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Abstract

The problem of efficiency versus fairness is considered in relation to the splitting of costs for shared facilities between the users. This is considered as a result of a problem of sharing the cost of the provision of central computing facilities between different faculties in a large university, but the basic problem is widespread. A Linear Programming model is considered in order to minimise cost. The dual of this model is shown to correspond to an efficient allocation of costs. An alternative optimal dual solution is shown to give a 'fair' solution according to criteria resulting from cooperative game theory.

Keywords: *Cost Allocation, Efficiency, Fairness, Linear Programming, Duality, Game Theory*

1. INTRODUCTION

When different consumers share the use of common facilities the problem of how much to charge each consumer for its share of the cost of each facility arises. While it is reasonable that each consumer be charged the *marginal* cost of its usage it is not clear how much of the fixed cost it should be charged. There is no unambiguous answer. Notions of ‘fairness’ and ‘equity’ (themselves ambiguous, see, for example Rawls [1]) may suggest one division. Economic efficiency may suggest another. These aims are not, however, diametrically opposed as will be seen in subsequent sections. They are, however, clearly not the same.

The problem we are considering arose in the allocation of central computing facilities and services in a large university. Technological change over the last decade has resulted in decentralisation of computing power to personal computers, although there is still some need among specialist users for very powerful mainframes. However there are still arguments in favour of some centralisation of services and the provision of services. These considerations gave rise to some faculties in the university considering ‘going it alone’ with their own provision (possibly from private companies) or forming consortia with other faculties with similar needs and tendering for the provision of external services. The final solution was however, both on grounds of cost and control, to continue with one centrally provided service. This was argued to be the most *efficient* solution. However it was considered unacceptable for individual faculties, or groups of faculties, to pay more than they would by going it alone. This was the *fairness* criterion. The full problem is described in section 4. Some aspects and monetary figures have been changed to preserve confidentiality. The problem is, however, part of a more general problem which we will discuss.

The general problem of finding an efficient allocation of facilities to different ‘customers’ so as to maximise revenue or overall utility or to minimise cost can be formulated as a Linear (LP) or Integer (IP) Programming model. The dual of such a model gives an allocation of costs to the customers which can be regarded as the most ‘efficient’ (see, for example Williams [2, 3] and Biddle and Steinberg [4]).

For a ‘fair’ allocation we can use a model from Cooperative Game Theory. This model can itself be solved by Linear Programming. It is similar to the dual model mentioned above but has a different objective function.

Examples where problems of a similar nature arise, and have been studied, are:

Allocating the costs of railway stations to the lines that use them. (Rhys [5], also discussed in Williams [6]).

Allocating the fixed (set-up) costs of electricity generation to different classes of consumer (discussed in Williams [6]).

Allocating the costs of airport runways to the different airlines that use them. (Littlechild and Thompson [7]).

Allocating the cost of building a dam, to create a reservoir, among the different uses to which it might be put (hydro-electric generation, recreation, flood control etc.). This is discussed in relation to the Tennessee Valley Authority by Ransmeier [8].

Dividing Credit Card costs fairly (Thomas [9]).

The whole problem can be seen as a microcosm of a central problem of political organisation; reconciling fairness and equity of wealth with economic efficiency.

2. THE BASIC PROBLEM

We have a set of Facilities **F = {1,2,..,m}** serving a set of 'customers' **C = {1,2,..,n}.**

Customer **j** requires one of the facilities in $\boldsymbol{F}_j^k \subset \mathbf{F}$ for each $\mathbf{k} \ \varepsilon \ \mathbf{K}_j$**,** ie customer **j** requires at least one of $\boldsymbol{F}_j^1$, **and** at least one of $\boldsymbol{F}_j^2$ etc.

The fixed cost of $\mathbf{i} \ \varepsilon \ \mathbf{F}$ is $\mathbf{f}_i$.

Customer **j** produces a benefit (revenue) $\mathbf{b}_j$.

It may not be desirable to cater for all the customers or provide all the facilities.

F could be a set of railway stations, electric generators, airport runways of different lengths, dams of possible heights, factories, swimming pools or cash card dispensers. Corresponding to each of these **C** would be a set of railway lines, electricity consumers, aircraft types, reservoir users, domestic consumers, swimming pool users or brands of credit card.

If a facility **i** is provided its cost $\mathbf{f}_i$ must be split up among the customers **j** that use it in an acceptable or desirable way.

3. AN OPTIMISATION MODEL

Primal Model

We formulate a 0-1 Integer Programming model.

Variables

δ_i = 1 if facility **i** is provided
= 0 otherwise

γ_j = 1 if customer **j** is catered for
= 0 otherwise

Objective

$$\textbf{Maximise} \quad \sum_j \mathbf{b}_j \gamma_j - \sum_i \mathbf{f}_i \delta_i \qquad (1)$$

Constraints

$$\gamma_j - \sum_{i \varepsilon F_j^k} \delta_i \leq \mathbf{0} \quad \text{all j } \varepsilon \text{ C, all k } \varepsilon \text{ } K_j \qquad (2)$$

$$\gamma_j \leq \mathbf{1} \quad \text{all j } \varepsilon \text{ C} \qquad (3)$$

$$\delta_i \geq \mathbf{0} \quad \text{all i } \varepsilon \text{ F} \qquad (4)$$

We refer to the above model as **P.**

Constraints (2) force at least one of F_j^k to be provided, for each **j** and **k**, if customer **j** is to be served. It is not necessary to impose non-negativity conditions on the γ variables or append upper bounds of 1 on the δ variables. These conditions are guaranteed by the structure of the model.

There is no guarantee that this general model will produce integer solutions if solved as a Linear Programme. In certain special cases, however, integral solutions to the LP are guaranteed. For example if for all **j** and **k** $|F_j^k| = 1$ then this is the case. This is discussed in, for example, Williams [6] and illustrated there by the example of Rhys [5]. In that case a railway line requires exactly two terminal stations. Integrality is also guaranteed (trivially) when there are no distinct benefits **b**, only costs, the costs are *subadditive* ie sharing of facilities will not produce an increase in cost, but usually a decrease, and there is a facility which can serve all customers. Such a condition often applies in practice and is the case in the example which we consider in section 4 of this paper.

When the LP solution is *integer* then there is a well defined *dual* LP model (see eg Dantzig [10]). If this is not the case then there are a number of possible duals, all lacking some of the properties of the LP dual. For convenience of exposition we postpone discussion of this issue. It is covered by, for example, Williams [2].

We consider the dual of the LP relaxation of the above model. (For convenience we have reversed the direction of some of the constraints in the formal dual model)

Dual Model (of LP Relaxation)

$$\textbf{Minimise} \qquad \sum_{j \varepsilon C} \mathbf{u}_j \tag{5}$$

$$\textbf{Subject to:} \qquad \mathbf{u}_j + \sum_{k \varepsilon K_j} \mathbf{v}_j^k = \mathbf{b}_j \qquad \text{all } j \; \varepsilon \; C \tag{6}$$

$$\sum_{\substack{j:\, j \varepsilon F_i^k \\ k \varepsilon K_j}} \mathbf{v}_j^k \le \mathbf{f}_i \qquad \text{all } i \; \varepsilon \; F \tag{7}$$

$$\mathbf{u}_j,\ \mathbf{v}_j^k \ge \mathbf{0} \qquad \text{all } j \; \varepsilon \; C,\ k \; \varepsilon \; K_j \tag{8}$$

We refer to the above model as **D.**
$\mathbf{v}_j^k$ can be interpreted as the portion of the fixed cost $\mathbf{f}_i$ of each of the facilities **i** (for $\mathbf{k} \, \varepsilon \, \mathbf{K}_j$) that **j** requires. In the case that a customer is not catered for the $\mathbf{v}_j^k$ can be ignored.

$\mathbf{u}_j$ can be interpreted as the excess benefit (revenue) which **j** obtains after contributing all the required costs. The objective is to minimise the total excess.

Constraints (7) split the cost $\mathbf{f}_i$ between the customers using the facility. Should the full cost not be met (the constraint is non-binding) the orthogonality result of LP guarantees that $\delta_i = 0$, ie that the facility not be built. Hence there is no question of customers contributing more, in total, than they would if they were sharing an alternative facility. This is one of the conditions of 'fairness' discussed in the next section.

Constraints (6) split the benefits to customers between the imputed costs and excesses.
Generally there will be a number of alternate optimal dual solutions. Among these alternate solutions will be, a solution that could be regarded as, the *fairest* solution, in a sense to be defined in section 5. That the 'fairest' solution is among those cost

allocations that are consistent with the most *efficient* solution is of significance. This is proved in section 5 and illustrated by the example in section 6.

4. THE ALLOCATION OF COMPUTING FACILITIES

This problem arose in a large university where decisions had to be made about the provision of academic computing facilities. The university has nine faculties:

Veterinary Science
Medicine
Architecture
Engineering
Arts
Commerce
Agriculture
Science
Social Science

The university wanted to retain one computer services department providing support, servicing individual personal computers and laboratories as well as networking and providing specialist high powered computing and software. It also turned out that this was the most cost effective solution for the university as a whole. Individual faculties, however, argued, with some justification, that they could sometimes obtain these services more cheaply using other suppliers. Also certain faculties had similar needs and could form consortia to get even better deals.

The yearly fixed costs for the individual faculties ‘going it alone’ (ie seeking outside provision) were estimated to be (in £100 000).

Veterinary Science	6
Medicine	7
Architecture	2
Engineering	10
Arts	18
Commerce	30
Agriculture	11
Science	29
Social Science	7

The possible consortia with their yearly estimated fixed costs of obtaining independent provision were:

1.	(Veterinary Science, Medicine)	11
2.	(Architecture, Engineering)	14
3.	(Arts, Social Science)	22
4.	(Agriculture, Science)	37
5.	(Veterinary Science, Medicine, Agriculture, Science)	46
6.	(Arts, Commerce, Social Science)	50

Central provision (consortium 7) for all faculties would have a yearly fixed cost of 96.

No individual ‘benefits’ (over and above cost savings) to faculties from going it alone or joining consortia were quantified.

The model **P** can be simplified, for this example, by setting all the γ variables to 1. There are no **b** coefficients and $|\mathbf{K}_j| = 1$ for all j, making such sets (and indices **k**) superfluous.

It becomes

Minimise $$\sum_i \mathbf{f}_i \delta_i \qquad (9)$$

Subject to:

$$\sum_{i \varepsilon F_j} \delta_i \geq \mathbf{1} \quad \text{all } j \; \varepsilon \; C \text{ (Faculties)} \qquad (10)$$

$$\delta_i \geq \mathbf{0} \quad \text{all } i \; \varepsilon \; F \text{ (Individual provisions and consortia)} \qquad (11)$$

It is clear from the above figures that the most efficient solution, from the university’s point of view, was to ‘force’ every faculty to use the central provision, resulting in the *integer* solution to the (LP) model where only the δ variable corresponding to this provision is 1.

Of interest, however, is the corresponding *dual* model. **D**.

Maximise $$\sum_{j \varepsilon C} \mathbf{v}_j \qquad (12)$$

Subject to: $$\sum_{j: i \varepsilon F_j} \mathbf{v}_j \leq \mathbf{f}_i \quad \text{all } i \; \varepsilon \; F \qquad (13)$$

$$\mathbf{v}_j \geq \mathbf{0} \quad \text{all } j \; \varepsilon \; C \qquad (14)$$

Interpreting the dual variables $\mathbf{v}_j$ as the allocation of part of the cost of the central provision to faculty **j,** we see that the constraints guarantee that no faculty, or consortium of faculties, is charged more, in total, than they would by going it alone. This is one (but not the only) requirement of fairness. The objective is to maximise the total cost allocation. *An* optimal solution to this dual model is to make the following allocation of the cost of central provision:

Faculty	Cost Allocation	Saving from individual provision
Veterinary Science	6	0
Medicine	3	4
Architecture	2	0
Engineering	0	10
Arts	11	7
Commerce	30	0
Agriculture	8	3
Science	29	0
Social Science	7	0

Consortium	Savings
(Veterinary Science, Medicine)	2
(Architecture, Engineering)	9
(Arts, Social Science)	7
(Agriculture, Science)	0
(Veterinary Science, Medicine, Agriculture, Science)	0
(Arts, Commerce, Social Science)	5

Not surprisingly (as a result of the duality theorem of LP) the optimal solution exactly meets the cost (96) of the (optimal) central provision (the corresponding constraint (13) is binding). It can be observed that no faculty or consortium is charged more than would be the case with its alternative provision. It also follows from the duality theorem of LP that since the primal model is feasible and not unbounded there will be a feasible dual solution ie there will be an allocation of costs which is ‘fair’ in the above, restricted, sense. There are, however, many alternate dual solutions some of which might be ‘fairer’. The existence of alternate dual solutions would be expected from a model with this structure (the primal model exhibits ‘degeneracy’).

We now explore the issue of ‘fairness’ and demonstrate that among the alternate cost allocations we can find a fairer one.

Such a fairer one will, however, almost certainly not be a ‘basic’ one (in the terminology of LP).

5. A FAIRNESS MODEL

The issue of fairness in cost allocation has been explored by Shubik [12], and in the book edited by Peyton Young [13], using concepts from Cooperative Game Theory. For an allocation to be ‘fair’ it must lie in the ‘core’. This amounts to satisfying the constraints of the dual model above. Such an allocation would give players in a cooperative game an incentive to join a coalition as their overall payoff (negated cost) would be improved. Within the core there will generally be at least one, and normally many, solutions. A number of criteria exist for choosing a solution from the core. The most popular is probably the Shapley Value (see Shubik [12]). We, alternatively, use the *nucleolus*. This is an attempt to find the solution which is most 'central' to the core (see eg Young [13]) in the sense that the savings to the 'customers' from not using alternative coalitions (aggregated over the coalitions) is as nearly as possible equal.

To make this concept precise we *maximise the minimum saving* over each coalition. We also include all the possible fictional coalitions, whose costs can be regarded as the minimal sum of the costs of any coalitions and the alternative individual provision of the members of the coalition. If there are alternatives then we minimise the number of coalitions with this saving, repeating the procedure with the coalition with next smallest saving and so on (giving what is known as the *lexicographic* minimum).

Since a *fair* solution is not something about which there is ever likely to be consensus we anticipate objections which might be raised to the solution given below. A variety of fairness criteria are discussed in the seminal paper of Yaari and Bar-Hillel [13]. They conducted a series of experiments to see which of nine possible criteria were considered most fair by a sample of people questioned. In relation to *needs* (as opposed to the more subjective qualities of *tastes* and *beliefs)* an allocation based on minimising the maximum inequality was overwhelmingly considered the most fair. This accords with the criterion we are using here.

Formally, relating this to the dual model in section 3, we introduce *slack* variables **s** into the constraints (7) and create a new objective to give the model

$$\textbf{Maximise}\left(\underset{\mathbf{i}}{\textbf{Minimum}}\ \mathbf{s}_i\right) \qquad (16)$$

$$\textbf{Subject to: } \mathbf{u}_j + \sum_{k \varepsilon K_j} \mathbf{v}_j^k = \mathbf{b}_j \quad \text{all } j \ \varepsilon\ C \qquad (17)$$

$$\sum_{\substack{j:\, i\, \varepsilon\, F_j^k \\ k \varepsilon K_j}} \mathbf{v}_j^k + \mathbf{s}_i = \mathbf{f}_i \quad \text{all } i \ \varepsilon\ F \qquad (18)$$

$$\mathbf{u}_j,\ \mathbf{v}_j^k \geq \mathbf{0} \quad \text{all } j \ \varepsilon\ C,\ k \ \varepsilon\ K_j \qquad (19)$$

In the dual model, in section 3, it was not necessary to create constraints for the fictional coalitions since they must be satisfied automatically and therefore be redundant.

The objective can be dealt with by introducing a variable **z** and constraints

$$\mathbf{z} \leq \mathbf{s}_i \quad \text{all } i \ \varepsilon\ \mathbf{F} \qquad (20)$$

The objective then becomes

Maximise $\quad$ **z**

We refer to the above model as **F.**

A variant, which could be considered fairer, is to seek the *per capita nucleolis*. To do this we inversely weight the slack variables by the size of each coalition to which they correspond.

It is interesting to note that whereas the dual model to **D** is obviously **P** (reflexivity of the duality relation), even when **P** gives an integral solution, the dual model to **F** will, in general, not give an integral solution and has no obvious interpretation. This means that obtaining a 'fair' allocation of costs has to be modelled directly and does not emerge from the dual values of any obvious facility location model.

When we have found an optimal solution to **F** we will need to fix the objective at the optimal value and further optimise on sections of the model to find the lexicographic minimum.

Theorem

An optimal solution to **F** will also be an optimal solution to **D.**

Proof:

Suppose we have an optimal solution $\mathbf{v}_j^*$ to **F** which is not an optimal solution $\mathbf{v}_j^+$ to **D**

ie $\quad \sum_j \mathbf{v}_j^* \; < \; \sum_j \mathbf{v}_j^+ \qquad (21)$

Consider each of those constraints (18) in which the maximum $\mathbf{s}_i$ is attained. Each of these constraints must contain at least one variable $\mathbf{v}_j^*$ which occurs only in constraints involving a subset of constraints (7) which are not binding. Otherwise maximising (12) over all the constraints (18) containing this variable would contradict the subadditivity condition on the $\mathbf{f}_i$.

It is therefore possible to increase the value of each of these variables reducing the values of the maximal $\mathbf{s}_i$ and contradicting our hypothesis.

This result shows that the seeking of a *fair* solution is not at variance with the seeking of an *optimal* solution. Rather, among the different cost allocations which are consistent with the optimal solution, we must seek that one which minimises the savings in the manner described above. We illustrate this by means of our example.

6. A FAIR ALLOCATION OF THE COSTS OF COMPUTER PROVISION

We specialise model **F** to the example.
This gives the following cost allocation:

Faculty	**Cost Allocation**	**Savings from own Provision**
Veterinary Science	4	2
Medicine	1	6
Architecture	0	2
Engineering	8	2
Arts	15	3
Commerce	28	2
Agriculture	8	3
Science	27	2
Social Science	5	2

Consortium	**Savings**
(Veterinary Science, Medicine)	6
(Architecture, Engineering)	6
(Arts, Social Science)	2
(Agriculture, Science)	2
(Veterinary Science, Medicine, Agriculture, Science)	6
(Arts, Commerce, Social Science)	2

Savings for the fictional coalitions are not given.

Note that the total cost allocation is again 96 (the cost of central provision) and that no faculty or consortium would pay more than it would by alternative provision. What is more the allocation of costs is more equitable in that there is less disparity in savings.

If we weight the savings inversely by the sizes of the coalitions (the per capita nucleolis) we obtain the following allocation:

Faculty	**Cost Allocation**	**Savings from own Provision**
Veterinary Science	1.83	4.17
Medicine	5	2
Architecture	0	2
Engineering	7.5	2.5
Arts	16	2
Commerce	24.67	5.33
Agriculture	9	2
Science	27	2
Social Science	5	2

Consortium	Savings
(Veterinary Science, Medicine)	4.17
(Architecture, Engineering)	6.5
(Arts, Social Science)	1
(Agriculture, Science)	1
(Veterinary Science, Medicine, Agriculture, Science)	3.17
(Arts, Commerce, Social Science)	4.33

7. CONCLUSIONS AND EXTENSIONS

We demonstrate that, in certain problems, considerations of efficiency (optimality) and fairness in the allocation of the costs of shared facilities are not necessarily opposed. However a 'fair' allocation may be only one of a number of efficient allocations. We are not asserting that the solutions produced are 'fair' in any absolute sense but hoping that they help to clarify the concept.

This result arises when the Linear Programming formulation of the facility allocation problem has an integer solution, and therefore a well-defined dual. When this is not the case there are a number of alternative ways of defining a dual with more limited properties. The value of these duals in providing a satisfactory allocation of costs will form the subject of another paper.

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