, but all choices are equivalent. Both the Leech lattice and the E_8 root lattices are self–dual, *i.e.* $\Lambda^* = \Lambda$. in these cases we may apply (5.14) directly:

$$c_T \;=\; 2^{d/2}, \qquad (\Lambda^* \;=\; \Lambda).$$

Since $\widetilde{V}_0^T(\alpha)$ does not annihilate the ground state ($\alpha \in \Lambda_2$)

$$\widetilde{V}_0^T(\alpha) \mid 0\rangle \otimes \phi = \mid 0\rangle \otimes \frac{1}{4} C_T(\alpha)\phi. \tag{5.15}$$

Therefore, the twisted operator cocycles must themselves generate g_0, on the positive roots of g:

$$[C_T(\alpha), C_T(\beta)] \;=\; \begin{cases} 2\varepsilon(\alpha,\beta)C_T(\alpha+\beta) & \alpha.\beta = -1 \\ -2\varepsilon(\alpha,-\beta)C_T(\alpha-\beta) & \alpha.\beta = 1 \\ 0 & \text{otherwise.} \end{cases}$$

This is the g_0 representation which we have called $\mathcal{R}_0$. In particular, the maximal set of commuting operators generate the Cartan Subalgebra of g_0. This is the set

$$C_T(\alpha), \qquad \alpha \;\in\; \Lambda_2^+ \cap A. \tag{5.16}$$

Notice that $\Lambda_2^+ \cap A$ is just the maximal set of orthogonal positive roots of g, which explains our earlier statement about the rank of g_0.

For the reflection twist, the twisted operator cocycles may be constructed rather concisely from elements of an appropriate Clifford algebra [6,42]. Let $C(N)$ be the Clifford algebra generated by the unit matrix and the elementary, hermitian **gamma matrices**, γ^i $i = 1, ..., N$, which satisfy the anticommutation relations

$$\{\gamma^i, \gamma^j\} = 2\delta^{ij}, \quad (\gamma^i)^\dagger = \gamma^i.$$

The algebra $C(N)$ is then spanned by the 2^N antisymmetric products

$$\gamma^i\gamma^j........\gamma^k.$$

including γ^i and $\mathbb{1}$, the unit matrix. The gamma matrices in an irreducible representation are $2^{N/2} \times 2^{N/2}$ dimensional (we are taking N to be even only). It is useful to notice that if the number of labels of an antisymmetric product is either $4\mathbb{Z}$ or $4\mathbb{Z}+1$ then the product is hermitian, otherwise it is antihermitian. As usual we define the product

$$\gamma^{N+1} = \prod_{i=1}^{N} \gamma^i,$$

with

$$\left\{\gamma^{N+1}, \gamma^i\right\} = 0.$$

The **even**(**odd**) elements of $C(N)$ are those which (anti–)commute with γ^{N+1}.

The rationale for constructing a representation of the twisted operator cocycles. involves identifying a set of independent elements of an appropriate Clifford algebra with a set of basis vectors for Λ (α_i, $i = 1, ..., d$). Let us suppose that we have managed to assign products $\hat{\gamma}^i$ to the basis vectors α_i, so that

$$\hat{\gamma}^i\hat{\gamma}^j = \Omega(\alpha_i, \alpha_j)\hat{\gamma}^j\hat{\gamma}^i.$$

The operator cocycles for the whole lattice are then built up using the projective group property (3.39). For a vector $\alpha = \sum n_i\alpha_i$ we have

$$C_T(\alpha) = \zeta_\alpha \prod_{i=1}^{d} \left(\hat{\gamma}^i\right)^{n_i}, \tag{5.17}$$

where the phase ζ_α is determined by the hermiticity constraint in equation (5.13).

One advantage of constructing the twisted operator cocycles out of gamma matrices is that the charge conjugation operation has a well known representation in the Clifford algebra. The charge conjugate of a state is

$$\phi^c = \mathrm{C}^{-1}\phi^*, \tag{5.18}$$

where C is the charge conjugation matrix appropriate to the Clifford algebra in which the operator cocycles lie. C has the following action on the algebra

$$\eta_\alpha C_T^*(\alpha) = \mathrm{C}C_T(\alpha)\mathrm{C}^{-1}, \tag{5.19}$$

where η_α is ± 1 depending on whether ζ_α is real or complex, respectively. The Clifford algebras we use are all appropriate to a Euclidean space, in the sense that δ^{ij} is a Euclidean metric, in this case [59]

$$\mathrm{C}\mathrm{C}^\dagger = \mathrm{C}^\dagger\mathrm{C} = 1, \quad \text{and } \mathrm{C}\mathrm{C}^* = \epsilon, \tag{5.20}$$

where $\epsilon = 1$ if N is $8\mathbb{Z}$ or $8\mathbb{Z}+2$, otherwise $\epsilon = -1$. The significance of ϵ is that

$$(\phi^c)^c = \epsilon\phi.$$

We now give a case by case construction for all the situations that interest us. When Λ is the root lattice of a simply–laced Lie algebra it is expedient to choose the α_i's to be the simple roots; the assignment of elements of the Clifford algebra can then be illustrated on the Dynkin diagram.

When evaluating the cosets Λ/A, it is useful to notice that the set B, for the reflection twist, is precisely 2Λ. We will label the cosets Λ/A by a representative n_i, $i = 1, ..., c_T$, and we will introduce the notation

$$[\beta_1, \beta_2, ..., \beta_n] \equiv \{0, \beta_i + \beta_j (i \neq j), \beta_i + \beta_j + \beta_k (i \neq j \neq k), ..., \beta_1 + ... + \beta_n\}.$$

for arbitrary vectors β_i. This set has dimension 2^n.

<u>$1 : \Lambda = \Lambda_R(A_{N-1})$</u>

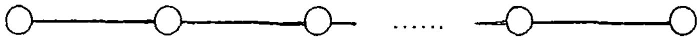
(i) N is even.

$$\begin{aligned} A \bmod B &= [\alpha_1, \alpha_3, \ldots, \alpha_{N-1}], \\ n_i &\in [\alpha_2, \alpha_4, \ldots, \alpha_{N-2}], \\ A \cap \Lambda_2^+ &= \{\alpha_1, \alpha_3, \ldots, \alpha_{N-1}\}. \end{aligned}$$

$\mathrm{rank}(g_0) = \mid A \cap \Lambda_2^+ \mid = N/2$, which is consistent with $g_0 = D_{N/2}$, and

$$c_T = \mid \Lambda/A \mid = \mid [\alpha_2, \alpha_4, \ldots, \alpha_{N-2}] \mid = 2^{(N-2)/2}.$$

After this preliminary, we now go on to assign elements of the Clifford algebra $C(N)$ to the simple roots.



$\gamma^1\gamma^2 \qquad \gamma^2\gamma^3 \qquad \gamma^3\gamma^4 \qquad \qquad \gamma^{N-1}\gamma^N$

The dimension of this representation is $2^{N/2}$, *i.e.* the representation is a reducible $\mathbf{s} + \mathbf{c}$.[‡] On the roots the set $\{\gamma^i\gamma^j\}$ is generated, while on the whole lattice all the even elements of the Clifford algebra are generated. As expected we can consistently take an irreducible spinor representation of dimension $c_T = 2^{N/2-1}$, by imposing a chiral projection on the representation space:

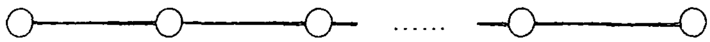
$$\gamma^{N+1}\phi = \phi.$$

(ii) N is odd.

$$\begin{aligned} A \bmod B &= [\alpha_1, \alpha_3, \ldots, \alpha_{N-2}], \\ n_i &\in [\alpha_2, \alpha_4, \ldots, \alpha_{N-1}], \\ A \cap \Lambda_2^+ &= \{\alpha_1, \alpha_3, \ldots, \alpha_{N-2}\}. \end{aligned}$$

$\mathrm{rank}(g_o) = (N-1)/2$, which is consistent with $g_0 = B_{(N-1)/2}$, and $c_T = 2^{(N-1)/2}$.

The appropriate Clifford algebra is $C(N-1)$:



$\gamma^1\gamma^2 \qquad \gamma^2\gamma^3 \qquad \gamma^3\gamma^4 \qquad \qquad \zeta\gamma^{N-1}\gamma^N$

where ζ is a phase chosen so that $C_T(\alpha_{N-1})$ is antihermitian, *i.e.* $\zeta = i$ when $N = 4\mathbb{Z}+3$, and is 1 otherwise. The representation is a spinor of $B_{(N-1)/2}$, of dimension $2^{(N-1)/2}$. On the roots the set $\{\gamma^i\gamma^j, \gamma^i\gamma^N\}$ is generated, while on the whole lattice it is the entire Clifford algebra.

‡ In our notation $\mathbf{s}$ is a spinor, while $\mathbf{c}$ is the spinor of opposite chirality.

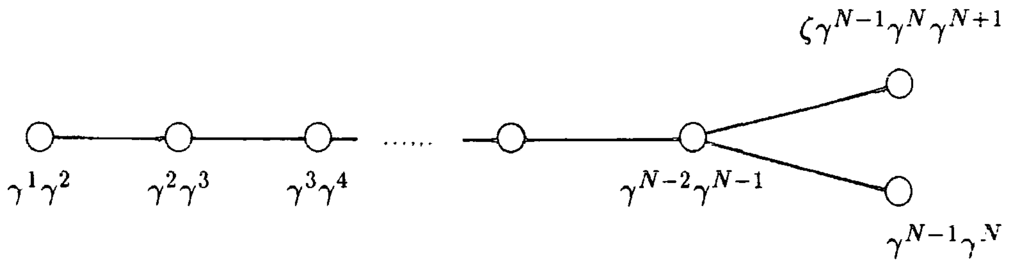
$2 : \Lambda = \Lambda_R(D_N)$

(i) N is even.

$$
\begin{aligned}
A \bmod B &= [\alpha_1, \alpha_3, ..., \alpha_{N-3}, \alpha_{N-1}, \alpha_N], \\
n_i &\in [\alpha_2, \alpha_4, ..., \alpha_{N-2}], \\
A \cap \Lambda_2^+ &= \{\alpha_1, \alpha_3, ..., \alpha_{N-3}, \alpha_{N-1}, \alpha_N, \\
&\qquad \alpha_j + 2\sum_{i=j+1}^{N-2} \alpha_i + \alpha_{N-1} + \alpha_N, \;\; 1 \le j \le N-3, \; j \in 2\mathbb{Z}+1\}.
\end{aligned}
$$

$\mathrm{rank}(g_0) = N$, which is consistent with $g_0 = D_{N/2} + D_{N/2}$, and $c_T = 2^{N/2-1}$.

The appropriate Clifford algebra is $C(N)$. One possible assignment is



The dimension of this representation is $2^{N/2}$; a reducible $(\mathbf{s}, 1) + (1, \mathbf{s})$ of g_0. The phase ζ is chosen to ensure that $\zeta(\gamma^{N-1}\gamma^N\gamma^{N+1})$ is antihermitian, *i.e.* $\zeta = i$, when $N = 4\mathbf{Z} + 2$, and is 1 otherwise. On the roots the set $\{\gamma^i\gamma^j, \gamma^i\gamma^j\gamma^{N+1}\}$ of elements of $C(N)$ is generated, whilst on the lattice all the even elements are generated. In view of this, as before, we can impose a chiral projection to end up with an irreducible $(\mathbf{s}, 1)$ of g_0, with dimension c_T. To see this explicitly let us identify the two $D_{N/2}$ algebras, which we can do as follows. In terms of an orthogonal basis e_i, $i = 1, ..., N$, the roots of D_N are $\pm e_i \pm e_j$, $(i \neq j)$. Splitting the positive roots into the two mutually orthogonal sets, $\{\alpha_{ij}\} = \{e_i - e_j\}$ and $\{\tilde{\alpha}_{ij}\} = \{e_i + e_j\}$, the two $D_{N/2}$ are identified with the zeroth moments of

$$
V^{\pm}(\alpha_{ij}, z) = \frac{1}{2}\gamma^i\gamma^j\left(V^T(\alpha_{ij}, z) \pm \gamma^{N+1}V^T(\tilde{\alpha}_{ij}, z)\right),
$$

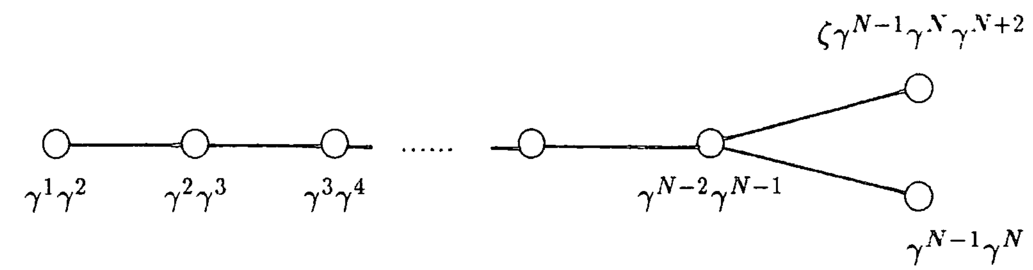
where we have written the operator cocycles explicitly. It is now apparent that $V_0^-(\alpha_{ij})$ annihilates the chiral spinor, consequently the irreducible representation is a singlet of one of the $D_{N/2}$'s.

(ii) N is odd.

$$
\begin{aligned}
A \bmod B &= [\alpha_2, \alpha_4, ..., \alpha_{N-3}, \alpha_{N-1}, \alpha_N], \\
n_i &\in [\alpha_1, \alpha_3, ..., \alpha_{N-2}], \\
A \cap \Lambda_2^+ &= \{\alpha_2, \alpha_4, ..., \alpha_{N-3}, \alpha_{N-1}, \alpha_N, \\
&\quad \alpha_j + 2\sum_{i=j+1}^{N-2} \alpha_i + \alpha_{N-1} + \alpha_N, \;\; 2 \leq j \leq N-2, \; j \in 2\mathbb{Z}\}.
\end{aligned}
$$

$\mathrm{rank}(g_0) = N-1$, which is consistent with $g_0 = B_{(N-1)/2} + B_{(N-1)/2}$, and $c_T = 2^{(N-1)/2}$.

The appropriate Clifford algebra is $C(N+1)$. One possible assignment is



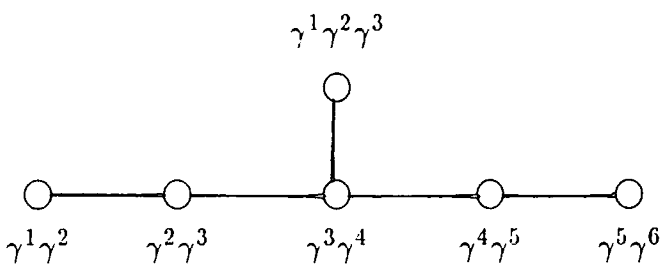
The dimension of this representation is $2^{(N+1)/2}$, a reducible $(\mathbf{s}, 1) + (1, \mathbf{s})$ of g_0. The phase ζ is i, when $N = 4\mathbb{Z}+1$, and 1 otherwise. On the roots the set $\{\gamma^i\gamma^j, \gamma^i\gamma^j\gamma^{N+2}\}$, with $i, j \neq N+1$, is generated, whilst on the lattice a subset of the even elements is generated. By imposing a chiral projection the representation becomes an irreducible $(\mathbf{s}, 1)$ as before, but notice that the set generated on the lattice is not complete, even in the restricted sense of all the even elements.

<u>$3 : \Lambda = \Lambda_R(E_6)$</u>

$$
\begin{aligned}
A \bmod B &= [\alpha_1, \alpha_3, \alpha_5], \\
n_i &\in [\alpha_2, \alpha_4, \alpha_6], \\
A \cap \Lambda_2^+ &= \{\alpha_1, \alpha_3, \alpha_5, \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6\}.
\end{aligned}
$$

$\mathrm{rank}(g_0) = 4$, which is consistent with $g_0 = C_4$, and $c_T = 8$.

The appropriate Clifford algebra is $C(6)$, and one possible assignment is



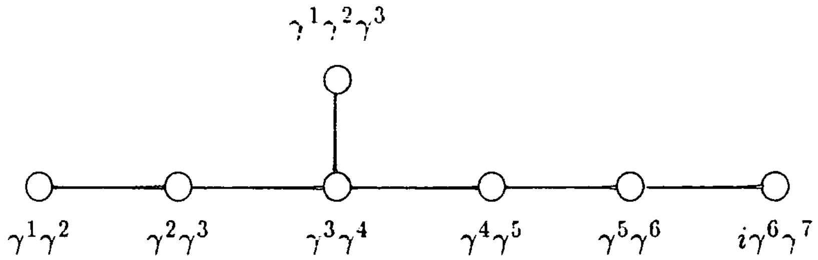
The dimension of the representation is 8; an irreducible $\mathbf{8}$ of C_4. On the roots the set $\{\gamma^i\gamma^j, \gamma^i\gamma^j\gamma^k, \gamma^7\}$ is generated, whilst on the lattice the entire Clifford algebra is generated.

$\underline{4 : \Lambda = \Lambda_R(E_7)}$

$$
\begin{aligned}
A \bmod B &= [\alpha_2, \alpha_4, \alpha_6, \alpha_7], \\
n_i &\in [\alpha_1, \alpha_3, \alpha_5], \\
A \cap \Lambda_2^+ &= \{\alpha_2, \alpha_4, \alpha_6, \alpha_7, \alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_7, \\
&\quad \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7, \\
&\quad 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + 2\alpha_7\}.
\end{aligned}
$$

rank$(g_0) = 7$, which is consistent with $g_0 = A_7$, and $c_T = 8$.

The appropriate Clifford algebra is $C(6)$, and one possible assignment is



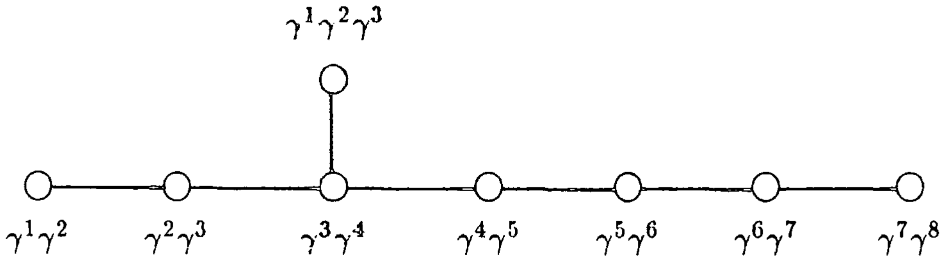
The representation is an irreducible **8** of A_7. On the roots, the whole Clifford algebra, except for **1**, is generated, whilst on the whole lattice the entire Clifford algebra is generated.

$\underline{6 : \Lambda = \Lambda_R(E_8)}$

$$
\begin{aligned}
A \bmod B &= [\alpha_1, \alpha_3, \alpha_5, \alpha_7], \\
n_i &\in [\alpha_2, \alpha_4, \alpha_6, \alpha_8], \\
A \cap \Lambda_2^+ &= \{\alpha_1, \alpha_3, \alpha_5, \alpha_7, \\
&\quad \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_8, \\
&\quad \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7 + 2\alpha_8, \\
&\quad \alpha_1 + 2\alpha_2 + 4\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 + 2\alpha_8, \\
&\quad 2\alpha_1 + 4\alpha_2 + 5\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 + 2\alpha_8\}.
\end{aligned}
$$

rank$(g_0) = 8$, which is consistent with $g_0 = D_8$, and $c_T = 16$.

The appropriate Clifford algebra is $C(8)$, and one possible assignment is



The dimension of the representation is 16; an irreducible $\mathbf{16}_v$ (vector) of D_8. The set $\{\gamma^i\gamma^j, \gamma^i\gamma^j\gamma^9, \gamma^i\gamma^j\gamma^k, \gamma^i\gamma^9\}$ is generated on the roots, in fact all the antihermitian elements of $C(8)$. On the lattice the entire Clifford algebra is generated.

$7 : \Lambda = \Lambda(spin(32)/\mathbb{Z}_2)$

The $spin(32)/\mathbb{Z}_2$ lattice is the union of the root lattice of D_{16} with one of the spinor cosets of $\Lambda_R^*(D_{16})$. It is, in fact, one of the two self–dual lattices in sixteen dimensions, the other being the root lattice of $E_8 + E_8$ [11,74]. Since the spinor weights have length squared 4, Λ_2 is the same as that for $\Lambda_R(D_{16})$ and so $g = D_{16}$. We may apply equation (5.14) directly to calculate

$$c_T = 2^8.$$

We take the following basis for the lattice:

$$\alpha_i, \quad (i = 1, \ldots, 15), \quad \lambda = \frac{1}{2}\sum_{i=1}^{7} \alpha_{2i-1} + \frac{1}{2}\alpha_{16},$$

where $\{\alpha_i\}$ are the simple roots of D_{16}, and λ is a spinor weight. The reason why this forms a bona–fide basis for Λ is due to the fact that we can express α_{16} in terms of it:

$$\alpha_{16} = 2\lambda - \sum_{i=1}^{7} \alpha_{2i-1}.$$

We have already assigned elements of the Clifford algebra $C(16)$ to the simple roots α_i, $i = 1, \ldots, 15$, of D_{16}, so our task is to select an element for λ, consistent with the inner products

$$\begin{aligned} \lambda.\alpha_i &= \pm 1, \quad 0 \leq i \leq 14 \\ \lambda.\alpha_{15} &= 0. \end{aligned}$$

One possible choice for $C_T(\lambda)$ is

$$C_T(\lambda) = \left(\prod_{i=1}^{8} \gamma^{2i-1}\right) \gamma^{16}.$$

This clearly anticommutes with $C_T(\alpha_i)$, $i = 1, ..., 14$, but commutes with $C_T(\alpha_{15})$, as the inner products require. It is also hermitian as required by (5.13). $C_T(\lambda)$ is an odd element of $C(16)$, *i.e.* it anticommutes with γ^{17}, this has the consequence that the whole Clifford algebra is generated on the lattice, in contrast to the D_{16} case, and one cannot impose the chiral projection. The ground state in the twisted sector is a reducible $(\mathbf{128}_s, 1) + (1, \mathbf{128}_s)$ of g_0, with dimension $c_T = 2^8$.

$\underline{8 : \Lambda = \Lambda_L}$

In order to discuss the Leech lattice it is convenient to consider its relationship to the unique even self–dual Lorentzian lattice $II^{25.1}$ in $\mathbf{R}^{25.1}$. This lattice can be defined as the set of vectors $\{x\}$ in $\mathbf{R}^{25.1}$ for which [42,75]:

1. $x \in \mathbf{Z}^{25.1}$ or $x - l \in \mathbf{Z}^{25.1}$ and
2. $x.l \in \mathbf{Z}$,

where $l = (1/2, 1/2, ...; 1/2)$, and the appropriate metric is $\mathrm{diag}((+)^{25}, (-))$.

To construct the Leech lattice consider the light–like vector

$$k = (0, 1, 2, ..., 24; 70) \in II^{25.1}.$$

Form the sublattice $II_k^{25.1} \subset II^{25.1}$ orthogonal to k (this includes the subspace K spanned by k itself). The quotient $II_k^{25.1}/K$ defines an even self–dual 24 dimensional Euclidean lattice, which with our choice of k is isomorphic to the Leech lattice.

We will now use the fact that it is relatively easy to assign a set of gamma matrices from $C(26)$ to each vector in $II^{25.1}$, consistent with the symmetry factor $(-)^{x.y}$, to infer a set for the Leech lattice. This representation is in fact 2^{13} dimensional. However, on the subspace $II_k^{25.1}$ the operator cocycle $C_T(k)$ commutes with all the $C_T(\alpha)$'s, and so the representation is reducible. To show this explicitly it is more convenient to work with $C(24) \otimes C(2)$ rather than $C(26)$.

Introducing a set of unit orthogonal vectors in $\mathbb{R}^{25,1}$, $\{e_i\}$, $i = 1, ..., 26$:

$$e_i . e_j = \mathrm{diag}\Big((+)^{25}, (-)\Big),$$

it is easily verified that the following vectors form a basis for $II^{25,1}$,

$$\begin{aligned} \alpha_i &= e_{i+2} - e_{i+3} \quad i = 1, ..., 23 \\ \alpha_{24} &= -l \\ \alpha_{25} &= e_{25} + e_{26} \\ \alpha_{26} &= k. \end{aligned}$$

To simplify matters we have deliberately included k in our basis.

Let $C(24)$ and $C(2)$ be spanned by products of the elementary gamma matrices $\{\gamma^i\}$, $i = 1, ..., 24$ and $\{\sigma^i\}$. $i = 1, 2$, respectively. It is now a relatively straightforward matter to assign operator cocycles to the above basis. One consistent choice is:

$$\begin{aligned} C_T(\alpha_i) &= \gamma^i\gamma^{i+1} \otimes \sigma^2 \quad i = 1, ..., 22 \\ C_T(\alpha_{23}) &= \gamma^{23}\gamma^{24} \otimes \mathbb{1} \\ C_T(\alpha_{24}) &= \gamma^{24}\gamma^{25} \otimes \sigma^2 \\ C_T(\alpha_{25}) &= \gamma^{23}\gamma^{24}\gamma^{25} \otimes \mathbb{1} \\ C_T(\alpha_{26}) &= \mathbb{1} \otimes \sigma^1. \end{aligned}$$

Notice that $C_T(k) = \mathbb{1} \otimes \sigma^1$, therefore on the sublattice $II_k^{25,1}$ the operator cocycles are of the form $A \otimes (\sigma^1)^n$, for $A \in C(24)$. This exposes the reducibility explicitly: the operator cocycles on the quotient $II_k^{25,1}/K$ are simply given by the $C(24)$ piece. Therefore we have succeeded in assigning elements of $C(24)$ to the Leech lattice consistent with the symmetry factor $(-)^{\alpha.\beta}$. In fact our construction can be generalized to the other 23 even self-dual Euclidean (Niemeier) lattices which can be generated from $II^{25,1}$ via other (inequivalent) light-like vectors in an analogous fashion. We shall discuss these other lattices and their relationship to the Leech lattice in the last chapter.

6. THE REFLECTION TWIST II.

In this chapter we continue our investigation of the reflection twist, by evaluating the operator algebra of the TSEVs for the ground and first excited states in the twisted sector.

6.1 The Twisted–Untwisted Operator Product Expansion.

The cases of interest are when the TSEVs have unit conformal dimension for the Lie algebras, and dimension two for the cross-bracket algebra, *i.e.*

1. $d = 16$, in which case the ground state has unit dimension.
2. $d = 8$, in which case the first excited state has unit dimension.
3. $d = 24$, in which case the first excited state has dimension two.

The VW OPE has been computed in section 4.2. To use the expression there, we must calculate

$$\widetilde{V}_n^T(\alpha) \mid \Phi\rangle, \quad \text{with } n \geq 1 - \frac{1}{2}\alpha^2.$$

For the Lie algebra cases $\alpha \in \Lambda_2$.

1. $d = 16$

$$\widetilde{V}_n^T(\alpha) \mid 0\rangle \otimes \phi = \begin{cases} \mid 0\rangle \otimes \frac{1}{4}C_T(\alpha)\phi & n = 0 \\ 0 & n \geq 1. \end{cases}$$

2. $d = 8$

$$\widetilde{V}_n^T(\alpha)e.c_{-\frac{1}{2}} \mid 0\rangle \otimes \phi = \begin{cases} (e - 2(e.\alpha)\alpha).c_{-\frac{1}{2}} \mid 0\rangle \otimes \frac{1}{4}C_T(\alpha)\phi & n = 0 \\ 0 & n \geq 1. \end{cases}$$

For the cross-bracket algebra case $\alpha \in \Lambda_4$. In addition, we do not need to know the simple pole in the expansion. in other words, we do need to calculate the $n = -1$ term.

$$\widetilde{V}_n^T(\alpha)e.c_{-\frac{1}{2}} \mid 0\rangle \otimes \phi = \begin{cases} (e - 2(e.\alpha)\alpha).c_{-\frac{1}{2}} \mid 0\rangle \otimes 2^{-4}C_T(\alpha)\phi & n = 0 \\ 0 & n \geq 1. \end{cases}$$

From these results we deduce the OPEs

$\underline{\alpha \in \Lambda_2}$

$$\begin{aligned} W(\phi, z)\tilde{V}(\alpha, \omega) &= \frac{1/4}{z-\omega} W(C_T(\alpha)\phi, \omega) + Reg \\ W(e, \phi, z)\tilde{V}(\alpha, \omega) &= \frac{1/4}{z-\omega} W(e - 2(e.\alpha)\alpha, C_T(\alpha)\phi, \omega) + Reg. \end{aligned} \tag{6.1}$$

$\underline{\alpha \in \Lambda_4}$

$$W(e, \phi, z)\tilde{V}(\alpha, \omega) = \frac{2^{-4}}{(z-\omega)^2} W(e - 2(e.\alpha)\alpha, C_T(\alpha)\phi, \omega) + O(z-\omega)^{-1}. \tag{6.2}$$

In addition, for the cross–bracket algebra we also require the operator product of the TSEV with $V(\epsilon, \eta, z)$. From

$$V_n(\epsilon, \eta) e.c_{-\frac{1}{2}} \mid 0\rangle \otimes \phi = \begin{cases} \frac{1}{2}((e.\eta)\epsilon + (e.\epsilon)\eta).c_{-\frac{1}{2}} \mid 0\rangle \otimes \phi & n = 0 \\ 0 & n \geq 1, \end{cases}$$

we deduce

$$W(e, \phi, z)V(\epsilon, \eta, \omega) = \frac{1/2}{(z-\omega)^2} W((e.\eta)\epsilon + (e.\epsilon)\eta, \phi, \omega) + O(z-\omega)^{-1}. \tag{6.3}$$

OPEs involving $\overline{W}$ follow from the above OPEs by conjugation. For example

$$\tilde{V}(\alpha, \omega)\overline{W}(\phi, z) = \frac{1/4}{z-\omega} \overline{W}(C_T(\alpha)\phi, \omega) + Reg.$$

6.2 The Twisted–Twisted Operator Product Expansion.

In this section we will establish, and to some extent verify, the OPE of a pair of TSEVs, for the reflection twist. There are two expansions that must be considered,

$$(a) \qquad F = \overline{W}(\Psi, z)W(\Phi, \omega) \tag{6.4}$$

and

$$(b) \qquad \overline{F} = W(\Psi, z)\overline{W}(\Phi, \omega). \tag{6.5}$$

with the following states

(1) $d = 16$, Ψ and Φ are both ground states, $\Delta = 1$:

$$|\,\Psi\rangle = |\,0\rangle \otimes \psi, \quad |\,\Phi\rangle = |\,0\rangle \otimes \phi.$$

(2) $d = 8$ and 24. Ψ and Φ are both first excited states with $\Delta = 1$ and 2, respectively:

$$|\,\Psi\rangle = e.c_{-\frac{1}{2}}\,|\,0\rangle \otimes \psi, \quad |\,\Phi\rangle = f.c_{-\frac{1}{2}}\,|\,0\rangle \otimes \phi.$$

In the later case when $d = 24$, as we remarked before. we will not require the simple pole, which does not contribute to the cross–bracket algebra.

We have not discovered a way to evaluate the OPEs directly, due to the complicated form of the operators, however, a direct attack is likely to benefit from the formalism developed in refs.[35,36]. Our route will be more circuitous, and we shall only be able to infer the OPEs. Nevertheless, our final answers are those which are expected from the factorization of two twisted strings onto an untwisted string, and this corroborates our conclusions.

For (a) we evaluate the product with a vacuum state on the right, this greatly simplifies the manipulation and allows us to extract the expansion. For product (b), all we have been able to do is to calculate its expectation value between two twisted ground states. From the expectation value, one is able to isolate the poles and hence infer the OPE. This calculation is far from trivial, and is interesting because, as we shall see, it will place some restrictions on the lattice in order that the operator algebra be well defined. The calculation of a four twist correlation function, for the reflection twist, has been done in refs.[33,34], from a path integral approach; we shall compare the results of our operator approach in chapter 9.

Product (a). The technique we use is essentially a generalization of that described in section 2.5. Consider

$$\begin{aligned}\overline{W}(\Psi, z)W(\Phi, \omega)\,|\,0\rangle &= \overline{W}(\Psi, z)e^{\omega L^T_{-1}}\,|\,\Phi\rangle \\ &= e^{\omega L_{-1}}\overline{W}(\Psi, z-\omega)\,|\,\Phi\rangle.\end{aligned}$$

In the above, we used the fact that $\overline{W}$ and W are generalized conformal fields. The problem is now reduced to that of evaluating

$$\overline{W}(\Psi,\varepsilon)\mid\Phi\rangle, \qquad \varepsilon = z-\omega.$$

Case 1 :

$$\overline{W}(\psi,\varepsilon)\mid\phi\rangle = \varepsilon^{-2}\sum_{\alpha\in\Lambda}\left\{\psi^T C C_T^\dagger(\alpha)\phi\right\} e^{A(1/\overline{\varepsilon})^\dagger}\mid\alpha\rangle.$$

Since

$$\exp\left(\frac{1}{2}\alpha^2 A_{00}(1/\overline{\varepsilon})^\dagger\right) = \left(-\frac{\varepsilon}{4}\right)^{\frac{1}{2}\alpha^2},$$

the only singular terms come from $\alpha = 0$ or $\alpha\in\Lambda_2$. Using the hermiticity of the operator cocycle, we find

$$\overline{W}(\psi,\varepsilon)\mid\phi\rangle = \varepsilon^{-2}\left\{\psi^T C\phi\right\}\mid 0\rangle + \frac{1}{4}\varepsilon^{-1}\sum_{\alpha\in\Lambda_2}\left\{\psi^T C C_T(\alpha)\phi\right\}\mid\alpha\rangle + Reg. \tag{6.6}$$

Case 2 :

$$\begin{aligned}\overline{W}(e,\psi,\varepsilon)\mid f,\phi\rangle &= \varepsilon^{-d/8-1}\sum_{\alpha\in\Lambda}\left\{\psi^T C C_T^\dagger(\alpha)\phi\right\} e^{A(1/\overline{\varepsilon})^\dagger}\\ &\qquad\times\langle 0\mid e.c_{\frac{1}{2}} : e^{B(1/\overline{\varepsilon})^\dagger} : f.c_{-\frac{1}{2}}\mid 0\rangle\mid\alpha\rangle.\end{aligned}$$

The twisted expectation value can easily be evaluated:

$$\langle 0\mid ....\mid 0\rangle = e.f + \sum_{n,m=0}^{\infty}(e.a_{-n})P_{nm}(f.a_{-m}),$$

where we have introduced the matrix P_{nm},

$$\begin{aligned}P_{nm} &= B^+_{n\frac{1}{2}}(1/\overline{\varepsilon})^\dagger B^-_{m\frac{1}{2}}(1/\overline{\varepsilon})^\dagger\\ &= \begin{cases} -2\sqrt{nm}\binom{-\frac{1}{2}}{n}\binom{\frac{1}{2}}{m}(-\varepsilon)^{n+m} & n,m\neq 0\\ -2\sqrt{n}\binom{-\frac{1}{2}}{n}(-\varepsilon)^{n} & n\neq 0,\ m=0\\ -2\sqrt{m}\binom{\frac{1}{2}}{m}(-\varepsilon)^{m} & n=0,\ m\neq 0\\ -2 & n=m=0.\end{cases}\end{aligned}$$

$d=8$: In this case there is a double pole and a single pole, coming from $\alpha = 0$ and

$\alpha \in \Lambda_2$, respectively.

$$\begin{aligned}\overline{W}(e,\psi,\varepsilon) \mid f.\phi\rangle &= \varepsilon^{-2}\left\{\psi^T C\phi\right\}(e.f) \mid 0\rangle \\ &+ \frac{1}{4}\varepsilon^{-1}\sum_{\alpha\in\Lambda_2}\left\{\psi^T C C_T(\alpha)\phi\right\}(e.f - 2(e.\alpha)(f.\alpha)) \mid \alpha\rangle + Reg.\end{aligned} \tag{6.7}$$

$\underline{d = 24:}$ Our sole interest in $d = 24$ is when Λ is the Leech lattice, so Λ_2 is empty. Using

$$\begin{aligned}\exp\left(A(1/\overline{\varepsilon})^\dagger\right) \mid 0\rangle &= \left(1 + \frac{\varepsilon^2}{16}(a_{-1}.a_{-1}) + O(\varepsilon^3)\right) \mid 0\rangle \\ \exp\left(A(1/\overline{\varepsilon})^\dagger\right) \mid \alpha\rangle &= \left(-\frac{\varepsilon}{4}\right)^2(1 + O(\varepsilon)) \mid \alpha\rangle, \quad \alpha \in \Lambda_4,\end{aligned}$$

it is relatively straightforward to calculate

$$\begin{aligned}\overline{W}(e,\psi,\varepsilon) \mid f.\phi\rangle &= \varepsilon^{-4}\left\{\psi^T C\phi\right\}(e.f) \mid 0\rangle \\ &+ \varepsilon^{-2}\Big\{\left\{\psi^T C\phi\right\}\left(\frac{1}{16}(e.f)(a_{-1}.a_{-1}) + \frac{1}{2}(e.a_{-1})(f.a_{-1})\right) \mid 0\rangle \\ &+ \frac{1}{16}\sum_{\alpha\in\Lambda_4}\left\{\psi^T C C_T(\alpha)\phi\right\}(e.f - 2(e.\alpha)(f.\alpha)) \mid \alpha\rangle\Big\} + O(\varepsilon^{-1}).\end{aligned} \tag{6.8}$$

Now that we have evaluated $\overline{W}(\Psi,\varepsilon) \mid \Phi\rangle$ in all the cases of interest, we can extract the OPEs. Using the fact that

$$e^{zL_{-1}} \mid \phi\rangle = V(\phi, z) \mid 0\rangle,$$

enables us to restore the vacuum to the right hand side of equations (6.6), (6.7) and (6.8). Since $C_T(-\alpha) = (-)^{\alpha^2/2}C_T(\alpha)$, it is clear that the products only factorize onto twist invariant untwisted states. Summarizing

$\underline{d = 16}$

$$F_1^{16} = \varepsilon^{-2}\left\{\psi^T C\phi\right\} + \varepsilon^{-1}\sum_{\alpha\in\Lambda_2^+}\frac{1}{2}\left\{\psi^T C C_T(\alpha)\phi\right\}\widetilde{V}(\alpha,\omega) + Reg. \tag{6.9}$$

$\underline{d = 8}$

$$\begin{aligned}F_2^8 &= \varepsilon^{-2}\left\{\psi^T C\phi\right\}(e.f) \\ &+ \varepsilon^{-1}\sum_{\alpha\in\Lambda_2^+}\frac{1}{2}\left\{\psi^T C C_T(\alpha)\phi\right\}(e.f - 2(e.\alpha)(f.\alpha))\widetilde{V}(\alpha,\omega) + Reg.\end{aligned} \tag{6.10}$$

$\underline{d = 24}$

$$
\begin{aligned}
F_2^{24} &= \varepsilon^{-4}\left\{\psi^T C\phi\right\}(e.f) + \varepsilon^{-2}\Big\{\left\{\psi^T C\phi\right\} \\
&\times \left(-\frac{1}{16}(e.f) : \partial X.\partial X(\omega) : -\frac{1}{2} : (e.\partial X)(f.\partial X)(\omega) :\right) \\
&+ \sum_{\alpha\in\Lambda_4^+} \frac{1}{8}\left\{\psi^T C C_T(\alpha)\phi\right\}(e.f - 2(e.\alpha)(f.\alpha))\tilde{V}(\alpha,\omega)\Big\} + O(\varepsilon^{-1}).
\end{aligned}
\tag{6.11}
$$

<u>Product (b).</u> This computation is less straightforward and quite lengthy. The idea is to compute the expectation value of $\overline{F}$ between two twisted ground states $\langle\mu \mid$ and $\mid \lambda\rangle$. where, μ and λ refer to vectors in the zero-mode space.

First of all we use the conformal properties of the fields W and $\overline{W}$, to write the expectation value more symmetrically in terms of $u = \omega/z$.

$$
\langle\overline{F}\rangle = (z\omega)^{-\Delta}\langle\mu \mid W(\Psi, 1/\sqrt{u})\overline{W}(\Phi,\sqrt{u}) \mid \lambda\rangle. \tag{6.12}
$$

L_1^T and L_{-1}^T annihilate to the right and left, respectively. At this point it is advantageous to introduce some notation. First of all, we define the matrices

$$
\mathbb{A} = A_{nm}(1/\sqrt{u}), \quad \mathbb{N} = n\delta_{n,m}, \quad \mathbb{M} = \mathbb{N}^{1/2}\mathbb{A}\mathbb{N}^{-1/2}, \tag{6.13}
$$

where $n, m \in [1, \infty]$, and secondly we introduce the vectors

$$
\begin{aligned}
\mathbf{x} &= A_{n0}(1/\sqrt{u}), \quad \mathbf{y} = B^+_{n\frac{1}{2}}(1/\sqrt{u}), \quad \mathbf{v} = \frac{1}{\sqrt{2}}\mathbb{N}^{1/2}\mathbf{x}, \\
\xi &= \frac{1}{1-\mathbb{M}}\mathbf{v}, \quad \mathbf{a} = a_n, \quad \mathbf{a}^* = a_{-n}.
\end{aligned}
\tag{6.14}
$$

where again $n \in [1, \infty]$. Our definitions of $\mathbb{M}$, ξ and $\mathbf{v}$, agree with those of Schwarz and Wu [72].

We now consider cases 1 and 2 separately.

Case 1.

Here the quantities $B^{\pm}$, in each of the vertices, disappear, and we are left with

$$\omega^{-d/8}\sum_{\alpha}\left\{\mu^{\dagger}C_T(\alpha)\psi\right\}\left\{\phi^T C C_T^{\dagger}(\alpha)\lambda\right\}e^{\alpha^2 A_{00}(1/\sqrt{u})}G(\alpha\mathbf{x},\alpha\mathbf{x}),$$

where we have defined the function

$$G(\mathbf{w},\mathbf{w}') = \langle 0 \mid e^{\frac{1}{2}\mathbf{a}^T\mathbb{A}\mathbf{a}+\mathbf{a}^T\mathbf{w}}e^{\frac{1}{2}\mathbf{a}^\dagger\mathbb{A}\mathbf{a}^*+\mathbf{a}^\dagger\mathbf{w}'} \mid 0\rangle. \tag{6.15}$$

The techniques for computing such an expectation value are well known [66,72], in fact

$$\begin{aligned} G(\mathbf{w},\mathbf{w}') =& \det(1-\mathbb{A}^2)^{-d/2}\exp\left\{\frac{1}{2}\mathbf{w}^T\frac{\mathbb{A}}{1-\mathbb{A}^2}\mathbf{w}\right. \\ &\left.+\frac{1}{2}\mathbf{w}'^T\frac{\mathbb{A}}{1-\mathbb{A}^2}\mathbf{w}'+\mathbf{w}^T\frac{1}{1-\mathbb{A}^2}\mathbf{w}'\right\}, \end{aligned} \tag{6.16}$$

so that

$$G(\alpha\mathbf{x},\alpha\mathbf{x}) = \det(1-\mathbb{A}^2)^{-d/2}\exp\left\{\alpha^2\left(\mathbf{x}^T\frac{1}{1-\mathbb{A}}\mathbf{x}\right)\right\}.$$

We now draw on the results of Schwarz and Wu, who calculated

$$\Delta(u) = \det(1-\mathbb{A}^2) = \det(1-\mathbb{M}^2) = \frac{2}{\pi}K(\sqrt{u})(1-u)^{1/4}, \tag{6.17}$$

and

$$\mathbf{x}^T\frac{1}{1-\mathbb{A}}\mathbf{x} = \ln\left(\frac{4}{\sqrt{u}}\right) - \frac{\pi K'(\sqrt{u})}{2K(\sqrt{u})}. \tag{6.18}$$

In the above $K(\sqrt{u})$ is the complete elliptic integral of the first kind [76,77], and $K'(\sqrt{u}) = K(\sqrt{1-u})$.

In the operator product expansion, we are interested in the behaviour as $z \to \omega$, *i.e.* $u \to 1$. To facilitate the discussion it is useful to define the variable q:

$$\begin{aligned} \ln q \equiv -\frac{\pi K'(\sqrt{u})}{K(\sqrt{u})}, \quad \sqrt{u} = \left(\frac{\theta_2}{\theta_3}\right)^2, \quad \sqrt{1-u} = \left(\frac{\theta_4}{\theta_3}\right)^2 \\ K(\sqrt{u}) = \frac{\pi}{2}\theta_3^2, \quad K(\sqrt{1-u}) = -\frac{\ln q}{2}\theta_3^2, \end{aligned} \tag{6.19}$$

where the θ_i are as usual [76,77] given by

$$\begin{aligned}
\theta_1 &= 2q_0 q^{\frac{1}{4}} \prod_n \left(1 - q^{2n}\right)^2 \\
\theta_2 &= 2q_0 q^{\frac{1}{4}} \prod_n \left(1 + q^{2n}\right)^2 \\
\theta_3 &= q_0 \prod_n \left(1 + q^{2n-1}\right)^2 \\
\theta_4 &= q_0 \prod_n \left(1 - q^{2n-1}\right)^2
\end{aligned} \tag{6.20}$$

and

$$q_0 = \prod_n \left(1 - q^{2n}\right) . \tag{6.21}$$

Inserting (6.16) into (6.15), we finally obtain

$$\langle \overline{F}_1 \rangle = \omega^{-d/8} \left(\sum_\alpha \left\{ \mu^\dagger C_T(\alpha)\psi \right\} \left\{ \phi^T C C_T^\dagger(\alpha)\lambda \right\} q^{\frac{1}{2}\alpha^2} \right) \frac{\theta_3(q)^{-d}}{(1-u)^{d/8}}, \tag{6.22}$$

Case 2.

For this case the following twisted expectation values are required:

$$\begin{aligned}
\langle 0 \mid e^{B^+} e.c_{-\frac{1}{2}} \mid 0 \rangle &= e.\mathbf{a}^T \mathbf{y} + e.pB^+_{0\frac{1}{2}} \\
\langle 0 \mid f.c_{\frac{1}{2}} e^{(B^+)^\dagger} \mid 0 \rangle &= f.\mathbf{a}^\dagger \mathbf{y} + f.pB^+_{0\frac{1}{2}} .
\end{aligned}$$

Using the above we find

$$\begin{aligned}
\langle \overline{F}_2 \rangle = &\omega^{-d/8-1} \sum_{\alpha \in \Lambda} \left\{ \mu^\dagger C_T(\alpha)\psi \right\} \left\{ \phi^T C C_T^\dagger(\alpha)\lambda \right\} e^{\alpha^2 A_{00}} \\
&\times \left\{ \langle 0 \mid e^{\frac{1}{2}\mathbf{a}^T \mathbf{A}\mathbf{a} + \alpha \mathbf{a}^T \mathbf{x}} (e.\mathbf{a}^T \mathbf{y} + e.\alpha B^+_{0\frac{1}{2}})(f.\mathbf{a}^\dagger \mathbf{y} + f.\alpha B^+_{0\frac{1}{2}}) e^{\frac{1}{2}\mathbf{a}^\dagger \mathbf{A}\mathbf{a}^* + \alpha \mathbf{a}^\dagger \mathbf{x}} \mid 0 \rangle \right\} .
\end{aligned} \tag{6.23}$$

We now leave the main thread of the calculation to evaluate the quantity in the large braces above. This quantity can be expressed in terms of $G(\mathbf{w}, \mathbf{w}')$ using the auxiliary variables σ and τ:

$$\left\{\cdots\right\} = \left(\frac{\partial}{\partial\sigma} + (e.\alpha)B^{+}_{0\frac{1}{2}}\right)\left(\frac{\partial}{\partial\tau} + (f.\alpha)B^{+}_{0\frac{1}{2}}\right) G(\alpha\mathrm{x} + \sigma e\mathrm{y}, \alpha\mathrm{x} + \tau f\mathrm{y})\big|_{\sigma=\tau=0}.$$

Using the expression for G in equation (6.16) we find that this equals

$$G(\alpha\mathrm{x}, \alpha\mathrm{x})\left\{\mathrm{y}^T\frac{1}{1-\mathbb{A}^2}\mathrm{y}(e.f) + \left(\mathrm{y}\frac{1}{1-\mathbb{A}}\mathrm{x} + B^{+}_{0\frac{1}{2}}\right)^2 (e.\alpha)(f.\alpha)\right\}. \tag{6.24}$$

Our problem is now reduced to that of evaluating

$$(i) \quad \mathrm{y}^T\frac{1}{1-\mathbb{A}^2}\mathrm{y}$$
$$(ii) \quad \mathrm{y}^T\frac{1}{1-\mathbb{A}}\mathrm{x}.$$

(i) First of all let us make contact with the notation of ref.[72], by rewriting the expression in terms of $\mathbb{M}$ and v.

$$\begin{aligned}
\mathrm{y}^T\frac{1}{1-\mathbb{A}^2}\mathrm{y} &= -4x\left(\mathrm{v}^T\frac{1}{1-\mathbb{M}^2}\mathbb{N}\mathrm{v}\right) \\
&= -2x\mathrm{v}^T\left(\frac{1}{1-\mathbb{M}} + \frac{1}{1+\mathbb{M}}\right)\mathbb{N}\mathrm{v},
\end{aligned}$$

where $x = \sqrt{u}$. Now

$$\mathbb{N}\mathrm{v}\mathrm{v}^T = \frac{1}{2}x\frac{d}{dx}\mathbb{M},$$

therefore

$$\begin{aligned}
\mathrm{v}^T\frac{1}{1\pm\mathbb{M}}\mathbb{N}\mathrm{v} &= \mathrm{Tr}\left(\frac{1}{1\pm\mathbb{M}}\mathbb{N}\mathrm{v}\mathrm{v}^T\right) \\
&= \pm\frac{1}{2}x\mathrm{Tr}\left(\frac{d}{dx}\ln(1\pm\mathbb{M})\right) \\
&= \pm\frac{1}{2}x\frac{d}{dx}\ln\left(\det(1\pm\mathbb{M})\right).
\end{aligned}$$

If we define $\Delta_{\pm} = \det(1\pm\mathbb{M})$, so that $\Delta = \Delta_{+}\Delta_{-}$, then we may write

$$\mathrm{y}^T\frac{1}{1-\mathbb{A}^2}\mathrm{y} = -2x\left(2\mathrm{v}\frac{1}{1+\mathbb{M}}\mathbb{N}\mathrm{v} - \frac{1}{2}x\frac{d}{dx}\ln\Delta\right).$$

Now we follow a manipulation which appears in ref.[69] for the matrices associated with the fermion emission vertex (although these matrices are actually different from ours).

to show that the identity

$$\mathbb{MN}+\mathbb{NM} = 2\mathbb{N}\mathrm{v}\mathrm{v}^T,$$

implies

$$\mathrm{v}^T\frac{1}{1+\mathbb{M}}\mathbb{N}\mathrm{v} = \frac{\mathrm{v}^T\mathbb{N}\xi}{1+2\mathrm{v}^T\xi},$$

where $\xi = (1-\mathbb{M})^{-1}\mathrm{v}$, in the notation of ref.[72]. Our problem is now reduced to the evaluation of $\mathrm{v}^T\mathbb{N}\xi$ and $\mathrm{v}^T\xi$, which we have relegated to appendix C. The results are

$$\mathrm{v}^T\xi = \frac{1}{2}\left\{\frac{\pi}{2(1-u)^{\frac{1}{2}}K(\sqrt{u})}-1\right\},$$

and

$$\mathrm{v}^T\mathbb{N}\xi = \frac{\pi u}{16(1-u)^{\frac{3}{2}}K(\sqrt{u})}.$$

Therefore

$$\mathrm{y}^T\frac{1}{1-\mathbb{A}^2}\mathrm{y} = \frac{-u^{\frac{3}{2}}}{1-u}+2u^{\frac{3}{2}}\frac{d}{du}\ln K(\sqrt{u}). \tag{6.25}$$

(ii) This quantity can be rewritten in terms of a known expression:

$$\begin{aligned}\mathrm{y}^T\frac{1}{1-\mathbb{A}}\mathrm{x} &= -\,2\sqrt{2}(-x)^{\frac{1}{2}}\mathrm{v}^T\xi\\ &= -\,\sqrt{2}(-\sqrt{u})^{\frac{1}{2}}\left\{\frac{\pi}{2(1-u)^{\frac{1}{2}}K(\sqrt{u})}-1\right\},\end{aligned}$$

so that

$$\left(\mathrm{y}\frac{1}{1-\mathbb{A}}\mathrm{x}+B^+_{0\frac{1}{2}}\right)^2 = \frac{-2u^{\frac{1}{2}}}{(1-u)\left(\frac{2}{\pi}K(\sqrt{u})\right)^2}. \tag{6.26}$$

Returning to the main thread of the calculation, we substitute all the expressions we have evaluated into (6.23), and introduce the variable q as before,

$$\begin{aligned}\langle\overline{F}_2\rangle = \omega^{-d/8-1}&\sum_{\alpha\in\Lambda}\left\{\mu^\dagger C_T(\alpha)\nu\right\}\left\{\phi^T CC_T^\dagger(\alpha)\lambda\right\}q^{\frac{1}{2}\alpha^2}\\ &\times\left\{\frac{u^{\frac{3}{2}}\theta_3(q)^{-d}}{(1-u)^{d/8}}\left(-\frac{1}{(1-u)}+4\frac{d}{du}\ln\theta_3(q)\right)(e.f)\right.\\ &\qquad\left.-\frac{2u^{\frac{1}{2}}}{(1-u)^{d/8+1}}\theta_3(q)^{-d-4}(e.\alpha)(f.\alpha)\right\}.\end{aligned} \tag{6.27}$$

Now two problems face us. Firstly, both (6.22) and (6.27) are expressed in a form manifestly good for small q (or $u \to 0$). However, on the contrary, we require the behaviour as $q \to 1$ ($u \to 1$). The other obstacle is that we actually require a sum of terms of the form $\psi^T C C_T(\alpha)\phi$, and not the 'crossed' form apparent in both (6.22) and (6.27). A hint as to how these two problems remedy each other is contained in the work of ref.[69], where an analogous problem was encountered with the fermion emission vertex. There the duality transformation, needed to cross the four fermion amplitude, induces a Fierz transformation of just the right type to cross the spinors associated with the four fermions. In that case the sums were finite. Here however, they run over the whole lattice Λ. Remarkably, we shall find that the **modular transformation** of the variable q, which crosses the amplitude over to the dual region, causes a Fierz–like rearrangement of the ground state vectors of just the right kind.

As in the fermion emission vertex case, an important ingredient in the calculation is the completeness relation for the twisted operator cocycles. It is crucial that the set of elements of the Clifford algebra generated on the lattice is complete, at least in some restricted sense. By restricted we mean all the even elements, say. For the moment let us suppose that there exists such a set $\Lambda_0 \subset \Lambda$ of vectors (not necessarily having the same length), with $\dim\Lambda_0 = c_T^2$, such that

$$\sum_{\tau\epsilon\Lambda_0} C^\dagger(\tau)_{ij} C(\tau)_{kl} \;=\; c_T \delta_{il}\delta_{jk}. \tag{6.28}$$

When we consider the case where Λ_0 is complete only in the sense of all the even elements of the Clifford algebra $C(N)$, one should really append projection operators $\frac{1}{2}(1+\gamma^{N+1})$ to the right hand side of the above completeness relation. We shall consider them implicit.

Using the completeness relation we may rearrange the inner products in equations (6.22) and (6.27), giving

$$\begin{aligned}
&\left\{\mu^\dagger C_T(\alpha)\psi\right\}\left\{\phi^T C C_T^\dagger(\alpha)\lambda\right\} \\
&\quad = c_T \sum_\tau \left\{\mu^\dagger C_T^\dagger(\tau)\phi\right\}\left\{\lambda^T C_T^\star(\alpha) C^T C_T(\tau) C_T(\alpha)\psi\right\} \\
&\quad = c_T \sum_\tau \left\{\mu^\dagger C_T^\dagger(\tau)\phi\right\}\left\{\lambda^T C^T C_T(\tau)\psi\right\}(-)^{\alpha\cdot\tau}(-)^{\frac{1}{2}\alpha^2},
\end{aligned} \tag{6.29}$$

where we have made use of the properties of the properties of the charge conjugation matrix, equation (5.19), and the twisted operator cocycles, equations (5.10) and (5.13).

The dependence on α is now manageably contained in the partition–like functions

$$\begin{aligned} \Gamma_1(q) &= \sum_{\alpha\in\Lambda} (-q)^{\frac{1}{2}\alpha^2}(-)^{\alpha.\tau} \\ \Gamma_2(q) &= \sum_{\alpha\in\Lambda} (-q)^{\frac{1}{2}\alpha^2}(-)^{\alpha.\tau}(e.\alpha)(f.\alpha). \end{aligned} \tag{6.30}$$

We are now ready to perform the duality transformation. The appropriate transformation is in fact a variant of the *Jacobi Imaginary Transformation*. Since

$$\ln q = -\frac{\pi K(\sqrt{1-u})}{K(\sqrt{u})},$$

it is apparent that the relevant transformation is

$$\ln q \ln q' = \pi^2, \tag{6.31}$$

so that

$$\ln q' = -\frac{\pi K(\sqrt{u})}{K(\sqrt{1-u})},$$

and $q \to 1$ becomes $q' \to 0$. In appendix D we demonstrate that under this transformation:

$$\Gamma_1(q) = (-)^{\frac{1}{2}\tau}\left(\frac{\ln q'}{i\pi}\right)^{d/2}\Gamma_1(q'), \tag{6.32}$$

and

$$\Gamma_2(q) = (-)^{\frac{1}{2}\tau^2}\left\{\frac{e.f}{i\pi}\left(\frac{\ln q'}{i\pi}\right)^{d/2+1}\Gamma_1(q') + \left(\frac{\ln q'}{i\pi}\right)^{d/2+2}\Gamma_2(q')\right\}, \tag{6.33}$$

for *any* lattice vector τ, provided that the dual lattice Λ^* is '**integral**', in the sense that each vector in Λ^* has integer length squared† . We recall that the theta function pieces also transform conveniently:

$$\theta_3(q) = \left(\frac{\ln q'}{\pi}\right)^{\frac{1}{2}}\theta_3(q'). \tag{6.34}$$

Before we put all the pieces together it is worth tidying up the cocycle contribution. One of the effects of the modular transformation is to produce the factor $(-)^{\tau^2/2}$. This

† Warning: this is actually an abuse of the usual use of the term *integral* as applied to lattices.

factor can be used to reverse the quantity

$$\begin{aligned}\lambda^T C^T C_T(\tau)\psi(-)^{\frac{1}{2}\tau^2} &= \psi^T C_T^T(\tau) C\lambda(-)^{\frac{1}{2}\tau^2} \\ &= \psi^T C C_T(\tau)\lambda.\end{aligned}$$

We now use the completeness property again, noticing that the effect of the extra factor $(-)^{\tau^2/2}$ is to swop $\psi \leftrightarrow \lambda$, and replace C by its transpose C^T. Therefore we end up with

$$\left\{\phi^T C^T C_T^\dagger(\alpha)\psi\right\}\left\{\mu^\dagger C_T(\alpha)\lambda\right\}$$

Finally the ϕ—ψ inner product can be reversed using equations (5.13) and (5.19).

$$\phi^T C^T C_T^\dagger(\alpha)\psi = \psi^T C C_T(\alpha)\phi. \tag{6.35}$$

So we end up with exactly the desired term.

Collecting all the results together we have:

<u>Case 1.</u>

The potentially embarrassing factors of $\ln q'$ and i cancel, to leave

$$\langle \overline{F}_1\rangle = \omega^{-d/8}\sum_{\alpha\in\Lambda}\left\{\psi^T C C_T(\alpha)\phi\right\}\left\{\mu^\dagger C_T(\alpha)\lambda\right\}(q')^{\frac{1}{2}\alpha^2}\frac{\theta_3(q')^{-d}}{(1-u)^{d/8}}. \tag{6.36}$$

<u>Case 2.</u>

First let us deal with the derivative term:

$$\begin{aligned}\frac{d}{du}\ln\theta_3(q) &= \frac{dq'}{du}\frac{d}{dq'}\ln\left(\left(\frac{\ln q'}{\pi}\right)^{\frac{1}{2}}\theta_3(q')\right) \\ &= -\frac{q'\theta_3(q')^{-4}}{u(1-u)}\left\{\frac{1}{2q'\ln q'}+\frac{d}{dq'}\ln\theta_3(q')\right\},\end{aligned}$$

where we have used the fact that

$$\frac{dq'}{du} = -\frac{q'\theta_3(q')^{-4}}{u(1-u)}.$$

Therefore

$$\langle \overline{F}_2\rangle = \omega^{-d/8-1} \sum_{\alpha\in\Lambda} \left\{\psi^T C C_T(\alpha)\phi\right\} \left\{\mu^\dagger C_T(\alpha)\lambda\right\} (q')^{\frac{1}{2}\alpha^2} (i)^{-\frac{d}{2}} \frac{u^{\frac{1}{2}}\theta_3(q')^{-d}}{(1-u)^{d/8+1}}$$
$$\times \left\{ - \left(u + 4q'\theta_3(q')^{-4} \left[\frac{d}{dq'} \ln\theta_3(q') + \frac{1}{2q'\ln q'} \right] \right) (e.f) \right.$$
$$\left. + 2\theta_3(q')^{-4} \left(\frac{1}{\ln q'}(e.f) + (e.\alpha)(f.\alpha) \right) \right\}.$$

The magic now unfolds; the two offending terms with factors of $\ln q'$ cancel! Also $(i)^{-d/2} = 1$ for $d = 8$ and 24. The derivative of a theta function is evaluated in ref.[77] (page 470),

$$G(q') = \frac{q'}{2}\frac{d}{dq'}\ln\theta_3(q') = \sum_{n=1}^{\infty} \frac{q'^{2n-1}}{(1+q'^{2n-1})^2}. \tag{6.37}$$

We have managed to work $\langle \overline{F}_1\rangle$ and $\langle \overline{F}_2\rangle$ into a form suitable for expanding in powers of $z-\omega$. One can easily verify that

$$\begin{aligned} q' &= \frac{1}{16}(1-u) - \frac{1}{32}(1-u)^2 + O(1-u)^3 \\ \theta_3(q')^a &= 1 + 2aq' + 2a(a-1)q'^2 + O(q')^3. \end{aligned}$$

Using these results we calculate:

<u>Case 1.</u>

$$\frac{\omega^{-d/8}\theta_3(q')^{-d}}{(1-u)^{d/8}} = \frac{1}{(z-\omega)^{d/8}}(1+O(1-u)^2).$$

So that when $d=16$ we have

$$\langle \overline{F}_1^{16}\rangle = \frac{\{\psi^T C\phi\}\{\mu^\dagger\lambda\}}{\varepsilon^2} + \frac{\omega^{-1}}{16\varepsilon}\sum_{\alpha\in\Lambda}\left\{\psi^T C C_T(\alpha)\phi\right\}\left\{\mu^\dagger C_T(\alpha)\lambda\right\} + Reg. \tag{6.38}$$

Case 2.

We now prove the identity

$$\begin{aligned}\omega^{-d/8-1}\frac{u^{1/2}}{(1-u)^{d/8+1}}\theta_3(q')^{-d}\left(u+8\theta_3(q')^{-4}G(q')\right) \\ =\frac{1}{(z-\omega)^{d/8+1}}\left(1+\frac{2d+16}{64}(1-u)^2+O(1-u)^3\right).\end{aligned} \tag{6.39}$$

Proof: Writing $(z-\omega)^{d/8+1}\times$LHS in terms of the $\theta_i(q')$'s;

$$\begin{aligned}&[\theta_3(q')\theta_4(q')]^{-d/2-2}\left(\theta_4(q')^4+8G(q')\right) \\ &=(1+(2d+8)q'^2+O(q')^4)(1+8q'^2+O(q')^4) \\ &=(1+(2d+16)q'^2+O(q')^4) \\ &=\left(1+\frac{2d+16}{64}(1-u)^2+O(1-u)^3\right).\end{aligned}$$

When $d=8$:

$$\begin{aligned}\langle\overline{F}_2^8\rangle&=\frac{\{\psi^TC\phi\}\{\mu^\dagger\lambda\}(e.f)}{\varepsilon^2} \\ &-\frac{\omega^{-1}}{16\varepsilon}\sum_{\alpha\in\Lambda_2}\left\{\psi^TCC_T(\alpha)\phi\right\}\left\{\mu^\dagger C_T(\alpha)\lambda\right\}(e.f-2(e.\alpha)(f.\alpha))+Reg.\end{aligned} \tag{6.40}$$

When $d=24$, and Λ_2 is empty:

$$\begin{aligned}\langle\overline{F}_2^{16}\rangle&=\frac{\{\psi^TC\phi\}\{\mu^\dagger\lambda\}(e.f)}{\varepsilon^4} \\ &+\frac{1}{\varepsilon^2}\left\{\frac{\omega^{-2}}{2^8}\sum_{\alpha\in\Lambda_4}\left\{\psi^TCC_T(\alpha)\phi\right\}\left\{\mu^\dagger C_T(\alpha)\lambda\right\}(e.f-2(e.\alpha)(f.\alpha))\right. \\ &\left.+\frac{\omega^{-2}}{4}\{\psi^TC\phi\}\{\mu^\dagger\lambda\}(e.f)\right\}+O(\varepsilon^{-1}).\end{aligned} \tag{6.41}$$

After these tortuous calculations let us take stock of the results. For the terms we have evaluated it is true that in the three cases

$$F\simeq\overline{F}. \tag{6.42}$$

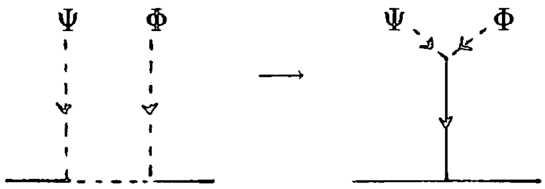
i.e. the $\overline{W}W$ and $W\overline{W}$ OPEs are isomorphic. Although we emphasize, this is only true when Λ^* is an 'integral' lattice, *i.e.* all the vectors in Λ^* have an integer squared length, otherwise the $W\overline{W}$ OPE is ill-defined. For cases 1 and 2, with $d = 16$ and 8, respectively, the isomorphism is manifest. For case 2, $d = 24$, we can quickly convince ourselves of its truth by recalling the normal ordering subtlety from section 3.3,

$$
\begin{aligned}
{}_T\langle 0 |: \left\{ -\frac{1}{16}(e.f)\partial X_T.\partial X_T(\omega) - \frac{1}{2}e.\partial X_T f.\partial X_T(\omega)\right\} :| 0\rangle_T & \\
= \left\{ \frac{1}{8}\left(\frac{d}{16}\right) + \frac{1}{16}\right\}(e.f)\omega^{-2} & \\
= \frac{1}{4}(e.f)\omega^{-2}, \quad \text{if } d = 24. &
\end{aligned}
$$

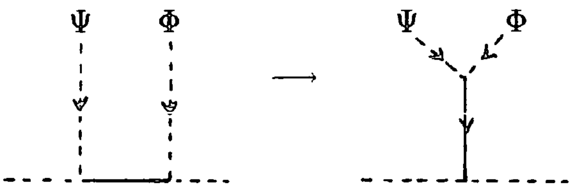
This accounts for the last term in the large braces of equation (6.41).

Now that we have evaluated the structure constants of the operator algebra, we can go on to investigate the their duality properties. The OPEs that we have calculated correspond to the following factorizations:

$\overline{W}(\Psi)W(\Phi)$:



$W(\Psi)\overline{W}(\Phi)$:



Notice that our expressions are consistent with factorization, in the sense that the structure constants indeed are just the $SL(2,\mathbb{C})$ invariant 3-point functions. For example:

$$\psi^T C C_T(\alpha)\phi = \langle \phi^c \mid V^T(-\alpha, 1) \mid \psi\rangle,$$

for $d = 16$. and

$$\left\{\psi^T C C_T(\alpha)\phi\right\}(e.f - 2(e.\alpha)(f.\alpha)) = \langle f, \phi^c \mid V^T(-\alpha, 1) \mid e, \psi\rangle,$$

for $d = 8$. These results are only valid when $C = C^T$, which holds in $d = 0, 2 \bmod 8$. In addition the structure constants are consistent with exchange duality when $d =$

0, 2 mod 8:

$$
\begin{aligned}
\psi^T C C_T(\alpha)\phi &= \phi^T C_T^T(\alpha) C^T \psi \\
&= \phi^T C^T C_T^\dagger(\alpha)\psi \\
&= (-)^{\frac{1}{2}\alpha^2} \phi^T C C_T(\alpha)\psi.
\end{aligned}
$$

Here, the phase $(-)^{\alpha^2/2}$ matches the phase resulting from interchanging the order of the two operators in the OPE.

All these facts suggest that, in certain cases, we can generalize the FKS mechanism with TSEVs. We shall investigate this possibility in the next chapter.

7. THE ALGEBRA ENHANCEMENT MECHANISM.

In this chapter we build on the remarks made in the last chapter, and investigate the possibility of generalizing the Frenkel–Kac–Segal mechanism, and the cross–bracket algebra, by involving the twisted string emission vertices in the operator algebra. A generalization is clearly a possibility if there exist TSEVs with dimension one and two, for the Lie and cross–bracket algebras, respectively. For the reflection twist such level matching can occur in $d = 0 \bmod 8$. In particular, when $d = 8$ the first excited state in the twisted sector has unit dimension, whereas in $d = 16$ it is the ground state. The relevant state in $d = 24$ is the first excited state, which has dimension two.

In these special dimensions, and if the OPEs of the TSEVs are well defined, *i.e.* Λ^* is 'integral', then the TSEVs act as additional generators which intertwine between the untwisted and twisted sectors. The twist invariant subalgebras $\hat{g}_0$ and $\hat{S}_0$ become *enhanced*. We will call the enhanced algebras $\hat{g}^\natural$ and $\hat{S}^\natural$, following the notation of ref.[28]. Compare this situation with the NSR spinning string in which the fermion emission vertex (at zero momentum) becomes the generator of supersymmetry [9,78–81]. What we will describe is essentially a purely bosonic analogue of this phenomenon. Our formalism illuminates some of the statements made in the literature about the relationship between the $E_8 \times E_8$ and $spin(32)/\mathbf{Z}_2$ heterotic strings [2,31,37,38]. We will also find a explicit operator representation of the algebra discovered by Frenkel Lepowsky and Meurman [27,28], which is related to Griess's algebra [82], whose automorphism group is the Fischer–Griess Monster group [83].

7.1 THE ENHANCED ALGEBRA.

We will assume for the moment that the OPEs for all the TSEVs are well defined. In the last chapter we evaluated all the OPEs relevant to the enhanced algebra. Recall that the twist invariant algebras $\hat{g}_0$ and $\hat{S}_0$ are generated by the diagonal operators

$$\mathbf{V}(\alpha, z) = \begin{pmatrix} \widetilde{V}(\alpha, z) & 0 \\ 0 & \widetilde{V}^T(\alpha, z) \end{pmatrix}, \tag{7.1}$$

where $\alpha \in \Lambda_2^+$ and Λ_4^+, respectively. In addition $\hat{S}_0$ also includes the generators

$$\mathbf{V}(\epsilon, \eta, z) = \begin{pmatrix} - : (\epsilon.\partial X)(\eta.\partial X)(z) : & 0 \\ 0 & - : (\epsilon.\partial X_T)(\eta.\partial X_T)(z) : \end{pmatrix}.$$

The TSEVs contribute the off–diagonal generators

$$\mathbb{V}(\Phi, z) = \begin{pmatrix} 0 & \overline{W}(\Phi, z) \\ W(\Phi, z) & 0 \end{pmatrix},$$

which have the effect of mixing states in the two sectors.

The additional generators contribute the following commutators/cross–brackets to the algebras:

$\underline{d=8}$

$$\begin{aligned} [\mathbb{V}_n(e,\phi), \mathbb{V}_m(f,\psi)] &= \frac{1}{2}\sum_{\alpha\in\Lambda_2^+} (\phi^T C C_T(\alpha)\psi)(e.f - 2(e.\alpha)(f.\alpha))\mathbb{V}_{n+m}(\alpha) \\ &\quad + n(\phi^T C\psi)(e.f)\delta_{n+m.0} \\ [\mathbb{V}_n(e,\phi), \mathbb{V}_m(\alpha)] &= \frac{1}{4}\mathbb{V}_{n+m}(e - 2(e.\alpha)\alpha, C_T(\alpha)\phi). \end{aligned} \tag{7.2}$$

$\underline{d=16}$

$$\begin{aligned} [\mathbb{V}_n(\phi), \mathbb{V}_m(\psi)] &= \frac{1}{2}\sum_{\alpha\in\Lambda_2^+} (\phi^T C C_T(\alpha)\psi)\mathbb{V}_{n+m}(\alpha) + n(\phi^T C\psi)\delta_{n+m.0} \\ [\mathbb{V}_n(\phi), \mathbb{V}_m(\alpha)] &= \frac{1}{4}\mathbb{V}_{n+m}(C_T(\alpha)\phi). \end{aligned} \tag{7.3}$$

$\underline{d=24}$ (Cross–bracket algebra)

$$\begin{aligned} [\mathbb{V}_n(e,\phi) \times \mathbb{V}_m(f,\psi)] &= \frac{1}{8}\sum_{\alpha\in\Lambda_4^+} (\phi^T C C_T(\alpha)\psi)(e.f - 2(e.\alpha)(f.\alpha))\mathbb{V}_{n+m}(\alpha) \\ &\quad + (\phi^T C\psi)\Big(\frac{1}{2}\mathbb{V}_{n+m}(e,f) + \frac{1}{8}(e.f)\mathbb{L}_{n+m} + \frac{n^2}{2}(e.f)\delta_{n+m.0}\Big) \\ [\mathbb{V}_n(e,\phi) \times \mathbb{V}_m(\alpha)] &= 2^{-4}\mathbb{V}_{n+m}(e - 2(e.\alpha)\alpha, C_T(\alpha)\phi) \\ [\mathbb{V}_n(e,\phi) \times \mathbb{V}_m(\epsilon,\eta)] &= \frac{1}{2}\mathbb{V}_{n+m}((e.\eta)\epsilon + (e.\epsilon)\eta, \phi). \end{aligned} \tag{7.4}$$

These expressions for the cross–bracket algebra are identical with the results obtained in refs.[27,28].

For the Lie algebra cases, we can determine $g^\natural$ by considering its decomposition under $g_0 \subset g^\natural$. The relevant twisted states transform as a representation $\mathcal{R}_0$ ($\mathcal{R}_1$) of g_0, for $d = 16$ ($d = 8$). So the g_0 content of $g^{\natural}$ is

$$\text{adjoint}\left(g^{\natural}\right) = \text{adjoint}\left(g_0\right) + \mathcal{R}_0\left(\mathcal{R}_1\right). \tag{7.5}$$

In particular this implies

$$\dim\left(g^{\natural}\right) = \begin{cases} \dim(g_0) + c_T & d = 16 \\ \dim(g_0) + 8c_T & d = 8. \end{cases} \tag{7.6}$$

Similarly for the cross–bracket algebra defined by the Leech lattice:

$$\dim\left(S^{\natural}\right) = \dim\left(S_0\right) + 24.2^{12}. \tag{7.7}$$

The only difficulty is to find how the twisted states transform under g_0. The weights of $\mathcal{R}_0$ were determined in equation (3.62). Recall for an inner automorphism the Cartan subalgebra, in the twisted sector, is generated by

$$\tilde{V}_0^T(\beta_i),$$

where $\{\beta_i\} = \Lambda_2^+ \cap A$, is a set of d mutually orthogonal positive roots of g which we found in chapter 5 for each of the relevant algebras. The twisted operator cocycles are diagonal on $\{\beta_i\}$. Suppose the eigenstates are ϕ_μ, $\mu = 1, ..., c_T$, so that

$$\tilde{V}_0^T(\beta_i) \mid 0\rangle \otimes \phi_\mu = \lambda_\mu^i \mid 0\rangle \otimes \phi_\mu.$$

The vectors $\underline{\lambda}_\mu$ are simply the weights of $\mathcal{R}_0$. Since the symmetry factor for the reflection twist is $(-)^{\beta.n_\mu}$, we deduce from equation (3.62)

$$\lambda_\mu^i = (-)^{\beta_i.n_\mu},$$

where the n_μ's are the representatives of the cosets Λ/A, $\mu = 1, ..., c_T$. These representatives were determined in chapter 5. The weights $\underline{\lambda}_{(\mu i)}$ of the first excited state are most efficiently determined by choosing the 'polarisation vectors' from the set $\{\beta_i\}$, in which case

$$\begin{aligned}\widetilde{V}_0^T(\beta_i)\beta_j.c_{-\frac{1}{2}} \mid 0\rangle \otimes \phi_\mu &= (\beta_j - 2(\beta_i.\beta_j)\beta_i).c_{-\frac{1}{2}} \mid 0\rangle \otimes \lambda^i_\mu \phi_\mu \\ &= \lambda^i_\mu (1 - 4\delta_{ij})\beta_j.c_{-\frac{1}{2}} \mid 0\rangle \otimes \phi_\mu,\end{aligned}$$

so that the weights of $\mathcal{R}_1$ are simply

$$\lambda^i_{(\mu j)} = \lambda^i_\mu (1 - 4\delta_{ij}). \tag{7.8}$$

When $g = g^1 + g^2 + ...$ is not simple the ground state representation is

$$\mathcal{R}_0(g) = \left(\mathcal{R}_0(g^1), \mathcal{R}_0(g^2), ...\right),$$

where the $\mathcal{R}_0(g^i)$'s are the representations for each of the component algebras g^i.

7.2 Scope for Enhancement.

In order to ensure that the $W\overline{W}$ OPE is well-defined, we require:

1. All the vectors in the dual lattice to Λ have integer squared length.
2. The operator cocycles on Λ are complete, in the sense that they span the whole Clifford algebra, or the subalgebra of even elements.

7.2.1. Lie Algebras.

The only simple simply-laced Lie algebras with weight lattices whose vectors all have integral squared length and rank$\leq$ 16 are:

$$E_8, \quad D_4, \quad D_8, \quad D_{12}, \quad D_{16}.$$

In addition to these root lattices we can also consider the $spin(32)/\mathbf{Z}_2$ lattice which is even and self-dual. The equivalent lattice in 8 dimensions is just the E_8 root lattice, because the spinor weights of D_8 have length squared two. The twisted operator cocycles corresponding to these algebras are also complete, in the sense required for the calculation of the $W\overline{W}$ OPE, see section 6.2. In particular, for E_8 and $spin(32)/\mathbf{Z}_2$ the twisted operator cocycles generate the whole Clifford algebra on the root lattice, whilst it is the set of even elements for D_4, D_8, D_{12} and D_{16}. In these latter cases we must take the twisted ground state to be an irreducible representation of g_0, *i.e.* the chiral projection on the twisted zero-mode space is imposed. In this case we have:

$$\mathcal{R}_0(E_8) = \mathbb{16}_v \quad \text{of} \quad D_8$$
$$\mathcal{R}_0(D_{2n}) = \begin{cases} (2^{n-1}, \mathbb{1}) & n > 2 \ \text{ of } \ D_n + D_n \\ (\mathbb{2}, \mathbb{1}, \mathbb{1}, \mathbb{1}) & n = 2 \ \text{ of } \ A_1^4. \end{cases}$$

Using these results we can determine the g_0 content of adjoint($g^\natural$) and hence deduce $g^\natural$ uniquely. The relevant branching rules are taken from ref.[85]. Below we list the only cases where Lie algebra enhancement can occur for the reflection twist.

$\underline{d=8}$

$$E_8 \qquad D_8 \qquad D_4 + D_4$$

$\underline{d=16}$

$E_8 + E_8$	$E_8 + D_8$	$E_8 + D_4 + D_4$
D_{16}	$D_{12} + D_4$	$D_8 + D_8$
$D_8 + D_4 + D_4$	$D_4 + D_4 + D_4 + D_4$	$spin(32)/\mathbb{Z}_2$

We now give a case–by–case discussion for each of the above algebras.

$\underline{d=16}$

(1) $E_8 + E_8$

The twist invariant algebra is $D_8 + D_8$. The ground state, in the twisted sector, is a $(\mathbb{16}_v, \mathbb{16}_v)$ of g_0. Therefore the g_0 content of $g^\natural$ is

$$(\mathbb{120}, \mathbb{1}) + (\mathbb{1}, \mathbb{120}) + (\mathbb{16}_v, \mathbb{16}_v),$$

this is D_{16}. We conclude

$$(E_8 + E_8)^\natural = D_{16}.$$

Later, we will find that the $\hat{g}^\natural$–module is in fact isomorphic to the untwisted $spin(32)/\mathbb{Z}_2$ module. This case is relevant to the original heterotic string model [11,12].

(2) $E_8 + D_8$

The twist invariant algebra is $D_8 + D_4 + D_4$. The ground state in the twisted sector is a $(\mathbf{16}_v, \mathbf{8}_s, \mathbf{1})$ of g_0. The relevant branching rule is $\mathrm{ad}(D_{12}) = \mathrm{ad}(D_8 + D_4) + (\mathbf{16}_v, \mathbf{8}_s)$. Therefore the g_0 content of $g^\natural$ implies

$$(E_8 + D_8)^\natural \;=\; D_{12} + D_4.$$

(3) $E_8 + D_4 + D_4$

The twist invariant algebra is $D_8 + A_1^8$. The ground state in the twisted sector is a $(\mathbf{16}_v, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1})$ of g_0. The relevant branching rule is $\mathrm{ad}(D_{10}) = \mathrm{ad}(D_8 + A_1^2) + (\mathbf{16}_v, \mathbf{2}, \mathbf{2})$. Therefore

$$(E_8 + D_4 + D_4)^\natural \;=\; D_{10} + A_1^6.$$

(4) D_{16}

The twist invariant subalgebra is $D_8 + D_8$. The ground state is an irreducible $(\mathbf{128}_s, \mathbf{1})$ of g_0. The relevant branching rule is $\mathrm{ad}(E_8) = \mathrm{ad}(D_8) + \mathbf{128}_s$. Therefore

$$(D_{16})^\natural \;=\; E_8 + D_8.$$

(5) $D_{12} + D_4$

The twist invariant subalgebra is $D_6^2 + A_1^4$. The ground state is a $(\mathbf{32}_s, \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1})$ of g_0. The relevant branching rule is $\mathrm{ad}(E_7) = \mathrm{ad}(D_6 + A_1) + (\mathbf{32}_s, \mathbf{2})$. Therefore

$$(D_{12} + D_4)^\natural \;=\; E_7 + D_6 + A_1^3.$$

(6) $D_8 + D_8$

The twist invariant subalgebra is D_4^4. The ground state is a $(\mathbf{8}_s, \mathbf{1}, \mathbf{8}_s, \mathbf{1})$ of g_0. The relevant branching rule is $\mathrm{ad}(D_8) = \mathrm{ad}(D_4 + D_4) + (\mathbf{8}_s, \mathbf{8}_s)$. Therefore

$$(D_8 + D_8)^\natural \;=\; D_8 + D_4 + D_4.$$

(7) $D_8 + D_4 + D_4$

The twist invariant subalgebra is $D_4 + D_4 + A_1^8$. The ground state in the twisted sector is a $(\mathbb{8}_s, \mathbb{1}, \mathbb{2}, \mathbb{1}, \mathbb{1}, \mathbb{1}, \mathbb{2}, \mathbb{1}, \mathbb{1}, \mathbb{1})$ of g_0. The relevant branching rule is $\mathrm{ad}(D_6) = \mathrm{ad}(D_4 + A_1 + A_1) + (\mathbb{8}_s, \mathbb{2}, \mathbb{2})$. Therefore

$$(D_8 + D_4 + D_4)^\natural = D_6 + D_4 + A_1^6.$$

(8) $D_4 + D_4 + D_4 + D_4$

The twist invariant algebra is A_1^{16}. The ground state is a $\mathbb{2}$ of four of the A_1's and a singlet of the others. The relevant branching rule is $\mathrm{ad}(D_4) = \mathrm{ad}(A_1^4) + (\mathbb{2}, \mathbb{2}, \mathbb{2}, \mathbb{2})$. Therefore

$$(D_4 + D_4 + D_4 + D_4)^\natural = D_4 + A_1^{12}.$$

(9) $spin(32)/\mathbb{Z}_2$, $g = D_{16}$

The twist invariant subalgebra is $D_8 + D_8$. Recall that the ground state of the twisted sector had to be a *reducible* $(\mathbb{128}_s, \mathbb{1}) + (\mathbb{1}, \mathbb{128}_s)$ of g_0. Therefore

$$(spin(32)/\mathbb{Z}_2)^\natural = E_8 + E_8.$$

Before we go on to discuss the $d = 8$ cases, we pause to remark upon the significance of two of the above results for the heterotic string. It has previously been noted, in the fermionic construction of the gauge degrees of freedom, that the two original heterotic string theories [11,12], based on the $E_8 + E_8$ and $spin(32)/\mathbb{Z}_2$ lattices, are twisted versions of one another [2,31,37,38]. We have managed to show this correspondence in the bosonic picture by constructing the algebras explicitly. The characters, or partition functions, of the $E_8 + E_8$ and $spin(32)/\mathbb{Z}_2$ theories are in fact equal, and modular invariant. Later in this chapter we will compute the twisted characters to show that the $(E_8 + E_8)^\natural$ and $spin(32)/\mathbb{Z}_2{}^\natural$ modules also have the same character as the untwisted theories, and hence lead to modular invariant theories. It is remarkable that a theory constructed on an ordinary torus can be equivalent to one constructed on an orbifold [86,87].

$\underline{d=8}$

(1) E_8

The twist invariant subalgebra is D_8. The ground state in the twisted sector is a vector $\mathbf{16}$ of D_8. We must now determine how the first excited state transforms under g_0. Since σ is an inner automorphism for E_8, we can identify the Cartan subalgebra with 8 mutually orthogonal positive roots $\{\beta_i\}$. The suitably normalized weights of the ground state representation $\mathcal{R}_0$ can be extracted from equation (3.62):

$$\lambda^i_\mu = \frac{1}{2}(-)^{\beta_i . n_\mu}.$$

By using the explicit form for the sets $\{\beta_1\}$ and $\{n_\mu\}$, which we wrote down in chapter 5, we find that these weights form 8 orthogonal vectors, which along with their negatives, are the weights of the vector representation of D_8. Using equation (7.8), the 120 weights of the first excited state $\mathcal{R}_1$ are

$$\lambda^i_{(\mu j)} = \frac{1}{2}(-)^{\beta_i . n_\mu}(1-\delta_{ij}),$$

which have squared length two. In fact, the inner products of these weight vectors are consistent with one of the spinor representations of D_8. Later we shall discover that it is the spinor ($\mathbf{c}$), if the original E_8 was formed from the spinor ($\mathbf{s}$). Therefore

$$(E_8)^\natural = E_8.$$

The fact that the reflection twisted E_8 model allows the construction of another E_8 which mixes the sectors, was first pointed out by Frenkel, Lepowsky and Meurman [26–28]; we now have an explicit operator representation of their construction. The importance of this phenomenon is that it gives a simple analogue to the more complicated cross–bracket algebra case, based on the Leech lattice.

(2) D_8

The twist invariant algebra is D_4+D_4. The ground state in the twisted sector is an $(\mathbf{8}_s, \mathbf{1})$ of g_0. Recall that the two D_4's, in the twisted sector, were identified with the zeroth moments of

$$V^\pm(\alpha_{ij},z) = \frac{1}{2}\gamma^i\gamma^j\left(V^T(\alpha_{ij},z) \pm \gamma^9 V^T(\tilde{\alpha}_{ij},z)\right),$$

where the positive roots have been split into the two mutually orthogonal sets $\{\alpha_{ij}\} = \{e_i - e_j\}$, $\{\tilde{\alpha}_{ij}\} = \{e_i + e_j\}$, $i,j = 1,\ldots,8$. The ground state is a chiral spinor with

$\gamma^9\phi = \phi$, therefore

$$V_0^+(\alpha_{ij}) \mid 0\rangle \otimes \phi = \mid 0\rangle \otimes \gamma^i\gamma^j\phi$$
$$V_0^-(\alpha_{ij}) \mid 0\rangle \otimes \phi = 0,$$

which confirms that $\mathcal{R}_0 = (8_s, 1)$. The Cartan subalgebra of g_0 can be identified with the 8 mutually orthogonal roots $e_i \pm e_{i+1}$, $i = 1, 3, 5, 7$. On the first excited state we calculate:

$$V_0^\pm(\alpha_{ij})e.c_{-\frac{1}{2}} \mid 0\rangle \otimes \phi = \left\{\frac{1}{2}(1\pm 1)e - (e.\alpha_{ij})\alpha_{ij} \mp (e.\tilde{\alpha}_{ij})\tilde{\alpha}_{ij}\right\}.c_{-\frac{1}{2}} \mid 0\rangle \otimes \gamma^i\gamma^j\phi.$$

Choosing e from the set $\{\alpha_{ij}, \tilde{\alpha}_{ij}\}$, $i = 1, 3, 5, 7$, allows us to calculate the weights efficiently. The factor in braces takes the following values for V^+ and V^- respectively:

$$\begin{array}{ll} e \neq \alpha_{ij},\ \tilde{\alpha}_{ij} & e,\ 0 \\ e = \alpha_{ij} & -e,\ -2e \\ e = \tilde{\alpha}_{ij} & -e,\ 2e. \end{array}$$

Studying these factors shows that the weights of $\mathcal{R}_1$ are those of a $(8_c, 8_v)$ of $D_4 + D_4$. Therefore

$$(D_8)^\natural = D_8.$$

Notice that the new D_8 has a different g_0 content than the original D_8, but nevertheless it is equivalent due to the triality of the **v**, **s** and **c** representations of D_4.

(3) $D_4 + D_4$

The twist invariant subalgebra is A_1^8. The Cartan subalgebra is spanned by $C_T(\beta_i)$, where the β_i's are 8 orthogonal positive roots of $D_4 + D_4$:

$$\beta_i \in \left\{\alpha_1, \alpha_3, \alpha_4, \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4, \alpha_1', \alpha_3', \alpha_4', \alpha_1' + 2\alpha_2' + \alpha_3' + \alpha_4'\right\}.$$

The suitably normalized weights of the ground state are

$$\frac{1}{\sqrt{8}}(-)^{\beta_i.n_\mu},$$

where

$$n_\mu \in \left\{0, \alpha_2, \alpha_2', \alpha_2 + \alpha_2'\right\}.$$

The ground state is therefore a $(\mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1})$ of g_0. The weights of the first excited state are

$$\frac{1}{\sqrt{8}}(-)^{\beta_i \cdot n_\mu}(1-\delta_{ij}).$$

These weights have length squared 2, and when examined in detail imply

$$\mathcal{R}_1 \;=\; (\mathbb{1},\mathbb{2},\mathbb{2},\mathbb{2},\mathbb{2},\mathbb{1},\mathbb{1},\mathbb{1}) + (\mathbb{2},\mathbb{1},\mathbb{1},\mathbb{1},\mathbb{1},\mathbb{2},\mathbb{2},\mathbb{2}),$$

of g_0. Therefore we deduce

$$(D_4 + D_4)^\natural \;=\; D_4 + D_4.$$

7.2.2. The Cross–Bracket Algebra.

The enhanced module for the $d = 24$ theory based on the Leech lattice, is constructed by taking the twist invariant untwisted space and the integer graded twisted space:

$$\mathcal{H}^\natural \;=\; \mathcal{H}_e^+ \oplus \mathcal{H}_T[\mathbb{Z}],$$

This is the 'moonshine module' of Frenkel, Lepowsky and Meurman (FLM). The character of $\mathcal{H}^\natural$ turns out to be almost the same as the untwisted module $\mathcal{H}_e$, the important difference is the constant term 24 is not present, *i.e.*

$$\chi^\natural(q) \;=\; J(q).$$

This yields the clue that $\mathcal{H}^\natural$ is the *natural* (hence the use of the '$\natural$' label) graded F_1–module. In a series of remarkable papers FLM showed that this is indeed the case [26–28]. In fact, the cross–bracket algebra $\hat{B} \;=\; \hat{S}^\natural$ is *not* isomorphic to the untwisted algebra $\hat{S}$, unlike the Lie algebra example involving E_8. The commutative non–associative horizontal algebra B, the zero–graded subalgebra of $\hat{B}$, has dimension 196884, and is precisely Griess's original algebra [82], with an adjoined identity ($= \mathbf{L}_0$):

$$B \;=\; B_0 \oplus \mathbb{1}.$$

With an explicit construction of B_0, in terms of vertex operators acting on an infinite dimensional Fock space, some aspects of the phenomenon of 'monstrous moonshine' are rather elegantly explained. The Fischer–Griess Monster group [83] is identified as the automorphism group of B_0. Each level in the Fock space $\mathcal{H}^\natural$ will therefore be a representation of F_1; the fact which launched the 'monstrous game' in the first place [53].

To identify the monster group explicitly the automorphisms of B_0 must be determined. For completeness we now explain the nature of $F_1 = \mathrm{Aut}(B_0)$ following refs.[27,28].

The automorphism group of S, the untwisted algebra, is, in analogy with the Lie algebra examples, the lift of $\mathrm{Aut}(\Lambda_L) = \cdot 0$ into the Hilbert space. $\cdot 0$, the group of automorphisms of the Leech lattice which leave the origin fixed, is known as the Conway group. $\mathrm{Aut}(S)$ as isomorphic to the automorphism group C_0 of $\hat{\Lambda}_L$. Recall that $\hat{\Lambda}_L$ is the central extension of Λ_L by $\{\pm 1\}$, with product

$$\hat{\alpha} \circ \hat{\beta} = (\xi, \alpha) \circ (\zeta, \beta) = (\xi \zeta \varepsilon(\alpha, \beta), \alpha + \beta).$$

C_0 has the following action on $\hat{\Lambda}_L$:

$$\hat{u} \in C_0 : \quad (\xi, \alpha) \longmapsto (\xi u_\alpha, u(\alpha)).$$

where $u_\alpha \in \{\pm 1\}$ and $u \in \cdot 0$. To ensure that C_0 is a group of automorphisms we require:

$$\hat{u}(\hat{\alpha} \circ \hat{\beta}) = \hat{u}(\hat{\alpha}) \circ \hat{u}(\hat{\beta}).$$

This implies equation (3.52):

$$u_\alpha u_\beta \varepsilon(u\alpha, u\beta) = u_{\alpha+\beta} \varepsilon(\alpha, \beta).$$

C_0 is the extension of $\cdot 0$ by the abelian group $X \simeq \mathbb{Z}_2^{24}$. An element of the subgroup X is of the form:

$$\hat{x} \in X : \quad (\xi, \alpha) \longmapsto (\xi u_\alpha, \alpha).$$

For such an element, the phases satisfy

$$u_\alpha u_\beta = u_{\alpha+\beta}.$$

Notice that C_0 is *not* a central extension of $\cdot 0$, as X does not commute with $\cdot 0$.

Now consider the 'covering' theory $\mathcal{H}_c \oplus \mathcal{H}_T$. We have to discover how C_0 acts in the twisted sector. The twisted operator cocycles generate a projective representation of Λ_L, *i.e.*

$$C_T(\alpha)C_T(\beta) = \varepsilon(\alpha,\beta)C_T(\alpha+\beta),$$

corresponding to the central extension of Λ_L by $\{\pm 1\}$, as in the untwisted sector. Actually the situation in the twisted sector is somewhat different. Equation (3.50) implies that with suitable normalization

$$C_T(\alpha) = \mathbf{1}, \qquad \forall\, \alpha \in B = 2\Lambda_L.$$

Therefore to be more precise the twisted operator cocycles generate a projective representation of the 2–group $Y = \Lambda_L/2\Lambda_L \simeq \mathbb{Z}_2^{24}$. This representation derives from the central extension of $\Lambda_L/2\Lambda_L$ by $\{\pm 1\}$:

$$1 \longrightarrow \{\pm 1\} \longrightarrow E \longrightarrow \Lambda_L/2\Lambda_L \longrightarrow 1.$$

E is an *extra–special group*, and is usually denoted 2_+^{1+24}. The twisted operator cocycles generate a representation π of E over $\hat{\Lambda}_L/2\Lambda_L$:

$$\pi\left(\hat{\alpha} = (\xi,\alpha)\right) = \xi C_T(\alpha), \tag{7.9}$$

of dimension 2^{12}. In chapter 5 this representation was constructed in terms of gamma matrices (see ref.[84] for a discussion of the connection between extra–special groups and Clifford algebras).

C_0 induces automorphisms of 2_+^{1+24}:

$$\begin{aligned} \phi :\ \hat{u} &\longmapsto \hat{u}_T \\ \hat{u}_T :\ \pi(\hat{\alpha}) &\longmapsto \pi(\hat{u}(\hat{\alpha})). \end{aligned}$$

We denote these induced automorphisms $\hat{u}_T$ to show that they act in the twisted sector. In fact these induced automorphisms are generated by matrices:

$$\pi(\hat{u}(\hat{\alpha})) = M\pi(\hat{\alpha})M^{-1},$$

where $M \in GL(2^{12})$. C_0 has a diagonal action on $\mathcal{H}_c \oplus \mathcal{H}_T$:

$$\begin{pmatrix} \hat{u} & 0 \\ 0 & \hat{u}_T \end{pmatrix}.$$

Under the homomorphism ϕ the subgroup X of C_0 is isomorphic to the subgroup $Y =$

$\Lambda_L/2\Lambda_L$ of 2_+^{1+24}, where the action of Y on 2_+^{1+24} is given by

$$\pi(\hat{\alpha}) \longmapsto C_T(\beta)\pi(\hat{\alpha})C_T(\beta)^{-1}, \quad \beta \in \Lambda_L/2\Lambda_L.$$

To see the isomorphism $X \simeq Y$ explicitly, we compute:

$$\begin{aligned} C_T(\beta)\pi(\xi,\alpha)C_T(\beta)^{-1} &= \pi(\varepsilon(\beta,\alpha)\varepsilon(\alpha+\beta,-\beta)\xi,\alpha) \\ &= \pi\left((-)^{\alpha\cdot\beta}\xi,\alpha\right) \\ &= \pi\left(\hat{x}(\hat{\alpha})\right), \quad \hat{x} \in X. \end{aligned}$$

where we have taken $\varepsilon(\alpha,0) = \varepsilon(\alpha,-\alpha) = 1$.

In addition to these automorphisms there is also an involution g that assigns twisted states -1 and untwisted states 1:

$$g = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1}_T \end{pmatrix},$$

where $\mathbb{1}$ is the identity of C_0. g commutes with C_0. Let us define $\hat{C}$ to be the central extension of C_0 by the $\mathbb{Z}_2$ group generated by g.

Now we perform the projection

$$\mathcal{H}_c \oplus \mathcal{H}_T \longrightarrow \mathcal{H}^\natural = \mathcal{H}_c^+ \oplus \mathcal{H}_T[\mathbb{Z}],$$

by demanding that states be invariant under $\hat{\sigma}$, the reflection twist. Clearly the $\mathbb{Z}_2$ subgroup of $\hat{C}$ generated by

$$\begin{pmatrix} \hat{\sigma} & 0 \\ 0 & \hat{\sigma}_T \end{pmatrix},$$

which we denote Z, acts trivially on $\mathcal{H}^\natural$. Let C be the quotient of $\hat{C}$ which acts faithfully on $\mathcal{H}^\natural$:

$$C = \hat{C}/Z.$$

C is the extension of $\cdot 1$ by the extra-special group 2_+^{1+24}. Here $\cdot 1 = \cdot 0/\{\mathbb{1},\sigma\}$ is intuitively the automorphism group of the Leech lattice with $\alpha \sim -\alpha$. An element of the subgroup $2_+^{1+24} \subset C$ has a form

$$\begin{pmatrix} \hat{x} & 0 \\ 0 & \pi(\hat{\beta}) \end{pmatrix},$$

where

$$\pi(\hat{\beta})\pi(\hat{\alpha})\pi(\hat{\beta})^{-1} = \pi\left(\hat{x}(\hat{\alpha})\right).$$

C is the centre of the involution g in F_1. Notice that the automorphism group C preserves the sector. To complete the description of the Monster group it is necessary to consider an additional involution which mixes the twisted and untwisted sectors:

$$\begin{pmatrix} 0 & h \\ h & 0 \end{pmatrix}.$$

The construction of this additional involution is described in refs.[27,28].

The question as to whether this construction can be incorporated in a physically realistic string model remains a rather distant whim [88], although it is possible to construct a heterotic string model in two dimensions, with a 24 dimensional internal space which is the $\mathbf{Z}_2$ orbifold:

$$\frac{\mathbf{R}^{24}}{\mathbf{Z}_2 \times \Lambda_L}.$$

The resulting theory has no continuous gauge symmetries, since there are no dimension one operators, however, it does have a discrete F_1 symmetry [30].

7.3 Enhanced Module Characters.

In this section we shall compute some of the character functions associated with the enhanced algebras to verify that the whole representation spaces are in fact $\hat{g}^\natural$ modules.

Recall that $\mathcal{H}_e$ is the level one basic $\hat{g}$–module. We have already written down its character in equation (2.49):

$$\chi_g^0(q) = q^{-d/24} \frac{\theta_\Lambda(q)}{\prod_{n=1}^{\infty}(1-q^n)^d},$$

where the label '0'=basic, and

$$\theta_\Lambda(q) = \sum_{\alpha \in \Lambda} q^{\alpha^2/2}.$$

For the rest of the discussion it is useful to use the theta functions defined in equation

(6.20), however, in this chapter we put $q \to q^{1/2}$, so that

$$\theta_2(q) = 2q^{1/12}\eta(q)\prod_{n=1}^{\infty}(1+q^n)^2,$$

etc, where $\eta(q)$ is the Dedekind function:

$$\eta(q) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n).$$

In this notation

$$\chi_g^0(q) = \eta(q)^{-d}\theta_\Lambda(q). \tag{7.10}$$

Under the reflection twist the untwisted space decomposes:

$$\mathcal{H}_e = \mathcal{H}_e^+ \oplus \mathcal{H}_e^-,$$

where $\mathcal{H}_e^\pm$ are level one $\hat{g}_0$–modules, with characters

$$\chi_g^\pm(q) = \frac{1}{2}\left\{\chi_g^0 \pm \left(2\eta\theta_2^{-1}\right)^{d/2}\right\}. \tag{7.11}$$

The twisted sector is also a direct sum of two $\hat{g}_0$–modules:

$$\mathcal{H}_T = \mathcal{H}_T[\mathbb{Z}] \oplus \mathcal{H}_T[\mathbb{Z}+1/2],$$

with characters

$$\begin{aligned}\chi_g^{\mathbf{Z}}(q) &= \frac{1}{2}c_T\eta^{d/2}\left\{\theta_4^{-d/2} + (-)^{d/8}\theta_3^{-d/2}\right\}\\ \chi_g^{\mathbf{Z}+1/2}(q) &= \frac{1}{2}c_T\eta^{d/2}\left\{\theta_4^{-d/2} - (-)^{d/8}\theta_3^{-d/2}\right\}.\end{aligned} \tag{7.12}$$

The enhanced module is

$$\mathcal{H}^\natural = \mathcal{H}_e^+ \oplus \mathcal{H}_T[\mathbb{Z}].$$

We now turn our attention to the evaluation of the character

$$\chi_g^\natural(q) = \chi_g^+(q) + \chi_g^{\mathbf{Z}}(q), \tag{7.13}$$

for a number of examples.

$\underline{E_8}$

Using the character formulae of ref.[89] the authors of ref.[26] show that $\mathcal{H}_e^+$, $\mathcal{H}_e^-$, $\mathcal{H}_T[\mathbb{Z}]$ and $\mathcal{H}_T[\mathbb{Z}+1/2]$ are the level one $D_8^{(1)}$–modules, basic (0), spinor (s), $\overline{\text{spinor}}$ (c) and vector (v), respectively. Since

$$\chi_{D_8}^{s}(q) \;=\; \chi_{D_8}^{c}(q), \tag{7.14}$$

they conclude that $\chi_{E_8}^{-}(q) = \chi_{E_8}^{\mathbb{Z}}$, this means

$$\chi_{E_8}^{0}(q) \;=\; 2^4\eta^4\left(\theta_2^{-4} - \theta_3^{-4} + \theta_4^{-4}\right),$$

and implies that

$$\chi_{E_8}^{\natural}(q) \;=\; \chi_{E_8}^{0}(q). \tag{7.15}$$

The enhanced space indeed has the character of the basic level one $E_8^{(1)}$–module. Notice, however, that the original E_8 was built from a spinor of D_8 while the 'enhanced' E_8 is built from a $\overline{\text{spinor}}$.

$\underline{D_8}$

We can equate $\chi_{D_8}^{0}(q)$ to $\chi_{E_8}^{+}(q)$, giving

$$\chi_{D_8}^{0}(q) \;=\; 2^3\eta^4\left(2\theta_2^{-4} - \theta_3^{-4} + \theta_4^{-4}\right).$$

The character of the enhanced module is

$$\begin{aligned}\chi_{D_8}^{\natural}(q) &= \chi_{D_8}^{+}(q) + \chi_{D_8}^{\mathbb{Z}}(q)\\ &= 2^2\eta^4\left(4\theta_2^{-4} - \theta_3^{-4} + \theta_4^{-4}\right) + 2^2\eta^4\left(\theta_4^{-4} - \theta_3^{-4}\right)\\ &= 2^3\eta^4\left(2\theta_2^{-4} - \theta_3^{-4} + \theta_4^{-4}\right)\\ &= \chi_{D_8}^{0}(q),\end{aligned}$$

so the enhanced space has the character of the basic level one $D_8^{(1)}$–module.

$\underline{D_4 + D_4}$

Equating

$$\chi^0_{D_4^2}(q) = \chi^+_{D_8}(q),$$

gives

$$\chi^0_{D_4^2}(q) = 2^2\eta^4\left(4\theta_2^{-4} - \theta_3^{-4} + \theta_4^{-4}\right).$$

the character of the enhanced module is

$$\begin{aligned}\chi^{\natural}_{D_4^2}(q) &= 2\eta^4\left(8\theta_2^{-4} - \theta_3^{-4} + \theta_4^{-4}\right) + 2\eta^4\left(\theta_4^{-4} - \theta_3^{-4}\right) \\ &= 2^2\eta^4\left(4\theta_2^{-4} - \theta_3^{-4} + \theta_4^{-4}\right) \\ &= \chi^0_{D_4^2}(q),\end{aligned}$$

so the enhanced space has the character of the basic level one $(D_4 + D_4)^{(1)}$–module.

$\underline{E_8 + E_8 \text{ and } spin(32)/\mathbb{Z}_2}$

$\mathcal{H}_e$ is the sum of two basic level one $E_8^{(1)}$ modules, therefore

$$\chi^0_{E_8+E_8}(q) = \left(\chi^0_{E_8}(q)\right)^2.$$

We now pause to prove that

$$\chi^{\mathbf{Z}}_{E_8+E_8}(q) = 2\chi^{\mathbf{Z}}_{E_8}\left(\chi^{\mathbf{Z}}_{E_8} + 2^4\eta^4\theta_2^{-4}\right).$$

Proof :

$$\begin{aligned}2\left(\chi^{\mathbf{Z}}_{E_8}(q)\right)^2 &= 2^7\eta^8\left(\theta_4^{-4} - \theta_3^{-4}\right)^2 \\ &= 2^7\eta^8\left(\theta_4^8 - 2\theta_3^{-4}\theta_4^{-4} + \theta_3^{-8}\right),\end{aligned}$$

and

$$2^5\eta^4\theta_2^{-4}\chi^{\mathbf{Z}}_{E_8} = 2^8\eta^8\left(\theta_4^{-4} - \theta_3^{-4}\right)\theta_2^{-4},$$

therefore

$$2\chi^{\mathbf{Z}}_{E_8}\left(\chi^{\mathbf{Z}}_{E_8} + 2^4\eta^4\theta_2^{-4}\right) = \chi^{\mathbf{Z}}_{E_8+E_8} + 2^8\eta^4\left(\theta_2\theta_3\theta_4\right)^{-4}\left(\theta_3^4 - \theta_4^4 - \theta_2^4\right).$$

The use of the '*aequatio identico satis abstrusa*' of Jacobi:

$$\theta_2^4 + \theta_4^4 \;=\; \theta_3^4,$$

completes the proof.

The enhanced character is

$$\chi^{\natural}_{E_8+E_8} \;=\; \frac{1}{2}\left\{\left(\chi^0_{E_8}\right)^2 + 2^8\eta^8\theta_2^{-8}\right\} + 2\chi^{\mathbb{Z}}_{E_8}\left(\chi^{\mathbb{Z}}_{E_8} + 2^4\eta^4\theta_2^{-4}\right),$$

but from (7.14) we have

$$\chi^{\mathbb{Z}}_{E_8} \;=\; \frac{1}{2}\left(\chi^0_{E_8} - 2^4\eta^4\theta_2^{-4}\right),$$

therefore

$$\chi^{\natural}_{E_8+E_8}(q) \;=\; \left(\chi^0_{E_8}(q)\right)^2 \;=\; \chi^0_{E_8+E_8}(q). \tag{7.16}$$

We have already discovered that

$$(E_8+E_8)^{\natural} \;=\; D_{16},$$

which together with the above result implies that the enhanced module is actually a reducible basic+spinor of $D_{16}^{(1)}$. That is the module associated with the $spin(32)/\mathbb{Z}_2$ lattice. The result relies on the fact that the characters of the E_8+E_8 and $spin(32)/\mathbb{Z}_2$ theories are in fact equal [11]. Since this is true we do not need to repeat the character analysis for $spin(32)/\mathbb{Z}_2$. We have already remarked that the two possible algebras in the original heterotic string models are related by the reflection twist, we have now shown that the whole spectra are consistently related.

It is straightforward to complete the character analysis for the other cases we have not considered.

8. THE SHIFTED PICTURE AND HIGHER ORDER TWISTS.

There is an alternative way to set up the Hilbert space and calculate correlation functions of twisted emission vertices for certain classes of lattices and twists. In effect, one can 'rebosonize' the theory in such a way that the TSEVs become simple exponentials of free fields. The idea relies on the fact that when Λ is the root lattice of a simply-laced Lie algebra g and $\hat{W}$ is a cyclic group of order N, generated by an inner automorphism of the algebra g, then it is possible to represent $\hat{W}$ as a **shift** on the maximal torus of g [49,58]. This means $\hat{W} : g \to g$ such that for $\hat{u} \in \hat{W}$:

$$\begin{aligned}
\hat{u} : E_\alpha &\longmapsto e^{2\pi i \delta_u . H} E_\alpha e^{-2\pi i \delta_u . H} \\
&= e^{2\pi i \delta_u . \alpha} E_\alpha \\
\hat{u} : H &\longmapsto H,
\end{aligned}$$

i.e. $\hat{u}$ is represented by the shift operator:

$$e^{2\pi i \delta_u . H}.$$

We call this representation the **shifted picture**. This rebosonization of the twisted model is analogous to the bosonization of the NSR spinning string, which has the advantage that the fermion emission vertex becomes a simple exponential of the bosonizing field [9]. The benefit of dealing with exponentials of free fields is that the correlation functions are easily calculated. The phenomenon of algebra enhancement, in the shifted perspective, is also greatly simplified. The isomorphism between the shifted and twisted pictures has also been exploited to determine the twist invariant algebras g_0 and ground state degeneracies c_u for inner automorphisms of all the simply–laced exceptional Lie algebras [4,54].

In this chapter we also discuss the possibility of algebra enhancement for higher order twists, concentrating on a third order in E_6 and E_8.

Before we go on to discuss shifting we find it convenient to pause and consider the regrading of the basic $g^{(1)}$–module we constructed in chapter 2 via the FKS mechanism.

8.1 REGRADINGS OF $g^{(1)}$.

In section 2.7 we showed how the untwisted sector was a $g^{(1)}$-module. With L_0 as the derivation, we had the **homogeneous gradation** of $g^{(1)}$, characterized by the horizontal algebra g. Other gradings of $g^{(1)}$ can be formed by choosing new derivations d. Picking a new derivation corresponds to the freedom of shifting L_0 by an element of the Cartan subalgebra:

$$d = L_0 + \lambda.p.$$

This corresponds to a new stress–energy tensor

$$T^\lambda(z) = -\frac{1}{2} : \partial X(z).\partial X(z) : -i\lambda.\partial^2 X(z),$$

which satisfies the Virasoro algebra with central charge

$$c = d - 12\lambda^2.$$

The shifting of $T(z)$ is an important feature of the bosonization of the reparameterization ghosts in the bosonic string, and the super–conformal ghosts in the spinning string [9].

The fields have their dimension altered due to the shift. For example the highest weight momentum field $V(\alpha, z)$ satisfies

$$T^\lambda(z)V(\alpha,\omega) = \frac{\alpha^2/2 + \alpha.\lambda}{(z-\omega)^2}V(\alpha,\omega) + \frac{1}{z-\omega}\partial V(\alpha,\omega) + Reg,$$

i.e. its new dimension is

$$\Delta^\lambda = \Delta + \alpha.\lambda.$$

This leads to a regrading of the fields:

$$V(\alpha, z) = \sum_{n \in \mathbf{Z} - \alpha.\lambda} V_n^\lambda(\alpha) z^{-n-\Delta^\lambda}. \tag{8.1}$$

Since $V(\alpha, z)$ has an integral expansion, $n \in \mathbf{Z} - \alpha.\lambda$ and

$$V_n^\lambda(\alpha) = V_{n+\alpha.\lambda}(\alpha).$$

To sum up, with derivation $d = L_0 + \lambda.p$ the $\{V_n^\lambda(\alpha)\}$, $n \in \mathbf{Z} - \alpha.\lambda$ and $\alpha \in \Lambda_2$, along with $\alpha_n + \lambda$, generate a **regraded** representation of $g^{(1)}$. In particular, the horizontal finite Lie algebra $\overline{g}[\lambda]$, generated by

$$g[\lambda] \sim \begin{cases} V_0^\lambda(\alpha) & \alpha.\lambda \in \mathbb{Z} \\ p + \lambda, & \end{cases}$$

is a subalgebra of g.

8.2 The Shifted Picture.

As we remarked in the introduction to this chapter when

1. Λ is the root lattice of a simply–laced Lie algebra g, and
2. $\hat{W} \simeq \mathbb{Z}_N$, the cyclic group generated by an *inner* automorphism of g

then we can represent $\hat{u} \in \hat{W}$ by

$$\hat{u} = e^{2\pi i \delta_u . p}. \tag{8.2}$$

For $\hat{u}$ to be a non–trivial automorphism the shift vector must be of the form:

$$\delta_u = \frac{1}{N_u}\omega_u, \quad \omega_u \in \Lambda^\star,$$

where, as usual, N_u is the order of $\hat{u}$. This ensures that $(\hat{u})^{N_u} = \mathbf{1}$, the identity, up to a trivial automorphism. Notice that δ_u' and δ_u are equivalent if $\delta_u' - \delta_u \in \Lambda^\star$.

The untwisted sector is constructed as before, except the lift $\hat{W}$ into $\mathcal{H}_c$ is now given by (8.2). The difference occurs in the twisted sectors, which now have a much simpler construction in terms of a shifted lattice:

$$\mathcal{H}_u^s = \mathcal{F} \otimes \mathcal{P}(\Lambda + \delta_u), \tag{8.3}$$

where $\mathcal{F}$ is the conventional untwisted Fock space. The equivalence of the twisted and shifted pictures, for inner automorphisms, is embodied in the equality of the characters [2,49,58]:

$$c_u \frac{q^{\Theta^u} \theta_{\Lambda^u}(q)}{\prod_{n=1}^{\infty} (1 - q^{n/N_u})^{d_n}} = \frac{\theta_{\Lambda+\delta_u}(q)}{\prod_{n=1}^{\infty} (1 - q^n)^d}. \tag{8.4}$$

In particular the equality of the conformal weight of the ground state implies $\delta_u^2/2 = \Theta^u$, for suitable δ_u.

In the shifted picture the vertices which insert a twisted state are simply normal untwisted vertices but with momenta lying on the shifted lattices $\Lambda + \delta_u$. The complicated matrix structure of the operators is replaced by a much simpler structure: each operator has just one canonical representation in all sectors. The effect of changing sectors is simply to regrade the operators. In particular the algebra generated by the untwisted operators in a twisted sector is a regrading of the algebra generated in the untwisted sector:

$$\begin{aligned} V^u(\alpha, z) \mid \delta_u\rangle &\equiv V(\alpha, z) \mid \delta_u\rangle \\ &= \exp\left(i\alpha.X^-(z) + i\alpha.q\right) z^{\alpha.\delta_u} \mid \delta_u\rangle \\ &= \sum_{n\in\mathbb{Z}-\alpha.\delta_u} V^u_n(\alpha) z^{-n-\alpha^2/2} \mid \delta_u\rangle. \end{aligned}$$

The operators $V^u(\alpha, z)$ and $i\epsilon.\partial X(z)$ generate an algebra isomorphic to a regraded representation of $g^{(1)}$, explicitly

$$\begin{aligned} V^u_n(\alpha) &\simeq V^{\delta_u}_n(\alpha), \qquad n \in \mathbb{Z} - \alpha.\delta_u \\ \alpha_n &\simeq \alpha_n + \delta_u, \qquad n \in \mathbb{Z}, \end{aligned}$$

i.e. the algebra generated by the untwisted operators $V(\alpha, z)$, $\alpha \in \Lambda_2$, and $i\epsilon.\partial X(z)$, is $g^{(1)}$ in *all* sectors; the twisted sectors being regraded $g^{(1)}$–modules. The twist invariant horizontal subalgebra $g_0[u]$ is generated by

$$g_0[u] \sim \begin{cases} V_0(\alpha) & \alpha.\delta_u \in \mathbb{Z} \\ p. & \end{cases}$$

$g_0[u] \simeq g[\delta_u]$ is actually rather easily determined from the extended Dynkin diagram by a process of 'knocking out spots' [4,49]. We now consider this in more detail. It has been shown by Kac [49] that all possible finite inner automorphisms of a Lie algebra are given by labelling the Dynkin diagram of its corresponding Kac–Moody algebra (the extended diagram) with coprime integers. Let $s = (s_0, s_1, ..., s_d)$ be the sequence of non–negative relatively prime integers ('0' refers to the extra spot). Set

$$N = \sum_{i=0}^{d} k_i s_i,$$

where the indices k_i are the **Kac labels**, which are given by the decomposition of the highest root α_H of g:

$$\alpha_H = \sum_{i=1}^{d} k_i\alpha_i, \quad k_0 = 1.$$

The particular automorphism is now described by the shift vector

$$\delta = \frac{1}{N}\sum_{i=1}^{d} s_i\omega_i, \tag{8.5}$$

where the $\{\omega_i\}$ are the fundamental weights. This δ defines a unique automorphism of g of order N. Furthermore, such a choice of δ is *dominant* as it lies in, or along, the wall of the fundamental Weyl Chamber, and it is the shortest vector in the equivalence class ($\delta' \sim \delta$ if $\delta' - \delta \in \Lambda^*$). The subalgebra left invariant by this automorphism is simply obtained by knocking out the spots of the extended Dynkin diagram corresponding to non–zero s_i's, and adding enough $u(1)$'s to preserve the rank.

Since all the vertex operators in the theory have a simple form the operator algebra is rather straightforward to elucidate. If $\hat{W}$ is generated by $\hat{u}$ then the N_u sectors are based on the lattices $\Lambda + n\delta_u$, $0 \leq n < N_u$. The operator algebra has the following action on the sectors:

$$\mathcal{H}^s_{u^n} \times \mathcal{H}^s_{u^m} \longrightarrow \mathcal{H}^s_{u^{n+m}}. \tag{8.6}$$

If the operator algebra is to close then we require

$$N_u\delta_u \in \Lambda. \tag{8.7}$$

In general $N_u\delta_u \in \Lambda^*$, so the above condition is *not* always guaranteed. For the reflection twist we will find that the above constraint seems to play an analogous rôle in the shifted picture that Λ^* having to be 'integral' played in the twisted picture. We conjecture that this statement generalizes to arbitrary twists. The conclusion is that the operator algebra is ill–defined, in the sense that it does not close, unless (8.7) is satisfied. Let us consider this in more detail for the inner reflection twists.

$\underline{D_{2n},\ n \in \mathbf{Z}}$

The appropriate shift vector is

$$\delta = \frac{1}{2}\omega_n,$$

where the $\{\omega_i\}$ are the fundamental weights. In terms of the simple roots:

$$\delta = \frac{1}{2}\Big\{\sum_{p=1}^{n-1} p\alpha_p + n\sum_{p=n}^{2n-2}\alpha_p + \frac{n}{2}(\alpha_{2n-1}+\alpha_{2n})\Big\}.$$

Notice that $2\delta \in \Lambda$ only when $n \in 2\mathbb{Z}$, but this is precisely the cases when Λ^* is 'integral'†.

$\underline{E_7}$

The appropriate shift vector is

$$\delta = \frac{1}{2}\omega_7.$$

In terms of the simple roots:

$$\delta = \frac{1}{2}\Big\{2\alpha_1 + 4\alpha_2 + 6\alpha_3 + \frac{9}{2}\alpha_4 + 3\alpha_5 + \frac{3}{2}\alpha_6 + \frac{7}{2}\alpha_7\Big\},$$

so $2\delta \notin \Lambda$, which matches the fact that Λ^* is not 'integral'.

$\underline{E_8}$

The appropriate shift vector is

$$\delta = \frac{1}{2}\omega_1.$$

In terms of the simple roots:

$$\delta = \frac{1}{2}\Big\{4\alpha_1 + 7\alpha_2 + 10\alpha_3 + 8\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 + 5\alpha_8\Big\},$$

so that $2\delta \in \Lambda$, which matches the fact that Λ^* is 'integral'. In fact for E_8 all automorphisms satisfy the constraint (8.6) because $\Lambda^* = \Lambda$.

† Recall that the spinors of D_{2n} have length squared $n/2$.

8.3 Algebra Enhancement in the Shifted Picture.

We turn our attention to the operator algebra for the reflection twist in the shifted picture to investigate algebra enhancement. We have already established that the shifted operator algebra is only defined for D_{4n}, $n \in \mathbb{Z}$, and E_8. We will be led to a simple explanation of why, for the rank 8 cases, E_8, D_8 and $D_4 + D_4$, the enhanced algebra is always equal to the original algebra g. The discussion of the rank 16 cases requires a more detailed analysis.

Rank 8

Choosing δ to be *primitive*, for rank 8 we have:

$$\delta^2 = 1,$$

which matches $\Theta = 1/2$ in the twisted picture. Also it is useful to note that $\alpha.\delta \in \{0, \pm 1/2, \pm 1\}$ for $\alpha \in \Lambda_2$. The invariant algebra $\hat{g}_0$ is generated by

$$\hat{g}_0 \sim \begin{cases} V(\alpha, z) & \alpha.\delta \in \{0, \pm 1\},\ \alpha \in \Lambda_2 \\ i\epsilon.\partial X(z). & \end{cases}$$

The TSEVs which contribute to the enhanced algebra are those of the form:

$$V(\alpha + \delta, z) \quad (\alpha + \delta)^2 = 2.$$

The crucial point to notice is that the vectors on the shifted lattice of squared length 2 are in one–to–one correspondence with the vectors $\alpha \in \Lambda_2$ with $\alpha.\delta = \pm 1/2$. The correspondence being given by

$$\alpha \longleftrightarrow \alpha - 2(\alpha.\delta)\delta.$$

Notice that

$$(\alpha - 2(\alpha.\delta)\delta)^2 = \alpha^2,$$

since $\delta^2 = 1$, so that the invariant generators can equally well be written as

$$V(\alpha - 2(\alpha.\delta)\delta, z), \quad \alpha.\delta = 0, \pm 1, \quad \alpha \in \Lambda_2,$$

which is consistent because $2\delta \in \Lambda$. The enhanced algebra is therefore spanned by the generators:

$$\hat{g}^{\natural} \sim \begin{cases} V(\alpha - 2(\alpha.\delta)\delta, z) & \alpha \in \Lambda_2 \\ i\epsilon.\partial X(z). \end{cases}$$

Since $\delta^2 = 1$ the generators of $g^{\natural}$ are defined on the root system of g Weyl–reflected in the vector 2δ. It is therefore manifest that $g^{\natural} = g$ for these cases.

For the rank 16 cases $\delta^2 = 2$ and we cannot interpret the shift as a Weyl reflection, however, we can evaluate the shifted lattices explicitly. Below we consider the E_8, D_8, $E_8 + E_8$, $E_8 + D_8$, D_{16}, $D_8 + D_8$, $spin(32)/\mathbb{Z}_2$ cases in detail.

In general under the shift Λ decomposes:

$$\Lambda = \Lambda^+ \cup \Lambda^-,$$

where generically

$$\begin{aligned} \Lambda^+.\delta &\in \mathbb{Z} \\ \Lambda^-.\delta &\in \mathbb{Z} + \frac{1}{2}. \end{aligned}$$

The shifted lattice decomposes into vectors of even and odd length

$$\Lambda^{\delta} = \Lambda + \delta = \Lambda^e \cup \Lambda^o.$$

The enhanced module is constructed on the lattice:

$$\Lambda^{\natural} = \Lambda^+ \cup \Lambda^e,$$

or equivalently

$$\Lambda^{\natural} = \begin{cases} \Lambda^+ \cup (\Lambda^- + \delta) & \text{rank } 8 \\ \Lambda^+ \cup (\Lambda^+ + \delta) & \text{rank } 16. \end{cases}$$

In what follows we shall use the following notation for the cosets Λ^*/Λ of D_n:

$$\begin{aligned} \Lambda &= (0) \qquad & \Lambda + \text{vector} &= (v) \\ \Lambda + \text{spinor} &= (s) \qquad & \Lambda + \overline{\text{spinor}} &= (c). \end{aligned}$$

In general the shift vectors are weights of g_0.

$\underline{E_8}$ $(\overline{g_0} = D_8)$

The E_8 root lattice can be written in terms of D_8 weight lattice cosets as

$$\Lambda = (0) \cup (s).$$

The shift vector $\delta = \omega_1/2$ in the orthogonal basis (see appendix A for notation) is

$$\delta = -e_1 = \omega_v,$$

i.e. δ is a vector weight of D_8. Using the fact that

$$\begin{aligned}(0) + \omega_v &= (v)\\ (s) + \omega_v &= (c),\end{aligned}$$

we deduce

$$\Lambda^\delta = (v) \cup (c),$$

and therefore

$$\Lambda^\natural = (0) \cup (c).$$

This is an E_8 root lattice again, but with a different decomposition; the s and c have been exchanged.

For the other cases we simply quote the relevant results.

$\underline{D_8}$ $(g_0 = D_4 + D_4)$

$$\begin{aligned}\delta &= \frac{1}{2}\sum_{i=1}^{4} e_i = (\omega_s, 0)\\ \Lambda &= (0,0) \cup (v,v)\\ \Lambda^\delta &= (s,0) \cup (c,v)\\ \Lambda^\natural &= (0,0) \cup (c,v).\end{aligned}$$

Because of the triality of the **v**, **s** and **c** representations of D_4 this is the root lattice of D_8, but with a different g_0 decomposition.

$\underline{E_8 + E_8}$ $(g_0 = D_8 + D_8)$

$$\begin{aligned}
\delta &= (\omega_v, \omega_v) \\
\Lambda &= (0,0) \cup (s,0) \cup (0,s) \cup (s,s) \\
\Lambda^\delta &= (v,v) \cup (c,v) \cup (v,c) \cup (c,c) \\
\Lambda^\natural &= (0,0) \cup (v,v) \cup (s,s) \cup (c,c) \\
&= \Lambda\,(spin(32)/\mathbb{Z}_2)\,.
\end{aligned}$$

$\underline{E_8 + D_8}$ $(g_0 = D_8 + D_4 + D_4)$

$$\begin{aligned}
\delta &= (\omega_{v'}, \omega_s, 0) \\
\Lambda &= (0,0,0) \cup (s',0,0) \cup (0,v,v) \cup (s',v,v) \\
\Lambda^\delta &= (v',s,0) \cup (c',s,0) \cup (v',c,v) \cup (c',c,v) \\
\Lambda^\natural &= (0,0,0) \cup (s',v,v) \cup (v',s,0) \cup (c',c,v) \\
&= \Lambda(D_{12} + D_4) \cup (\Lambda_s(D_{12}), \Lambda_v(D_4))\,.
\end{aligned}$$

In this case the enhanced module is in fact a reducible representation of $\hat{g}^\natural$, being a basic+(spinor, vector).

$\underline{D_{16}}$ $(g_0 = D_8 + D_8)$

$$\begin{aligned}
\delta &= \frac{1}{2}\omega_7 \;=\; \frac{1}{2}\sum_{i=1}^{8} e_i \;=\; (\omega_s, 0) \\
\Lambda &= (0,0) \cup (v,v) \\
\Lambda^\delta &= (s,0) \cup (c,v) \\
\Lambda^\natural &= (0,0) \cup (s,0) \\
&= \Lambda\,(E_8 + D_8)\,.
\end{aligned}$$

$\underline{D_8 + D_8}$ $(g_0 = D_4 + D_4 + D_4 + D_4)$

$$\begin{aligned}
\delta &= (\omega_s, 0, \omega_s, 0) \\
\Lambda &= (0,0,0,0) \cup (v,v,0,0) \cup (0,0,v,v) \cup (v,v,v,v) \\
\Lambda^\delta &= (s,0,s,0) \cup (c,v,s,0) \cup (s,0,c,v) \cup (c,v,c,v) \\
\Lambda^\natural &= (0,0,0,0) \cup (v,v,v,v) \cup (s,0,s,0) \cup (c,v,c,v) \\
&= \Lambda(D_8 + D_4 + D_4) \cup (\Lambda_v(D_8), \Lambda_v(D_4), \Lambda_v(D_4)) \,.
\end{aligned}$$

In this case the enhanced module is in fact a reducible representation of $\hat{g}^\natural$ composed of basic+(vector, vector, vector).

$\underline{spin(32)/\mathbb{Z}_2}$ $(g_0 = D_8 + D_8)$

$$\begin{aligned}
\delta &= (\omega_s, 0) \\
\Lambda &= (0,0) \cup (v,v) \cup (s,s) \cup (c,c) \\
\Lambda^\delta &= (s,0) \cup (c,v) \cup (0,s) \cup (v,c) \\
\Lambda^\natural &= (0,0) \cup (0,s) \cup (s,0) \cup (s,s) \\
&= \Lambda\,(E_8 + E_8) \,.
\end{aligned}$$

The other cases that we have not considered here can be analysed in a similar way.

8.4 Algebra Enhancement and other Twists.

An interesting question to pose at this point is: does algebra enhancement occur for other higher order twists? We will not attempt a systematic investigation of this question here, since when automorphisms which are not NFPAs are included the number of possibilities to consider is quite large [4,54,90]. Rather we shall concentrate on a couple of examples, both of third order. A quick glance at equation (3.25) shows that the conformal weight of the twisted ground state is $\Theta = d/18$, where d is the effective number of dimensions that the twist acts in. This shows that the relevant dimensions are 6, 12 and 18, for a NFPA. The simply–laced Lie algebras which admit such an automorphism are A_2, D_4, E_6 and E_8 [4,90]. The twist invariant subalgebras are $u(1) + u(1)$, A_2, $A_2 + A_2 + A_2$ and A_8, respectively. Out of the g simple examples only E_6 has a suitable rank. We shall consider this case in detail. As an example of a non–NFPA we shall also consider the third order twist in E_8 which has $g_0 = E_6 + A_2$ with $\Theta = 1/3$, and so is also a suitable candidate for enhancement.

8.4.1. E_6 in the Twisted Picture.

It is advantageous to discuss E_6 in terms of D_4 weights. The roots of E_6 are conveniently written as

$$\begin{array}{cc} (\alpha, 0) & (s, \pm a_2) \\ (v, \pm a_1) & (c, \pm a_3), \end{array}$$

where α is any root of D_4; v, s and c are any of the vector or two kinds of spinor weights, respectively. a_1, a_2 and $a_3 = -(a_1+a_2)$ are the roots of an A_2 scaled to have unit length. In this basis the inner automorphism we are concerned with has a decomposition:

$$\sigma = (\sigma_1, \sigma_2),$$

where σ_1 is one of the third order NFPAs of D_4 which takes $v \to s \to c$ [90], and σ_2 is a 120° rotation in the scaled A_2 subspace, permuting $\{a_1, a_2, a_3\}$. In the orthogonal basis (see appendix A for notation) σ_1 is explicitly:

$$\sigma_1 = \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \end{pmatrix}. \tag{8.8}$$

There are in fact three cubic automorphisms of D_4 (up to conjugation), one of which is (8.8) and is outer, the other two are like the ones discussed in ref.[15], with one inner and one outer [90].

There are 24 orbits of roots under σ, portraying the fact that $g_0 = A_2^3$. In order to represent the twist invariant algebra in the two twisted sectors we have to construct the operator cocycles, which are just finite matrices. From section 3.5.1 and using the fact that σ is inner we deduce

$$c_\sigma = |\det(1-\sigma)|^{1/2}\,\frac{1}{\mathrm{vol}(\Lambda)},$$

which we can compute directly because $\mathrm{vol}(\Lambda) = \sqrt{3}$, and σ has three sets of conjugate eigenvalues $\exp(\pm 2\pi i/3)$. We find

$$c_\sigma = 3.$$

This will be the ground state degeneracy in both twisted sectors.

Since $\Theta = 1/3$ the TSEVs with unit dimension correspond to the states:

$$c^{a}_{-2/3} \mid 0\rangle \otimes \phi, \quad c^{a}_{-1/3}c^{b}_{-1/3} \mid 0\rangle \otimes \phi,$$

where $a, b = 1, 2, 3$. There are in fact 27 of these states in each of the two twisted sectors. These 54 states combine with the 24 invariant states of unit dimension in the untwisted sector to give 78 states in all. Therefore we suspect that $g^{\natural}$ is E_6 again. Later we shall find from the shifted point of view that this is indeed the case. Unfortunately in the twisted picture the algebra of the TSEVs is very complicated and we have not managed to elucidate it fully, however, we can at least verify that $g^{\natural}$ has the correct g_0 content to be E_6. To accomplish this we need to know how g_0 is generated in the twisted sectors, this requires us to construct the twisted operator cocycles.

Recall that the twisted operator cocycles are invariant under σ, *i.e*

$$\sigma_{\alpha}\hat{C}_{\sigma}(\sigma\alpha) = \hat{C}_{\sigma}(\alpha),$$

in addition we have

$$\begin{aligned} \hat{C}_{\sigma}(-\alpha) &= \hat{C}_{\sigma}(\alpha)^{-1} \\ \hat{C}_{\sigma^2}(\alpha) &= \hat{C}_{\sigma}(\alpha)^{\star}. \end{aligned}$$

Therefore we need only consider each orbit in one of the sectors. The 24 orbits may be labelled by the representatives:

$$\begin{aligned} \Omega_i^0 &= (\beta_i, 0) \\ \Omega_i^+ &= (\pm e_i, a_1) \\ \Omega_i^- &= (\pm e_i, -a_1), \end{aligned}$$

where $i = 1, 2, 3, 4$, and their negatives. In the above $\beta_i = (\alpha_3, \alpha_4, \alpha_H, \alpha_1)$ where the $\{\alpha_i\}$ are simple roots of D_4 and α_H is the highest root of D_4. Our problem is to assign 3×3 matrices to these orbits consistent with the symmetry factor $\Omega_{\sigma}(\alpha, \beta)$. The labelling of the $\{\beta_i\}$ was done deliberately so that vectors from orbits Ω_i^0, Ω_i^+ and Ω_i^- are in fact orthogonal, so it is consistent to take

$$\hat{C}_{\sigma}(\Omega_i^0) = \hat{C}_{\sigma}(\Omega_i^+) = \hat{C}_{\sigma}(\Omega_i^-).$$

We have reduced the problem to assigning matrices to just four of the orbits, say Ω_i^0.

Explicit calculation gives

$$\Omega_\sigma(\beta_i,\beta_j) = \begin{pmatrix} 1 & \omega^2 & \omega & \omega^2 \\ \omega & 1 & \omega^2 & \omega^2 \\ \omega^2 & \omega & 1 & \omega^2 \\ \omega & \omega & \omega & 1 \end{pmatrix},$$

where $\omega = e^{2\pi i/3}$. Therefore we may take

$$\hat{C}_\sigma(\beta_1) = h \qquad \hat{C}_\sigma(\beta_2) = gh$$
$$\hat{C}_\sigma(\beta_3) = hg^{-1} \qquad \hat{C}_\sigma(\beta_4) = g$$

where

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad h = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

and

$$gh = \omega hg.$$

The three A_2's are generated by

$$V^n(\Omega_i^0) = \frac{1}{3}\Big(V_0(\Omega_i^0) + \omega^n V_0(\Omega_i^+) + \omega^{2n} V_0(\Omega_i^-)\Big), \qquad (8.9)$$

and their hermitian conjugates. Here $n = 0, 1, 2$, so (8.9) does indeed define three mutually commuting sets of eight generators. The Cartan subalgebra is generated by $V^n(\Omega_4^0)$, and its conjugate, therefore on the ground state:

$$V^n(\Omega_4^0) \mid 0\rangle \otimes \phi = \mid 0\rangle \otimes \frac{1}{3}\left(1 + \omega^n + \omega^{2n}\right) g\phi,$$

from which we deduce that the ground state is a $(\mathbf{3}, \mathbf{1}, \mathbf{1})$ of g_0.

Now consider the first excited state. First of all let us define the projections

$$\epsilon[n] = \frac{1}{3}\Big(\epsilon + \omega^n \sigma\epsilon + \omega^{2n}\sigma^2\epsilon\Big).$$

A convenient basis for $V_{2n/3}$ (see section 3.2) is provided by $\epsilon[n]$ where $\epsilon \in \{\Omega_4^0, \Omega_4^+, \Omega_4^-\}$.

Using this basis we compute

$$V_0(\alpha)\epsilon[1].c_{-1/3} \mid 0\rangle \otimes \phi = \frac{1}{3}\Big\{\epsilon[1] - 3(\epsilon[1].\alpha[2])\alpha[1]\Big\}.c_{-1/3} \mid 0\rangle \otimes \hat{C}_\sigma(\alpha)\phi.$$

By taking $\alpha = \Omega_4^0$ we can extract the weights of the first excited state. For $\epsilon = \Omega_4^0, \Omega_4^+, \Omega_4^-$ we find weights

$$\frac{1}{3}\Big(-2 + \omega^n + \omega^{2n}, 1 - 2\omega^n + \omega^{2n}, 1 + \omega^n - 2\omega^{2n}\Big)g,$$

showing that the first excited states transform as a $(\mathbf{1}, \overline{\mathbf{3}}, \overline{\mathbf{3}})$ of g_0.

The computation for the second excited state is quite tedious and we omit some of the details. The relevant slice of the calculation is:

$$\begin{aligned}
&V_0(\alpha)\epsilon[1].c_{-1/3}\tau[1].c_{-1/3} \mid 0\rangle \otimes \phi \\
&= \Bigg\{\frac{1}{3}\Big(\epsilon^a[1]\tau^b[1] - 3(\tau[1].\alpha[2])\epsilon^a[1]\alpha^b[1] - 3(\epsilon[1].\alpha[2])\alpha^a[1]\tau^b[1]\Big)c^a_{-1/3}c^b_{-1/3} \\
&\qquad + \frac{3}{2}(\epsilon[1].\alpha[2])(\tau[1].\alpha[2])\Big((\alpha[1].c_{-1/3})^2 + 3\alpha[2].c_{-2/3}\Big)\Bigg\} \mid 0\rangle \otimes \hat{C}_\sigma(\alpha)\phi,
\end{aligned}$$

and

$$\begin{aligned}
&V_0(\alpha)\lambda[2].c_{-2/3} \mid 0\rangle \otimes \phi \\
&= \Bigg\{\frac{1}{3}\Big(\lambda[2] - \frac{3}{2}(\lambda[2].\alpha[1])\alpha[2]\Big).c_{-2/3} - \frac{3}{2}(\lambda[2].\alpha[1])\Big(\alpha[1].c_{-1/3}\Big)^2\Bigg\} \mid 0\rangle \otimes \hat{C}_\sigma(\alpha)\phi.
\end{aligned}$$

The only obstacle to calculating the weights is that states of the form

$$\begin{gathered}
\Big(\epsilon[1].c_{-1/3}\Big)^2 \mid 0\rangle \otimes \phi \\
\epsilon[2].c_{-2/3} \mid 0\rangle \otimes \phi,
\end{gathered}$$

become mixed and need to be diagonalized with respect to $V_0(\Omega_4^0)$. This is fairly easily accomplished and the weights extracted. The conclusion is that the second excited state transforms as a $(\overline{\mathbf{3}}, \mathbf{3}, \mathbf{3})$ of g_0. The other twisted sector will contribute the conjugate representation $(\mathbf{3}, \overline{\mathbf{3}}, \overline{\mathbf{3}})$.

The g_0 content of $g^\natural$ is therefore

$$\text{adjoint } \big(A_2^3\big) + (\overline{\mathbf{3}}, \mathbf{3}, \mathbf{3}) + (\mathbf{3}, \overline{\mathbf{3}}, \overline{\mathbf{3}}).$$

This is E_6 again, but with a different g_0 content than the original E_6.

8.4.2. E_6 and E_8 in the Shifted Picture.

As with the reflection twist, this enhancement looks much simpler in the shifted picture. The relevant shift vector is

$$\delta = \frac{1}{3}\omega_3 = \frac{1}{3}(2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 4\alpha_4 + 2\alpha_5 + 3\alpha_6),$$

with $\delta^2 = 2/3 = 2\Theta$. Notice that $3\delta \in \Lambda$ so the operator algebra is well defined. We can interpret the enhanced algbera as a Weyl reflection in δ, as for the rank 8 reflection twists. The generators of $\hat{g}^{\natural}$ are

$$\hat{g}^{\natural} \sim \begin{cases} V(\alpha - 3(\alpha.\delta)\delta, z) & \alpha \in \Lambda_2 \\ i\epsilon.\partial X(z). & \end{cases}$$

The TSEVs correspond to generators for which $\alpha.\delta \in \{\pm 1/3, \pm 2/3\}$, and $\hat{g}_0$ is generated on the subset of vertices for which $\alpha.\delta \in \mathbb{Z}$.

This argument is perfectly general and holds for any third order automorphism for which $\delta^2 = 2/3$. In particular, it means that the third order automorphism of E_8, with $g_0 = E_6 + A_2$ and $\delta^2 = 2/3$ [54], has $g^{\natural} = E_8$. We will now consider these two cases in more detail.

E_6

The E_6 root lattice has a A_2^3 decomposition:

$$\Lambda = (0,0,0) \cup (3,3,3) \cup (\overline{3},\overline{3},\overline{3}).$$

The shift vector is a weight of g_0:

$$\delta = \frac{1}{3}\omega_3 = (\lambda_{\overline{3}}, 0, 0),$$

where $\lambda_{\overline{3}}$ is a weight of the $\overline{\mathbf{3}}$ of one of the A_2's, therefore

$$\begin{aligned} \Lambda^{\delta} &= (\overline{3},0,0) \cup (0,3,3) \cup (3,\overline{3},\overline{3}) \\ \Lambda^{2\delta} &= (3,0,0) \cup (\overline{3},3,3) \cup (0,\overline{3},\overline{3}). \end{aligned}$$

The enhanced lattice consists of the invariant sublattice of Λ and the even sublattices of Λ^{δ} and $\Lambda^{2\delta}$:

$$\Lambda^{\natural} = (0,0,0) \cup (\overline{3},3,3) \cup (3,\overline{3},\overline{3}),$$

which is the root lattice of E_6.

$\underline{E_8}$

The E_8 root lattice can be written in terms of $E_6 + A_2$ cosets as

$$\Lambda = (0,0) \cup (27,3) \cup (\overline{27},\overline{3}).$$

The shift vector is a weight of g_0:

$$\delta = \frac{1}{3}\omega_6 = (0,\lambda_3),$$

where λ_3 is a weight of the $\mathbf{3}$ representation of A_2. The shifted lattices are therefore

$$\begin{aligned} \Lambda^{\delta} &= (0,3) \cup (27,\overline{3}) \cup (\overline{27},0) \\ \Lambda^{2\delta} &= (0,\overline{3}) \cup (27,0) \cup (\overline{27},3). \end{aligned}$$

The enhanced lattice is

$$\Lambda^{\natural} = (0,0) \cup (27,\overline{3}) \cup (\overline{27},3),$$

which is the root lattice of E_8 again.

9. FINAL COMMENTS AND COMPARISONS.

In this final chapter we wish to consider generalizations of the formalism which we have developed in the preceding chapters, and also we want to make contact with other recent research in the field. In particular we will discuss **asymmetric orbifolds** [32], which include both left and right movers (analytic and anti–analytic degrees of freedom) propagating on two, possibly different, orbifolds. These models subsume symmetric orbifold models on which both left and right movers move on the *same* orbifold, and the totally asymmetric models we have been considering in the previous chapters. The reason why it is consistent to consider an asymmetric orbifold is because the analytic and anti–analytic sectors are completely decoupled, as we mentioned in chapter 2 for toroidal compactifications. For symmetric orbifolds we will find that the zero–mode space in the twisted sectors can indeed be identified with the singularities of the orbifold, and the twisted operator cocycles have a nice geometrical interpretation. We then go on to consider interactions. In particular, we compare our expression for the four twisted string interaction for the reflection twist with the work of refs.[33,34].

As an instructive and amusing exercise we apply the twisting/shifting formalism to the analytic toroidal models generated by the even self–dual Euclidean lattices in twenty–four dimensions. It is hoped that such considerations may eventually lead to a more complete understanding of the moonshine module. The main unsolved problem in this context is the question of the existence of a twenty–six dimensional niche for the FLM F_1 construction: surely it cannot be a coincidence that the bosonic string is anomaly free precisely in this dimension?

9.1 Asymmetric Orbifolds.

We saw in chapter 2 how an asymmetric toroidal compactification involved the even self–dual Lorentzian lattice $\Gamma = (\Lambda_L, \Lambda_R)$, with metric $\mathrm{diag}((+)^p, (-)^q)$, where $p = \dim(\Lambda_L)$ and $q = \dim(\Lambda_R)$. An orbifold generalization follows in an obvious way:

$$\Upsilon = \frac{\mathbf{R}^{p,q}}{2\pi\Gamma \times W}, \tag{9.1}$$

where the twist now acts separately on the left and right sectors:

$$W = (W_L, W_R)\,. \tag{9.2}$$

Although we are dealing with a Lorentzian lattice most of the analysis of chapter 3 is valid. Applying equation (3.44) and the self–duality of Λ we deduce that the zero–mode space in the $U = (u_L, u_R)$ twisted sector will have dimension:

$$c_U = \left| \frac{K_U}{(1-U)\Gamma} \right|^{1/2} . \tag{9.3}$$

In addition equation (3.7) gives the number of singularities of Υ as

$$F_U = c_U^2 .$$

At this point we remark that the requirement of modular invariance also determines the degeneracy c_U [32]. It is a remarkable unexplained 'coincidence' why the required degeneracy is exactly the same as (9.3) above. This hints of a deep connection between modular invariance and the irreducibility of the operator algebra which has yet to be understood.

Notice that for a symmetric model c_U is simply the number of fixed point singularities on the *single* orbifold on which both the left and right degrees of freedom move: in this case the zero–mode space *is* just a set of position eigenstates corresponding to the fixed points singularities. For a symmetric model the twisted operator cocycles also have a neat geometrical interpretation as we now show. Consider a U–twisted string at a fixed point q_1 absorbing an untwisted string which has a winding vector l on the torus Λ. Classically the initial twisted sting satisfies the boundary condition

$$X_U(2\pi) = UX_U(0) + 2\pi\alpha_1,$$

where

$$(1-U)q_1 = 2\pi\alpha_1, \quad \alpha_1 \in \Lambda.$$

The untwisted string satisfies the boundary condition $X(2\pi) = X(0) + 2\pi l$, where $l \in \Lambda$. On interacting $X(0) = X_U(2\pi)$ and so in the final state we have a U–twisted string at a fixed point q_2, where

$$(1-U)q_2 = 2\pi(\alpha_1 + l),$$

so the fixed point has been flipped from†

$$q_1 \longrightarrow q_2 = q_1 + 2\pi(1-U)^{-1}l,$$

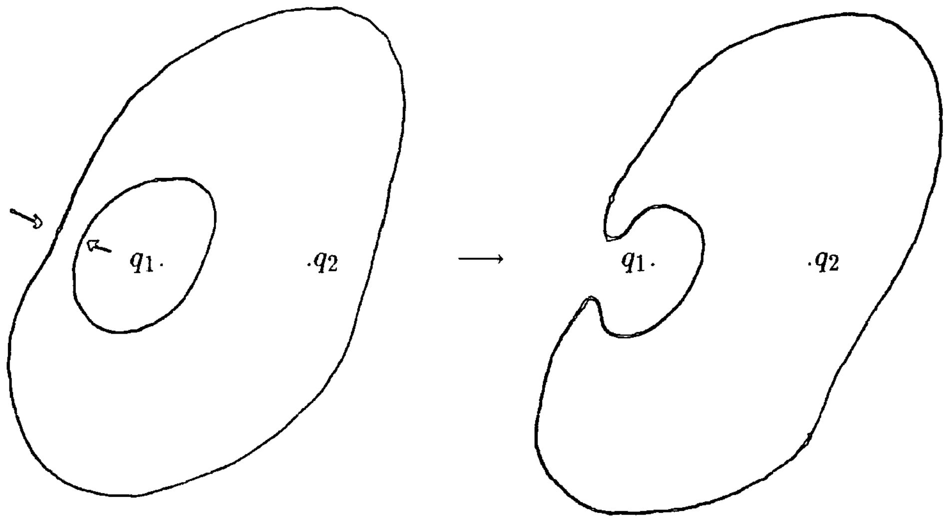
i.e. winding states couple twisted strings moving around different fixed points. From an

† For simplicity we assume that U is a NFPA.

operator point of view the twisted operator cocycle $\hat{C}_U(k,l)$, where k is the momentum of the state, is responsible for this:

$$\hat{C}_U(k,l) \mid q_1\rangle \propto \mid q_2\rangle = \mid q_1 + 2\pi(1-U)^{-1}l\rangle. \tag{9.4}$$

Pictorially:



Notice that pure momentum states do not change the fixed point, *i.e.* $\hat{C}_U(k,0)$ should be diagonal. This is in fact the case; vectors of the form $(k,0)$, which are light-like from the point of view of Γ (see section 2.6), are in the set A_U (on which the operator cocycles are diagonal, see section 3.5.1), because

$$\Omega_U\Big((k,0),(k',0)\Big) = 1.$$

The twisted operator cocycles, although essential for the consistency of the theory, appear to have been included in a somewhat ad-hoc manner. However recently, for a symmetric orbifold, they have been shown to arise from a proper canonical quantization of the string degrees of freedom taking particular care of the constraints involved [91]. In this picture they have the following action on the fixed point position eigenstates:

$$\hat{C}_U(k,l) \mid q\rangle = e^{ik.(q+\pi(1-U)^{-1}l)} \mid q + 2\pi(1-U)^{-1}l\rangle. \tag{9.5}$$

The zero–modes of orbifold models have also been considered in ref.[94].

9.2 GENERALIZED VERTEX OPERATORS.

The vertices which insert untwisted momentum states in a twisted sector are just generalizations of (3.38). We write them in terms of the lattice Γ:

$$V^U(A,z) \;=\; z^{-(\alpha_1^{\prime\prime L})^2/2}\overline{z}^{-(\overline{\alpha}_1^{\prime\prime R})^2/2} : \exp\Big(i\alpha.X_{u_L}(z) + i\overline{\alpha}.\overline{X}_{u_R}(\overline{z})\Big) : C_U(A), \qquad (9.6)$$

where $A = (\alpha, \overline{\alpha}) \in \Gamma$. The operator cocycles define a 2–cocycle on Γ:

$$\hat{C}_U(A)\hat{C}_U(B) \;=\; \varepsilon_U(A,B)\hat{C}_U(A+B),$$

where

$$\Omega_U(A,B) \;=\; \frac{\varepsilon_U(A,B)}{\varepsilon_U(B,A)} \;=\; \prod_{q=0}^{N_U-1}\left(-e^{-2\pi i q/N_U}\right)^{A.U^q B}.$$

The twisted string emission vertices also generalize in an obvious way. The oscillator pieces are as before, except that now there is an anti–analytic contribution. The zero–mode dependence is contained in the sum over momentum/winding states, written in terms of the lattice Γ:

$$\langle v \mid \longrightarrow \sum_{B\in\Gamma} \langle B \mid \hat{C}_U(B).$$

9.3 THE FOUR TWISTED STRING INTERACTION.

In chapter 6 we calculated the four twisted ground state string correlation function for the reflection twist. We now use these results to discuss the interaction of four twisted ground state strings, and compare with refs.[33,34] who perform the calculation from a path integral point of view. To make contact with these other works, we suppose that the vertices have momenta in some uncompactified space, *i.e.* the twisted vertices are of the form:

$$W_i(z,\overline{z}) \;=: e^{iK_i.Y(z,\overline{z})} : W(\phi_i, z, \overline{z}),$$

where to avoid confusion we have labelled the string field in the uncompactified dimension by $Y(z,\overline{z})$. The twisted string emission vertex has analytic and anti–analytic pieces which are a generalization of (5.4). The vertices are on shell, *i.e.* they have unit analytic and anti–analytic conformal dimensions:

$$\Delta_i = 1, \quad \overline{\Delta}_i = 1.$$

The scattering amplitude for four twisted strings is constructed in the usual way from the correlation function [40]:

$$\mathcal{A}_4 = \int \prod_{i=1}^{4} d^2 z_i \,\mathrm{Vol}\Big(SL(2,\mathbf{C})\Big)^{-1} \langle 0 \mid \overline{W}_1(z_1,\overline{z}_1) W_2(z_2,\overline{z}_2) \overline{W}_3(z_3,\overline{z}_3) W_4(z_4,\overline{z}_4) \mid 0 \rangle.$$

We take the $SL(2,\mathbf{C})$ volume element to be

$$\mathrm{Vol}\Big(SL(2,\mathbf{C})\Big) = \frac{d^2 z_1\, d^2 z_2\, d^2 z_4}{\mid z_1 - z_2 \mid^2 \mid z_1 - z_4 \mid^2 \mid z_2 - z_4 \mid^2},$$

and $(z_1, z_2, z_3, z_4) \to (\infty, 1, x, 0)$ giving

$$\mathcal{A}_4 = \int d^2 x \mid 1 - x \mid^{k_2.k_3/2} \mid x \mid^{k_1.k_3/2} F(x, \overline{x}),$$

where $F(x,\overline{x})$ is an expectation value of the type calculated in chapter 6, now with an anti–analytic contribution. Generalizing equation (6.22) in just such a way gives:

$$\begin{aligned} F(x,\overline{x}) &= \mid x(1-x) \mid^{-d/4} \\ &\times \mid \theta_3(q) \mid^{-2d} \sum_{\substack{k\in\Lambda^* \\ l\in\Lambda}} \Big\{ \mu^\dagger C_T(k,l)\psi \Big\} \Big\{ \phi^\dagger C_T^\dagger(k,l)\lambda \Big\}\, q^{(k/2+l)^2/2} \overline{q}^{(k/2-l)^2/2}. \end{aligned}$$

The vectors μ, ψ, ϕ and λ now correspond to the fixed point singularities of the $\mathbf{Z}_2$–orbifold.

The partition–like function in the amplitude can be written as a sum over the lattice Γ. Putting $q = e^{\pi i(2\sigma - 1)}$, as we are required to do if we want to check the duality of the amplitude, the sum becomes (for an arbitrary twist)

$$\mathcal{Z}_{cl} = \sum_{A\in\Gamma} \Big\{ \mu^\dagger \hat{C}_U(A)\psi \Big\} \Big\{ \phi^\dagger \hat{C}_U^\dagger(A)\lambda \Big\}\, e^{\pi i(2\sigma-1)A.A}. \tag{9.7}$$

Since Γ is even and self–dual Γ^* is 'integral' and so we expect the analysis of chapter 6 to be valid, at least for the reflection twist, *i.e.* the amplitude has poles as $x \to 1$, corresponding to the crossed channel, and hence the correct duality properties. To make contact with refs.[33,34] we use the representation of the operator cocycle in ref.[91], which we wrote down in equation (9.5). Let the fixed points represented by μ, ψ, ϕ and λ be q_1, q_2, q_3 and q_4, respectively. For an arbitrary twist we find:

$$\begin{aligned}\mu^\dagger \hat{C}_U(k,l)\psi &= e^{ik.(q_2+\pi(1-U)^{-1}l)}\langle q_1 \mid q_2 + 2\pi(1-U)^{-1}l\rangle \\ &= e^{ik.(q_2+(q_1-q_2)/2)}\delta_\Lambda\big(q_1 - q_2 - 2\pi(1-U)^{-1}l\big),\end{aligned}$$

and similarly

$$\phi^\dagger \hat{C}_U^\dagger(k,l)\lambda = e^{-ik.(q_4+(q_3-q_4)/2)}\delta_\Lambda\big(q_4 - q_3 - 2\pi(1-U)^{-1}l\big),$$

where $\delta_\Lambda(v) = 1$ when $v \in \Lambda$ and is 0 otherwise. For a non–zero contribution we deduce

$$q_1 - q_2 + q_3 - q_4 \in \Lambda.$$

The sum becomes

$$\mathcal{Z}_{cl} = \sum_{\substack{k\in\Lambda^* \\ l\in\Lambda'}} e^{ik.(q_2-q_3)} q^{(k/2+l)^2/2}\bar{q}^{(k/2-l)^2/2} \tag{9.8}$$

where Λ' is a sublattice of Λ defined as follows:

$$\Lambda' \equiv (1-U)\Big(\frac{1}{2\pi}(q_1 - q_2) + \Lambda\Big).$$

The final answer in this form is exactly the same as the amplitudes computed in refs.[33,34]. In their approach, via the path integral, $\mathcal{Z}_{cl}$ is the classical instanton contribution to the amplitude, computed by evaluating the classical contribution to the partition function from 'stretched string states'. In the operator formalism the zero–modes keep track of these classical contributions automatically. It is comforting to find that the operator and path integral formalisms are in complete agreement, at least for the reflection twist.

The generalization of the four twisted ground state string interaction to arbitrary twists has been given in ref.[33]. We have already considered the classical instanton contribution for an arbitrary twist, the other factors in the amplitude generalize as follows:

$$\begin{aligned} |\, x(1-x)\,|^{-d/4} &\longrightarrow |\, x(1-x)\,|^{-4\Theta^U} \\ \theta_3(q) &\longrightarrow \prod_{n=0}^{N_U-1} {}_2F_1\Big(\frac{n}{N_U}, 1-\frac{n}{N_U}, 1; x\Big)^{\dim(V_{n/N_U})/2}, \end{aligned}$$

where ${}_2F_1(a,b,c;x)$ is a hypergeometric function (see appendix C for its definition).

Even self-dual Euclidean lattices only exist in dimensions which are a multiple of 8. In 24 dimensions they were classified by Niemeier [95]. There are in fact 24 such lattices. Below we review their construction [96,97]. One of the lattices is unique in the sense that $\Lambda_2 = \emptyset$. This is the Leech lattice. All the other (Niemeier) lattices contain the root system (Λ_2) of a semi-simple simply-laced Lie algebra g. In fact the 23 lattices correspond precisely to the possible semi-simple simply-laced Lie algebras of rank 24, whose component subalgebras all have equal *Coxeter number h*. Suppose

$$g = \bigoplus_{a=1}^{m} g_a.$$

Let $\{\omega_i^a\}$ be the set of fundamental weights of the a^{th} component and $\Lambda_i^*(g_a)$ be the corresponding coset of $\Lambda^*(g_a)/\Lambda(g_a)$ which contains ω_i^a. The Niemeier lattice $\Lambda_N(g)$ is constructed by taking the root lattice of g together with certain combinations of weight lattice cosets from the different components:

$$\Lambda_N(g) = \bigcup_{f\in\mathcal{G}} f_{i_1,\ldots,i_m}\left(\Lambda_{i_1}^*(g_1)\oplus\ldots\oplus\Lambda_{i_m}^*(g_m)\right)\cup\Lambda(g). \tag{9.9}$$

The *glue vectors* $f_{i_1,\ldots,i_m}$ generate the *glue code* $\mathcal{G}$. A list of the 23 lattices together with their glue codes is given in refs.[96,97].

Each of the 23 Niemeier lattices defines an analytic conformal field theory based on the torus

$$\frac{\mathbf{R}^{24}}{\Lambda_N(g)},$$

in the way we described in chapter 2. It is a remarkable fact that by shifting or twisting each of these theories in a particular way we arrive at the theory defined by the Leech lattice.

Consider the shifted picture first. The appropriate shift vector which maps the theory defined on the Niemeier lattice $\mathcal{H}(\mathbf{R}^{24}/\Lambda_N)$ to the theory defined on the Leech lattice $\mathcal{H}(\mathbf{R}^{24}/\Lambda_L)$ is given by

$$\delta = \frac{1}{h}\sum_{a,i}\omega_i^a, \tag{9.10}$$

where h is the Coxeter number of the component subalgebras and ω_i^a are the fundamental weights as before. δ can also be written as

$$\delta = \frac{1}{2h} \sum_{\alpha\in(\Lambda_N)_2^+} \alpha.$$

An important result for what follows is the Freudenthal–de Vries 'Strange Formula' which states

$$\delta^2 = \frac{(h+1)}{12h} d, \tag{9.11}$$

where d is the rank of g (=24 in this case). The shift δ obviously has order h.

Let us define the following subsets of Λ_N:

$$\Lambda_N^{(i)} \equiv \{\alpha \in \Lambda_N \mid \alpha.\delta = i/h \bmod \mathbb{Z}\}.$$

To construct the shifted theory we must project onto the invariant sublattice $\Lambda_N^{(0)}$ and add in the even components of the shifted lattices $\Lambda_N + n\delta$, $1 \leq n \leq h-1$. Consider the length squared of the vector $\alpha + n\delta$:

$$(\alpha + n\delta)^2 = \alpha^2 + 2n\alpha.\delta + 2n^2\frac{(h+1)}{h},$$

where we used the 'Strange Formula'. So $\alpha + n\delta$ only has even length squared length if $\alpha.\delta = -n/h \bmod \mathbb{Z}$, *i.e.* $\alpha \in \Lambda_N^{(-n)}$. The new (enhanced) theory is therefore based on a lattice

$$\Lambda_N' = \bigcup_{n=0}^{h-1} \left(\Lambda_N^{(n)} - n\delta\right). \tag{9.12}$$

We now prove that Λ_N' is an even self-dual lattice itself.

<u>*Proof*</u> : Suppose $v = \alpha - n\delta$ and $w = \beta - m\delta$, where $\alpha \in \Lambda_N^{(n)}$ and $\beta \in \Lambda_N^{(m)}$, respectively. Consider the inner product

$$v.w = \alpha.\beta - n\frac{m}{h} - m\frac{n}{h} + 2nm\frac{(h+1)}{h} \bmod \mathbb{Z} \in \mathbb{Z}.$$

Therefore Λ_N' is an integral lattice. In addition

$$v^2 = \alpha^2 + 2n^2 - 2 \times \text{integer} \in 2\mathbb{Z},$$

so it is also an even lattice. Furthermore Λ_N' is self-dual because it contains the same density of points in $\mathbb{R}^{24}$ as Λ_N, *i.e.* the volumes of the unit cells are equal.

It is a remarkable result of Conway and Sloane [97] that for each of the 23 Niemeier lattices Λ'_N is a copy of the Leech lattice. The proof of this result involves showing that $(\Lambda'_N)_2 = \emptyset$.

The shift can also be realized as a twist of order h. The relevant twist σ_c is known as the Coxeter element [98]. In order to describe σ_c we introduce the notion of the Weyl reflection of a vector β in the root α:

$$r_\alpha(\beta) = \beta - 2\frac{\alpha.\beta}{\alpha^2}\alpha.$$

The Coxeter element is defined in terms of a product of Weyl reflections. The simple roots of g are first decomposed into two sets $(\alpha_1, ..., \alpha_n)$,$(\alpha'_1, ..., \alpha'_m)$ with $\alpha_i.\alpha_j = 0$ if $i \neq j$ and similarly for the second set. The Coxeter element is then given by

$$\sigma_c = \left(r_{\alpha_1} ....r_{\alpha_n}\right)\left(r_{\alpha'_1} ....r_{\alpha'_m}\right). \tag{9.13}$$

It can be shown that the ordering in (9.13) is not relevant; different ordering are conjugate.

Of course in the twisted picture the fact that the resulting theory is equivalent to one based on the Leech lattice is not manifest. Schematically we have the following isomorphism between theories:

$$\mathcal{H}\left(\frac{\mathbf{R}^{24}}{\Lambda_L}\right) \simeq \mathcal{H}\left(\frac{\mathbf{R}^{24}}{\Lambda_N \times \mathbf{Z}_h^T}\right) \simeq \mathcal{H}\left(\frac{\mathbf{R}^{24}}{\Lambda_N \times \mathbf{Z}_h^S}\right),$$

where $\mathbf{Z}_h^{T.S}$ is the cyclic group generated by the Coxeter twist/shift. Notice, in going from $\Lambda_N(g)$ to Λ_L the Kac–Moody algebra is broken from $g^{(1)}$ to $(u(1)^{24})^{(1)}$.

If we argued in analogy with the bosonic string then we might be temped to suppose that the 24 dimensional theories are in some senses the 'physical' or light–cone degrees of freedom of some 26 dimensional Lorentzian model. Experience with the bosonic string shows that there are many advantages in dealing with the 26 dimensional theory directly. For instance Lorentz invariance is manifest. The drawback is the Hilbert space contains negative norm ghost states. In the bosonic string these states do not couple singly to physical states and can therefore be projected out the theory via the Virasoro conditions:

$$L_n \mid \text{phys}\rangle = \begin{cases} 0 & n \geq 1 \\ \mid \text{phys}\rangle & n = 0. \end{cases}$$

A further advance has been the development of a BRST formalism using Fadeev–Popov ghosts to enlarge the Hilbert space. The physical subspace is identified as the equivalence class of states in the kernel of the BRST charge Q, where two states are equivalent if there difference lies in the image of Q. For consistency Q must be nilpotent, which requires the dimension of space–time to be 26.

Can the 24 dimensional torus models be imbedded in some Lorentzian model in 26 dimensions? We have already mentioned that all the 24 dimensional even self–dual Euclidean lattices are sublattices of the unique even self–dual lattice in $\mathbb{R}^{25,1}$. It is natural, therefore, to start with a theory based on the torus

$$\frac{\mathbb{R}^{25,1}}{II^{25,1}}.$$

The zero–modes of the dimension one operators of the resulting theory $\mathcal{H}_0$ will generate a rank 26 representation of a Lorentzian Lie algebra. This algebra is called the **monster Lie algebra** and is denoted L_∞, see refs.[42,99]. The lattice $II^{25,1}$ has an infinite number of roots, *i.e.* vectors of length squared 2, however, it is remarkable that the analogues of the simple roots (the so–called 'Leech' roots) are related to the Leech lattice. The set of Leech roots turns out to be

$$\{r \in II^{25,1} \mid r^2 = 2,\ r.\Delta = 1\}.$$

Here, Δ is the light–like vector which picks out the Leech lattice in $II^{25,1}$. The Leech roots are related to the Leech lattice via

$$r = \alpha + (1 - \alpha^2/2)\Delta + \overline{\Delta}, \quad \alpha \in \Lambda_L,$$

where $\overline{\Delta}^2 = 0$, $\overline{\Delta}.\Delta = 1$. The Dynkin diagram for L_∞ consequently has a spot for each vector in the Leech lattice.

L_∞ has 23 subalgebras corresponding to the affinizations of the 23 finite Lie algebras associated to the Niemeier lattices. These affinizations are not the same as those which have been considered in previous chapters. In these affinizations, which we denote g_k, the root lattice of g is enlarged by a light-like vector k. The two types of affinization are in fact isomorphic. The isomorphism $g_k \simeq \hat{g}$ is given by

$$\begin{aligned} V_0(\alpha + nk) &\longmapsto V_n(\alpha) \\ V_0(\mu.\alpha_{-1} \mid nk\rangle) &\longmapsto \mu.\alpha_n \\ V_0(k.\alpha_{-1} \mid nk\rangle) &\longmapsto \begin{cases} 1 & n = 0 \\ 0 & n \neq 0. \end{cases} \end{aligned} \tag{9.14}$$

In the above equation μ is any vector in the real span of the root space of g.

An important ingredient in showing that $g_k \subset L_\infty$, and for considering connections with the moonshine module, is the fact that the 2-cocycle ε_{26} defined on $II^{25,1}$ can be factorized in 24 different ways:

$$\varepsilon_{26} \longrightarrow \varepsilon_{24}\varepsilon_2,$$

corresponding to $II^{25,1} = \Lambda \oplus II^{1,1}$ where Λ is the Leech lattice or any one of the Niemeier lattices.

Consider this in more detail. The 2-cocycle ε_{26} defines the central extension of $II^{25,1}$ by $\{\pm 1\}$. When the symmetry factor is specified it is unique up to coboundaries [13], *i.e.* multiplication by

$$\frac{u(r)u(s)}{u(r+s)},$$

for some function $u : II^{25,1} \to \{\pm 1\}$. Here, we require

$$\frac{\varepsilon_{26}(r,s)}{\varepsilon_{26}(s,r)} = (-)^{r.s}.$$

Consider the decomposition $II^{25,1} = \Lambda \oplus II^{1,1}$. Since Λ and $II^{1,1}$ are integral lattices we can define two 2-cocycles ε_{24} and ε_2 with symmetry factors $(-)^{\alpha.\beta}$ and $(-)^{x.y}$, for $\alpha, \beta \in \Lambda$ and $x, y \in II^{1,1}$, respectively. The product

$$\varepsilon_{24}(\alpha,\beta)\varepsilon_2(x,y)$$

is clearly a 2-cocycle with a symmetry factor

$$(-)^{\alpha.\beta + x.y} = (-)^{(\alpha+x).(\beta+y)},$$

since $\alpha.y = \overline{\beta}.x = 0$. By uniqueness $\varepsilon_{24}\varepsilon_2$ must differ from ε_{26} by a coboundary:

$$\varepsilon_{26}(\alpha + x, \beta + y) = \frac{u(\alpha + x)u(\beta + y)}{u(\alpha + \beta + x + y)}\varepsilon_{24}(\alpha, \beta)\varepsilon_2(x, y). \tag{9.15}$$

We conclude that there are 24 ways of factorizing ε_{26} corresponding to the decomposition of $II^{25,1}$ into each of the even self–dual Euclidean lattices in 24 dimensions.

For each of the $II^{1,1}$ sublattices we can choose

$$\varepsilon_2(n_1 k + n_2\overline{k}, m_1 k + m_2\overline{k}) = (-)^{n_1 m_2},$$

so that $\varepsilon_2(nk, mk) = 1$. The algebra L_∞ is generated by vertex operators of the form

$$V_0(\alpha_{-1} \mid \lambda\rangle), \qquad V_0(r = \alpha + nk + m\overline{k}),$$

where $r^2 = 2$ and $\lambda^2 = 0$. The subalgebra g_k is spanned by the generators

$$V_0(\mu.\alpha_{-1} \mid nk\rangle), \qquad u(\alpha + nk)^{-1}V_0(\alpha + nk),$$

where $\alpha \in \Lambda_2$, $n \in \mathbb{Z}$ and μ is in the real span of Λ. The phase u ensures that the structure constants are simply $\varepsilon_{24}(\alpha, \beta)$ and do not involve k.

In a similar way, by considering the orbifold

$$\frac{\mathbb{R}^{25,1}}{\mathbb{Z}_2 \times \Lambda_L},$$

where the twist is generated by the reflection:

$$\alpha + n\Delta + m\overline{\Delta} \longmapsto -\alpha + n\Delta + m\overline{\Delta},$$

the affine algebra $B_\Delta \simeq \hat{B}$, associated with the moonshine module, can be imbedded in a larger cross–bracket algebra. However, this construction does not give any additional insights into the rôle of the monster group.

9.5 DISCUSSION.

The string for all its interesting features still remains remote from a description of 'low energy' four dimensional physics. However, it is only by investigating as many ways as possible to deal with the extra dimensions that a more complete understanding of the generation of four dimensional models will emerge. Orbifolds are only one such attempt to increase the class of theories available. Conformal field theory on orbifolds is a rewarding structure in its own right, providing many links with mathematics, as we have illuminated in this work. For example it is remarkable that the Moonshine Module and the Monster Group are directly related to string theory on a particular $\mathbb{Z}_2$–orbifold. In the theory of affine Lie algebras, we have seen that certain orbifold models lead to new representations of the algebras, involving mixing between the twisted sectors. It is not yet clear whether this algebra enhancement mechanism plays any rôle in string models, apart from connecting the two original heterotic models.

One of the main omissions from this work is the construction of the component vertex operators $V^{vu,u}(\Phi_v, z)$ for $u, v, vu \neq e$. Such operators intertwine two twisted sectors, and presumably will have a form similar to the three 'Reggeon' vertex [92,93], involving the three twisted Fock spaces $\mathcal{H}_u$, $\mathcal{H}_v$ and $\mathcal{H}_{vu}$. However, the 3–point functions for arbitrary twists follow from a factorization of the four twisted string interaction that we discussed in the last section. This factorization has been carried out in ref.[33].

Another omission is the extension of these ideas to the superstring, this has been dealt with to some extent elsewhere [2,33,34]. It is interesting that recent attempts to construct four dimensional models have relied on the bosonization of the internal fermionic degrees of freedom [10]. This construction revolves around a non–Euclidean lattice. A question immediately arises: can twisted versions of this 'covariant lattice' construction be considered?

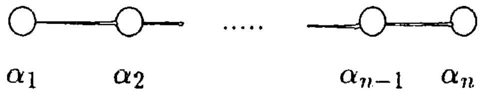
Another intriguing gap in this work is the failure to synthesize a twenty–six dimensional setting for the moonshine module. It may be that the search for this construction is itself a *moonshine*, however, it appears that many pieces of the jigsaw are there.

Even though the string is a long way off from being the theory of everything, it is encouraging to find that it is establishing bridges between many diverse areas of physics and mathematics. The concept of twisting the string is but one example of such a connection, and seems to have an interesting future ahead of it.

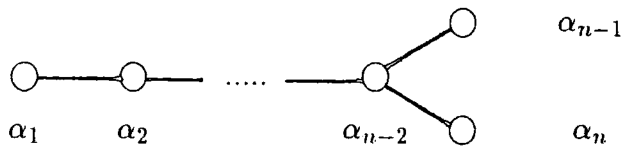
APPENDIX A

In this appendix we establish our notation for the roots of the simply-laced Lie algebras.

$\underline{A_n}$



$\underline{D_n}$



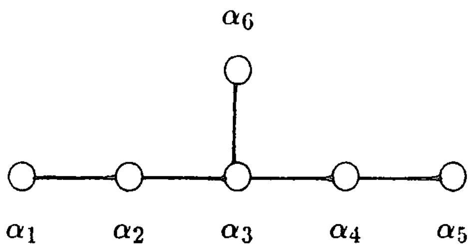
We will also have the occasion to use the orthogonal basis, which defines the roots in terms of the orthogonal set $\{e_i\}$, with $e_i.e_j = \delta_{ij}$:

$$\pm e_i \pm e_j \qquad 1 \leq i < j \leq n.$$

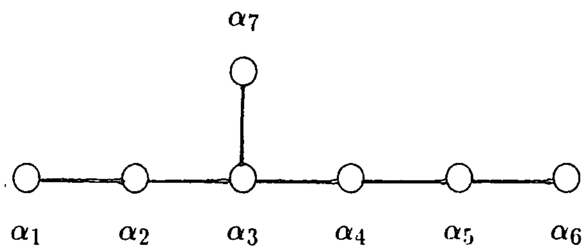
The simple roots can be chosen to be:

$$\alpha_i = e_i - e_{i+1} \quad 1 \leq i \leq n-1, \quad \alpha_n = e_{n-1} + e_n.$$

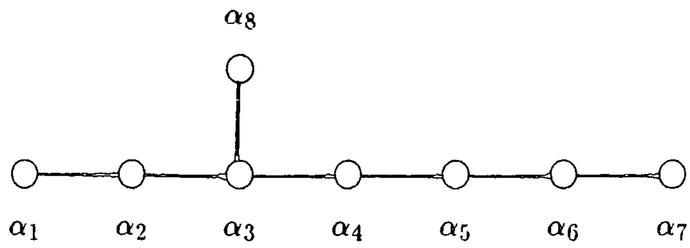
$\underline{E_6}$



$\underline{E_7}$



$\underline{E_8}$



APPENDIX B

In this appendix we prove that equation (3.55):

$$u_\alpha \hat{C}_u(u\alpha) = \hat{C}_u(\alpha) e^{\pi i (\alpha_1^u)^2 - 2\pi i \alpha_0^u \cdot p^u}, \tag{B1}$$

is consistent with the properties of the twisted operator cocycles.

From the projective representation (3.39) we have

$$\hat{C}_u(\alpha)\hat{C}_u((u-1)\alpha) = \varepsilon_u(\alpha,(u-1)\alpha)\hat{C}_u(u\alpha). \tag{B2}$$

Recall, from equation (3.50), that

$$\hat{C}_u((u-1)\alpha) = g_u((u-1)\alpha)e^{-2\pi i \alpha_0^u \cdot p^u}\mathbb{1}, \tag{B3}$$

where $g_u(\beta)$ is a one dimensional projective representation on A_u ($B_u \subset A_u$):

$$g_u(\beta)g_u(\gamma) = \varepsilon_u(\beta,\gamma)g_u(\beta+\gamma), \quad \beta,\gamma \in A_u. \tag{B4}$$

Substituting (B3) into (B2), we find:

$$u_\alpha \hat{C}_u(u\alpha) = \frac{u_\alpha g_u((u-1)\alpha)}{\varepsilon_u(\alpha,(u-1)\alpha)}\hat{C}_u(\alpha)e^{-2\pi i \alpha_0^u \cdot p^u}.$$

Comparing the above with equation (B1) shows that we need to prove that

$$g_u((u-1)\alpha) = \frac{\varepsilon_u(\alpha,(u-1)\alpha)}{u_\alpha}e^{i\pi(\alpha_1^u)^2}$$

is consistent with (B4). This is proved by making judicious use of the 2–cycle property of ε_u, and the equations relating the phases $\{u_\alpha\}$, (3.52).

In the following $\alpha' = (u-1)\alpha$ and $\beta' = (u-1)\beta$. Consider

$$\begin{aligned} g_u(\alpha')g_u(\beta') &= \frac{\varepsilon_u(\alpha,\alpha')\varepsilon_u(\beta,\beta')}{u_\alpha u_\beta} e^{i\pi(\alpha_1^u)^2 + i\pi(\beta_1^u)^2} \\ &= \frac{\varepsilon_u(u\alpha,u\beta)\varepsilon_u(\alpha,\alpha')\varepsilon_u(\beta,\beta')}{u_{\alpha+\beta}\varepsilon_u(\alpha,\beta)} e^{i\pi(\alpha_1^u)^2 + i\pi(\beta_1^u)^2} \\ &= \frac{\varepsilon_u(\alpha,\alpha')\varepsilon_u(u\alpha+\beta,\beta')\varepsilon_u(u\alpha,\beta)}{u_{\alpha+\beta}\varepsilon_u(\alpha,\beta)} e^{i\pi(\alpha_1^u)^2 + i\pi(\beta_1^u)^2}. \end{aligned}$$

We now use the fact

$$\varepsilon_u(u\alpha,\beta) = \Omega_u(u\alpha,\beta)\varepsilon_u(\beta,u\alpha),$$

and the 2–cocycle property again to get:

$$\Omega_u(u\alpha,\beta)\frac{\varepsilon_u(u\alpha+\beta,\beta')\varepsilon_u(\alpha+\beta,\alpha')\varepsilon_u(\beta,\alpha)}{u_{\alpha+\beta}\varepsilon_u(\alpha,\beta)} e^{i\pi(\alpha_1^u)^2 + i\pi(\beta_1^u)^2}.$$

Now we use the fact that

$$\frac{\varepsilon_u(\beta,\alpha)}{\varepsilon_u(\alpha,\beta)} = \Omega_u(\beta,\alpha) = \Omega_u(-\alpha,\beta),$$

and

$$\Omega_u(u\alpha,\beta)\Omega_u(-\alpha,\beta) = \Omega(\alpha',\beta) = e^{2\pi i\alpha_1^u.\beta_1^u}.$$

Finally, making use of the 2–cocycle property once more we obtain:

$$\begin{aligned} g_u(\alpha')g_u(\beta') &= \frac{\varepsilon_u(\alpha',\beta')\varepsilon_u(\alpha+\beta,\alpha'+\beta')}{u_{\alpha+\beta}} e^{\pi i(\alpha_1^u+\beta_1^u)^2} \\ &= \varepsilon_u(\alpha',\beta')g_u(\alpha'+\beta'), \end{aligned}$$

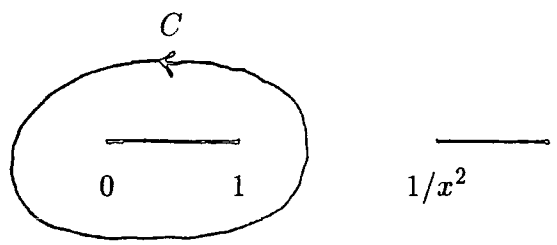
this completes the proof.

APPENDIX C

In this appendix we compute the quantities $\mathbf{v}^T\xi$ and $\mathbf{v}^T\mathbb{N}\xi$. Central to the calculations is the result of Schwarz and Wu [72]

$$\xi_n(x) \;=\; \frac{1}{4\sqrt{2}K(x)} \oint_C \frac{(tx)^n\,dt}{[t(1-t)(1-tx^2)]^{\frac{1}{2}}}, \tag{C1}$$

where $x = \sqrt{u}$ and the contour C encircles the cut from 0 to 1,



$\mathbf{v}$ is explicitly

$$v_n(x) \;=\; \frac{1}{\sqrt{2}} \binom{-\frac{1}{2}}{n} (-x)^n. \tag{C2}$$

Using these two definitions we can perform the infinite sums and so evaluate $\mathbf{v}^T\xi$ and $\mathbf{v}^T\mathbb{N}\xi$ explicity. Doing $\mathbf{v}^T\xi$ first,

$$\begin{aligned}
\mathbf{v}^T\xi \;&=\; \frac{1}{8K(x)} \sum_{n=1}^{\infty} \binom{-\frac{1}{2}}{n} \oint_C \frac{(-tx^2)^n\,dt}{[t(1-t)(1-tx^2)]^{\frac{1}{2}}} \\
&=\; \frac{1}{8K(x)} \left\{ \oint_C \frac{dt}{t^{\frac{1}{2}}(1-t)^{\frac{1}{2}}(1-tx^2)} - \oint_C \frac{dt}{[t(1-t)(1-tx^2)]^{\frac{1}{2}}} \right\}.
\end{aligned}$$

We recognise the last term in the braces as $4K(x)$, where $K(x)$ is the complete elliptic integral of the first kind [76]. To evaluate the first term we find it convenient to introduce the **hypergeometric function** [100]

$$_2F_1(a,b,c;x^2) \;=\; \frac{\Gamma(c)}{2\Gamma(b)\Gamma(c-b)} \oint_C dt\, t^{b-1}(1-t)^{c-b-1}(1-x^2t)^{-a}.$$

In terms of this we may write

$$\mathbf{v}^T\xi = \frac{1}{4K(x)}\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(1)}\ {}_2F_1\left(1,\frac{1}{2},1;x^2\right) - \frac{1}{2}.$$

Making use of the well known Euler identity for hypergeometric functions:

$${}_2F_1(a,b,c;x^2) = (1-x^2)^{c-a-b}\ {}_2F_1(c-a,c-b,c;x^2),$$

we deduce

$${}_2F_1\left(1,\frac{1}{2},1;x^2\right) = (1-x^2)^{-\frac{1}{2}}\ {}_2F_1\left(0,\frac{1}{2},1;x^2\right).$$

${}_2F_1(0,\frac{1}{2},1;x^2)$ is in fact independent of x, and is easily shown to be $\pi\Gamma(1)/\Gamma(\frac{1}{2})^2$. Therefore

$$\mathbf{v}^T\xi = \frac{1}{2}\left\{\frac{\pi}{2(1-x^2)^{\frac{1}{2}}K(x)} - 1\right\}. \tag{C3}$$

Essentially the same procedure is followed in the evaluation of $\mathbf{v}^T\mathbb{N}\xi$,

$$\begin{aligned}
\mathbf{v}^T\mathbb{N}\xi &= \frac{1}{8K(x)}\sum_{n=1}^{\infty} n\binom{-\frac{1}{2}}{n}\oint_C \frac{(-tx^2)^n\,dt}{[t(1-t)(1-x^2t)]^{\frac{1}{2}}} \\
&= \frac{x^2}{16K(x)}\oint_C \frac{dt\,t^{\frac{1}{2}}}{(1-t)^{\frac{1}{2}}(1-x^2t)^2} \\
&= \frac{x^2}{8K(x)}\frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})}{\Gamma(2)}\ {}_2F_1\left(2,\frac{3}{2},2;x^2\right).
\end{aligned}$$

Using the Euler relation again we can re–express the hypergeometric function to arrive at

$$\mathbf{v}^T\mathbb{N}\xi = \frac{x^2}{8K(x)}\frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})}{\Gamma(2)}(1-x^2)^{-\frac{3}{2}}\ {}_2F_1\left(0,\frac{1}{2},2;x^2\right).$$

${}_2F_1(0,\frac{1}{2},2;x^2)$ is independent of x, and is in fact equal to 1. Therefore

$$\mathbf{v}^T\mathbb{N}\xi = \frac{\pi x^2}{16(1-x^2)^{\frac{3}{2}}K(x)}. \tag{C4}$$

APPENDIX D

Here we prove the transformation properties of the partition functions encountered in chapter 6, equations (6.32) and (6.33).

Let us first introduce the auxiliary partition function:

$$\Gamma(q,\zeta,\tau) = \sum_{\alpha\in\Lambda}(-q)^{\frac{1}{2}\alpha^2+\zeta.\alpha}(-)^{\alpha.\tau}.$$

Recall the fundamental relation relating a lattice sum to its dual:

$$\frac{1}{|\Lambda|}\sum_{\omega\in\Lambda^*} e^{2\pi i\beta.\omega} = \sum_{\alpha\in\Lambda}\delta(\beta-\alpha). \tag{D1}$$

where $|\Lambda|$ is the volume of the unit cell of Λ. Multiplying each side by $e^{-2\pi i\beta.\beta'-z\beta^2}$ and integrating over all β yields:

$$\frac{1}{|\Lambda|}\left(\frac{\pi}{z}\right)^{d/2}\sum_{\omega\in\Lambda^*} e^{-\pi^2(\omega-\beta')^2/z} = \sum_{\alpha\in\Lambda} e^{-2\pi i\alpha.\beta'}e^{-\alpha^2 z}. \tag{D2}$$

Now, writing $z = -i\pi\sigma$, $\beta' = \frac{1}{2}\tau - \zeta\sigma$ and $q = e^{i\pi(2\sigma-1)}$, we recognise the right hand side of (D2) to be

$$\sum_{\alpha\in\Lambda}(-q)^{\frac{1}{2}\alpha^2+\zeta.\alpha}(-)^{\alpha.\tau} = \Gamma(q,\zeta,\tau). \tag{D3}$$

Next, define σ', q' via,

$$\sigma' = \frac{\sigma}{1-2\sigma}, \quad q' = e^{i\pi(2\sigma'+1)},$$

then

$$\ln q \ln q' = -\pi^2(2\sigma-1)(2\sigma'+1) = \pi^2,$$

which represents the transformation we are trying to achieve.

Consider

$$\begin{aligned}\Gamma(q,\zeta,\tau) &= \sum_{\alpha\in\Lambda}(-)^{\alpha.\tau}\exp\left\{i\pi(\alpha^2+2\zeta.\alpha)\frac{\sigma'}{1+2\sigma'}\right\}\\ &= \frac{1}{|\Lambda|}\left(i\frac{1+2\sigma'}{\sigma'}\right)^{d/2}\sum_{\omega\in\Lambda^*}\exp\left\{-i\pi\left(\omega-\frac{1}{2}\tau+\zeta\sigma\right)^2\frac{1+2\sigma'}{\sigma'}\right\},\end{aligned}$$

using (D2). Provided Λ^* is an 'integer' lattice, (in the sense that each vector in Λ^* has an integer squared length), we notice

$$e^{2i\pi(\omega-\frac{1}{2}\tau)^2} = e^{2i\pi(\omega^2-\omega.\tau+\frac{1}{4}\tau^2)} = (-)^{\frac{1}{2}\tau^2},$$

therefore,

$$\Gamma(q,\zeta,\tau) = \frac{1}{|\Lambda|}\left(i\frac{1+2\sigma'}{\sigma'}\right)^{d/2} e^{i\pi\zeta^2(\sigma'-\sigma)}(-)^{\frac{1}{2}\tau^2}\sum_{\omega\in\Lambda^*}\exp\left\{-\frac{i\pi}{\sigma'}(\omega-\frac{1}{2}\tau+\zeta\sigma')^2\right\},$$

and we can use equation (D2) again to get

$$\begin{aligned}\Gamma(q,\zeta,\tau) &= \left(i\frac{1+2\sigma'}{\sigma'}\right)^{d/2}(-)^{\frac{1}{2}\tau^2}(-i\sigma')^{d/2}e^{2i\pi\sigma\sigma'\zeta^2}\\ &\quad\times\sum_{\alpha\in\Lambda}(-)^{\alpha.\tau}\exp\left\{i\pi(\alpha^2+2\zeta.\alpha)\sigma'\right\}\\ &= (1+2\sigma')^{d/2}(-)^{\frac{1}{2}\tau^2}e^{2i\pi\sigma\sigma'\zeta^2}\Gamma(q',\zeta,\tau).\end{aligned}$$

Firstly, if we take $\zeta = 0$ then we arrive at

$$\Gamma_1(q) = \left(\frac{\ln q'}{i\pi}\right)^{d/2}(-)^{\frac{1}{2}\tau^2}\Gamma_1(q'). \qquad \text{(D4)}$$

This is precisely equivalent to (6.32). Secondly, if we act on (D4) with

$$\left(e.\frac{\partial}{\partial\zeta}\right)\left(f.\frac{\partial}{\partial\zeta}\right),$$

and evaluate the resulting expression at $\zeta = 0$ we get

$$\begin{aligned}-4\pi^2\sigma^2\Gamma_2(q) &= (1+2\sigma')^{d/2}(-)^{\frac{1}{2}\tau^2}\\ &\quad\times\left\{4\pi i\sigma\sigma'(e.f)\Gamma_1(q') - 4\pi^2(\sigma')^2\Gamma_2(q')\right\}.\end{aligned}$$

Using the fact that

$$\sigma'/\sigma = 1+2\sigma' = \frac{1}{i\pi}\ln q',$$

we find

$$\Gamma_2(q) = (-)^{\frac{1}{2}\tau^2}\left\{\frac{e.f}{i\pi}\left(\frac{\ln q'}{i\pi}\right)^{d/2+1}\Gamma_1(q') + \left(\frac{\ln q'}{i\pi}\right)^{d/2+2}\Gamma_2(q')\right\}. \qquad \text{(D5)}$$

This is precisely equation (6.33).

It is interesting that the arguments seem to be perfectly good even if Λ is not self-dual. Both the Fierz rearrangement and the modular transformation require two stages, the dual lattice only occurring at an intermediate step.

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