On Root's Barriers and Their Applications in Robust Pricing and Hedging of Variance Options

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Abstract

The variance option has recently drawn great attention in financial research because it provides an investor with protection on exposure to volatility risk. In this paper I will reexamine the result of Cox and Wang [3] which has made a remarkable contribution to this area. They explored the Root's barrier in terms of seeking the solution of a variational inequality and constructed subhedging strategies for variance options through an alternative proof of the optimal property of Root's barrier. In order to give a brief review of their work I instantiate their results by conducting both numerical implementation and theoretical calculation. To backup my result, I backtest my numerical solution and compare the subhedging strategies proposed in Cox and Wang [3] with the subhedging strategies separately suggested in Carr and Lee [17]. The comparison will well explain and demonstrate the generalization done in Cox and Wang [3].

Keywords: variance option, Root's barrier, Skorokhod embedding, obstacle problem, subhedging

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Chapter 1

Introduction

In this paper, I mainly go through the results in Cox and Wang [3] on their extension on the financial application of Root's barrier and the construction of subhedging strategies. The motivation comes from the recent increasing demand on the variance swaps and variance options to control over exposure to volatility risk. Since the market for such products is not well established, hedging and pricing of such products is academically done in a model-independent fashion to avoid risks of model misspecification. However, model-independent approaches usually can not lead us to the exact price of a variance option. What is done is to loosen the hedging problem to the problem of exploring the model-independent upper and lower bounds on prices by constructing superhedging and subhedging strategies. By narrowing the gap between the upper bound and the lower bound for a variance option, we can obtain an close approximation to the price of the variance option. Cox and Wang [3]'s work is mainly on the construction of an optimal lower bound for a variance option, same with them, I will only focus on the lower bound and the subhedging strategy of a variance call in this paper. Academically, the solution of this problem is connected to the well-known Skorokhod embedding problem (SEP) or sometimes called the Skorokhod stopping problem.

To further illustrate this issue in a mathematical manner, first consider a simplified risk-neutral environment with zero interest rate and assume there is no arbitrage in the market. The stock price process $S_t (0 < t < T)$ satisfies the following SDE with an unspecified σ_t :

$$dS_t = \sigma_t S_t dW_t \tag{1.1}$$

The log of stock price process then satisfies:

$$d\ln(S_t) = -\frac{1}{2}\sigma_t^2 dt + \sigma_t dW_t \tag{1.2}$$

By integrating both sides of (1.2), we have:

$$\langle \ln S \rangle_T = \int_0^T \sigma_t^2 dt \tag{1.3}$$

It means that the payoff of a variance swap or a variance option can be seen as a function of the realized logarithmic quadratic variation of the stock price process at maturity T . After writing the payoff in form of $F(\langle \ln S \rangle_T)$, the variance option with strike K is then a call or a put with $F(x) = (x - K)^+$ or $F(x) = (K - x)^+$ and the variance swap has $F(x) = x - K$.

The next step is to adopt a continuous time change τ_t to the stock price evolution, which originates from Dubins and Schwarz [14]. By Dubins-Schwarz theorem, take $\tau_t = \langle \ln S \rangle_t$ and letting $X_{\tau_t} = S_t$, we have:

$$d\tau_t = \sigma_t^2 dt \tag{1.4}$$

SDE (1.4) means that the law of $(S_T, \langle \ln S \rangle_T)$ is just equivalent to the law of (X_{τ_T}, τ_T). Therefore finding the lower bound of a variance call will be equivalent to finding an optimal τ_T to minimize $E(\tau_T - K)^+$ subject to the condition that the law of X_{τ_T} is equal to the law of S_T. The law of S_T, demoted by μ, or the risk neutral probability density of S_T, can be easily obtained through call prices. By taking the second derivative with respect to the strike price of call options on S_T, we are able to get a knowledge of the distribution of S_T provided that we have prices of call options on all strikes. This problem ensembles SEP. The main issue of SEP is to find to minimal stopping time τ with a given underlying stochastic process X_t and a given probability measure μ such that X_τ obeys the measure μ.

Root [9] has shown that when X_t is a Brownian motion with $X_0 = 0$ and μ being a centered law, the solution of SEP is the first hitting time of a barrier $B(\mu)$. This barrier B, which is called the Root's barrier, is defined as a set (x,t) in $R_+ \times R$ such that if $(x,t) \in B$ then $(x,s) \in B$ for every $s > t$. Precisely, Root's barrier is defined as follows:

"A closed subset B of $[-\infty, +\infty] \times [0, +\infty]$ is a barrier if

1) $(x, +\infty) \in B$ for all $x \in [-\infty, +\infty]$;
2) $(-\infty, t) \in B$ for all $t \in [0, +\infty]$;
3) if $(x,t) \in B$ then $(x,s) \in B$ whenever $s > t$. "

Root [9]'s result is later greatly extended by Rost [10] [11] to loosen the restriction on the underlying stochastic process X_t. Rost's result only requires X_t being a Markov process and right continuous. Furthermore, he proved the existence and uniqueness of a stopping time of minimal residual expectation to the Root's barrier. By saying a stopping time is of minimal residual expectation we mean that:

"A stopping time τ is of minimal residual expectation if for each $t \in \mathbb{R}^+$, it minimizes the following quantity:

$$E(\tau - t)^+ = E \int_{\tau \wedge t}^{\tau} ds = \int_t^{\infty} P(\tau > s) ds$$

over all $\tau \in \{\tau : X_\tau \sim \mu\}$ where we assume $X_0 \sim \nu$ and $X_\tau \sim \mu$. "

The result in Rost [10] [11] requires certain restrictions on both ν and μ. Literally, the condition requires that for all $x \in \mathbb{R}$ the expectation of -$|y - x|$ with $y \sim \nu$ is greater than or equal to the expectation of -$|y - x|$ with $y \sim \mu$ and both of the expectations must be strictly less then infinity. Mathematically, for all $x \in \mathbb{R}$:

$$\infty > U_\nu(x) \triangleq -\int_{\mathbb{R}} |y - x| \nu(y) dy \le -\int_{\mathbb{R}} |y - x| \mu(y) dy \triangleq U_\mu(x) \tag{1.5}$$

Cox and Wang [3] makes further contribution to the solution of SEP based on the work of Root [9] and Rost [10] [11]. They represented and proved an alternative method to solve the

Root's barrier problem with a generalized underlying stochastic process. The alternative is basically an obstacle approach but they further applied the popular variational inequalities approach to generalize the basic obstacle problem. One of advantages of transforming the SEP to an obstacle problem is that the transformed problem enables easier and straightforward numerical implementation, especially with a finite difference method. Another important result of Cox and Wang [3] is their construction of a dynamic subhedging strategy based on the optimal characterization of Root's barrier. These together intrigue me to make a brief review to the extension made in Cox and Wang [3] and illustrate some typical examples by instantiating their results.

To test the validity of Cox and Wang [3], I will backtest the subhedging strategies they proposed by applying the strategies to a variance swap and a variance call and make a comparison with the subhedging strategies suggested in Carr and Lee [17]. The biggest difference of Cox and Wang [3] and Carr and Lee [17] is that they look at the same problem from different heights. Generally, the relevant academic researches on variance options are divided into two groups. The first group investigates the usage of SEP and Root's barrier to study model independent hedging strategies while another group aims to construct model independent super-replication or sub-replication directly using European options and other financial products. Cox and Wang [3] is in the first group and the latest works on this aspect include Cox, Hobson and Obłój [2], Dupire [7] and Obłój [13]. This angle develops quickly in recent times because it enables subreplicating the variance option in very general settings. Carr and Lee [17] belongs to the second group. This group forms the basis for the construction of hedging strategies of variance options. Their work on this aspect mainly includes Neuberger [4] [5] and Carr and Lee [17]. However the work of the first group is actually a development and re-formulization of the second group. By discovering and proposing the important link of the Skorokhod embedding problem and the model-independent in their study by Dupire [6] and Carr and Lee [17], they facilitate the research of the first group.

The rest of this paper is organized as follows: Chapter 2 review the methodology and major results in Cox and Wang [3] related to Root's barrier and the process of constructing the subhedging strategies. In Chapter 3, I perform a numerical implementation to solve the obstacle problem with a Monte Carlo simulation to backtest the results. Chapter 4 is two instantiations of subhedging strategies in Cox and Wang [3] and comparison with results in Carr and Lee [17]. Chapter 5 concludes the paper.

In Chapter 2, I will summarize the main deducing steps and results in Carr and Lee [17]. For the first half of this chapter, the effort is on the alternative to SEP and for the latter half of this chapter how to build the subhedging strategy from Root's barrier is explained.

Chapter 3 is also composed by two parts. Section 3.1 shows the algorithm and results from numerically implementing the alternative to SEP. The target distribution includes uniform distribution, normal distribution, log-normal distribution, uniform distribution with atoms and exponential distribution. Section 3.2 shows the result by testing the barrier in Section 3.1 in Monte Carlo simulation.

In the forth chapter of this paper, I demonstrate in detail that the result of Carr and Lee [17], a latest representative paper in subreplication of variance options, is an instantiation of Root's barrier and Carr and Lee [17]'s subhedging strategy is included in the general strategy provided in

Cox and Wang [3]. The approach I adopt is to apply the hedging strategies of Carr and Lee [17] and Cox and Wang [3] in a variance swap and a variance call. The comparison is made and the difference is well explained in the last part of Chapter 4.

Chapter 2

Methodology and Results of Cox & Wang

The main contribution of Cox and Wang [3] is twofold. First, they proved a one-to-one correspondence of SEP and an obstacle problem (or strictly speaking a variational inequality) and secondly, they constructed subhedging strategies for variance options with the optimal property of Root's barrier. Therefore in this chapter, I will summaries their results in separate two sections.

2.1 Alternative of SEP on Solving Root's Barrier

Before moving forward to summarize the alternative method of SEP on solving Root's barrier, I restate the Root's barrier problem formally.

Generally, the stochastic process X_t under study can be written as:

$$dX_t = \sigma(X_t)dW_t \tag{2.1}$$

with W_t being a Brownian motion and X_0 following a certain law ν. Cox and Wang [3] makes certain restrictions on $\sigma : \mathbb{R} \to \mathbb{R}$:

"For some positive constant K,

$$|\sigma(x) - \sigma(y)| \le K|x - y|$$

$$0 < \sigma^2(x) < K(1 + x^2)$$

$$\sigma \text{ is smooth}$$ "

And we require for some strictly positive constant ε, $\sigma^2 > \varepsilon$.

As in a financial context, we are most interested in two major applications of $\sigma : \mathbb{R} \to \mathbb{R}$, namely $\sigma(x) = \sigma$ and $\sigma(x) = \sigma x$, which stand for the Brownian motion and geometric Brownian motion. As for $\sigma(x) = \sigma$, it is naturally included as the most trivial situation. For the case $\sigma(x) = \sigma x$, though excluded by the above restriction, Cox and Wang [3] (Section 4.6) employed a simple transformation by letting $v(x,t) = u(e^z, t)$ to ensure that all conclusions discussed in the rest of this paper can cover the case $\sigma(x) = \sigma x$.

With the above restrictions on $\sigma : \mathbb{R} \to \mathbb{R}$, the Root's Skorokhod embedding problem is a problem as follows (quoted from Cox and Wang [3]):

“**SEP** (σ, ν, μ): Find a lower semi-continuous function $R(x)$ such that the domain defined by $D = \{(x,t) : 0 < t < R(x)\}$ has $X_{\tau_D} \sim \mu$, and $(X_{t \wedge \tau_D})_{t \geq 0}$ is a uniformly integrable process, where ν is the initial law of X_t and σ the diffusion coefficient.”

where $R(x)$ is a lower semi-continuous function $R : \mathbb{R} \to [0, \infty]$ suggested by Loynes [19] to describe the Root’s barrier by rewriting B as $B = \{(x,t) : t > R(x)\}$.

As discussed before, Rost [11] and Loynes [19] have shown the existence and uniqueness of the Root’s barrier to SEP. However, attempt to solve the SEP directly is time-consuming, complex and only results in an explicit expression of the domain $D = \{(x,t) : 0 < t < R(x)\}$ in a few simple cases. Therefore, we hope that we could find an alternative method exactly equivalent to SEP but is more feasible and can be easily implemented in a numerical way. The alternative would be even better if it could provide a solution that is one-to-one corresponding to the domain $D = \{(x,t) : 0 < t < R(x)\}$ and has a very general setup.

The alternative suggested by Cox and Wang [3] meets all requirement mentioned above. It is basically an obstacle problem and they have shown that for every solution of the following obstacle problem, we can construct a domain D that solves the associated SEP. The description of the obstacle problem is from Cox and Wang [3]:

“**OBS** (σ, ν, μ): Find a function $u(x,t) \in \mathbb{C}^{1,1}(\mathbb{R} \times \mathbb{R}^+)$ such that

$$U_\nu(x) = u(x,0) \tag{2.2}$$

$$U_\mu(x) \leq u(x,t) \tag{2.3}$$

$$\frac{\partial u}{\partial t}(x,t) - \frac{1}{2}\sigma^2(x)\frac{\partial^2 u}{\partial x^2}(x,t) \leq 0 \tag{2.4}$$

$$(\frac{\partial u}{\partial t}(x,t) - \frac{1}{2}\sigma^2(x)\frac{\partial^2 u}{\partial x^2}(x,t))(U_\mu(x) - u(x,t)) = 0 \tag{2.5}$$

”

where $U_\nu(x)$ and $U_\mu(x)$ is the functions defined in (1.5).

Condition (2.4) can be understood as requiring the following in a probabilistic manner:

$$\int_{\mathbb{R}} (\phi(x)\frac{\partial u}{\partial t}(x,t) + \frac{1}{2}\sigma^2(x)\frac{\partial u}{\partial x}(x,t)\phi'(x))dx \leq 0 \tag{2.6}$$

And condition (2.5) indicates that whenever $U_\mu(x) < u(x,t)$:

$$\frac{\partial u}{\partial t}(x,t) - \frac{1}{2}\sigma^2(x)\frac{\partial^2 u}{\partial x^2}(x,t) = 0 \tag{2.7}$$

In other words, the function $u(x,t)$ will evolve with time according to the PDE (2.7) and will stop immediately whenever it hits the corresponding boundary specified by $U_\mu(x)$ for all $x \in \mathbb{R}$. Then we can define the domain D as $D \triangleq \{(x,t): U_{\mu}(x) < u(x,t), t>0\}$ based on Theorem3.1 in Cox and Wang [3]:

"Suppose D is a solution to **SEP** (σ, ν, μ) and is such that:

$$u(x,t) = -E\,|\,X_{t\wedge\tau_D} - x\,| \in \mathbb{C}^{1,1}(\mathbb{R}\times\mathbb{R}^+) \tag{2.8}$$

Then u solves **OBS** (σ, ν, μ)."

However there are two main drawbacks of this obstacle problem. First it restricts $u(x,t) \in \mathbb{C}^{1,1}(\mathbb{R}\times\mathbb{R}^+)$ to make sure that the second derivative of $u(x,t)$ is well defined. This obviously excludes many cases. In reality, we usually do not expect a very smooth law of S_T since it is unlikely to obtain prices of European calls for all strikes K. Even if we have access to this knowledge, the distribution of S_T derived from the prices of Europeans calls will not performs perfectly due to the unavoidable imperfection in the market. Therefore μ may contain atoms which makes $u(x,t)$ not first order continuous in x. Furthermore, Theorem3.1 in Cox and Wang [3] only ensures a one-way projection from the solution of SEP to OBS. It neither ensures the existence nor uniqueness of the solution of OBS. It prevents us from constructing a one-to-one correspondence between SEP and OBS.

The drawback of the obstacle problem leads Cox and Wang to reformulate the problem and further generalize the setup. They reformulate the obstacle problem through the popular approach called variational inequalities where existence and uniqueness of the solution are both ensured inherently. By coordinating the setup with SEP, the variational inequalities problem is described as follows (briefly quoted from Cox and Wang [3]):

"**VI** (σ, ν, μ): Find a function $v: \mathbb{R}\times[0,T]\to\mathbb{R}$ satisfying:

$$v \in L^{\infty}(0,T;H^{1,\lambda}) \text{ and } \frac{\partial v}{\partial t} \in L^{2}(0,T;H^{0,\lambda}) \tag{2.9}$$

$$(\frac{\partial v}{\partial t}, \omega - v)_{\lambda} + a_{\lambda}(t; v, \omega - v) \geq 0, \forall \omega \in H^{1,\lambda} \text{ such that } \omega \geq \psi(t) \textbf{ a.e. } t \in (0,T) \tag{2.10}$$

$$v(\cdot,t) \geq \psi(t) \textbf{ a.e. } t \in (0,T) \tag{2.11}$$

$$v(\cdot,0) = \overline{v} \tag{2.12}$$

"

where $H^{m,\lambda}$ and $L^2(0,T;H^{m,\lambda})$ are Hilbert spaces with natural inner products. And all the coefficients are given by the following definitions:

$$a_{\lambda}(t; v, \omega) = \int_{\mathbb{R}} e^{-2\lambda|x|}[a(x,t)\frac{\partial v}{\partial x}\frac{\partial \omega}{\partial t} + b(x,t)\frac{\partial v}{\partial x}\omega]dx \tag{2.13}$$

where $a(x,t)=\frac{\sigma^2(x)}{2}$ and $b(x,t)=\sigma(x)\sigma'(x)-\lambda\sigma^2(x)\operatorname{sgn}(x)$

$$\psi(x)=U_\mu(x) \tag{2.14}$$

$$\bar{v}=U_\nu(x) \tag{2.15}$$

Note that in order to ensure the existence and uniqueness of solution to **VI** (σ,ν,μ), some similar restrictions as those in obstacle problem are added to $\sigma:\mathbb{R}\to\mathbb{R}$. But as mentioned earlier, the results can cover $\sigma(x)=\sigma x$ with transformation of $v(x,t)=u(e^z,t)$. The formulation of **VI** (σ,ν,μ) then leads to the main result of Cox and Wang [3] Theorem 4.2, which ensures a one-to-one correspondence of SEP and VI:

"Let D and v be the solutions to **SEP** (σ,ν,μ) and **VI** (σ,ν,μ) respectively, define:

$$u(x,t)=-E^\nu\,|X_{t\wedge\tau_D}-x| \tag{2.16}$$

And

$$D^T\triangleq\{(x,t)\in\mathbb{R}\times[0,T]:v(x,t)>\psi(x)\} \tag{2.17}$$

then we have $D^T=D\cap\mathbb{R}\times[0,T]$ and for all $(x,t)\in\mathbb{R}\times[0,T]$,

$$u(x,t)=v(x,t) \tag{2.18}$$

"

The above theorem gives a perfect link between SEP and VI, which shifts the attempts to solve SEP to attempts to solve the system with VI. If μ is not too weird, implementation of **VI** (σ,ν,μ) or **OBS** (σ,ν,μ) will usually give out a nice approximation of Root's barrier. Note that our purpose of solving the SEP is to find the lower bound on the price of variance options. Hence here with the approximation of Root's barrier we could be able to do it. The connection of solution of SEP to the lower bound of variance call is described by the minimal residual expectation characterization of Root's barrier. Although Rost [11] has given a complete proof on the minimal residual expectation characterization, Cox and Wang [3] explored an alternative way to prove it in which they also found a submartingale G_t and a function $H(x)$ to form a subhedging strategy for variance calls.

2.2 The Subhedging Strategy

Suppose μ is a given centered distribution, X_t evolves as the process in (2.1) and $X_0\sim\nu$, the minimal residual expectation characterization of Root's barrier is equivalent to minimizing $E[F(\tau)]$ subject to $X_\tau\sim\mu$ where τ is a stopping time and function F is convex, increasing with $F(0)=0$. Denote its right derivative f. The subhedging strategy is based on definition of four functions, namely $M(x,t)$, $Z(x)$, $G(x,t)$ and $H(x)$.

$M(x,t)$, $Z(x)$, $G(x,t)$ and $H(x)$ are defined with a given domain D that solves a corresponding SEP:

$$M(x,t) = E^{(x,t)} f(\tau_D) \tag{2.19}$$

$$Z(x) = 2\int_0^x \int_0^y \frac{M(z,0)}{\sigma^2(z)} dz dy \tag{2.20}$$

$$G_t = G(X_t, t) \quad \text{and} \quad G(x,t) = \int_0^t M(x,s) ds - Z(x) \tag{2.21}$$

$$H(x) = \int_0^{R(x)} (f(s) - M(x,s)) dx + Z(x) \tag{2.22}$$

where τ_D is the first hitting time of the given domain D and $R(x)$ is the function used to describe the barrier.

There are three main results here given by Cox and Wang [3]:

(1) For all $(x,t) \in \mathbb{R} \times \mathbb{R}^+$, we have $G(x,t) + H(x) \leq F(t)$.

(2) $G(X_{t\wedge\tau_D}, t \wedge \tau_D)$ is a martingale and $G(X_t, t)$ is a sub-martingale if:

$$E[\int_0^T Z'(X_s)^2 \sigma^2(X_s) ds] < \infty \quad \text{and} \quad E[Z_0] < \infty$$

(3) $E[F(\tau_D)] \leq E[F(\tau)]$ where $X_\tau \sim \mu$

Proposition (1) and (2) together show that $G(x,t)$ plays the role of a dynamic subreplication of the variance option with payoff function F. If D is Root's barrier, this subreplicating strategy will provide an optimal lower bound for the price of the variance option. This lower bound is given by $E[F(\tau_D)]$ in Proposition (3). And Proposition (3) shows that it is indeed the lower bound and is optimal when equality in (3) holds.

With definitions in (2.19)-(2.22), the specific subhedging strategy is constructed through the following process assuming now we have a stock price path following an arbitrary stochastic process with initial price S_0 .The process can be similarly expressed as that in (2.1):

$$dS_t = \sigma(S_t) dW_t \tag{2.23}$$

And I assume that I have obtained the Root's barrier from the distribution of S_T implied by the call prices of all available strikes. Denote the corresponding domain of this Root's barrier as D.

As discussed before, the subhedging strategy in Cox and Wang [3] consists of finding a submartingale denoted by $G(S_t, t)$ and a function denoted by $H(S_t)$ to satisfy the inequality:

$$G(S_t, t) + H(S_t) \leq F(t) \tag{2.24}$$

where the equality holds when $t = \tau_D$. Here τ_D denotes the first hitting time of the obtained Root's barrier. After obtaining $G(S_t, t)$ and $H(S_t)$, it is able to build up a sub-replication portfolio separately subhedges $G(S_t, t)$ and fully hedges $H(S_t)$.

The process begins with a time change which is similar to that in Section Introduction but is more general:

$$X_{\langle \ln(S) \rangle_t} = S_t$$

which enables us to rewrite (2.23) as:

$$dX_t = X_t d\tilde{W}_t \tag{2.25}$$

where $\tilde{W}_t$ is a Brownian motion under the new filtration. This time change is based on theorem of Dubins and Schwarz [14]. By Ito's lemma, we have:

$$d\ln(S_t) = \frac{1}{S_t} dS_t - \frac{1}{2S_t^2} d\langle S \rangle_t = \frac{1}{S_t} dS_t - \frac{1}{2} d\langle \ln(S) \rangle_t$$

(2.23) indicates that S_t is a continuous local martingale hence so is Y_t where:

$$Y_t \triangleq \int_0^t \frac{1}{S_s} dS_s = \ln(S_t) + \frac{1}{2}\langle \ln(S) \rangle_t$$

By Dubins-Schwarz theorem, there exists a Brownian motion $\tilde{W}$ with:

$$\tilde{W}_{\langle \ln(S) \rangle_t} = Y_t$$

and hence we have:

$$\ln(S_t) = \tilde{W}_{\langle \ln(S) \rangle_t} - \frac{1}{2}\langle \ln(S) \rangle_t$$

This leads to SDE (2.25).

With this time change, inequality (2.24) becomes:

$$G(X_t, t) + H(X_t) \le F(t) = F(\langle \ln(S) \rangle_{A_t}) \tag{2.26}$$

where A_t is defined through $S_{A_t} = X_t$.

First consider the subhedging strategy for $G(X_t, t)$. The submartingale part of $G(X_t, t)$ can be subhedged through the Martingale Representation Theorem. By the Martingale Representation Theorem, there exists a process ϕ_t satisfying the following inequality:

$$G(X_t, t) \ge G(X_0, 0) + \int_0^t \phi_s dX_s \tag{2.27}$$

Since $G(X_{t\wedge\tau_D}, t \wedge \tau_D)$ is a martingale, the equality in (2.27) holds when $t = \tau_D$ and moreover:

$$G(X_{t\wedge\tau_D}, t\wedge\tau_D) = G(X_0, 0) + \int_0^{t\wedge\tau_D} \frac{\partial G}{\partial x}(X_{t\wedge\tau_D}, t\wedge\tau_D) dX_s \tag{2.28}$$

Therefore with some certain process ϕ_t, it is able to sub-replicate $G(S_t, t)$ through:

$$G(X_{\tau_t},\tau_t) \ge G(X_0,0) + \int_0^{\tau_t} \phi_s dX_s = G(S_0,0) + \int_0^t \phi_{\tau_s} dS_s \quad (2.29)$$

or:

$$G(S_t,\tau_t) \ge G(S_0,0) + \int_0^t \varphi_s dS_s \quad (2.30)$$

with $\varphi_s = \phi_{\tau_s}$. Hence the subreplicating portfolio can be constructed as holding φ_s units of stock and the initial investment equals $G(S_0,0) - \varphi_0 S_0$.

As for $H(S_t)$, Cox and Wang [3] fully replicates it through the following decomposition:

$$H(S_T) = H(S_0) + H'_+(S_0)(S_T - S_0) + \int_{(S_0,\infty)} (S_T - K)^+ H''(dK) + \int_{(0,S_0]} (K - S_T)^+ H''(dK) \quad (2.31)$$

(2.31) indicates that $H(S_T)$ can be replicated by putting $H(S_0) - H'_+(S_0)S_0$ in the bank account, holding $H'_+(S_0)$ unites of shares, holding $H''(dK)$ unites of calls with strike K for K greater than S_0 and holding $H''(dK)$ unites of puts with strike K for K smaller than or equal to S_0.

Based on the sub-replication or replication of $G(S_t,t)$ and $H(S_t)$, the lower bound for the price of the variance option with payoff $F(\langle \ln S \rangle_T)$ is then:

$$G(S_0,0) + H(S_0) + \int_{(S_0,\infty)} C(K)H''(dK) + \int_{(0,S_0]} P(K)H''(dK) \quad (2.32)$$

Chapter 3

Implementation of the Obstacle Problem

In this section, my aim is twofold. First, I will perform a numerical implementation to solve the Root's barrier in terms of an obstacle problem as described in Section 2. There are two reasons that I choose to perform under the system of **OBS** (σ,ν,μ) instead of **VI** (σ,ν,μ). The first reason is that I only solve the Root's barrier with simple assumptions of (σ,ν,μ) and they essentially meet the restrictions put on **OBS** (σ,ν,μ). Therefore there is no need to generalize from **OBS** (σ,ν,μ) to **VI** (σ,ν,μ). The second reason is that variational inequalities are just an extension of **OBS** (σ,ν,μ) and do not change its basic algorithm. Solving **VI** (σ,ν,μ) and solving **OBS** (σ,ν,μ) is essentially the same, which both provide us with a barrier $B = D^C$ with respect to a specified group of (σ,ν,μ).

My second object in this section is to verify the conclusion of Cox and Wang [3] that the barrier obtained from **OBS** (σ,ν,μ) is indeed the solution to the corresponding **SEP** (σ,ν,μ). In order to do this, I apply Monte Carlo simulation to execute a backtest. The idea is based on the following fact: If we use Monte Carlo method to simulate a sufficiently large number of paths of the underlying stochastic process X_t, stop them immediately when they hit the barrier and make record of the positions when they stop, we can get a close approximation of the distribution of X_τ. If the barrier is actually the Root's barrier that solves the SEP, this approximation of the distribution of X_τ will be an approximation of the given μ.

3.1 Finite Difference Method on the Obstacle Problem

In order to solve the **OBS** (σ,ν,μ) described in Section 2 numerically, I employ a forward explicit finite difference scheme. The advantage of a finite difference scheme is that we can stop whenever we want during the evolution. Since in finance we are most interested in cases where $\sigma(x)=\sigma$ and $\sigma(x)=\sigma x$, I only solve the **OBS** (σ,ν,μ) in this two representative cases. The solving process generally consists of the following two steps.

The first step is to build up the target distribution of X_τ denoted by μ. Here τ stands for the stopping time when $U_\mu(x)=u(x,t)$ and the whole evolution stops. To explain this, note that we have an explicit expression of $u(x,t)$ given by (2.8) as:

$$u(x,t) = -E\,|\,X_{t\wedge\tau_D} - x\,| \in \mathbb{C}^{1,1}(\mathbb{R}\times\mathbb{R}^+)$$

When $\sigma(x)=\sigma$, X_t is a Brownian motion and when $\sigma(x)=\sigma x$, X_t is a geometric

Brownian motion. These two processes both ensure that $E|X_{t\wedge\tau_D} - x|$ is non-decreasing in t. Therefore $u(x,t)$ is obviously a non-increasing function in t given x is fixed. Furthermore we have $U_\mu(x) \le u(x,0) = U_\nu(x)$ as a necessary assumption. Hence from the beginning $u(x,t)$ (if strictly greater than $U_\mu(x)$) will evolve according to PDE (2.7) and as the time goes forward, $u(x,t)$ will get closer and closer to $U_\mu(x)$. At some moment in time, $u(x,t)$ will eventually be equal to $U_\mu(x)$. From this point, $u(x,t)$ will no longer evolve with time according to PDE (2.7) but equal to $U_\mu(x)$ for all t greater than this moment. The moment when $u(x,t)$ hits $U_\mu(x)$ is just τ and the collection of values of x satisfying $u(x,t) = U_\mu(x)$ is then the barrier we get.

To illustrate the above argument, I plot the figure of $u(x,0)$, $U_\mu(x)$ and the corresponding evolution process in Figure 3.1.1 under the assumption of $\sigma(x) = \sigma$ and μ being the uniform distribution.

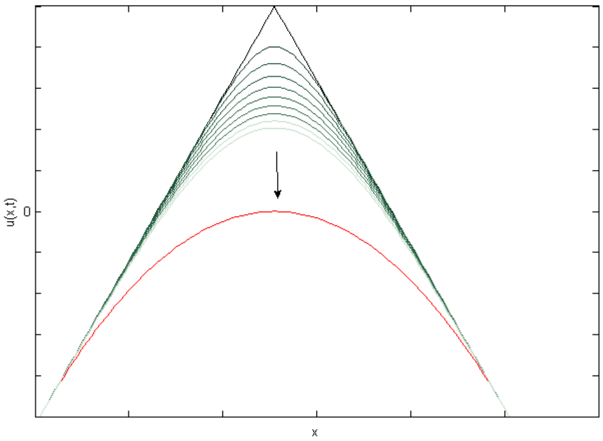


Figure 3.1.1: Evolution of $u(x,t)$ with time and $U_\mu(x)$

The black line on the top in Figure 3.1.1.represents $u(x,0)$ and the red parabola below is the target $U_\mu(x)$. We can see that with time increasing, the shape of $u(x,t)$ is pulled down smoothly and gets closer and closer to $U_\mu(x)$. Before the green line touches the red line, it will be continually pulled down by PDE (2.7). Once it touches the red line, it remains in the same place of the red line.

Obviously, setting up the target distribution of X_τ is equivalent to building up $U_\mu(x)$. From (1.5) we know that $U_\mu(x)$ is the expectation of $-|y-x|$ with $y \sim \mu$ .For simple μ like uniform, normal and log-normal distribution, $U_\mu(x)$ can be expressed explicitly by calculating out the integration of $-\int_{\mathbb{R}} |y-x|\,\mu(y)dy$. For cases where the integration can not be calculated out explicitly, we can adopt Monte Carlo simulation to get an approximation of the expectation for each x.

The second step in numerical implementation is to solve the system in **OBS** (σ,ν,μ). As discussed above, before $u(x,t)$ hits $U_\mu(x)$, $u(x,t)$ satisfies PDE (2.7):

$$\frac{\partial u}{\partial t}(x,t) - \frac{1}{2}\sigma^2(x)\frac{\partial^2 u}{\partial x^2}(x,t) = 0$$

Therefore the algorithm for solving the **OBS** (σ,ν,μ) goes like this: for every time step from $t=0$, I check whether $u(x,t)$ is smaller than or equal to $U_\mu(x)$. If it is, then make $u(x,t)$ evolves according to PDE (2.7) for the next time step. If it is not, then set $u(x,t)$ equal to $U_\mu(x)$ for the rest of the time steps and get record of the position of x immediately.

The forward explicit finite difference scheme to implement the above algorithm is formulated as follows. Assume we have a given maturity T, a large enough value $X_{\max}$ as the ceiling for x to be considered and a small enough value $X_{\min}$ as the floor for x to be considered. We then equally divide the space $[X_{\min}, X_{\max}]\times[0,T]$ into a $M\times N$ grid with $M+1$ points in x-grid and $N+1$ points in t-grid. For $1\le i\le N+1$ and $1\le j\le M+1$, denote $U(j,i)$ as the value of $u(x_j,t_i)$, where x_j is the j-th point in x-grid and t_i is the i-th grid in t-grid.

Define $\Delta t = T/N$ and $\Delta x = (X_{\max}-X_{\min})/M$. Then for every time-step Δt, PDE (2.7) can be approximated by the following equation:

$$\frac{U(j,i+1)-U(j,i)}{\Delta t}=\frac{1}{2}\sigma^2(x_j)\frac{U(j+1,i)-2U(j,i)+U(j-1,i)}{(\Delta x)^2} \tag{3.1}$$

for $1\le i\le N$ and $2\le j\le M$. Hence the value of $U(j,i+1)$ can be derived from values of $U(j+1,i)$, $U(j,i)$ and $U(j-1,i)$ which are already known. Precisely:

$$U(j,i+1)=\frac{\Delta t\sigma^2(x_j)}{2(\Delta x)^2}U(j+1,i)+(1-\frac{\Delta t\sigma^2(x_j)}{(\Delta x)^2})U(j,i)+\frac{\Delta t\sigma^2(x_j)}{2(\Delta x)^2}U(j-1,i) \quad (3.2)$$

for $1\le i\le N$ and $2\le j\le M$.

To sum up, the second step of the forward finite difference scheme starts by setting up the value for $U(1,j)$ for all j. Then by increasing the value of i one by one while calculating the value of $U(i,j)$ for all j, we are able to simulate the evolution process.

The boundary condition for the scheme is formulated according to the explicit expression of $u(x,t)$ given by (2.8). For values at $t=0$, $u(x,0)=-E\,|\,X_{0\wedge\tau_D}-x\,|$. Hence:

$$U(j,1)=u(x_j,0)=-|\,X_0-x\,| \quad \text{for all } 1\le j\le M+1 \quad (3.2)$$

and for values at $x=X_{\max}$ and $x=X_{\min}$:

$$U(1,i)=u(X_{\min},t_i)=\begin{cases}-E\,|\,X_{t_i}-X_{\min}\,| & \text{if } 0\le t_i<\tau\\ -E\,|\,X_\tau-X_{\min}\,| & \text{otherwise}\end{cases} \tag{3.3}$$

$$U(M+1,i)=u(X_{\max},t_i)=\begin{cases}-E\,|\,X_{t_i}-X_{\max}\,| & \text{if } 0\le t_i<\tau\\ -E\,|\,X_\tau-X_{\max}\,| & \text{otherwise}\end{cases} \tag{3.4}$$

Here we have obtained all necessary setting for implementing **OBS** (σ,ν,μ) numerically with a forward finite difference scheme. In the next section, I will represent the barriers I obtained from solving the obstacle problem. First I will represent the barrier by solving the **OBS** (σ,ν,μ) under $\sigma(x)=\sigma$ for μ being the uniform distribution and the normal distribution. Then I turn to the case where $\sigma(x)=\sigma x$ and explore the barrier with μ being the log-normal distribution. The reason why I choose these settings for μ is that we have already known what the shape of the barrier should be, which is very neat and regular. If the barrier I obtained is different from the shape it supposed to be, then I can quickly find that there must be something wrong either with my implementation or with the results given by Cox and Wang [3]. After this sanity check, I will implement two groups of (σ, ν, μ) where the explicit expression of Root's barrier can not be obtained immediately from theoretical calculation and see what the shape of barrier looks like. For simplicity, assume ν is a distribution that with probability one equal to a constant X_0 in the rest of this section.

3.1.1 Brownian Motion Case

There is no doubt that the simplest and most commonly used function for σ is that σ being a constant. In this case, the underlying stochastic process X_t evolves as a Brownian motion:

$$dX_t = \sigma dW_t \tag{3.5}$$

It has been shown that if μ is a uniform distribution centered at X_0, we should get a very nice barrier symmetric and concave to X_0. And if μ is set to be the normal distribution centered at X_0, we shall obtain a vertical straight line at some point in x-grid. This is a natural conclusion. If we stop the Brownian motion in (3.5) at any time t, X_t is equal to $X_0 + \sigma W_t$, which is normally distributed with mean X_0 and variance $\sigma^2 t$.

Figure 3.1.2 shows the barriers I obtained from numerical implementation. Obviously, except for some implementation errors near boundaries, they are both good approximations of the target Root's barriers and therefore give an initial supports for the result in Cox and Wang [3].

3.1.2 Geometric Brownian Motion Case

In the financial context, the underlying stochastic process under investigation is usually a geometric Brownian motion. In this case, $\sigma(x) = \sigma x$. Now for μ, the log-normal distribution becomes a trivial case. Because if we stop a geometric Brownian motion at time t, X_t is then log-normal distributed. Denote $Z_t = \ln(X_t)$, we have:

$$Z_t \sim N(\ln(X_0) - \frac{1}{2}\sigma^2 t, \sigma^2 t)$$

But as for X_t being a geometric Brownian motion, the finite difference scheme in equation (3.2) might encounter some problems when calculating $\Delta t \sigma^2(x_j) / 2(\Delta x)^2$ when x_j is very close to 0. In order to avoid this undesirable feature, I make a transformation to the issue by letting $u(x,t) = u(e^z, t) = v(z,t)$. Then PDE (2.7) changes to:

$$\frac{\partial v}{\partial t} - \frac{1}{2}\sigma^2 \frac{\partial^2 v}{\partial z^2} + \frac{1}{2}\sigma^2 \frac{\partial v}{\partial z} = 0 \tag{3.6}$$

In this way an M z-grid replaces the original M x-grid and the forward explicit finite difference scheme (3.1) changes to:

$$\frac{V(j,i+1) - V(j,i)}{\Delta t} = \frac{1}{2}\sigma^2 \frac{V(j+1,i) - 2V(j,i) + V(j-1,i)}{(\Delta x)^2} - \frac{1}{2}\sigma^2 \frac{V(j+1,i) - V(j-1,i)}{2\Delta x}$$

(3.7)

for $1 \le i \le N$ and $2 \le j \le M$ and where $V(j,i)$ stands for the value of $v(z_j, t_i)$. Similarly z_j is the j-th point in z-grid and t_i is still the i-th grid in t-grid, According to (3.7), value of $V(j, i+1)$ can also be derived from values of $V(j+1, i)$, $V(j,i)$ and $V(j-1,i)$ which are already known. Denote:

$$a = \frac{\sigma^2 \Delta t}{2(\Delta z)^2} \quad \text{and} \quad b = \frac{\sigma^2 \Delta t}{2\Delta z}$$

With the above denotation, (3.2) is transformed into:

$$V(j,i+1) = (a - \frac{1}{2}b)V(j+1,i) + (1-2a)V(j,i) + (a + \frac{1}{2}b)V(j-1,i) \qquad (3.8)$$

Figure 3.1.3 represents the barrier obtained in this case. It is a straight line and again accords with our prediction. It supports the idea that the result of Cox and Wang [3] can indeed be extended to the geometric Brownian motion.

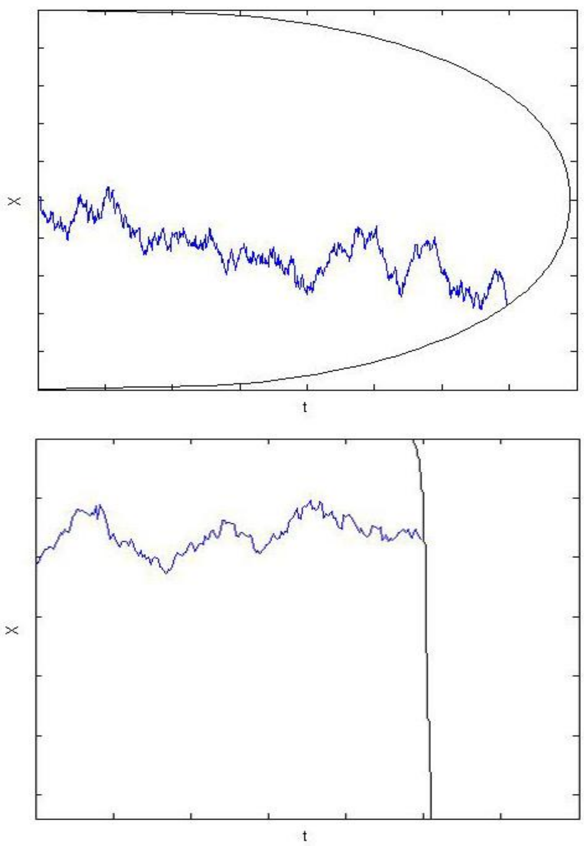


Figure 3.1.2: Barriers under uniform distribution (above) and normal distribution (below)

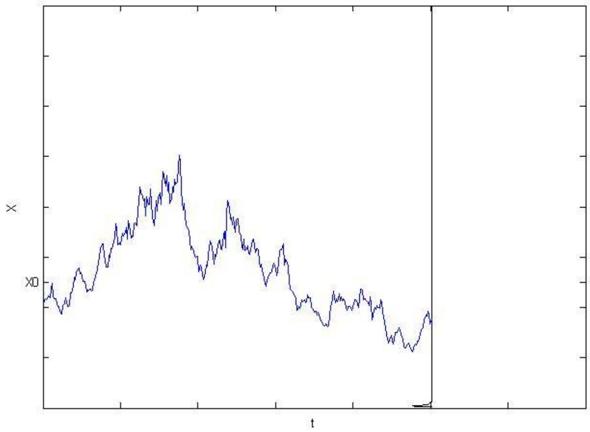


Figure 3.1.3: Barriers under log-normal distribution

3.1.3 Other Examples

Here I pick two distributions for μ that are not commonly seen and investigate what the barrier would look like. The aim to do this is to give more generalized examples since under most circumstances the barrier is either unsmooth or irregular. Discussion and representation in Section 3.1.1 and Section 3.1.2 only shows some very simple and trivial Root's barriers. There barriers usually serve as benchmarks as a sanity check or used as an inherent assumption in the construction of hedging strategies. One example is given by Carr and Lee [17]. Their result turns out to be a subhedging strategy based on a constant barrier. The lower graph in Figure 3.1.2 and Figure 3.1.3 are two most important constant barriers. However, in practice, the shape of Root's barrier will not be so perfect due to the deviation of the stochastic stock process from either a standard geometric Brownian motion or a Brownian motion.

The first distribution is a uniform distribution with an atom at X_0. It means that the law of μ has probability p to be a uniform distribution between 0 and $2X_0$ while it has probability 1-p to be at the point X_0. Here I set p=0.8 and get the barrier depicted in the upper figure in Figure 3.1.4.

It can be easily deduced that if μ is just an atom at X_0, the barrier will be nothing but a horizontal line beginning from X_0 to infinity. And it has already been shown in the upper figure in Figure 3.1.2 that the barrier of a uniform distribution under Brownian motion is a parabolic curve symmetrically concave to X_0. Therefore the barrier in the upper graph of Figure 3.1.4 can be viewed as a combination of the above two barriers. It is a parabolic curve concave to X_0 with a spike pointing to X_0. If we keep increasing the number of steps in finite difference evolution, the spike will eventually become a horizontal line. It is the shape of a spike due to the restriction on the obstacle problem. The barrier from solving the obstacle problem will be first order continuous in x and therefore we can only approximate the horizontal component smoothly by a continuous spike. Also, the shape of this barrier is connected to the value of p. If the value of p decreases, the length of this spike will be greater. It accords with intuition since if we set p=0, the barrier will become the one in the pure atom distribution case.

We can argue that the actual shape of Root's barrier will be the combination of a parabola and

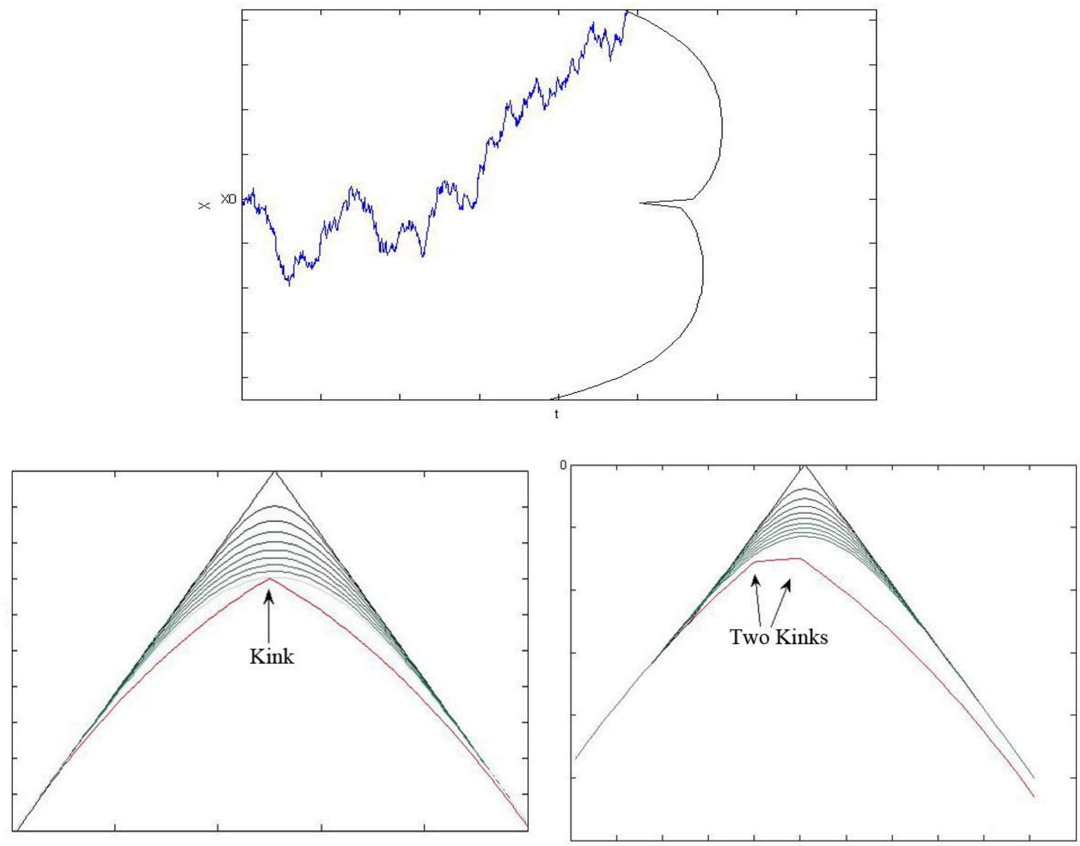


Figure 3.1.4: Barriers under uniform distribution with an atom at X_0 (above) and kinks in the target distribution (below)

a horizontal line pointing to the atom through the evolution process shown in the lower graph of Figure 3.1.4. The graph in the left below represents the situation where there exists only one atom at X_0 in the uniform distribution. The graph in the right below shows the situation where there exist two atoms, one at X_0 and another at $0.3X_0$. We can see that the number of atoms in a uniform distribution is shown by the number of kinks in the target distribution (red line in the graph). When the function $u(x,t)$ evolves with time and pulled down by the corresponding PDE, the points in x-grid near these kinks will be the first to hit the target red line. Therefore there will form a horizontal line in the Root's barrier which will be approximated by a spike.

The second distribution I choose is a target exponential distribution under geometric Brownian motion. The drawback of my numerical implementation here is that we abandoned z values that are smaller than the minimal value set on z-grid to avoid the negative infinity of log zero. By doing so, we directly altered the probability of hitting the barrier. Especially in this case, for an exponential distribution, the probability assigned to a smaller value of the underlying random variable is much greater than the probability assigned to those larger values. Therefore by cutting off smaller z values, the underlying stochastic process X_t will be more likely ending up hitting the floor set to z-grid as that shown in Figure 3.1.5. However we can still obtain a nice barrier.

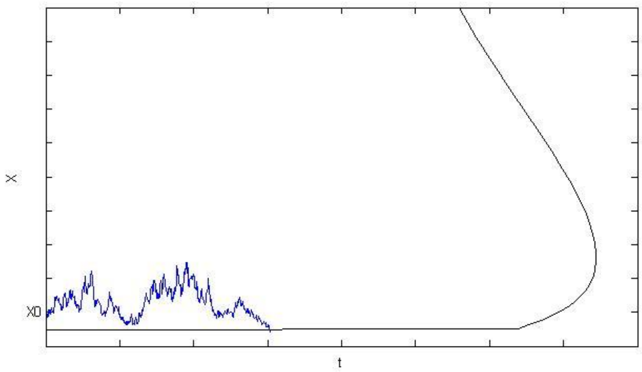


Figure 3.1.5: Barriers under exponential distribution with z-grid

3.2 Monte Carlo Backtest

Although the shape of Root's barrier of occasions with uniform, normal and log-normal distributions have already been proved theoretically, I implement a numerical backtest using Monte Carlo simulation to better illustrate the connection between SEP and OBS. The theoretical Root's barrier is a result of natural and mental induction of properties of the underlying stochastic process. However a Monte Carlo simulation dynamically reproduces the whole process of a SEP in which we are able to get a very close approximation of the distribution of X_τ .

I adopt the simple Euler method to stimulate the path of X_t . When X_t evolves as a Brownian motion in (3.5), the path is approximated by:

$$X_{j+1} = X_j + \sigma\sqrt{\Delta t}B_j \tag{3.9}$$

for $0 \le j \le M$. Here B_j is a standard normally distributed variable chosen independently for every step in (3.9).

When X_t evolves as a geometric Brownian motion, the path is approximated by:

$$X_{j+1} = X_j + X_j\sigma\sqrt{\Delta t}B_j \tag{3.10}$$

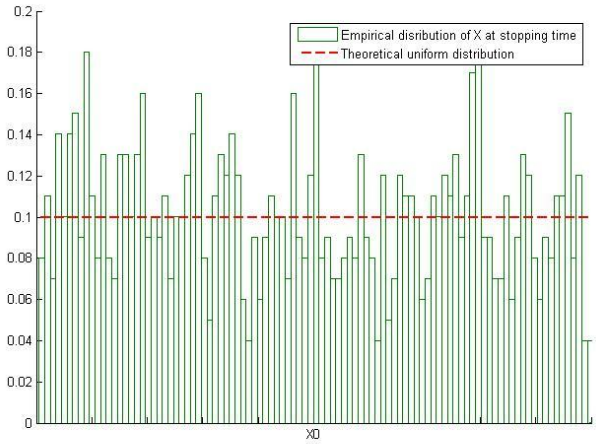
for $0 \le j \le M$. But here we adopt the z-grid and the Euler method for approximat8ing the path of Z_t is then:

$$Z_{j+1} = Z_j - \frac{1}{2}\sigma^2\Delta t + \sigma\sqrt{\Delta t}B_j \tag{3.11}$$

And in the Monte Carlo simulation, I keep the time step the same length as that in implementation of barriers in Section 3.1.

The hollow rectangles in the following histograms represent the distribution of X_τ with τ being the first hitting time of the barrier obtained in Section 3.1. The broken red line shows the theoretical probability density function of X_τ with Figure 3.2.3 as an exception. In Figure 3.2.3 the broken red line represents the probability density of a pure uniform distribution since the probability density of a uniform distribution with an atom is just the probability density of a pure uniform distribution with a jump at the atom.

The upper graph in Figure 3.2.1 shows the distribution of X_τ with τ being the first hitting time of the barrier represented in the upper figure of Figure 3.1.2. By comparing the distribution of rectangles and the broken red line we can see that it indeed approximates a uniform distribution between 0 and $2X_0$ under limited number of simulated paths. Also, the lower graph in Figure 3.1.2 shows a very close approximation of a normal distribution centered at X_0 with limited number of simulation paths.



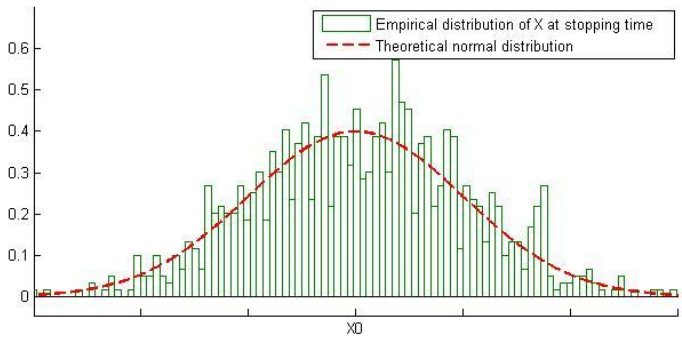


Figure 3.2.1: Distribution of X_τ under uniform distribution (above) and normal distribution (below)

Figure 3.2.2 shows the distribution of X_τ with τ being the first hitting time of the barrier in Figure 3.1.3. It is the shape of a log-normal distribution with the normal component centered at Z_0 which equals $ln(X_0)$.

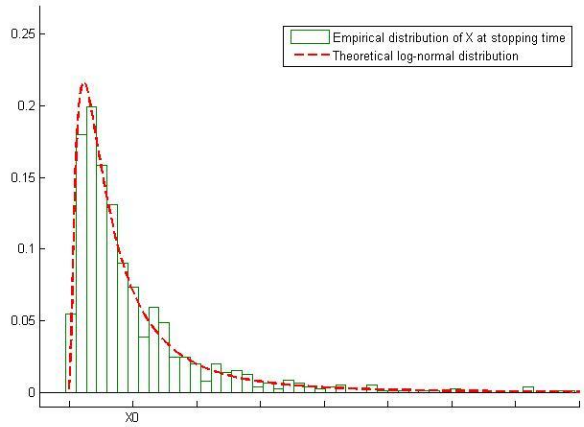


Figure 3.2.2: Distribution of X_τ under log-normal distribution

Figure 3.2.3 represents the Monte Carlo result associated with the barrier in the upper graph of Figure 3.1.4. We can see that the distribution shown in Figure 3.2.3 is actually a combination of a uniform distribution and an atom at X_0. This supports that the result of Cox and Wang [3] works on barriers that is not smooth.

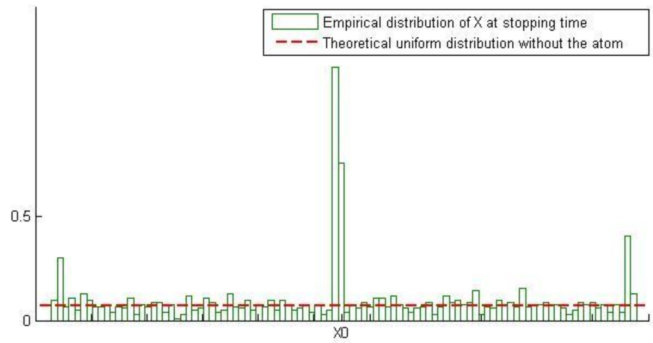


Figure 3.2.3: Distribution of X_τ under uniform distribution with an atom at X_0

Figure 3.2.4 shows the distribution of X_τ with τ being the first hitting time of the barrier constructed from the exponential distribution. The result in Figure 3.2.4 is obtained applying the z-grid. As mentioned in Section 3.1.3, the limit z-grid of numerical implementation twists the distribution of X_τ especially with small value of X_τ. It makes X_t end up with value closer to 0 with greater probability. Hence the result of Monte Carlo should be a twisted exponential distribution with more probability on values close to 0. However we can see that with limited grids in histogram this distortion is not obviously represented. Figure 3.2.4 also provides strong support for Cox and Wang [3].

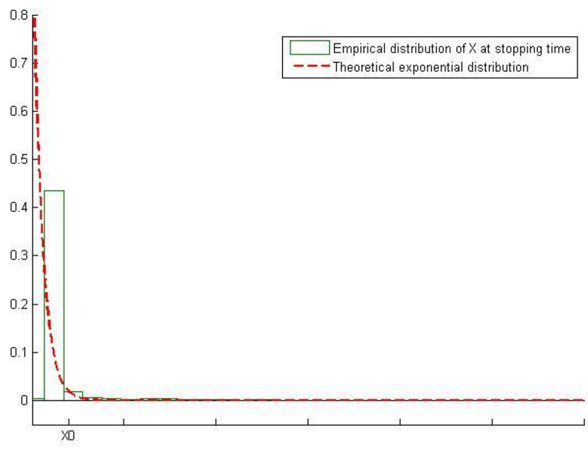


Figure 3.2.4: Distribution of X_τ under exponential distribution with z-grid

Chapter 4

Robust Subhedging Examples

In this section I re-examine the subhedging strategy given in Section 5 and Section 6 of Cox and Wang [3] and compare the result with that in Carr and Lee [17]. The strategy constructed in Cox and Wang [3] is generalized to any variance derivatives with final payoff satisfying the function $F(\langle \ln S \rangle_T)$ provided that F is a convex, increasing function with $F(0) = 0$. Its right derivative is denoted by f and is required to be bounded. But the subhedging strategy in Carr and Lee [17] (Proposition 2.6 and Proposition 3.1) only limits to variance swaps and variance calls. In fact, the strategy of Carr and Lee [17] (Proposition 3.1) can be viewed as an instantiation of the result in Cox and Wang [3] with a chosen constant barrier. I will illustrate this in detail by applying the strategy summarized in Section 2.2 in two specific examples. The first example is a variance swap with a simplified payoff $\langle \ln S \rangle_T$ and the second example is a variance call with final payoff $(\langle \ln S \rangle_T - K)^+$. The first example in variance swap is actually used as a sanity check which demonstrates the idea of the subhedging strategy in Cox and Wang [3]. The detailed comparison will be represented in the second example.

4.1 (Sub)Hedging Strategy for a Variance Swap

With a variance swap we have $F(t) = t$ for $t \geq 0$, which is perfectly linear in the variable. Hence we have a trivial case where $f(\tau_D) \equiv 1$ no matter what the barrier looks like.

According to definition:

$$M(x,t) = E^{(x,t)} f(\tau_D) = 1 \tag{4.1}$$

where $M(x,t)$ is surely bounded on $\mathbb{R} \times \mathbb{R}^+$. $Z(x)$ is then defined as:

$$Z(x)'' = 2\frac{M(x,0)}{\sigma(x)^2} = -\frac{2}{\sigma^2 x^2} \tag{4.2}$$

Equation (4.2) is equivalent to equation (2.20). So:

$$Z(x) = -\frac{2\ln(x)}{\sigma^2} \tag{4.3}$$

Note here I choose a particular normalization to $Z(x)$ to simplify the calculation. This normalization makes no difference to the final subhedging strategy because any two solutions of (4.3) only differ by a constant. Hence the final subhedging strategies derived from any two solutions of (4.3) are essentially the same.

$G(x,t)$ and $H(x)$ are then calculated through the definition given in (2.21) and (2.22):

$$G(x,t) = t + \frac{2\ln(x)}{\sigma^2} \tag{4.4}$$

$$H(x) = -\frac{2\ln(x)}{\sigma^2} \tag{4.5}$$

$$G(x,t) + H(x) \equiv F(t) \tag{4.6}$$

The identity in (4.6) means that we can perfectly replicates the payoff of the variance swap. In order to construct such a strategy, consider substitute the corresponding time-changed process X_t (2.25) into the expression of $G(x,t)$. The aim here is to find the admissible dynamic strategy $\varphi_s = \phi_{\tau_s}$ to satisfy:

$$t + 2\ln(X_t) = G(X_t,t) \geq G(X_0,0) + \int_0^t \varphi_s dX_s = 2\ln(X_0) + \int_0^t \varphi_s dX_s \tag{4.7}$$

Since X_t follows a unit-variance geometric Brownian motion, $\ln(X_t)$ can be explicitly expressed through $\ln(X_0) - t/2 + \tilde{W}_t$. Therefore it can be easily deduced that

$$\varphi_t = \phi_{\tau_t} = \frac{\partial G(X_{\tau_t},t)}{\partial X_{\tau_t}} = \frac{2}{X_{\tau_t}} = \frac{2}{S_t} \tag{4.8}$$

satisfies the inequality in (2.26).

This result accords with the traditional sub-replicating strategy for a variance swap. Neuberger [4] and Carr and Lee [17] have both deduced the same replicating strategy for a variance swap paying $\langle \ln(S) \rangle_T$ at some specific time T. Neuberger [4] argued by writing $\langle \ln(S) \rangle_T$ as:

$$\langle \ln(S) \rangle_T = \int_0^T \left(\frac{dS_\mu}{S_\mu}\right)^2 \tag{4.9}$$

and consider applying Ito's lemma on $\ln(S_t)$:

$$\ln(S_T) - \ln(S_0) = \int_0^T \frac{dS_\mu}{S_\mu} - \frac{1}{2}\int_0^T \left(\frac{dS_\mu}{S_\mu}\right)^2 \tag{4.10}$$

Compare (4.9) and (4.10), we have:

$$\langle \ln(S) \rangle_T = -2\ln(S_T) + 2\ln(S_0) + \int_0^T \frac{2}{S_\mu} dS_\mu \tag{4.11}$$

Based on (4.11) and assume that call options with all strikes are publicly traded, Neuberger [4] suggested that the variance contract with final payoff $\langle \ln(S) \rangle_T$ has a model independent initial price equal to $2(\ln(S_0) - \ln(S_T))$. By dynamically holding $\theta_t = 2/S_t$ units of the underlying stock we are able to replicate the final payoff. We can see that $\theta_t = \varphi_t$.

Carr and Lee [17] built up the replication strategy by defining a difference of convex functions $\lambda : \mathbb{R} \to \mathbb{R}^+$ which satisfies:

$$\lambda_{yy}(y) \leq \frac{2}{y^2}\omega(y) \tag{4.12}$$

where λ_{yy} is the second derivative of λ and $\omega : \mathbb{R}^+ \to [0,\infty)$ is a weight function. Proposition 2.16

in Carr and Lee [17] states that (I quote from Carr and Lee [17] Proposition 2.16 with some minor adjustment in their notation):

"Let τ be a stopping time. If claims on $\lambda(S_T)$ and $\lambda(S_{\tau\wedge T})$ are tradable, then the strategy of holding at each time $t \in (0, \tau \wedge T]$:

$$1 \text{ claim on } \lambda(S_T)$$

$$\text{I claim on } -\lambda(S_{\tau\wedge T})$$

and holding at each time $t \in (0, \tau \wedge T]$:

$$1 \text{ claim on } \lambda(S_T)$$

$$-\lambda_y(S_t) \text{ shares}$$

$$-\lambda(S_{\tau\wedge T}) - \int_{\tau\wedge T}^{T} \lambda_y(S_s)dS_s + S_t\lambda_y(S_t) \text{ bonds}$$

subreplicates the forward-starting weighted variance of $\ln(S)$, defined by:

$$\langle S\rangle_{\tau,T}^{\omega} = \int_{\tau\wedge T}^{T} \omega(S_s)d\langle \ln(S)\rangle_s$$

If the equality in (4.12) holds, then the strategy replicates $\langle S\rangle_{\tau,T}^{\omega}$ exactly."

If we set $\omega(y) = 1$ and $\tau = 0$, we reproduce the variance swap with payoff $\langle \ln(S)\rangle_T$ at T. And moreover, if we set the equality in (4.12) and solve the ODE, we have $\lambda(y) = -2\ln(y)$. Hence the exact replicating strategy consists of holding $-\lambda_y(S_t) = 2/S_t$ units of shares, which is again the same with the hedging strategy from Cox and Wang [3].

Note here the reason why we obtain the same trading strategy in both Cox and Wang [3] and Carr and Lee [17] or Neuberger [4] is that the hedging strategy in Cox and Wang [3] has nothing to do with the chosen barrier. The special form of the payoff function of a variance swap makes function $M(x,t)$ in (4.1) independent from τ_D, which results in a universal hedging strategy for a variance swap. This conceals the significance of the subhedging strategy in Cox and Wang [3]. If the first derivate of the payoff function is not independent of τ_D, we may obtain different subhedging strategies when we replace D by another barrier. Cox and Wang [3] shows that the subhedging strategy obtained by choosing Root's barrier is optimal among other strategies obtained using other barriers. In the next section, I move forward to consider a variance call with payoff $(\langle \ln(S)\rangle_T - K)^+$ and further discuss this point.

4.2 Subhedging Strategy for a Variance Call

As for a variance call, function $f(t)$ is dependent on the barrier we chose. Therefore in this case, we need to assume a barrier to calculate out $M(x,t)$, $Z(x)$, $G(x,t)$ and $H(x)$. As mentioned at the end of Section 4.1, Cox and Wang [3] shows that though we can apply any barrier to construct the subhedging strategy, we can only obtain the optimal one by choosing Root's barrier. Therefore I will choose a Root's barrier and another different barrier to see what the hedging strategies will be like and discuss the results I obtain.

The Root's barrier I choose is a constant one with $\tau_D = \inf\{t > 0 : t \geq R(x) = K\}$. It accords with the situation where S_t follows a geometric Brownian motion with volatility σ . Figure 3.1.3 has represented that if S_T is log-normal distributed with mean $\ln(S_0) - T/2\sigma^2$ and variance σ , the Root's barrier will be a vertical line.

First, from $F(t) = (t-K)^+$ we have:

$$f(t) = \begin{cases} 1 & if \quad t \geq K \\ 0 & if \quad t < K \end{cases} \tag{4.13}$$

According to (2.19), we have $M(x,t) = E^{(x,t)} f(\tau_D) = 1$ for all $t \geq 0$. It is because when $t \geq R(x) = K$, $\tau_D = t$ and when $0 \leq t < K$, $\tau_D = K$. Therefore $Z(x)$ can be calculated through (2.20):

$$Z(x) = -\frac{2\ln x}{\sigma^2} \tag{4.14}$$

which is the same as the $Z(x)$ for a variance swap after normalization.

$G(x,t)$ then follows as;

$$G(x,t) = \int_0^t M(x,s)ds - Z(x) = t + \frac{2\ln(x)}{\sigma^2} \tag{4.15}$$

which is also the same as the $G(x,t)$ for a variance swap. Finally, $H(x)$ is calculated though (2.22):

$$H(x) = \int_0^K (0-1)dx + Z(x) = -K - \frac{2\ln(x)}{\sigma^2} \tag{4.16}$$

Therefore we have:

$$G(x,t) + H(x) = t - K \leq (t-K)^+ = F(t) \tag{4.17}$$

which confirms the argument of Cox and Wang [3] Proposition 5.1.

Follow a similar deduction in variance swap case, substitute the corresponding time-changed process X_t into the expression of $G(x,t)$, we aim to find the admissible dynamic strategy $\varphi_s = \phi_{\tau_s}$ which satisfies:

$$t + 2\ln(X_t) = G(X_t, t) \geq G(X_0, 0) + \int_0^t \varphi_s dX_s = 2\ln(X_0) + \int_0^t \varphi_s dX_s \tag{4.18}$$

Here (4.18) is the same as (4.7) and hence choosing $\varphi_t = 2/S_t$ forms a subhedging strategy for the variance call. Note the subhedging strategy for a variance call is exactly the same as the replicating strategy for a variance swap. The idea is based on the fact that the payoff of a variance call is always greater or equal to the payoff of a variance swap with a same strike. Therefore by replicating the variance swap we get a natural subhedging for the corresponding variance call.

Now I substitute another barrier to see what the subhedging strategy will look like. Choose $\tau_D = \inf\{t > 0 : t \geq R(x) = Q\}$ with $Q < K$. Although the expression of $f(t)$ will not be changed, $M(x,t)$ will no longer equal one for all $t \geq 0$. When $0 \leq t < R(x) = Q$, $\tau_D = Q$ and hence $f(\tau_D) = 0$. When $t \geq R(x) = Q$, $\tau_D = t$ and there are two cases we need to consider. If $Q \leq t < K$, $f(\tau_D) = 0$ but if $t \geq K$, $f(\tau_D) = 1$. Therefore $M(x,t)$ now becomes:

$$M(x,t) = \begin{cases} 0 & if \quad 0 \leq t < K \\ 1 & if \quad t \geq K \end{cases} \tag{4.19}$$

By substituting (4.19) into definition of $Z(x)$, $G(x,t)$ and $H(x)$, we have $Z(x) = 0$, $H(x) = 0$ and:

$$G(x,t) = \begin{cases} 0 & if \quad 0 \leq t < K \\ t - K & if \quad t \geq K \end{cases} \tag{4.20}$$

Hence we have:

$$F(t) = G(x,t) + H(x) = \begin{cases} 0 & if \quad 0 \leq t < K \\ t - K & if \quad t \geq K \end{cases} \tag{4.21}$$

Obviously, expression of current $G(x,t)$ and $H(x)$ implies a different subhedging strategy from the one using Root's barrier. Here the independence of $G(x,t)$ and x means that we choose to hold zero units of shares to subhedge a variance call.

However, here we have the equality $E[F(\tau)] = E[F(K)] = E[F(Q)]$, which means that both subhedging strategies are optimal. This results from the special payoff form of a variance call. The realized volatility of a variance call is always a constant so we will obtain an optimal subhedging strategy if the barrier we choose is a constant.

Note here if we restrict our choice of $R(x)$ in D to be a constant, then the function λ in Carr and Lee [17] actually corresponds to $G(x,t)$. The subhedging strategy of Carr and Lee [17] consists of the following components (partly quoted from Proposition 3.1 from Carr and Lee [17] with minor adjustment in notation):

"At each time $0 \leq t < T$ hold

1 claim on $\lambda(S_T)$

N_t shares

$-BS(S_0, K; \lambda) + \int_0^t N_s dS_s - N_t S_t$ bonds

"

where N_t is defined as:

$$N_t = \begin{cases} -BS_y(S_t, K - \langle \ln(S) \rangle_t; \lambda) & if \quad t \le K \\ -\lambda_y(S_t) & if \quad t > K \end{cases} \tag{4.22}$$

And the function $BS(y, v; f)$ is defined as:

$$BS(y, v; f) = \begin{cases} \int_{-\infty}^{\infty} f(ye^z) \frac{1}{\sqrt{2\pi v}} \exp(-(z + \frac{v}{2})^2 / 2v) dz & if \quad v > 0 \\ f(y) & if \quad v = 0 \end{cases} \tag{4.23}$$

There are many choices for function λ. However we can see that the subhedging strategy obtained by using Root's barrier corresponds to choosing $\lambda(y) = -2\ln(y)$, which gives equality in (4.12). The subhedging strategy obtained by using $R(x) = Q < K$ corresponds to $\lambda \equiv 0$.

To see this, when $\lambda(y) = -2\ln(y)$, (4.23) becomes:

$$BS(y, v; f) = \begin{cases} \int_{-\infty}^{\infty} -2(\ln(y) + z) \frac{1}{\sqrt{2\pi v}} \exp(-(z + \frac{v}{2})^2 / 2v) dz & if \quad v > 0 \\ -2\ln(y) & if \quad v = 0 \end{cases} \tag{4.24}$$

and (4.22) becomes simply $N_t = 2/S_t$ for all t, which is exactly the φ_t in the subhedging strategy of applying Root's barrier in Cox and Wang [3]. And choosing $\lambda \equiv 0$ will make $N_t \equiv 0$ meaning that we hold zero units of shares to subhedge.

Generally, by solving the equality in (4.12) we are actually solving the equality in (2.28). It means that holding N_t shares where

$$N_t = \begin{cases} -BS_y(S_t, K - \langle \ln(S) \rangle_t; \lambda) & if \quad t \le K \\ -\lambda_y(S_t) & if \quad t > K \end{cases}$$

can be view as an optimal solution to:

$$G(X_t, t) \ge G(X_0, 0) + \int_0^t \phi_s dX_s$$

where ϕ_t is the subhedging strategy in Cox and Wang [3]. We can just substitute $\phi_t = N_t$ into the above inequality and check that it does satisfy the condition.

As mentioned in Remark 5.4 in Cox and Wang [3], by accurately choosing Root's barrier for the stock process with respect to each variance option, we are able to obtain the optimal subhedging strategy which gives the optimal lower bound for the price of the variance option. But the lower bound given by Carr and Lee [17] is merely by fixing the barrier as a constant. In general the subhedging strategy in Carr and Lee [17] will be suboptimal compared to that from Cox and Wang [3]'s construction. The reason why here choosing the optimal function of λ in Carr and Lee [17]'s construction corresponds to choosing Root's barrier in Cox and Wang [3] is the same as mentioned earlier. The special form of a variance call covers the fact that fixing the barrier as a constant will result in a suboptimal strategy. If in another case where the Root's barrier is not a constant, Carr and Lee [17] can not provide an optimal subreplicating strategy.

Generally speaking, the generalization of Cox and Wang [3] comes from two aspects. One is that they provide more choices for the shape of the barrier and another is that they enable us to apply the subhedging strategy on variance options different from a variance swap and a variance call. Carr and Lee [17]'s result mainly works on variance swaps and variance calls but Cox and Wang [3]'s result can work on more forms of the volatility derivatives.

Supplementary to the first point, Example 5.6 in Cox and Wang [3] demonstrates a barrier that is not a constant. They choose $D=\{(x,t): t<R(x)\}$ with $R(x)=-\lambda(x+\alpha)(x-\beta)1_{(-\alpha,\beta)}$. Here we require that $\lambda,\alpha,\beta>0$. Given the underlying process is a Brownian motion and payoff function as $F(t)=t^2/2$, $G(x,t)$ and $H(x)$ car calculated as:

$$G(x,t)=\begin{cases}\dfrac{\lambda}{\lambda+1}[\dfrac{t^2}{2}-t(x+\alpha)(x-\beta)]-Z(x) & if \quad 0\le t<R(x)\\ \dfrac{R^2(x)}{2(\lambda+1)}+\dfrac{t^2}{2}-Z(x) & if \quad t\ge R(x)\end{cases}$$

$$H(x)=-\frac{R^2(x)}{2(\lambda+1)}+Z(x)$$

We can see that:

$$G(x,t)+H(x)-F(t)\le 0$$

for all $t\ge 0$. This strategy obviously can not be deduced from results in Carr and Lee [17]. In reality, the barrier is most likely to be non-constant and some non-constant barrier examples are plotted in Section 3.1 Figure 3.1.2 (above), Figure 3.1.4 and Figure 3.1.5.

Chapter 5

Conclusions

I conclude this paper first by summarizing the steps I have taken to give a brief review of Cox and Wang [3]. Cox and Wang [3]'s work is based on the connection of Root's barrier and the Skorokhod embedding problem and their aim is to find a model-independent subhedging for variance options. Their work mainly consists of two parts. The first part is that they proved a one-to-one correspondence of Root's barrier and an obstacle problem or strictly, a variational inequality problem. And the second part is that they provided an alternative proof of the optimality property of Root's barrier leading to a construction of subhedging strategies for variance options.

As for the first part of Cox and Wang [3]'s work, I provide a brief review by solving the obstacle problem numerically and checking whether it gives the right result of the corresponding Skorokhod embedding problem. The obstacle problem is solved through an explicit forward finite difference scheme and the result is checked by performing a Monte Carlo simulation. The result from numerically solving the obstacle problem represents us a barrier and I have shown in Section 3.1 that the barrier is a close approximation of the suggested Root's barrier from Skorokhod embedding. Hence my numerical implementation gives an initial support to Cox and Wang [3]'s work. As for back checking the barrier, I simulated a sufficiently large number of paths to reproduce the stopping process. By reconstructing the distribution of the underlying stochastic process at the stopping time, I further represented that solving the obstacle problem actually renders us the same Root's barrier as solving the Skorokhod embedding problem.

To reexamine the second part of Cox and Wang [3], I substitute the subhedging strategy of Cox and Wang [3] into a variance swap and a variance call and compare the strategies with those given by Carr and Lee [17]. For a variance swap, no matter what barrier we choose in Cox and Wang [3]'s construction, we will end up in the unique replicating strategy since its payoff function is perfectly linear in the variable. Hence we obtain the same hedging strategy from Carr and Lee [17] and Cox and Wang [3]. For a variance call, I also obtained the same subhedging strategy as that in Carr and Lee [17] by using a constant Root's barrier in Cox and Wang [3]. However it is because the realized volatility of a variance call is always a constant and this conceals the fact that Cox and Wang [3]'s strategy is optimal while Carr and Lee [17] is only suboptimal. The tradeoff between Cox and Wang [3] and Carr and Lee [17] is that even though both papers work for any continuous positive martingale, Cox and Wang [3]'s method is less explicit than Carr and Lee [17] but is optimal as opposed to sub-optimal. Due to the limit of time I am unable to provide an example fully shows the optimality of Cox and Wang [3]. This will be taken as a further study in this subject.

For the second part of conclusion I would like to provide some other further steps that might be taken to give a more profound understanding of the subject. The first extension that comes to mind is to develop more Root's barriers by numerically implementing the obstacle problems. With barely some simple examples in this paper, I can not fully represent the strong connection of the obstacle problem and the Skorokhod embedding problem. Moreover, the subhedging strategy given by Cox and Wang [3] can be applied to other variance options besides a variance swap and a variance call. Also, the different approaches taken by Cox and Wang [3] and Carr and Lee [17] do not limit to what I have discussed here. The deduction process in Carr and Lee [17] can be viewed as solving a backward Kolmogorov equation while the obstacle problem in Cox and Wang [3] can be viewed as solving a forward Kolmogorov equation. I suppose there is much more to investigate in this area.

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