THE DENSITY FORM OF EQUILIBRIUM PRICES IN CONTINUOUS TIME AND BOITEUX'S SOLUTION TO THE SHIFTING-PEAK PROBLEM

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Abstract

Bewley's condition on production sets, imposed to ensure the existence of an equilibrium price density when L^{∞} is the commodity space, is weakened to allow applications to continuous-time problems, and especially to peak-load pricing when the users' utility and production function are Mackey continuous. A general form of the production sets with the required property is identified, and examples are given of technologies which meet the weakened but not the original condition: these include industrial use and storage of cyclically priced goods. General equilibrium results are supplemented by those for prices supporting individual consumer or producer optima. Also, to make clear the restriction implicit in Mackey continuity, we interpret it as interruptibility of demand; and we point out that, without this assumption, the equilibrium can feature pointed peaks with singular, instantaneous capacity charges.

Keywords: price density; continuous-time peak-loading pricing.

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1. Introduction

Cyclical problems which involve capacity costs or constraints, such as peak-load pricing, must be set up in a commodity space that consists entirely of bounded functions of time and any other relevant commodity characteristics. An obvious choice is the space of continuous functions $\mathcal{C}[0,1]$; and an immediate modelling benefit is that its norm-dual, the space of all Borel measures $\mathcal{M}[0,1]$ is available as the price space. This can accommodate the instantaneous capacity charges which arise in the case of firm point peaks. However, as is well known, $\mathcal{C}[0,1]$ is not a dual Banach space, and equilibrium analysis with $\mathcal{C}[0,1]$ as the commodity space is hampered by the consequent lack of a vector topology that would make the unit ball compact. Bewley [3] gets round this mathematical difficulty by using the larger commodity space $L^{\infty}[0,1]$ or in general $L^{\infty}(S,\sigma)$, the space of all σ-essentially bounded real-valued functions on a set S of commodity characteristics which carries a measure σ. Unlike $\mathcal{C}[0,1]$, $L^{\infty}[0,1]$ does have a Banach predual, which is $L^{1}(S,\sigma)$, the space of all σ-integrable functions on S. Bewley [3] uses this first to give an equilibrium existence result with a price system $p^{\star}$ in the norm-dual $L^{\infty *}$, and then to deduce the existence of an equilibrium price system in the subspace L^{1} under additional assumptions. This is done by showing that any singular part of $p^{\star}$ can be deleted without disturbing the equilibrium; hence the remaining density part, which belongs to L^{1}, is itself an equilibrium price.

For the L^{∞}-model the price density result is an integral part of the analysis: the singularities in $L^{\infty *}$ are mathematically intractable and therefore unsuitable for describing prices. This is a basic limitation of the L^{∞}-model because the L^{1}-price functions obviously cannot represent the instantaneous charges mentioned above; and these can arise in equilibrium if preferences are norm-continuous. For example, in the peak-load pricing problem the demand trajectory can have a firm, pointed peak; and in such a case the peak capacity charge is levied wholly at the peak instant. It is then a charge for the *rate* of consumption at that instant, and *not* a charge per unit of the good. In the context of electricity pricing this is a capacity charge in \$ per kW demanded at the peak instant; and it is additional to the marginal fuel charge, which is a price density, i.e., a price rate in \$/kWh. (In other words, there is a charge per unit of *power* taken at peak, as well a charge per unit of *energy* at any time.) Such a price system can be represented by the sum of a point measure and a measure with a density (with respect to the Lebesgue measure), but this requires restricting the commodity space to the space of continuous functions $\mathcal{C}[0,1]$ and pairing it with the price space of measures $\mathcal{M}[0,1]$, as we do in [11].

Bewley's model can, however, be adapted to peak-load pricing with Mackey continuous preferences; and in this case the price density can be seen as Boiteux's solution to the "shifting-peak problem". The type of equilibrium which the price space $L^{1}[0,1]$ *can* accommodate is one with a peak plateau in the output trajectory: the capacity charge is spread over the peak's duration so that the price is a density $p \in L^{1}[0,1]$.

This type of equilibrium arises if demand for the good in question is harmlessly interruptible, i.e., if a brief interruption of a consumption or input flow $x \in L^{\infty}[0,1]$ causes only a small loss of utility or output. In such a case the customer would switch off briefly rather than pay any concentrated or instantaneous charges; so, being ineffective, such charges cannot be part of an equilibrium tariff. In the context of peak-load pricing, a concentration of the capacity charge during a short-lived peak would cause the peak to shift; but a flattened, spread-out peak with the corresponding spreading of the capacity charge can remove the shifting-peak problem: see (9.1)–(9.3) and (9.7) in Section 9 or [8] for details.

To adapt Bewley's framework to continuous-time applications, one must relax his so-called "Exclusion Assumption" on the production sets [3, p. 524]: in the case of input demand, this is unacceptably stronger than the interruptibility requirement. Interruptibility means that, if F is the utility or production function on $L^{\infty}[0,1]$ and E is a subset of $[0,1]$, then $F\left(x 1_{[0,1]\setminus E}\right) \nearrow F(x)$ as $\operatorname{meas} E \searrow 0$. This condition on F follows from, and is actually equivalent to, the Mackey continuity of F. For households, Mackey continuity of preferences is exactly what Bewley assumes. But an industrial user of a continuous-time flow (with a production function F) must fail Bewley's assumption because it would require, in this case, that $F\left(x 1_{[0,1]\setminus E}\right)$ be exactly equal to $F(x)$, instead of only converging to it as $\operatorname{meas} E \searrow 0$. In general, Bewley postulates, roughly speaking, that the singularly priced commodities can be deleted from an input-output bundle y without rendering it infeasible (i.e., without moving it out of the production set Y). This would obviously follow from free disposal if the commodities in question were outputs, as in the case of a producer supplying the flow (or an otherwise differentiated output good); but it must fail when the commodities being deleted are inputs. When a "small" set of inputs $E \subset S$ is deleted, the rest of an efficient production bundle, $y 1_{S\setminus E}$, becomes infeasible; but all that actually *has* to be assumed is that it can be modified "slightly" so as to make it feasible again, e.g., by lowering the output. For an industrial user of a flow $x \in L^{\infty}[0,1]$, it therefore suffices that his production function F be Mackey continuous; and this condition is exactly in line with the continuity assumption on consumer preferences. Although a restriction on input demand is inevitable if the price is to take the form of a density function, it need not be any stricter than that on consumer demand.

To remedy this shortcoming of Bewley's assumption we formulate a weaker Exclusion Condition which serves the same purpose—viz., elimination of price singularities—but *is* met by users with Mackey continuous production functions (Example 5.2) and also, less obviously, by some producers who are neither pure users nor pure suppliers. An example in peak-load pricing is a pumped-storage plant (Example 5.4). The new condition has a wider applicability than might perhaps be expected at first. It also has some useful permanence properties. One is that the Exclusion Condition continues to hold after aggregation of commodities, e.g., the aggregation of an input bundle into a (scalar) input cost (Remark 9.3). Another is that the Exclusion Condition on a long-run production cone (i.e., a technology with constant returns to scale) implies the same for the short run (i.e., for a plant with fixed capacities): see Lemma 9.4.

Like continuity of preferences, our Exclusion Condition is a technical assumption without any single interpretation. Even within the context of continuous-time pricing, the economic meaning of the Exclusion Condition depends on the type of producer. As we have pointed out, in the case of a pure supplier of a flow $x \in L^{\infty}_{+}[0,1]$ it comes down to no more than free disposal (Example 5.1), whilst in the case of a pure user of such a flow it means interruptibility of his production processes. For other technologies meeting the Condition, its verification is more complex and rests on a combination of monotonicity and continuity arguments. For example, the pumped-storage technology can be described, as in (5.5), by the balance constraint $\int_0^1 x(t)\,\mathrm{d}t = 0$ and by two capacity requirements $\check{k}(x)$, one of which is Mackey continuous in x, whilst the other is monotone in $|x|$. A direct check based on these properties is made in [12]; here we use the ideas to identify a general class of production sets that meet the Exclusion Condition for the same mathematical reasons (Proposition 5.3). The result captures all of the afore-mentioned cases; and its application to storage is spelt out in Example 5.4.

Our restriction on the production sets is, then, significantly weaker than Bewley's but sufficient for the removal of singularities from equilibrium prices (Theorem 6.2). A similar result is given by Back [1, Theorem 1], but with no example of a production set that meets his condition ("Property E") but not Bewley's. What Back focuses on is the structure of consumption sets—which is of interest because removal of singularities requires a formally identical Exclusion Condition on the consumer trade sets. This is not explicit in Bewley's analysis because his consumption sets are orthants containing the initial endowments, for which the condition is obviously met (Example 4.1). Further examples of consumption sets with this property are given in [2, Section 4].

It is also useful to identify those instances in which a price singularity can be removed from prices supporting an *individual* optimum, instead of the general equilibrium; and in Section 7 this is shown to hold for both consumer and input demands (derived from Mackey continuous utility and production functions).

In some cases of central interest there is actually no price singularity to remove, i.e., the equilibrium price system is a pure density function to begin with. Bewley's result of this type [3] is for pure exchange; and although it extends to production economies, it relies on positivity of the initial endowment. It is therefore inapplicable when, as is typical, the produced differentiated good is absent from the endowments. What we give is a result that does apply to the purely produced goods (Proposition 8.1).

In Section 9 this last result is applied to peak-load pricing with or without storage (Theorems 9.1 and 9.2). For this problem the L^{∞}-model provides a technical setting in which Boiteux's peak-plateau conjecture can be examined, and the implicit economic assumption underlying this form of equilibrium is identified (as the interruptibility of consumption).

2. The commodity and price spaces

The set of commodity characteristics, S, is assumed to carry a sigma-finite, non-negative measure σ on a sigma-algebra $\mathcal{S}$ of subsets of S; and the commodity space is $L^{\infty}(S, \mathcal{S}, \sigma)$, the space of all the equivalence classes of essentially bounded functions on S with values in the real line $\mathbb{R}$. It is normed by the supremum norm; and its norm-dual, denoted by $L^{\infty *}$, serves as the price space. This contains $L^{1}(\sigma)$, the space of all σ-integrable functions. A linear functional $p \in L^{\infty *}(\sigma)$ defines a bounded, finitely additive set function $\imath(p)(E) := \langle p, 1_E \rangle$ which vanishes on every σ-null set $E \in \mathcal{S}$ (where 1_E is the 0-1 indicator of E). All such set functions can be obtained in this way, and furthermore p can be identified with $\imath(p)$. This is because the integral of any $x \in L^{\infty}$ with respect to (w.r.t.) such a set function defines a bounded linear functional on L^{∞}: see, e.g., [7, III.1–III.2 and IV.8.16] or [17, 2.3]. However, the integral lacks some basic properties unless p is *countably* additive, i.e., unless p is a measure; and the only measures in $L^{\infty *}$ are those having densities (i.e., those in L^{1}). Since we reserve the symbol $\int$ for integration w.r.t. measures (which are countably additive by definition), the value of a commodity bundle $x \in L^{\infty}$ at a general price system $p \in L^{\infty *}$ is denoted by $\langle p, x \rangle$.

Like any additive set function, a $p \in L^{\infty *}$ has the Hewitt-Yosida decomposition into $p_{\mathrm{CA}} + p_{\mathrm{FA}}$, the sum of its countably additive and purely finitely additive parts (c.a./p.f.a. parts): see, e.g., [3, Appendix I: (26)–(27)] or [7, III.7.8] or [17, 1.23 and 1.24]. The c.a. part of p is identified with its density w.r.t. σ, which exists by the Radon-Nikodym Theorem; so $p_{\mathrm{CA}} \in L^{1}(S, \sigma)$. A p.f.a. set function is one that is lattice-disjoint from every c.a. one. Since p_{FA} vanishes on σ-null sets (and is p.f.a.), it can be characterised as a singular element of $L^{\infty *}$, i.e., as one that is concentrated on a set of commodities with an arbitrarily small σ-measure (if σ is finite). Formally, a $p \in L^{\infty *}$ is *concentrated on*, or *supported by*, a measurable set $E \in \mathcal{S}$ if $\langle p, x \rangle = \langle p, x 1_E \rangle$ for every $x \in L^{\infty}$. A sequence of sets $E^n \in \mathcal{S}$ is *evanescent* if $E^{n+1} \subseteq E^n$ for every n and $\sigma\left(\bigcap_{n=1}^{\infty} E^n\right) = 0$; and p is called *singular* if there exists an evanescent (E^n) such that p is concentrated on E^n for each $n = 1, 2, \ldots$. A $p \in L^{\infty *}$ is p.f.a. if and only if it is singular: see [17, 3.1].

If $\sigma(S) < \infty$, then p is singular if and only if there exists a sequence (E^n) of sets supporting p with $\sigma(E^n) \to 0$ as $n \to \infty$. This gives p_{FA} the interpretation of an extremely concentrated charge (when σ is finite).

Comments:

1. Bewley [3, p. 516] asserts that the singular functionals "have no economic interpretation", and this may, by and large, be so in the two areas of application he outlines—viz., uncertainty (with a probability measure σ on a set of events) and discrete-time, infinite-horizon intertemporal problems (in which σ is the counting measure on the set of natural numbers). But, as we have indicated in the Introduction, any blanket argument against the presence of concentrated charges must be mistaken because these have an essential role in continuous-time

problems (in which σ is the Lebesgue measure on an interval of $\mathbb{R}$). Although in the L^∞-model—with the *whole* of L^∞ as the commodity space—there is no alternative but to impose restrictions which guarantee equilibria supported by pure density prices, this is because of the mathematical intractability of price singularities within the L^∞-framework, and *not* for any inherent lack of economic interpretation for singular prices.

2. With capacity constraints, $L^\infty(\sigma)$ is the *largest* possible commodity space whenever σ-equivalent functions (i.e., functions equal to each other σ-almost everywhere) must be regarded as representing one and the same commodity bundle. This is so with probabilistic uncertainty and, less obviously, with continuous-time problems as well: all that matters in a flow x of a good is its total amount, $\int_{t'}^{t''} x(t)\,\mathrm{d}t$, produced or consumed between *any* two instants t' and t''; and the integrals of equivalent functions are equal.
3. When the equilibrium allocation lies in a subspace of L^∞, a manageable mathematical representation of a price singularity may be achievable by restricting the price functional to this smaller commodity space. For example, if S is a compact topological space with a Borel measure σ, then the restriction of any $p \in L^{\infty*}(S,\sigma)$ to $\mathcal{C}(S)$ is a (countably additive) measure, by Riesz's Representation Theorem. If p is a singular element of $L^{\infty*}$ that is supported by a σ-evanescent sequence of *closed* sets $(E^n)_{n=1}^{\infty}$, then its restriction to $\mathcal{C}$ is a σ-singular measure, i.e., a measure concentrated on a σ-null set ($\bigcap_n E^n$ in this case).

3. The Exclusion Condition

For the existence of singularity-free equilibrium prices, the production sets and the consumers' trade sets are assumed to meet the condition formulated next. In this, $\mathrm{m}(L^\infty, L^1)$ denotes the Mackey topology on L^∞ for the duality with L^1; this is the strongest of those locally convex topologies on L^∞ which yield L^1 as the continuous dual.[1] The weakest of such topologies is the weak* topology, denoted by $\mathrm{w}(L^\infty, L^1)$. On every bounded subset of $L^\infty(S,\sigma)$, the topology $\mathrm{m}(L^\infty, L^1)$ is equivalent to the topology of convergence in the measure σ (on each subset of S with a finite measure). It follows that if (E^n) is an evanescent sequence, then $1_{E^n} \to 0$ in $\mathrm{m}(L^\infty, L^1)$ as $n \to \infty$.

Definition 3.1. *A set $Z \subseteq L^\infty$ meets the Exclusion Condition if for every $z \in Z$ and $p \in L^{\infty*}$ there exists a sequence $(z^n)_{n=1}^{\infty}$ in Z with $z^n \to z$ in $\mathrm{m}(L^\infty, L^1)$ and $\langle p_{\mathrm{FA}}, z^n \rangle \to 0$ as $n \to \infty$.*

This condition obviously implies that there is a sequence $z^n \in Z$ with

$$(3.1) \qquad \langle p, z^n \rangle = \langle p_{\mathrm{CA}}, z^n \rangle + \langle p_{\mathrm{FA}}, z^n \rangle \to \langle p_{\mathrm{CA}}, z \rangle,$$

[1]The other Mackey topology, $\mathrm{m}(L^\infty, L^{\infty*})$, is identical to the norm topology of L^∞.

and this is what is actually required of each production set.[2] For comparison, Bewley's Exclusion Assumption [3, p. 524] is that for every $z \in Z$ and $p \in L^{\infty *}$ there exists a sequence of sets E^n supporting p_{FA} with $z 1_{S \setminus E^n} \in Z$ for each n and with $p_{\mathrm{CA}}(E^n) \to 0$ as $n \to \infty$. If this holds, then $\langle p_{\mathrm{CA}}, z 1_{S \setminus E^n} \rangle \to \langle p_{\mathrm{CA}}, z \rangle$ and $\langle p_{\mathrm{FA}}, z 1_{S \setminus E^n} \rangle = 0$ for each n, which of course implies the required property (3.1) for $z^n := z 1_{S \setminus E^n}$. But in some cases of interest *this* particular sequence does *not* lie in Z even though there *is* a sequence in Z that satisfies (3.1); and this is why Bewley's assumption is significantly more restrictive than (3.1). This is so with the production sets of Examples 5.2 and 5.4, which fail Bewley's assumption but meet our Exclusion Condition (and hence (3.1)).

4. Consumption sets with the exclusion property

An orthant containing the origin is the basic example of a set that meets the Exclusion Condition and can be interpreted as a translated consumption set.

Example 4.1. *For any $d \in L^{\infty}_{+}(S)$, the set $-d + L^{\infty}_{+}$ meets the Exclusion Condition.*

Proof. For any $p \in L^{\infty *}$, take an evanescent sequence of sets $(E^n)_{n=1}^{\infty}$ supporting p_{FA}; and for any $z \geq -d$ define $z^n := z 1_{S \setminus E^n}$. This sequence has all the required properties: $z^n \geq -d$, $z^n \to z$ in $\mathrm{m}(L^{\infty}, L^1)$ as $n \to \infty$, and $\langle p_{\mathrm{FA}}, z^n \rangle = 0$ for each n.[3] ∎

Applied to a household with an initial endowment x^{En} and an orthant $\underline{x} + L^{\infty}_{+}$ as its consumption set X, this example means that if $x^{\mathrm{En}} \geq \underline{x}$ (i.e., $x^{\mathrm{En}} \in X$), then the consumer's set of feasible trades, $X - x^{\mathrm{En}} = -\left(x^{\mathrm{En}} - \underline{x}\right) + L^{\infty}_{+}$, meets the Exclusion Condition. This is assumed of each consumer h for the removal of singularities from equilibrium prices (Theorem 6.2). This particular assumption can sometimes be weakened by replacing the total endowment's *actual* distribution $\left(x_h^{\mathrm{En}}\right)$ with some hypothetical redistribution $\left(x_h^{\mathrm{Rd}}\right)$—when the other assumptions are strengthened so that singularities are absent from the *original* equilibrium prices (Theorem 8.1). This means that, when subsistence is defined by least requirements $\underline{x}_h$ (i.e., when $X_h = \underline{x}_h + L^{\infty}_{+}$), our exclusion assumptions about consumers are satisfied if survival is feasible without production or exchange (for Theorem 6.2), or with exchange but without production (for Theorem 8.1).

Example 4.1 can be extended by replacing the orthant with a Cartesian product, over a measure space T, of finite-dimensional sets. When a $t \in T$ is a time instant or a state of nature, this captures the case of trade-offs, between a finite number N of goods, in subsistence requirements at any time or in any state. In this context $S = T \times \{1, 2, \ldots, N\}$; for details see [2, p. 97].

[2] The Mackey convergence of z^n (to z) is needed in full only when Z is a consumption set.

[3] In all of our examples of sets meeting the Exclusion Condition, $\langle p_{\mathrm{FA}}, z^n \rangle = 0$ for each n.

Example 4.2. *When σ is the product of a measure τ on T and the counting measure on $\{1, 2, \ldots, N\}$, and $X(\cdot) : T \twoheadrightarrow \mathbb{R}^N$ is a measurable correspondence, define*

$$X := \{x : x(t) \in X(t) \text{ for } \tau\text{-almost every } t \in T\}.$$

This is a subset of $L^\infty\left(T, \mathbb{R}^N\right)$, which is identified with $L^\infty(T \times \{1, \ldots, N\})$. So $X \subseteq L^\infty(S)$; and $X - x$ meets the Exclusion Condition for every $x \in X$.[4]

The above examples are useful in verifying the Exclusion Condition on the consumers, which they clarify as a combination of survival assumptions and structural assumptions about the consumption sets. But the examples do not apply to production sets (except for the case of pure free disposal).

5. Production sets with the exclusion property

Production sets meeting the Exclusion Condition include two producer types in cyclical continuous-time pricing problems—viz., the pure suppliers and users of the flow in question (Examples 5.1 and 5.2). There is also the "mixed" type, i.e., a producer that uses the flow in some parts of the cycle to supply it in others. All of these types can be captured by a general form of a production set with the exclusion property (Proposition 5.3). In each case the production set, Y, is a subset of $L^\infty(T, \tau) \times \mathbb{R}^G$, where $1, 2, \ldots, G$ represent homogeneous goods, whilst τ is a sigma-finite measure on a space T of commodity characteristics that represents a differentiated good: e.g., in deterministic, continuous-time cyclical problems T is $[0, 1]$ with the Lebesgue measure, meas. In such a context S is the disjoint union of T and $\{1, 2, \ldots, G\}$, and σ is the direct sum of τ and the counting measure, so that $L^\infty(S, \sigma) = L^\infty(T, \tau) \times \mathbb{R}^G$. In this section and its application in Section 9, p means a price system for the differentiated good *only*, i.e., $p \in L^{\infty *}(T)$. A complete price system, corresponding to the $p \in L^{\infty *}(S)$ of the other sections, is denoted by $(p, r) \in L^{\infty *}(T) \times \mathbb{R}_G$. It is convenient to arrange prices into a row vector $r \in \mathbb{R}_G$ (whilst a $q \in \mathbb{R}^G$ is a column vector of quantities).

In the case of a pure supplier of a flow, who uses a finite number of homogeneous inputs, the Exclusion Condition follows from free disposal alone.

Example 5.1. *The production set of a supplier of the differentiated good is a $Y \subset L^\infty_+(T) \times \mathbb{R}^G_-$. It meets the Exclusion Condition if it includes free disposal of any produced output; i.e., if the conditions $(y, -a) \in Y$ and $y \geq y' \geq 0$ imply that $(y', -a) \in Y$.*

Proof. Given a $(y, -a) \in Y$ and a $(p, r) \in L^{\infty *}(T) \times \mathbb{R}_G$, take any evanescent sequence of measurable sets $(E^n)_{n=1}^\infty$ supporting p_{FA}, and define $y^n = y 1_{T \setminus E^n}$. Then $(y^n, -a) \in Y$, and the sequence has the required properties: $y^n \to y$ in $\mathrm{m}(L^\infty, L^1)$ as $n \to \infty$, and $\langle (p, r)_{\mathrm{FA}}, (y^n, -a) \rangle = \langle p_{\mathrm{FA}}, y^n \rangle = 0$ for each n. ∎

[4]Example 4.1 is a special case of Example 4.2, with $N = 1$ and $X(t) = \underline{x}(t) + \mathbb{R}_+$ for τ-almost every t.

Comment: Example 5.1 and its proof obviously extend to the case of *unlimited* free disposal of the output good, i.e., to the set $Y - L_+^\infty \times \{0\}$ in place of Y: if $(y, -a) \in Y - L_+^\infty \times \{0\}$, then $(y'', -a) \in Y$ for some nonnegative $y'' \geq y$; so $(y^+, -a) \in Y$ (since $y'' \geq y^+ \geq 0$). Since $y^+ \geq y^+ 1_{T \setminus E^n} \geq 0$, it also follows that $\left(y^+ 1_{T\setminus E^n}, -a\right) \in Y$, and hence $\left(y 1_{T\setminus E^n}, -a\right) \in Y - L_+^\infty \times \{0\}$. (A similar argument applies to the set $Y - L_+^\infty \times \mathbb{R}_+^G$.)

Example 5.1 is formally identical to Bewley's example of production under uncertainty [3, p. 527], with an uncertain output from a deterministic input: a supplier's production set Y satisfies even Bewley's Exclusion Assumption.[5] By contrast, Bewley's assumption cannot but fail in the case of a user (of the differentiated good), since it would mean that a "small" subset E^n of the *input* commodities could always be deleted from the production plan with no loss of output at all. This must be relaxed to the assumption that the resulting loss is small—as is the case if the user's production function, or correspondence, is Mackey continuous.[6] Such a user does meet *our* Exclusion Condition.

Example 5.2. *A production correspondence* $P\colon L_+^\infty(T) \twoheadrightarrow \mathbb{R}^G$ *defines the production set of an industrial user of the differentiated good as*

$$Y := \left\{(-z, b) \in L_-^\infty \times \mathbb{R}^G : b \in P(z)\right\}$$

i.e., any $b \in P(z)$ *is an input-output bundle (of the* G *homogeneous goods) that is feasible when the differentiated input is fixed at* z. *If* P *is (sequentially)* $\mathrm{m}(L^\infty, L^1)$*-lower hemicontinuous (l.h.c.) on* $L_+^\infty(T)$, *then* Y *meets the Exclusion Condition.*

Proof. Given a $(-z, b) \in Y$ and a $(p, r) \in L^{\infty *}(T) \times \mathbb{R}_G$, take any evanescent sequence of measurable sets $(E^n)_{n=1}^\infty$ supporting p_{FA}, and define $z^n = z 1_{T\setminus E^n}$; then $z^n \to z$ in $\mathrm{m}(L^\infty, L^1)$ as $n \to \infty$. Since P is l.h.c. and $b \in P(z)$, there exists a sequence $(b^n)_{n=1}^\infty$ with $b^n \in P(z^n)$ for each n and $b^n \to b$ as $n \to \infty$. The sequence $(-z^n, b^n)_{n=1}^\infty$ has all the required properties: $(-z^n, b^n) \in Y$, $(-z^n, b^n) \to (-z, b)$ in $\mathrm{m}(L^\infty, L^1)$ as $n \to \infty$, and $\langle (p, r)_{\mathrm{FA}}, (-z^n, b^n)\rangle = -\langle p_{\mathrm{FA}}, z^n\rangle = 0$ for each n. ∎

Comment: In the special case of a single-output producer with a production function $F\colon L_+^\infty(T) \times \mathbb{R}_+^{G-1} \to \mathbb{R}$, the production correspondence is

$$(5.1) \qquad P(z) := \left\{(b_1, \dots, b_{G-1}; b_G) \in \mathbb{R}_-^{G-1} \times \mathbb{R} : b_G \leq F(z; -b_1, \dots, -b_{G-1})\right\},$$

which is l.h.c. in z if $F(z; -b_1, \dots)$ is lower semicontinuous (l.s.c.) in $z \in L_+^\infty(T)$, given any $(b_1, \dots, b_{G-1}) \in \mathbb{R}_-^{G-1}$. This is because, for each $(b_1, \dots, b_{G-1})$, the l.s.

[5] Bewley's Exclusion Assumption holds also in his discrete-time, infinite-horizon intertemporal example [3, p. 527]: although the dated commodities cannot *a priori* be classed as net inputs or outputs, the sequential structure of production makes possible an argument similar to the Proof of Example 5.1, with $E^n = \{n, n+1, \dots\}$.

[6] A production correspondence, instead of a function, is useful in describing a technology with multiple output goods.

continuity of $F(\cdot\,; -b_1, \dots)$ is equivalent to the l.h. continuity of the correspondence $z \mapsto (-\infty, F(z; -b_1, \dots)]$—see, e.g., [15, 9.1.4 (i)]—and this implies the l.h.c. of $z \mapsto P(z)$.

For a unified formulation of the pure cases as well as the mixed one, it is useful to start by recognising that a supplier's production set $Y \subset L^{\infty}_{+}(T) \times \mathbb{R}^G_{-}$ can be represented as the graph of a correspondence D from $L^{\infty}_{+}(T)$, or a subset thereof, into $\mathbb{R}^G_{-}$. That is, $(y, -a) \in Y$ if and only if a is in $-D(y)$, the input requirement set for an output $y \geq 0$. The free-disposal assumption (of Example 5.1) translates into the condition that D is a decreasing map when its images are ordered by inclusion, i.e., that $y' \leq y''$ implies $D(y'') \subseteq D(y')$.

Such a monotone correspondence, D'', is used in (5.2) below to capture any joint constraints on the output part, x^+, of the flow and on the quantities, $q_{J''} = (q_g)_{g \in J''}$, of a subset J'' of the G homogeneous goods. Another monotone correspondence, D', similarly captures any constraints on the input flow x^- and on another subset, J', of the G goods. In particular, J' can contain any input that is a perfect complement to the input flow, and whose use therefore always increases with x^-, as in Example 5.4. (Recall that $x^- := \max\{-x, 0\}$ is the nonpositive part of x, whilst $x^+ := \max\{x, 0\}$ is the nonnegative part.)

The subsets J' and J'' may overlap, but they must be disjoint from a third subset I, which consists of those of the G goods whose quantities are constrained jointly with x by means of a Mackey lower hemicontinuous correspondence H. Any other constraints, on the flow x alone, are captured by a set $A \subseteq L^{\infty}(T)$.[7] Since $\mathbb{R}^G$ is used as an abbreviation for $\mathbb{R}^{\{1,2,\dots,G\}}$, a point in this space is a finite sequence q: $\{1, 2, \dots, G\} \to \mathbb{R}$; and q_I means the restriction of q to $I \subseteq \{1, 2, \dots, G\}$.

Proposition 5.3. *A production set of the form*

$$(5.2)\quad Y = \left\{(x, q) \in L^{\infty}(T) \times \mathbb{R}^G : x \in A, q_I \in H(x), q_{J'} \in D'(x^-), q_{J''} \in D''(x^+)\right\}$$

meets the Exclusion Condition if:

1. *For every $x \in A$ and every evanescent sequence of measurable sets $E^n \subset T$ there exists an evanescent sequence $V^n \supseteq E^n$ with $x 1_{T \setminus V^n} \in A$ for each $n = 1, 2, \dots$.*
2. *The correspondence H: $L^{\infty}(T) \cap A \twoheadrightarrow \mathbb{R}^I$ is (sequentially) $\mathrm{m}(L^{\infty}, L^1)$-lower hemicontinuous.*
3. *The correspondences D': $L^{\infty}_{+}(T) \cap A_{-} \twoheadrightarrow \mathbb{R}^{J'}$ and D'': $L^{\infty}_{+}(T) \cap A_{+} \twoheadrightarrow \mathbb{R}^{J''}$ are nonincreasing when their images are ordered by inclusion (i.e., $x' \leq x''$ implies that $D(x'') \subseteq D(x')$ for $D = D'$, D'').*
4. *$I \cap (J' \cup J'') = \emptyset$.*

[7] In addition to any further restrictions on the domains of D' and D'', this can express all the *closed* constraints on the domain of H (since any l.h.c. H: $A \twoheadrightarrow B$ whose domain A is closed in a topological space L has an l.h.c. extension to the whole of L, defined as $H(x) = B$ for $x \in L \setminus A$).

Proof. Given an $(x,q) \in Y$ and a $(p,r) \in L^{\infty *}(T) \times \mathbb{R}_G$, use Assumption 1 to take an evanescent sequence of measurable sets $(V^n)_{n=1}^{\infty}$ supporting p_{FA} with $x1_{T \setminus V^n} \in A$; and define $x^n = x1_{T \setminus V^n}$. Then $x^n \to x$ in $\mathrm{m}(L^\infty, L^1)$ as $n \to \infty$; also, $x_+^n \leq x_+$ and $x_-^n \leq x_-$. It follows by the l.h.c. of H (Assumption 2) that there exists a sequence $q_I^n \in \mathbb{R}^I$ with $q_I^n \in H(x^n)$ for each n and $q_i^n \to q_i$ for each $i \in I$ as $n \to \infty$. For each n, extend q_I^n to a $q^n \in \mathbb{R}^G$ by setting $q_g^n := q_g$ for $g \notin I$ (independently of n). Since $q_{J'} \in D'(x^n)$ and $q_{J''} \in D''(x^n)$ by Assumption 3, and since $I \cap (J' \cup J'') = \emptyset$, $(x^n, q^n)_{n=1}^{\infty}$ is a sequence in Y; and it has the required properties: $(x^n, q^n) \to (x,q)$ in $\mathrm{m}(L^\infty, L^1)$ as $n \to \infty$, and $\langle (p,r)_{\mathrm{FA}}, (x^n, q^n) \rangle = -\langle p_{\mathrm{FA}}, x^n \rangle = 0$ for each n. ∎

Formula (5.2) has been designed to accommodate the mixed cases as well as the pure ones; and all of its elements are used in the next example. The technology in question is pumped storage of a good with cyclical time-of-use prices, e.g., electricity. The signed outflow from the storage reservoir is a bounded function of time, $-\dot{s}(t) = -\mathrm{d}s/\mathrm{d}t$ for $t \in [0,1]$. The good is moved in and out of storage with a converter, which is taken to be perfectly efficient and symmetrically reversible: this means that in a unit time a unit converter can either turn a unit of the marketed good (electricity) into a unit of the stocked intermediate good (a storable form of energy), or *vice versa*. On this simplifying assumption, $-\dot{s}(t)$ equals the input/output rate for the good, $x(t) = (x^+ - x^-)(t)$. The converter's capacity is denoted by k_{Co} (in kW in the case of electricity). The reservoir's capacity is k_{St} (in kWh); stock can be held in storage at no running cost (or loss of stock). See [12] for further analysis of the model (including the case of imperfect conversion with a round-trip conversion efficiency $\eta < 1$).

Example 5.4. *The production set for pumped storage, which is*

$$Y_{\mathrm{PS}} := \left\{ (x, -k_{\mathrm{St}}, -k_{\mathrm{Co}}) \in L^\infty[0,1] \times \mathbb{R}_-^2 : |x| \leq k_{\mathrm{Co}}, \right. \\ \left. \exists s \quad \dot{s} = -x, s(0) = s(1), \text{and } 0 \leq s \leq k_{\mathrm{St}} \right\},$$

has the form (5.2), and therefore it meets the Exclusion Condition.

Proof. The input requirement set for an input/output flow x of the stored good is the orthant $\left(\check{k}_{\mathrm{St}}(x), \check{k}_{\mathrm{Co}}(x)\right) + \mathbb{R}_+^2$, where

$$\check{k}_{\mathrm{St}}(x) := \max_{t \in [0,1]} \int_0^t x(t)\,\mathrm{d}t + \max_{t \in [0,1]} \int_t^1 x(t)\,\mathrm{d}t \tag{5.3}$$

$$\check{k}_{\mathrm{Co}}(x) := \mathrm{EssSup}\,|x| = \operatorname*{ess\,sup}_{t \in [0,1]} |x(t)|. \tag{5.4}$$

(These are the minimum requirements for storage capacity and conversion capacity: see [12] for details.) In these terms, $(x, -k_{\mathrm{St}}, -k_{\mathrm{Co}}) \in Y_{\mathrm{PS}}$ if and only if

$$\int_0^1 x(t)\,\mathrm{d}t = 0, \ \check{k}_{\mathrm{St}}(x) \leq k_{\mathrm{St}} \text{ and } \check{k}_{\mathrm{Co}}(x) \leq k_{\mathrm{Co}}. \tag{5.5}$$

This shows that Y_{PS} has the form (5.2) with:

$A = \left\{ x \in L^{\infty} : \int_0^1 x(t)\, \mathrm{d}t = 0 \right\}$;
the two capital inputs $\{\mathrm{St}, \mathrm{Co}\}$ in place of $\{1, \ldots, G\}$;
$I = \{\mathrm{St}\}$ and $H(x) = \left[\check{k}_{\mathrm{St}}(x), +\infty\right)$;
$J' = J'' = \{\mathrm{Co}\}$, $D'(x^-) = [\mathrm{EssSup}(x^-), +\infty)$ and $D''(x^+) = [\mathrm{EssSup}(x^+), +\infty)$.[8]

The conditions of Proposition 5.3 are met because:

1. As is readily shown, if $\int_0^1 x(t)\, \mathrm{d}t = 0$, then for every sequence of sets $E^n \subset [0,1]$ with $\mathrm{meas}\, E^n \to 0$ there exists a sequence $V^n \supseteq E^n$ with $\mathrm{meas}\, V^n \to 0$ and $\int_{V^n} x(t)\, \mathrm{d}t = 0$ for each n;
2. As is shown in [12], $\check{k}_{\mathrm{St}}$ is an $\mathrm{m}(L^{\infty}, L^1)$-continuous function on $L^{\infty}[0,1]$;[9]
3. The function $\mathrm{EssSup} : L^{\infty}[0,1] \to \mathbb{R}$ is nondecreasing.[10]

Therefore Y_{PS} meets the Exclusion Condition. ∎

Comments:

1. Bewley's Exclusion Assumption cannot be verified in this way because it can be that $\check{k}_{\mathrm{St}}(x_n) > \check{k}_{\mathrm{St}}(x)$: an example can be constructed from an x with the sign pattern $(- + - + - +)$ over $[0,1]$.
2. In the storage example the continuous constraint H comes from an input requirement function ($\check{k}_{\mathrm{St}}$), instead of a production function or correspondence as in (5.1) or Example 5.2 (where $H(x)$ is $P(-x)$ for $x \in L^{\infty}_{-}$).

6. Removal of singularities from equilibrium prices

The Exclusion Condition of Section 2 serves to remove the singular term of an equilibrium price system for an Arrow-Debreu model with the commodity space $L^{\infty}(S, \sigma)$, where σ is a measure on S. In this abstract setting x, y and p denote functions on S which represent complete commodity bundles and price systems (whereas in Sections 5 and 9 these letters denote commodity bundles and price systems for the differentiated good alone).

The sets of producers and households (or consumers) are denoted by Pr and Ho. The production set of producer $i \in \mathrm{Pr}$ is Y_i, and the consumption set of household $h \in \mathrm{Ho}$ is X_h. Consumer preferences, taken to be complete and transitive, are given by a total (a.k.a. complete) weak preorder $\preccurlyeq_h$ on X_h. The corresponding strict preference is denoted by $\prec_h$. The household's initial endowment is denoted by x_h^{En}; and the household's share in the profits of producer i is $\varsigma_h^i \geq 0$, with $\sum_h \varsigma_h^i = 1$ for each i. (The ranges of running indices in summations, etc., are always taken to be the largest possible with any specified restrictions.)

[8] Formally $H(x) = \{\mathrm{St}\} \times \left[\check{k}_{\mathrm{St}}(x), +\infty\right)$, since this real half-line must be interpreted as a subset of $\mathbb{R}^{\{\mathrm{St}\}}$. Similarly $D' = \{\mathrm{Co}\} \times [\mathrm{EssSup}(\cdot), +\infty) = D''$.

[9] It is the u.s.c. of $\check{k}_{\mathrm{St}}$ that is relevant here: it means that $H := \left[\check{k}_{\mathrm{St}}(\cdot), +\infty\right)$ is l.h.c.

[10] So D' and D'', here both equal to $[\mathrm{EssSup}(\cdot), +\infty)$, are nonincreasing correspondences from L^{∞}_{+} into $\mathbb{R}$.

The following assumptions are made henceforth.

Set Closedness. The sets Y_i and X_h are $\mathrm{w}(L^\infty, L^1)$-closed, for each i and h.[11]

Preference Lower Semicontinuity. For each h the preorder $\preccurlyeq_h$ is Mackey lower semicontinuous, i.e., for every x' the set $\{x \in X_h : x \preccurlyeq_h x'\}$ is $\mathrm{m}(L^\infty, L^1)$-closed.

Local Nonsatiation. For each h, every $x \in X_h$ is in the $\mathrm{m}(L^\infty, L^1)$-closure of $\{x' \in X_h : x \prec_h x'\}$.[12]

Comment: The analysis requires only the *sequential* l.s. continuity of preferences; and this condition is easier to verify than full l.s. continuity because, unlike an uncountable net, a sequence in L^∞ that converges for $\mathrm{m}(L^\infty, L^1)$ is the same as a bounded sequence that converges in measure. However, for monotone preferences on L^∞_+, sequential Mackey semicontinuity is actually equivalent to full semicontinuity, as we show in [14].

Definition 6.1. *A competitive equilibrium consists of a price system, $p^\star \in L^{\infty *}$, and an allocation, $x^\star_h \in X_h$ and $y^\star_i \in Y_i$ for each household h and producer i, that meet the conditions:*

1. $\sum_h \left(x^\star_h - x^{\mathrm{En}}_h\right) = \sum_i y^\star_i$.
2. $\langle p^\star, y^\star_i\rangle = \Pi_i(p^\star) := \sup_y \{\langle p^\star, y\rangle : y \in Y_i\}$.
3. $\langle p^\star, x^\star_h\rangle = \hat{M}_h(p^\star) := \left\langle p^\star, x^{\mathrm{En}}_h + \sum_i \varsigma^i_h y^\star_i\right\rangle$.
4. *For every $x \in X_h$, if $\langle p^\star, x\rangle \le \langle p^\star, x^\star_h\rangle$, then $x \preccurlyeq_h x^\star_h$.*

The related concept of a *quasi-equilibrium* is defined by making the inequality sign strict in the antecedent of Condition 4. Every equilibrium is a quasi-equilibrium, but not *vice versa*: e.g., the zero price vector is trivially a quasi-equilibrium price (if a feasible allocation exists) but never an equilibrium (unless all households can be satiated).[13]

Theorem 6.2. *In addition to the Mackey lower semicontinuity and local nonsatiation of preferences, assume that the production set Y_i and the consumer's set of feasible trades $X_h - x^{\mathrm{En}}_h$ meet the Exclusion Condition, for each i and h. If a price system $p^\star \in L^{\infty *}$ supports an allocation $\left((x^\star_h)_{h \in \mathrm{Ho}}, (y^\star_i)_{i \in \mathrm{Pr}}\right)$ as a competitive equilibrium, then $p^\star_{\mathrm{CA}}$ is a quasi-equilibrium price that supports the same allocation.*

[11] For *convex* sets, this is equivalent to $\mathrm{m}\left(L^\infty, L^1\right)$-closedness.

[12] For the equilibrium results it suffices to assume this of the attainable consumption set (i.e., only of those x's which appear in feasible allocations).

[13] Another related concept is that of a *compensated equilibrium*, defined by replacing preference maximisation with expenditure minimisation, i.e., with the condition that, if $x^\star_h \preccurlyeq_h x \in X_h$, then $\langle p^\star, x\rangle \ge \langle p^\star, x^\star_h\rangle$. A compensated equilibrium is always a quasi-equilibrium; and the converse holds if each $\preccurlyeq_h$ is locally nonsatiated for a topology which makes p continuous. To see this, take any $x' \succcurlyeq_h x^\bullet_h$, where $x^\bullet_h$ quasi-maximises $\preccurlyeq_h$ (given prices p and income M_h, which therefore equals $\langle p, x^\bullet_h\rangle$ by local nonsatiation). Then $\langle p, x'\rangle \ge \inf_x \{\langle p, x\rangle : x \succ_h x'\} \ge \langle p, x^\bullet_h\rangle$. The first inequality is similarly employed after (6.4) below.

Proof. This is structured to identify, for use in Section 7, those parts of the analysis which apply not only to a general equilibrium but also to an individual optimum for a producer or consumer. Suppose that $y_i^\bullet$ is an input-output bundle that maximises the profit of producer i, given a price system $p \in L^{\infty *}$. For any $y \in Y_i$, use the Exclusion Condition to take a sequence $(y^n)_{n=1}^{\infty}$ in Y_i with $\langle p, y^n \rangle \to \langle p_{\mathrm{CA}}, y \rangle$ as $n \to \infty$. Since $\langle p, y_i^\bullet \rangle \geq \langle p, y^n \rangle$ for every n, it follows by passage to the limit that

$$\langle p, y_i^\bullet \rangle \geq \langle p_{\mathrm{CA}}, y \rangle . \tag{6.1}$$

For the case of $y = y_i^\bullet$ this gives that

$$\langle p_{\mathrm{FA}}, y_i^\bullet \rangle \geq 0. \tag{6.2}$$

Next suppose that $x_h^\bullet$ is a consumption bundle that maximises $\preccurlyeq_h$ subject to the budget constraint given by an income M_h and a price system $p \in L^{\infty *}$. For any $x \succ_h x_h^\bullet$, use the Exclusion Condition (on $X_h - x_h^{\mathrm{En}}$) to take a sequence (x^n) in X_h with $\langle p_{\mathrm{FA}}, x^n - x_h^{\mathrm{En}} \rangle \to 0$ and $x^n \to x$ in $\mathrm{m}(L^\infty, L^1)$, as $n \to \infty$. Then $\langle p, x^n - x_h^{\mathrm{En}} \rangle \to \langle p_{\mathrm{CA}}, x - x_h^{\mathrm{En}} \rangle$, i.e., $\langle p, x^n \rangle \to \langle p_{\mathrm{CA}}, x \rangle + \langle p_{\mathrm{FA}}, x_h^{\mathrm{En}} \rangle$ as $n \to \infty$. Also, $x^n \succ_h x_h^\bullet$ for every sufficiently large n by Preference Lower Semicontinuity; and therefore

$$M_h < \langle p, x^n \rangle \to \langle p_{\mathrm{CA}}, x \rangle + \langle p_{\mathrm{FA}}, x_h^{\mathrm{En}} \rangle . \tag{6.3}$$

By passage to the limit as $n \to \infty$, it follows that

$$\langle p_{\mathrm{CA}}, x \rangle + \langle p_{\mathrm{FA}}, x_h^{\mathrm{En}} \rangle \geq M_h \geq \langle p, x_h^\bullet \rangle$$

so

$$\langle p_{\mathrm{CA}}, x - x_h^{\mathrm{En}} \rangle \geq \langle p, x_h^\bullet - x_h^{\mathrm{En}} \rangle . \tag{6.4}$$

Although $x_h^\bullet$ cannot simply be substituted for x here (unlike the case of $y = y_i^\bullet$ in (6.1)), one can use Local Nonsatiation to approximate $x_h^\bullet$ in $\mathrm{m}(L^\infty, L^1)$ by *strictly* preferred x's, to which (6.4) does apply (and then use the $\mathrm{m}(L^\infty, L^1)$-continuity of p_{CA}). This gives

$$\langle p_{\mathrm{CA}}, x_h^\bullet - x_h^{\mathrm{En}} \rangle \geq \inf_x \left\{ \langle p_{\mathrm{CA}}, x - x_h^{\mathrm{En}} \rangle : x \succ_h x_h^\bullet \right\} \geq \langle p, x_h^\bullet - x_h^{\mathrm{En}} \rangle ,$$

so[14]

$$0 \geq \langle p_{\mathrm{FA}}, x_h^\bullet - x_h^{\mathrm{En}} \rangle . \tag{6.5}$$

Since $(p^\star, (x_h^\star), (y_i^\star))$ is an equilibrium, (6.2) and (6.5) can be applied to $y_i^\star$, $x_h^\star$ and $p^\star$ (in place of $y_i^\bullet$, $x_h^\bullet$ and p) to obtain, by adding up over i and over h, that

$$0 \geq \sum_h \langle p_{\mathrm{FA}}^\star, x_h^\star - x_h^{\mathrm{En}} \rangle = \sum_i \langle p_{\mathrm{FA}}^\star, y_i^\star \rangle \geq 0$$

[14]In heuristic terms, (6.2) and (6.5) mean that neither producers nor households would choose to spend on the commodities with singular prices—as Bewley [3, p. 523] puts it, singular prices would "make an arbitrarily small set of commodities extraordinarily expensive", so that consumers "would prefer to trade them for cheaper ones".

and therefore that

$$\langle p^\star_{\mathrm{FA}}, y^\star_i \rangle = 0 \tag{6.6}$$

$$\left\langle p^\star_{\mathrm{FA}}, x^\star_h - x^{\mathrm{En}}_h \right\rangle = 0 \tag{6.7}$$

for each i and h. This and (6.1) for $y^\star_i$ (in place of $y^\bullet_i$) give

$$\langle p^\star_{\mathrm{CA}}, y^\star_i \rangle = \langle p^\star, y^\star_i \rangle \geq \langle p^\star_{\mathrm{CA}}, y \rangle . \tag{6.8}$$

This shows that $y^\star_i$ yields a maximum profit not only at $p^\star$, but also at $p^\star_{\mathrm{CA}}$. And (6.8) also shows that the maximum profit at $p^\star_{\mathrm{CA}}$ is the same as at $p^\star$ (so the consumers' profit incomes are the same as well). Since $x^\star_h - x^{\mathrm{En}}_h$ costs the same at $p^\star_{\mathrm{CA}}$ as at $p^\star$ by (6.7), $x^\star_h$ is in the budget set of consumer h at $p^\star_{\mathrm{CA}}$ (since it is at $p^\star$). Formally,

$$\left\langle p^\star_{\mathrm{CA}}, x^\star_h - x^{\mathrm{En}}_h \right\rangle = \left\langle p^\star, x^\star_h - x^{\mathrm{En}}_h \right\rangle = \sum_i \varsigma^i_h \langle p^\star, y^\star_i \rangle = \sum_i \varsigma^i_h \langle p^\star_{\mathrm{CA}}, y^\star_i \rangle .$$

Finally, use (6.4) for $x^\star_h$ (in place of $x^\bullet_h$) and (6.7) to obtain that, for any $x \succ_h x^\star_h$,

$$\left\langle p^\star_{\mathrm{CA}}, x - x^{\mathrm{En}}_h \right\rangle \geq \left\langle p^\star, x^\star_h - x^{\mathrm{En}}_h \right\rangle = \left\langle p^\star_{\mathrm{CA}}, x^\star_h - x^{\mathrm{En}}_h \right\rangle$$

and therefore $\langle p^\star_{\mathrm{CA}}, x \rangle \geq \langle p^\star_{\mathrm{CA}}, x^\star_h \rangle$. This means that $x^\star_h$ is (weakly) preferable to every bundle which satisfies the budget constraint *strictly*; and this completes the proof that $p^\star_{\mathrm{CA}}$ is a quasi-equilibrium price. ∎

Comments:

1. In Theorem 6.2 it suffices to assume that $p^\star$ is a quasi-equilibrium price system: a weak inequality in (6.4) would suffice for the result that $p^\star_{\mathrm{CA}}$ is also a quasi-equilibrium price.
2. The quasi-equilibrium price $p^\star_{\mathrm{CA}}$ is actually an equilibrium price if, for each h, there is an $x^0_h \in X_h$ with $\langle p^\star_{\mathrm{CA}}, x^0_h \rangle < \langle p^\star_{\mathrm{CA}}, x^\star_h \rangle$: see, e.g., [5, p. 269].[15] The inequality holds for any x^0_h in the intersection of $X_h - x^{\mathrm{En}}_h$ and the norm-interior of the asymptotic cone of $\sum_i Y_i$; that the intersection be nonempty is the usual adequacy assumption for the existence of an equilibrium price system in $L^{\infty *}$.
3. Since L^∞ with $\mathrm{m}(L^\infty, L^1)$ is a topological vector lattice, the general approach of [16] applies; and that analysis also establishes the existence of an equilibrium price $p^\star \in L^1$ (when L^∞ is the commodity space). But the assumptions of [16]—viz., Mackey uniform properness of the technologies and preferences—are stronger than those needed for the two-stage approach in which an equilibrium existence result with a price $p^\star \in L^{\infty *}$, such as [3, Theorem 1], is followed by removal of $p^\star_{\mathrm{FA}}$ as per Theorem 6.2. For example, an additively separable function $F(z) = \int_0^1 f(z(t))\,\mathrm{d}t$ for $z \in L^\infty_+[0,1]$, with f concave and nondecreasing on $\mathbb{R}_+$, can serve as a Mackey continuous utility function [3, p. 535]; or it can

[15] To spell this out, take any $x \in X_h$ with $\langle p^\star_{\mathrm{CA}}, x_h \rangle \leq \langle p^\star_{\mathrm{CA}}, x^\star_h \rangle$, and introduce $x^\alpha := (1-\alpha)\, x^0 + \alpha x$ for $\alpha \in [0,1)$; then $\langle p^\star_{\mathrm{CA}}, x^\alpha \rangle < \langle p^\star_{\mathrm{CA}}, x^\star_h \rangle$, so $x^\alpha \preccurlyeq_h x^\star_h$. As $\alpha \nearrow 1$, it follows that $x \preccurlyeq_h x^\star_h$.

be a production function, in which case our Exclusion Condition holds by Example 5.2. But if $(\mathrm{d}f/\mathrm{d}z)(0+) = +\infty$, then F is *not* Mackey uniformly proper. For the sequence space l^∞, this is noted in [2, pp. 97–98].

7. Removal of singularities from prices supporting individual optima

When each consumption set X is an orthant $\underline{x} + L^\infty_+$ containing the endowment x^{En}, the assumption on $X - x^{\mathrm{En}}$ in Theorem 6.2 holds by Example 4.1: i.e., in this case singularity removal can be based on approximation of x with the sequence $x^n := x 1_{S \setminus E^n} + x^{\mathrm{En}} 1_{E^n}$. Another suitable sequence can be obtained by using the orthant's vertex $\underline{x}$ instead of x^{En} (when $p \geq 0$); and this gives an extra result on a consumer's *individual* optimum (and not only on the general equilibrium).

Proposition 7.1. *With an orthant $X = \underline{x} + L^\infty_+$ as the consumption set, if a bundle $x^\bullet$ maximises a Mackey lower semicontinuous preference preorder $\preccurlyeq$ on the budget set*

$$B(p, M) := \{x : x \geq \underline{x} \text{ and } \langle p, x \rangle \leq M\},$$

where $M \in \mathbb{R}$ and $p \in L^{\infty *}_+$, then $x^\bullet$ also maximises $\preccurlyeq$ on $B(p_{\mathrm{CA}}, M - \langle p_{\mathrm{FA}}, \underline{x} \rangle)$.

Proof. Take an evanescent sequence of sets $(E^n)_{n=1}^\infty$ supporting p_{FA}; and, for any $x \succ x^\bullet$ introduce $x^n := x 1_{S \setminus E^n} + \underline{x} 1_{E^n} \in X$. This is a nondecreasing sequence (since $x \geq \underline{x}$); so $x^n \nearrow x$ as $n \to \infty$ (and *a fortiori* $x^n \to x$ in $\mathrm{m}(L^\infty, L^1)$). So $x^n \succ x^\bullet$ for every sufficiently large n (as in the proof of (6.3)); and therefore

$$M < \langle p, x^n \rangle \nearrow \langle p_{\mathrm{CA}}, x \rangle + \langle p_{\mathrm{FA}}, \underline{x} \rangle . \tag{7.1}$$

Also, since $p_{\mathrm{FA}} \geq 0$ (and $\underline{x} \leq x^\bullet$),

$$\langle p_{\mathrm{CA}}, x^\bullet \rangle + \langle p_{\mathrm{FA}}, \underline{x} \rangle \leq \langle p, x^\bullet \rangle \leq M. \tag{7.2}$$

By (7.1) and (7.2),

$$\langle p_{\mathrm{CA}}, x \rangle > M - \langle p_{\mathrm{FA}}, \underline{x} \rangle \geq \langle p_{\mathrm{CA}}, x^\bullet \rangle , \tag{7.3}$$

as required. ∎

In such a case no further argument is needed to establish that, in Theorem 6.2, $p^\star_{\mathrm{CA}}$ is an equilibrium price (and not only a quasi-equilibrium price).

Corollary 7.2. *If, in Theorem 6.2, $p^\star \geq 0$ and the assumption on each X_h and x^{En}_h is strengthened to: $X_h = \underline{x}_h + L^\infty_+$ with $x^{\mathrm{En}}_h \geq \underline{x}_h$, then $p^\star_{\mathrm{CA}}$ is an equilibrium price.*

Proof. For each h, apply Proposition 7.1 to $x^\star_h$ (in place of $x^\bullet$), with $M = \langle p^\star, x^\star_h \rangle$. ∎

Comments:

1. Another application of Proposition 7.1 gives a concrete example of nonexistence of a consumer optimum when $p \in L^{\infty *} \setminus L^1$: see [10]. For $p \in L^1$ a consumer optimum does exist (after the consumption set has been truncated to make it weakly* compact), and it depends on p in a norm-to-weak* continuous way: see [10] or [13].

2. In Proposition 7.1, if $\preccurlyeq$ is Mackey locally nonsatiated, then $\langle p_{\rm FA}, x^\bullet \rangle = \langle p_{\rm FA}, \underline{x} \rangle$: if $\langle p_{\rm FA}, x^\bullet \rangle$ were above this minimum, then this part of the expenditure could be reallocated to a better use. Formally this is because, although $x^\bullet$ cannot be substituted for x in (7.3), it can be approximated in $\mathrm{m}\left(L^\infty, L^1\right)$ by *strictly* preferred x's, to which (7.3) does apply. This and (7.2) give

$$\begin{aligned} \langle p_{\rm CA}, x^\bullet \rangle &\geq \inf_x \{\langle p_{\rm CA}, x \rangle : x \succ x^\bullet\} \geq M - \langle p_{\rm FA}, \underline{x} \rangle \\ &\geq \langle p_{\rm CA}, x^\bullet \rangle + \langle p_{\rm FA}, x^\bullet \rangle - \langle p_{\rm FA}, \underline{x} \rangle \geq \langle p_{\rm CA}, x^\bullet \rangle, \end{aligned}$$

whence the result. And this gives another proof of (6.5), on the assumptions of Corollary 7.2: $\langle p_{\rm FA}, x_h^\bullet \rangle = \langle p_{\rm FA}, \underline{x}_h \rangle \leq \left\langle p_{\rm FA}, x_h^{\rm En} \right\rangle$.

The singular part can also be removed from a price system supporting a profit maximum for an industrial user of a differentiated commodity (who produces a finite number of homogeneous output goods).

Remark 7.3. *For a production set Y such as in Example 5.2, if a $y^\bullet \in Y$ maximises the profit at a $p \in L_+^{\infty *}$, then it also maximises the profit at $p_{\rm CA}$.*

Proof. With $p \geq 0$, in such a case one has $\langle p_{\rm FA}, y \rangle \leq 0$ for every $y \in Y$. Applied to $y^\bullet$ and combined with (6.1), this gives $\langle p_{\rm CA}, y^\bullet \rangle \geq \langle p, y^\bullet \rangle \geq \langle p_{\rm CA}, y \rangle$. ∎

8. Absence of singularities from equilibrium prices

Under additional assumptions price singularities, rather than just being removable, are simply absent from the *original* equilibrium. Two such results are given next; the first of these is applied to peak-load pricing in Section 9.

Theorem 8.1. *In addition to the Mackey lower semicontinuity and local nonsatiation of preferences, assume that the sets Y_i and $X_h - x_h^{\rm Rd}$ meet the Exclusion Condition for some $\left(x_h^{\rm Rd}\right)_{h \in {\rm Ho}}$ with $\sum_h x_h^{\rm Rd} = \sum_h x_h^{\rm En}$, and that a nonnegative price system $p^\star \in L_+^{\infty *}$ supports a competitive equilibrium allocation (with $(y_i^\star)_{i \in {\rm Pr}}$ as the production bundles). If, for some constant $\epsilon > 0$ and a subset $\mathcal{P}$ of producers, $\sum_{i \in \mathcal{P}} y_i^\star \geq \epsilon$ on some set that supports $p_{\rm FA}^\star$, then $p_{\rm FA}^\star = 0$, i.e., $p^\star \in L_+^1$.*

Proof. The same argument as in the Proof of Theorem 6.2, but with $x_h^{\rm Rd}$ in place of $x_h^{\rm En}$, establishes (6.6). From this and the assumptions,

$$0 = \sum_{i \in \mathcal{P}} \langle p_{\rm FA}^\star, y_i^\star \rangle \geq \langle p_{\rm FA}^\star, \epsilon \rangle = \epsilon \left\| p_{\rm FA}^\star \right\|_\infty^* \geq 0$$

so $p_{\rm FA}^\star = 0$. ∎

The other result on the absence of a price singularity is a straightforward extension of the case of pure exchange given in [3, Theorem 2] and [2, Theorem 4]. It rests, however, on the extra assumption of a strongly positive total endowment—and this is rather restrictive in the context of production (since the produced commodities are typically absent from the initial endowment).

Proposition 8.2. *In addition to the Mackey lower semicontinuity and local nonsatiation of preferences, assume that, for each i and h, the sets Y_i and $X_h - x'_h$ meet the Exclusion Condition for some $(x'_h)_{h \in \mathrm{Ho}}$ with $\sum_h x'_h \le \sum_h x_h^{\mathrm{En}} - \epsilon$ for some constant $\epsilon > 0$. If a nonnegative $p^\star \in L_+^{\infty *}$ is a competitive equilibrium price system, then $p^\star_{\mathrm{FA}} = 0$, i.e., $p^\star \in L_+^1$.*

Proof. With $(x_h^\star, y_i^\star)$ denoting the equilibrium allocation, the same argument as in the Proof of Theorem 6.2—but with x'_h in place of x_h^{En}—establishes (6.2) and the counterpart of (6.5), viz., that $0 \le \langle p_{\mathrm{FA}}, y_i^\star \rangle$ and $\langle p^\star_{\mathrm{FA}}, x_h^\star - x'_h \rangle \le 0$. From this and the assumptions,

$$0 \le \sum_i \langle p^\star_{\mathrm{FA}}, y_i^\star \rangle = \sum_h \left\langle p^\star_{\mathrm{FA}}, x_h^\star - x_h^{\mathrm{En}} \right\rangle \le \sum_h \left\langle p^\star_{\mathrm{FA}}, x'_h - x_h^{\mathrm{En}} \right\rangle \le -\epsilon \left\| p^\star_{\mathrm{FA}} \right\|_\infty^* \le 0$$

so $p^\star_{\mathrm{FA}} = 0$. ∎

9. A solution to the shifting-peak problem

In the context of continuous-time peak-load pricing—i.e., pricing a produced good with a cyclical demand and a capacity input (in addition to a variable input)—the price density results can be used to formalise and examine Boiteux's conjecture on the shifting-peak problem. The cyclically priced flow in question is referred to as electricity (since this is a typical example, although the model applies to other goods as well). When electricity is priced at long-run marginal cost (LRMC), capacity charges may be levied only at the times of peak demand; but if this principle is applied to an existing demand pattern, it may mean concentrating the capacity charges on peaks which are extremely brief. In such a case the users' response is likely to destroy these peaks and create new ones which are equally brief, so that the difficulty arises afresh. Boiteux [4, 3.4 and 3.3.3] conjectures that, nevertheless, there is an equilibrium solution which consists in spreading and timing the capacity charge in such a way that the resulting demand has a "reasonably extended" peak plateau which bears all of the capacity charge: the spreading reduces price differences sufficiently to remove the incentive to shift demand to the lower-priced times.

Boiteux's solution is by no means always valid: a firm, pointed peak is equally possible *a priori*. An implicit assumption underlying the peak-plateau type of equilibrium is that electricity consumption is interruptible, i.e., that the losses from an interruption vanish in the limit as its duration becomes arbitrarily short. And this is exactly what Mackey continuity (of a utility or production function) means in the context of continuous-time consumption: recall that, for a sequence $(E^n)_{n=1}^\infty$ of subsets of $[0,1]$, the condition $\mathrm{meas}\, E^n \to 0$ implies that $1_{E^n} \to 0$ in $\mathrm{m}\left(L^\infty, L^1\right)$, and hence that $U\left(x 1_{[0,1] \setminus E^n}\right) \to U(x)$ as $n \to \infty$ if U: $L_+^\infty [0,1] \to \mathbb{R}$ is Mackey continuous. On this assumption on the users' utility and production functions, the equilibrium time-of-use (TOU) tariff for electricity is a price density function $p^\star \in L^1 [0,1]$. The

LR equilibrium price has the form

$$p^{\star}(t) = w + r\kappa^{\star}(t), \tag{9.1}$$

where r and w are the unit capacity cost and the unit running cost, and $\kappa^{\star} \in L^1_+[0,1]$ is the equilibrium density of the capacity charge, which is concentrated on the peaks of the LR equilibrium output $y^{\star}$; i.e.,

$$\int_0^1 \kappa^{\star}(t)\,\mathrm{d}t = 1 \tag{9.2}$$

$$\kappa^{\star}(t) = 0 \quad \text{for almost every } t \in [0,1] \text{ with } y^{\star}(t) < \mathrm{EssSup}(y^{\star}). \tag{9.3}$$

It follows that the LR output has a peak plateau; i.e., the set $\{t : y^{\star}(t) = \mathrm{EssSup}(y^{\star})\}$ has a positive Lebesgue measure. The results hold also when thermal generation is supplemented by pumped storage (Theorem 9.2).

To present this application rigorously yet briefly, we assume that, as a result of aggregating commodities on the basis of some fixed relative prices, there are just two commodities apart from electricity—viz., a numeraire and a homogeneous final good whose production requires an input of electricity. A complete consumption bundle consists therefore of electricity, the produced final good and the numeraire. These quantities are written, in this order, as $(x;\gamma,m) \in L^{\infty}[0,1] \times \mathbb{R}^2$. A matching price system is $(p;\rho,1) \in L^{\infty *}[0,1] \times \mathbb{R}_2$. There is a finite set, Ho, of households; and for each $h \in \mathrm{Ho}$ the preference preorder $\preccurlyeq_h$ is $\mathrm{m}(L^{\infty} \times \mathbb{R}^2, L^1 \times \mathbb{R}_2)$-continuous (Mackey continuous) on the consumption set $L^{\infty}_+[0,1] \times \mathbb{R}^2_+$. Each household's initial endowment is a quantity $m^{\mathrm{En}}_h > 0$ of the numeraire only; and nonsatiation in the numeraire commodity is assumed.

There are two producers: one electricity supplier with constant returns to scale and one industrial user, who produces the final good from inputs of electricity and of the numeraire. (In the case of decreasing returns to scale, each households' share ς_h in the user industry's profits must also be specified.)

With the unit capacity cost and the unit running cost of thermal electricity generation denoted by $r > 0$ and $w \geq 0$, the LR cost of an output flow $y \in L^{\infty}[0,1]$ is[16]

$$C^{\mathrm{Th}}_{\mathrm{LR}}(y) = w\int_0^1 y^{+}(t)\,\mathrm{d}t + r\,\mathrm{ess}\sup_{t\in[0,1]} y^{+}(t). \tag{9.4}$$

With fixed and variable inputs aggregated into the numeraire, the LR production set is

$$Y^{\mathrm{Ag}}_{\mathrm{Th}} := \left\{(y;0,-m) \in L^{\infty}[0,1] \times \mathbb{R}^2 : C^{\mathrm{Th}}_{\mathrm{LR}}(y) \leq m\right\}. \tag{9.5}$$

The user's production function $F\colon L^{\infty}_+[0,1] \times \mathbb{R}_+ \to \mathbb{R}$ is assumed to be concave, nondecreasing and $\mathrm{m}(L^{\infty} \times \mathbb{R}, L^1 \times \mathbb{R})$-continuous. The firm is assumed to be capable of some productive activity with constant returns to scale which is not limited to

[16] So $C(y) = C(y^{+})$, which means assuming (unlimited) free disposal: see [9].

free disposal. Formally, the recession function of F is assumed not to be identically zero. In terms of the firm's production set, which is

$$(9.6) \qquad Y_{\mathrm{IU}} = \left\{ (-z; \gamma, -m) \in L_{-}^{\infty}[0,1] \times \mathbb{R} \times \mathbb{R}_{-} : F(z,m) \geq \gamma \right\},$$

this means that its recession (a.k.a. asymptotic) cone, $\mathrm{rec}\, Y_{\mathrm{IU}}$, is not just $L_{-}^{\infty} \times \mathbb{R}_{-}^{2}$.

On these assumptions, the equilibrium capacity charge is spread out as a price density over a peak output plateau.

Theorem 9.1. *The electricity pricing model has a long-run competitive equilibrium. Furthermore, if an equilibrium tariff $p^\star \in L_{+}^{\infty *}[0,1]$ supports (together with some price $\rho^\star \in \mathbb{R}_{+}$ for the other produced good) an equilibrium allocation with a nonzero electricity output $y^\star = z^\star + \sum_h x_h^\star > 0$, then $p^\star \in L_{+}^{1}[0,1]$. Therefore $y^\star$ has a peak plateau, i.e.,*

$$(9.7) \qquad \mathrm{meas}\left\{ t \in [0,1] : y^\star(t) = \mathrm{EssSup}(y^\star) \right\} > 0.$$

Proof. An equilibrium price system with $p^\star \in L_{+}^{\infty *}$ exists by [3, Theorem 1]. (The Adequacy Assumption of [3] holds here because $m_h^{\mathrm{En}} > 0$ for each h, $\mathrm{rec}\, C_{\mathrm{LR}}^{\mathrm{Th}} = C_{\mathrm{LR}}^{\mathrm{Th}} \neq 0$ and $\mathrm{rec}\, F \neq 0$, and because both functions are norm-continuous. The Boundedness Assumption of [3] holds because $C_{\mathrm{LR}}^{\mathrm{Th}}(y) > 0$ for every $y > 0$, and because F is finite everywhere. Also, the set $Y_{\mathrm{Th}}^{\mathrm{Ag}}$ of (9.5) is weakly* closed, since it is convex and Mackey closed. And this is because $C_{\mathrm{LR}}^{\mathrm{Th}}$ is $\mathrm{m}(L^\infty, L^1)$-l.s.c.:[17] each of its terms is the composition of an $\mathrm{m}(L^\infty, L^1)$-l.s.c. function ($\int$ or EssSup) and the map $y \mapsto y^+$, which is $\mathrm{m}(L^\infty, L^1)$-continuous: see, e.g., [3, p. 535]. Similarly the Mackey (u.s.) continuity of F implies that the set Y_{IU} of (9.6) is weakly* closed.)

To show that $p^\star \in L^1$, verify the assumptions of Theorem 8.1. The production sets $Y_{\mathrm{Th}}^{\mathrm{Ag}}$ and Y_{IU} meet the Exclusion Condition by Examples 5.1 and 5.2; and, for each h, the consumer's trade set $L_{+}^{\infty}[0,1] \times \mathbb{R}_{+}^{2} - \left(0; 0, m_h^{\mathrm{En}}\right)$ meets the Exclusion Condition by Example 4.1.

Note next that, for every constant $\epsilon > 0$, $p_{\mathrm{FA}}^\star$ is concentrated on the set of ϵ-near-peaks

$$(9.8) \qquad S_\epsilon(y^\star) := \left\{ t \in [0,1] : y^\star(t) \geq \mathrm{EssSup}(y^\star) - \epsilon \right\}.$$

One way to show this is to use the subdifferential of the supremum function: every $\kappa \in \partial\, \mathrm{EssSup}(y^\star)$ is concentrated on $S_\epsilon(y^\star)$: see, e.g., [6]. Hence so is κ_{FA}. And $p_{\mathrm{FA}}^\star = r\kappa_{\mathrm{FA}}^\star$ for some $\kappa^\star \in \partial\, \mathrm{EssSup}(y^\star)$, since $p^\star \in \partial C_{\mathrm{LR}}(y^\star)$, and since every subgradient of the integral in (9.4) belongs to L^1. (If y is strictly positive, then the integral term has an ordinary gradient, viz., the constant w.)

Since $\mathrm{EssSup}(y^\star) > 0$, for a small enough $\epsilon > 0$ one has $\epsilon \leq \mathrm{EssSup}(y^\star) - \epsilon \leq y^\star$ on the set $S_\epsilon(y^\star)$, which supports $p_{\mathrm{FA}}^\star$. Therefore $p_{\mathrm{FA}}^\star = 0$ by Theorem 8.1.

So $\kappa_{\mathrm{FA}}^\star = (1/r)\, p_{\mathrm{FA}}^\star = 0$, i.e., $\kappa^\star$ is countably additive. Since $S_\epsilon(y^\star)$ supports $\kappa^\star$ for each $\epsilon > 0$, it follows that $\kappa^\star(t) = 0$ outside the set $S_0(y^\star)$ of the *exact* peaks of $y^\star$,

[17] It follows that $C_{\mathrm{LR}}^{\mathrm{Th}}$ is also $\mathrm{w}\left(L^\infty, L^1\right)$-l.s.c. (since it is convex).

which is formally defined by (9.8) with $\epsilon = 0$. So $\int_{S_0(y^\star)} \kappa^\star(t)\,\mathrm{d}t = \int_0^1 \kappa^\star(t)\,\mathrm{d}t = 1$; and *a fortiori* meas $S_0(y^\star) > 0$, i.e., (9.7). (That $\langle \kappa, 1 \rangle = 1$ for every $\kappa \in \partial\,\mathrm{EssSup}(y) \subset L_+^{\infty *}$ and $y \in L^\infty$ is shown also in, e.g., [6].) ∎

Finally, pumped storage is added to the technology for electricity supply. Example 5.4 describes the technique's LR production set in terms of separate, unaggregated inputs, viz., the reservoir St and the converter Co. Given their unit costs $r^{\mathrm{St}} \geq 0$ and $r^{\mathrm{Co}} > 0$, the LR cost of a flow from storage, $y \in L^\infty[0,1]$ with $\int_0^1 y(t)\,\mathrm{d}t = 0$, can be expressed in terms of the capacity requirement functions (5.3)–(5.4) as

$$C_{\mathrm{LR}}^{\mathrm{PS}}(y) = r^{\mathrm{St}}\check{k}_{\mathrm{St}}(y) + r^{\mathrm{Co}}\check{k}_{\mathrm{Co}}(y).$$

With the two inputs aggregated into the numeraire, the production set is

$$Y_{\mathrm{PS}}^{\mathrm{Ag}} = \left\{ (y; 0, -m) \in L^\infty[0,1] \times \mathbb{R}^2 : C_{\mathrm{LR}}^{\mathrm{PS}}(y) \leq m \text{ and } \int_0^1 y(t)\,\mathrm{d}t = 0 \right\}.$$

Theorem 9.2. *The electricity pricing model with storage has a long-run competitive equilibrium. Furthermore, if an equilibrium tariff* $p^\star \in L_+^{\infty *}[0,1]$ *supports (together with some price* $\rho^\star \in \mathbb{R}_+$ *for the other produced good) an equilibrium allocation with a nonzero electricity output* $y_{\mathrm{Th}}^\star + y_{\mathrm{PS}}^\star$ *(from thermal generation and pumped storage), then* $p^\star \in L_+^1[0,1]$.

Proof. An equilibrium price system with $p^\star \in L_+^{\infty *}$ exists by [3, Theorem 1]. (The Adequacy Assumption always continues to hold after an addition to the technology. The Boundedness Assumption holds for the reasons given in the Proof of Theorem 9.1 together with the fact that $C_{\mathrm{LR}}^{\mathrm{PS}}(y) > 0$ for every $y \neq 0$.)

To show that $p^\star \in L^1$, the Proof of Theorem 9.1 is extended as follows: the production set $Y_{\mathrm{PS}}^{\mathrm{Ag}}$ meets the Exclusion Condition by Example 5.4 and Remark 9.3 below. Also, $y_{\mathrm{Th}}^\star + y_{\mathrm{PS}}^\star \neq 0$ implies that $y_{\mathrm{Th}}^\star + y_{\mathrm{PS}}^\star > 0$ (since $y_{\mathrm{Th}}^\star + y_{\mathrm{PS}}^\star > 0$ by market clearance), and it follows that $y_{\mathrm{Th}}^\star > 0$ (since $y_{\mathrm{Th}}^\star \geq 0$ and $\int_0^1 y_{\mathrm{PS}}^\star(t)\,\mathrm{d}t = 0$). So the previous argument applies from (9.8) on, with $y_{\mathrm{Th}}^\star$ in place of $y^\star$. This shows that $p^\star \in L^1$, and also that meas $S_0(y_{\mathrm{Th}}^\star) > 0$.[18] ∎

Remark 9.3. *Given a production set* $Y \subset L^\infty(T) \times \mathbb{R}_-^G$ *and input prices* $r = (r^1, \ldots, r^G) \geq 0$, *define the cost function*

$$C(y) := \inf_k \{rk : (y, -k) \in Y\},$$

and assume that the infimum is attained (except when $C(y) = +\infty$ *because the section of* Y *through* y *is empty). If* Y *meets the Exclusion Condition, then so does the set*

$$Y^{\mathrm{Ag}} := \{(y, -m) \in L^\infty \times \mathbb{R}_- : C(y) \leq m\}.$$

[18] It can also be shown that meas $S_0(y_{\mathrm{PS}}^\star) > 0$ (although this is less obvious because the conversion capacity can recover some of its cost at times other than $S_0(y_{\mathrm{PS}}^\star)$). What is more, meas $(S_0(y_{\mathrm{Th}}^\star) \cap S_0(y_{\mathrm{PS}}^\star)) > 0$; i.e., the storage output also has a peak plateau, which overlaps with the thermal peak plateau (to form the supply system's peak plateau).

Proof. Take any $p \in L^{\infty *}(T)$ and $(y, -m) \in Y^{\mathrm{Ag}}$. By assumption, $m - \delta = C(y) = rk$ for some $\delta \in \mathbb{R}_+$ and some k such that $(y, -k) \in Y$. So, by the Exclusion Condition on Y, there exists a sequence $(y^n, -k^n) \in Y$ with: $y^n \to y$ in $\mathrm{m}(L^\infty, L^1)$, $k^n \to k$ and $\langle p_{\mathrm{FA}}, y^n \rangle \to 0$ as $n \to \infty$. Then $C(y^n) \le rk^n$, i.e., $(y^n, -rk^n) \in Y^{\mathrm{Ag}}$; and *a fortiori* $C(y^n) \le rk^n + \delta$, i.e., $(y^n, -rk^n - \delta) \in Y^{\mathrm{Ag}}$. To establish that this sequence has all the properties required in the Exclusion Condition on Y^{Ag}, it suffices to note that $rk^n + \delta \to rk + \delta = m$. ∎

For the short-run (SR) equilibrium, the price density representation can be deduced from the LR result (by representing a SR equilibrium as a LR equilibrium for suitably chosen prices of the fixed inputs). A direct analysis of the SR model is also workable; and for this approach it is useful to note that, if the Exclusion Condition has been verified for a LR production cone Y, the same follows for the SR production sets (which are sections of Y through the fixed-input bundles).

Lemma 9.4. *Assume that Y is a cone in $L^\infty(T) \times \mathbb{R}^\Phi_- \times \mathbb{R}^\Sigma$ such that the set $\{k \in \mathbb{R}^\Phi_+ : (y, -k, -v) \in Y\}$ is comprehensive upwards,*[19] *for any $y \in L^\infty(T)$ and $v \in \mathbb{R}^\Sigma$ (where Φ and Σ are finite sets); and given any $k \in \mathbb{R}^\Phi_{++}$, define*

$$Y(-k) := \{(y, -v) \in L^\infty(T) \times \mathbb{R}^\Sigma : (y, -k, -v) \in \mathbb{Y}\}$$

(which is a SR production set when Φ and Σ are interpreted as the fixed and the variable inputs). If Y meets the Exclusion Condition, then so does $Y(-k)$.

Proof. Take any $p \in L^{\infty *}(T)$ and $(y, -v) \in Y(-k)$. By the Exclusion Condition on Y, there exists a sequence $(y^n, -k^n, -v^n) \in Y$ with: $y^n \to y$ in $\mathrm{m}(L^\infty, L^1)$, $k^n \to k$, $v^n \to v$ and $\langle p_{\mathrm{FA}}, y^n \rangle \to 0$ as $n \to \infty$. Since $k^n \in \mathbb{R}^\Phi_{++}$ for every large enough n, the scalar sequence $\alpha^n := \min_{\phi \in \Phi} (k_\phi / k^n_\phi) > 0$ is well defined; and $\alpha^n \to 1$ as $n \to \infty$. Since

$$(\alpha^n y^n, -k, -\alpha^n v^n) \le (y^n, -\alpha^n k^n, -v^n) \in Y,$$

$\alpha^n (y^n, -v^n)$ is a sequence in $Y(-k)$, and it has all the required properties. ∎

10. **Conclusions**

When $L^\infty[0,1]$ is a suitable commodity space for a continuous-time flow, the assumption on the production sets which is needed for an equilibrium price to take the form of a density function in $L^1[0,1]$ can be weakened to be no more restrictive than the Mackey continuity assumption on consumer preferences. This permits the inclusion of some industrial users of the flow in the model. Its application to peak-load pricing settles Boiteux's conjecture on the shifting-peak problem.

[19]That is, if $k' \le k''$ and $(y, -k', -v) \in Y$, then $(y, -k'', -v) \in Y$.

References

[1] Back, K. (1984): "Existence of equilibria in economies with subsistence requirements and infinitely many commodities", Discussion Paper No. 633, Center for Mathematical Studies in Economics and Management Science, Northwestern University, Evanston, IL.

[2] Back, K. (1988): "Structure of consumption sets and existence of equilibria in infinite-dimensional spaces", *Journal of Mathematical Economics*, 17, 89–99.

[3] Bewley, T. (1972): "Existence of equilibria in economies with infinitely many commodities", *Journal of Economic Theory*, 4, 514–540.

[4] Boiteux, M. (1964): "Peak-load pricing", in *Marginal cost pricing in practice* (Chapter 4), ed. by J. R. Nelson. Engelwood Cliffs, NJ: Prentice Hall. (A translation of "La tarification des demandes en pointe: application de la theorie de la vente au cout marginal", *Revue General de l'Electricite*, 58 (1949), 321–340.)

[5] Debreu, G. (1962): "New concepts and techniques for equilibrium analysis", *International Economic Review*, 3, 257–273.

[6] Dubovitskii, A. Ya., and A. A. Milutin (1965): "Extremum problems in the presence of restrictions", *USSR Computational Mathematics and Mathematical Physics*, 5, 1–80.

[7] Dunford, N., and J. T. Schwartz (1958): *Linear operators, Part I: General theory*. New York: Interscience.

[8] Horsley, A., and A. J. Wrobel (1989): "The existence of an equilibrium density for marginal cost prices, and a solution to the shifting-peak problem", STICERD Discussion Paper TE/89/186, LSE; CentER Discussion Paper 9012, Tilburg University.

[9] Horsley, A., and A. J. Wrobel (1991): "The closedness of the free-disposal hull of a production set", *Economic Theory*, 1, 386–391.

[10] Horsley, A. and A. J. Wrobel (1992): "Continuity of demand and the direct approach to equilibrium existence in dual Banach commodity spaces", STICERD Discussion Paper TE/92/246, LSE.

[11] Horsley, A., and A. J. Wrobel (1993): "Uninterruptible consumption, concentrated charges, and equilibrium in the commodity space of continuous functions", STICERD Discussion Paper TE/96/300, LSE (presented at ESEM93, Uppsala).

[12] Horsley, A., and A. J. Wrobel (1996): "Efficiency rents of storage plants in peak-load pricing, I: pumped storage", STICERD Discussion Paper TE/96/301, LSE.

[13] Horsley, A., T. Van Zandt, and A. J. Wrobel (1998): "Berge's Maximum Theorem with two topologies on the action set", *Economics Letters*, 61, 285–291.

[14] Horsley, A., and A. J. Wrobel (1999 or 2000): "Localisation of continuity to bounded sets for nonmetrisable vector topologies and its applications to economic equilibrium theory", to appear in *Indagationes Mathematicae* (Nederl. Akad. Wetensch. Proc. Ser. A).

[15] Klein, E., and A. C. Thompson (1984): *Theory of correspondences*. New York-Chichester-Brisbane-Toronto: Wiley.

[16] Richard, S. (1989): "A new approach to production equilibria in vector lattices", *Journal of Mathematical Economics*, 18, 41–56.

[17] Yosida, K., and E. Hewitt (1952): "Finitely additive measures", *Transactions of the American Mathematical Society*, 72, 46–66.

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