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Optimal Seedings in Elimination Tournaments

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Optimal Seedings in Elimination Tournaments:

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Abstract

We study an elimination tournament with heterogenous contestants whose ability is common-knowledge. Each pair-wise match is modeled as an all-pay auction where the winner gets the right to compete at the next round. Equilibrium efforts are in mixed strategies, yielding rather complex play dynamics: the endogenous win probabilities in each match depend on the outcome of other matches through the identity of the expected opponent in the next round. The designer can seed the competitors according to their ranks. For tournaments with four players we find optimal seedings with respect to three different criteria: 1) maximization of total effort in the tournament; 2) maximization of the probability of a final among the two top ranked teams; 3) maximization of the win probability for the top player. In addition, we find the seedings ensuring that higher ranked players have a higher probability to win the tournament. Finally, we compare the theoretical predictions with data from NCAA basketball tournaments.

JEL Classification Numbers: D72, D82, D44

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1 Introduction

Kentucky and Arizona, the highest-ranked teams to reach the Final Four during the 2003 National Collegiate Athletic Association (NCAA) Basketball March Madness, were on the same bracket and therefore could meet only in the semifinal. Not for the first time, an emotional debate began: should the Final Four teams be reseeded after the regional finals, placing the two top teams in separate national semifinals with the highest ranked team facing the lowest ranked?[1]

This paper offers a simple game-theoretic model of an elimination tournament which allows to study the effect of seedings on several performance criteria related to a tournament's outcome. For example, our model predicts that the probability of a final among the two top-ranked teams without reseeding (i.e., as occurring under random seeding) is in fact equal to the probability of such a final if reseeding is done according to the method described above[2]. But, we also predict that such a reseeding will increase the probability that the top-ranked team actually wins the tournament and show that there exists a reseeding - different to the one described above - which implies a final among the two top-ranked teams with probability one. We present these and other theoretical results and compare some of the obtained predictions with historical data from the NCAA basketball tournament.

In single elimination (or knockout) tournaments teams or individual players play pair-wise matches. The winner advances to the next round while the loser is eliminated from the competition. Many sportive events (or their respective final stages, sometimes called playoffs) are organized in such a way. Examples are the ATP tennis tournaments, professional playoffs in US-basketball, -football, -baseball and -hockey, NCAA college basketball, the FIFA (soccer) world-championship playoffs, the UEFA champions' league, Olympic disciplines such as fencing, boxing and wrestling, and top-

[1]For example, a similar event occured in 1996 where the top ranked Kentucky and Massachusetts also met in a semifinal that was thought to be the "real" final. For the full story see USA Today, March 25, 2003. We assume that all US readers are experts in the mechanics of this tournament. Ignorants (this group previously included the present authors) can find useful information at http://www.sportsline.com/collegebasketball/.

[2]That means, the top team meets the lowest ranked team in one semifinal, while the second and third ranked teams meet in the other.

level bridge and chess tournaments. There also numerous elimination tournaments among students that solve scientific problems, and even tournaments among robots or algorithms that perform certain tasks. Less rigidly structured variants of elimination tournaments are also used within firms, for promotions or budgeting decisions, and by committees who need to choose among several alternatives.

A widely observed procedure in elimination tournaments is to rank competitors based on some historically observed performance, and then to match them according to their ranks: the team or player that is historically considered to be best (or ranks first after some previous stage of the tournament) meets the lowest ranked player, the second best team meets the second lowest team and so on. In the second round, the winner of the highest ranked vs. lowest ranked match meets the (expected) lowest ranked winner from the first round, and so on[3]. The above design logic is deeply ingrained in our mind. For example, Webster's College Dictionary defines the relevant meaning of the verb "to seed" as:

> "**a.** to rank (players or teams) by past performance in arranging tournament pairings, so that the most highly ranked competitors will not play each other until later rounds. **b.** to arrange (pairings or a tournament) by means of such a ranking."

As the above quotation makes clear, the raison d'être of seeding is to protect top teams from early elimination: two teams ranked among the top 2^N should not meet until the field has been reduced to 2^N or fewer teams. In particular, the two best teams can meet only in the final, and, with the above seeding method, indeed meet there if there are no surprises along the way. Presumably, this delivers the most thrilling match in the final. An outcome where these teams meet in an earlier round greatly reduces further interest in the tournament and probably does not make financial sense.

From the large literature on contests, however, we know that expected effort and win probabilities in any two-player contest does not only solely depend on the absolute strength (or ability) of the respective players, but also on their relative strength (see

[3]This design is used, for example, in the professional basketball (NBA)- and ice hockey (NHL)-playoffs.

for example Baye et al. (1993)). For example, if the difference in ability between the best and second-best team is larger than the difference between the second and the third, a final between the second and third best teams may induce both more effort and "thrill" (in the sense of more symmetric expected probabilities of winning) than a final between the two strongest teams. Consequently, there might be (at least theoretically) rationales for various seedings.

There are many possible seedings in an elimination tournament. The reader may amuse herself/himself by calculating that, with 2^N players, there are $\frac{(2^N)!}{2^{(2^N-1)}}$ different seedings. This yields 3 seedings for 4 players, 315 seedings for 8 players, 638.510.000 seedings for 16 players and $6.126\,5\times 10^{25}$ seedings for 32 players.

There is a significant statistical literature on the design of various forms of elimination tournaments (the pioneering paper[4] is David (1959) who considered the win probability of the top player in a four player tournament with a random seeding). This literature assumes that, for each encounter among players i and j, there is a fixed, exogenously given probability that i beats j. In particular, this probability does not depend on the stage of the tournament where the particular match takes place, and does not depend on the identity of the expected opponent at the next stage[5]. Most results in that literature offer formulas for computing overall probabilities with which various players may win the tournament. For specific numerical examples it has been noted that the seeding where best meets worst, etc...yields for the top ranked player a higher probability of winning than a random seeding. Several papers (see for example, Hwang (1982), Horen and Reizman (1985) and Schwenk (2000)) consider various optimality criteria for choosing seedings. Given the sheer number of possible seedings and match outcomes, there are no general results for tournaments with more than four players. In particular, the optimal seeding for a given criterion may depends on the particular matrix of win probabilities (see Horen and Reizman (1985) who consider general (fixed) win probabilities and analyze tournaments with four and eight players).

In contrast to the above mentioned literature, we consider here a tournament model

[4]See also Glenn (1960) and Searles (1963) for early contributions.

[5]Additional assumptions are that $i's$ probability to win against j is larger than vice-versa if i is higher ranked than j (and thus it is at least 50%), and that the win probability decreases if the opponent's rank is increased. Probability matrices satisfying these conditions are called *doubly-monotone.*

where forward looking agents exert effort in order to win a match and advance to the next stage. We assume that players have different, common knowledge abilities and we model each match among two players as an all-pay auction: the prize for the winner of a particular match is either the tournament's prize if that match was the final, or else the right to compete at the next round. As a result, win probabilities in each match become *endogenous* - they result from mixed equilibrium strategies, and are positively correlated to ability. Moreover, the win probabilities depend on the stage of tournament where the match takes place and on the identity of the future expected opponents (which are determined in other parallel matches). Thus, in order to determine the tournament's outcome, we need to compute an intertwined set of pair-wise equilibria for each seeding[6]. In this paper we provide full analytic solutions for the case of four players (or three different seedings in the semifinals).

Since the players' ranking is assumed to be common-knowledge, it can be used by the designer in order to determine the tournament's seeding structure, and we look for the optimal seeding from the designer's point of view. In reality there are many possible designer's goals, tailored to the role and importance of the competition, to local idiosyncracies (such as fan support for a home team), and depending on commercial contracts with large sponsors (that may be also related to prominent competitors), or with media companies. We consider here three separate optimality criteria, and, additionally, a "fairness" criterion:

1. Find the seeding(s) that maximizes total expected effort in the tournament.
2. Find the seeding(s) that maximizes the probability of a final among the two highest ranked players.
3. Find the seeding(s) that maximizes the win probability of the highest ranked player.
4. Find the seeding(s) with the property that higher ranked players have a higher probability of winning the tournament.

Note that deterministic seedings are inherently "unfair" since they favor certain players, while handicapping others. Our first optimality criterion is "conservative"

[6]This requires solving for fixed points.

in the sense that it treats all matches in the tournament symmetrically, and it does not a-priori bias the decision in favor of top players. The statistical literature did not analyze this criterion since there are no strategic decisions (e.g., about effort) in their models. In contrast, the other two optimality criteria have been discussed in the statistical literature, and seem to be prevalent in practice. The last criterion (also discussed in the past) poses a constraint on the unfairness of a seeding by requiring that the overall win probabilities are naturally ordered according to the players' ranking. If this property does not hold for a given seeding, anticipating players have a perverse incentive to manipulate their ranking (e.g., by exerting less effort in a qualifying stage).

Our main findings are as follows: Let the four players be ranked in decreasing order of strength: 1,2,3,4. Seedings specify who meets whom in the semifinals. It turns out that the seeding most observed in practice[7], A:1-4,2-3, maximizes the win probability of the strongest player, and is the unique one with the property that strictly stronger players have a strictly higher probability of winning (criteria 3 and 4). On the other hand, seeding B:1-3,2-4 maximizes both total effort across the tournament and the probability of a final among the two top players (criteria 1 and 2). Seeding C:1-2,3-4, under which the two top players meet already in the semifinal, does not satisfy any of the optimality or fairness criteria, and the same holds for a random seeding[8].

An important feature of our findings is that the above results do not depend on cardinal differences between players' strength (which are hard to measure), but only on the ordinal ranking specifying who is stronger than whom. This allows us to compare the theoretical predictions to real-life tournaments where, in most cases, the remaining players in the semifinals need not be the a-priori highest ranked four players. To understand the nature of such an exercise, consider the four regional conferences of the NCAA basketball tournament. In each conference 16 ranked teams play in an elimination tournament whose winner goes on to play in the national semifinals. Needless to say, the conference semifinals are not necessarily played by the four originally highest ranked teams. For example, the 2002 Midwest semifinals where Kansas (1)-Illinois (4)

[7]This is also the seeding method proposed for the Final Four.

[8]In a recent paper Schwenk (2000) argues for *cohort randomized* seeding based on three fairness criteria. In cohort randomized seeding players are first divided in several cohorts according to strength (say top, middle, bottom) and players in the same cohort are treated symmetrically in the randomization.

and Oregon (2)-Texas (6). Since the top ranked team (Kansas) played against the third highest ranked team among the remaining ones (Illinois), this semifinal corresponds to our seeding B:1-3,2-4. The West semifinals were Oklahoma (2)-Arizona (3) and UCLA (8)-Missouri (12). Since the two top remaining teams (Oklahoma and Arizona) meet already in the semifinal, this corresponds to our seeding C:1-2,3-4 . The 2001 South semifinals were Tennessee (4)-N.Carolina (8) and Miami (6)-Tulsa (7). Since the highest ranked remaining team (Tennessee) plays against the lowest ranked remaining one (N.Carolina), this corresponds to seeding A. In this way, available data can generate observations for all three possible seedings, even if the initial method of seeding at the beginning of the tournament is fixed. We find that the data from college basketball tournaments are broadly in line with our predictions from the game theoretic model about optimal seeding.

We conclude our Introduction by mentioning several related papers from the economics literature. In a classical piece, Rosen (1986) looks for the optimal prize structure in an elimination tournament with homogeneous players where the probability of winning a match is a stochastic function of players' efforts. In the symmetric equilibrium, the winner of every match is completely determined by the exogenous stochastic terms[9]. In Section IV he also considers an example with four players that can be either "strong" or "weak". Rosen finds (numerically) that a random seeding yields higher total effort than the seeding where strong players meet weak players in the semifinals. He did not consider the seeding strong/strong and weak/weak in the semifinals, but, in his numerical example, it turns out, for example, that this seeding (which corresponds then to our seeding C:1-2,3-4) yields the highest total effort.

As a by-product of our analysis, we show that total expected effort in the elimination tournament where the two strongest players meet in the final with probability one (seeding B:1-3,2-4) equals total effort in the all-pay auction where all players compete simultaneously. This should be contrasted with the main finding of Gradstein and Konrad (1999) who study a rent-seeking setting à la Tullock with homogenous players. They found that simultaneous contests are strictly superior if the contest's rules are discriminatory enough (as in an all-pay auction). In a setting with heteroge-

[9]His main result is that rewards in later stages must be higher than reward in earlier stages in order to sustain a non-decreasing effort along the tournament.

nous valuations, our analysis indicates that, for the Gradstein-Konrad result to hold, it is necessary that the multistage contest induces a positive probability that the two strongest players do *not* reach the final (e.g., our seedings A:1-4,2-3 and C:1-2,3-4) with probability one.

Baye et al. (1993) look for the optimal set of contestants in an all-pay auction, and they find that it is sometimes advantageous to exclude the strongest player. These authors do not consider explicit mechanisms by which finalists are selected. Our analysis suggests that, given the constraints imposed by the structure of an elimination tournament, it never pays off to exclude the strongest player from the final[10].

While we ask which types of players should be in the final, Amegashie (1999) determines the optimal *number* of finalists in a two-stage contest à la Tullock with homogenous players[11].

The paper is organized as follows. We present the tournament model in section 2. In section 3 we present the optimality results, and briefly illustrate the employed techniques. In section 4 we compare the theoretical results with historical data from the NCAA tournament. In section 5 we gather several concluding remarks. All proofs and additional remarks concerning the empirical part are in an Appendix.

2 The Model

There are four players (or teams) $i = 1, ..., 4$ competing for a prize. The prize is allocated to the winner of a contest which is organized as an elimination tournament. First, two pairs of players simultaneously compete in two semifinals. The two winners (one in each semifinal) compete in the final, and the winner in the final obtains the prize. The losers of the semifinals do not compete further. We model each match among two players as an all-pay auction: both players exert effort, and the one exerting the higher effort wins.

[10] Clark and Riis (1998) show that it does not pay off to exclude the strongest player from a simultaneous moves contest if there are several prizes.

[11] Increasing the number of finalists has several effects: efforts in the final and also efforts at the previous stage are lower since the probability of winning the final decreases. But, there is also a positive effect on effort since the probability of getting to the final increases.

Player i values the prize at v_i, where $v_1 \geq v_2 \geq v_3 \geq v_4 > 0$. Valuations are common-knowledge. The heterogeneity in valuations should be interpreted as arising from heterogeneity in abilities: if effort is less costly for more able contestants, the players with higher valuations can be thought of as being more able.

We assume that each finalist obtains a payment $k > 0$, independent from his performance in the final[12], and we consider the limit behavior as $k \to 0$. This technicality is required in order to ensure that all players have positive present values when competing in the semifinals - this is a necessary condition for the existence of equilibria in the semifinals.

In a final between players i and j, the exerted efforts are e_i , e_j. Net of k, the payoff for player i is given by

$$u_i^F(e_i, e_j) = \begin{cases} -e_i & \text{if } e_i < e_j \\ \frac{v_i}{2} - e_i & \text{if } e_i = e_j \\ v_i - e_i & \text{if } e_i > e_j \end{cases} \tag{1}$$

and analogously for player j. Player $i's$ payoff in a semifinal between players i and j is given by

$$u_i(e_i^S, e_j^S) = \begin{cases} -e_i^S & \text{if } e_i^S < e_j^S \\ \dfrac{Eu_i^F + k}{2} - e_i^S & \text{if } e_i^S = e_j^S \\ Eu_i^F + k - e_i^S & \text{if } e_i^S > e_j^S. \end{cases} \tag{2}$$

and analogously for player j. Note that each player's payoff in a semifinal depends on the expected utility associated with a participation in the final. In turn, this expected utility depends on the expected opponent in the final. It is precisely this feature that can be "manipulated" by designing the seeding of the semifinals. The contest designer chooses the structure s of the semifinals out of the set of feasible seedings $\{A, B, C\}$, where:

- A:1-4,2-3
- B:1-3,2-4

[12]There are many examples where such a feature is indeed present.

- C:1-2,3-4.

The following well-known Lemma characterizes behavior in an all-pay auction among two heterogenous players.

Lemma 1. *Consider two players* i *and* j *with* $0 < v_j \leq v_i$ *that compete in an all-pay auction for a unique prize. In the unique Nash equilibrium both players randomize on the interval* $[0, v_j]$. *Player* i*'s effort is uniformly distributed, while player* j *'s effort is distributed according to the cumulative distribution function*[13] $G_j(e) = (v_i - v_j + e)/v_i$. *Given these mixed strategies, player* i'*s winning probability against* j *is given by* $q_{ij} = 1 - \frac{v_j}{2v_i}$. *Player* i*'s expected effort is* $\frac{v_j}{2}$, *and player* j*'s expected effort is* $\frac{v_j^2}{2v_i}$. *Total expected effort is therefore*

$$\frac{v_j}{2}(1 + \frac{v_j}{v_i}). \tag{3}$$

The respective expected payoffs are

$$u_i = v_i - v_j \tag{4}$$

and

$$u_j = 0. \tag{5}$$

Proof. See Hillman and Riley (1989) and Baye et al. (1993). □

[13] Note that this distribution has an atom of size $(v_i - v_j)/v_i$ at $e = 0$.

3 Semifinals Design

We provide here optimal seedings for the last crucial stage requiring design - the semi-finals. Recall that the proponents of a change in the March Madness rules suggest to design the national semifinals after the results of the previous stages become known- thus our model precisely provides tools for assessing a change in the rules.

We first verbally sketch the main intuition behind the results. We next offer an illustration for the simple special case where there are two equally strong and two equally weak players (this can be compared to Rosen's example mentioned in the Introduction). Finally, we present the general optimality results for the various criteria.

3.1 Intuition

Let us first look at design A:1-4,2-3. As k goes to zero, player 1 reaches the final with almost certainty. This happens because player 4 expects a limit payoff of zero no matter which player (either 2 or 3) she meets in a final. A-priori, players 2 and 3 are not in a symmetric position: while both would obtain a limit payoff of zero in a final against player 1, their expected payoffs are positive but different in a final against player 4 (with 2 having the higher valuation). But, since the event of meeting 4 in a final has a zero limit probability, the position of 2 and 3 becomes symmetric: since both know that they are going to meet the stronger player 1 in the final, their limit expected valuation for the final is zero. Hence, both reach the final with probability one-half and meet there player 1.

In design B:1-3,2-4, player 4 has a limit expected utility of zero in *any* final (where he meets either 1 or 3 - both stronger players), whereas player 2 has a positive expected value stemming from the event where he might meet player 3 in the final. In the limit, player 2 reaches the final with probability one. But then, player 3 does not expect a positive payoff in the final. Hence, player 1 reaches the final with probability one, and meets there player 2.[14]

[14]Of course, as k gets small, neither 2 nor 4 have a positive valuation for the final where they meet player 1 for sure; but player 2's valuation converges to zero at a slower rate, confirming the above logic.

Since the expected final in design B:1-3,2-4 (among players 1 and 2) is tighter than the expected final in design A:1-4,2-3 (where 1 meets either 2 or 3 , each with probability one-half), design B:1-3,2-4 dominates design A:1-4,2-3 with respect to total effort. However, the comparison between seedings B:1-3,2-4 and C:1-2,3-4 with respect to total effort is more subtle.

In design C:1-2,3-4 all four possible finals have a positive probability since both stronger players expect a positive payoff in a final , and both weak players expect a zero payoff. An important observation is that a semifinal among players 1 and 2 yields less total effort than a final among these players because both players anticipate that, in order to ultimately win the tournament, they need to exert an additional effort in the final. The decrease in effort caused by the fact that 1 and 2 meet already in the semifinal cannot be compensated by the additional effort in a final among one of the stronger players and one of the weaker players, and seeding B:1-3,2-4 also dominates seeding C:1-2,3-4 with respect to total effort.

Recall that in seeding B:1-3,2-4 there is a final among players 1 and 2 with limit probability one. Hence, player 1's overall win probability equals the probability with which he wins a final against player 2. In seeding A:1-4,2-3 player 1 also reaches the final with limit probability one, but meets there either player 2 or player 3 (with equal limit probabilities). Since player 1 is more likely to win a final against player 3 than a final against player 2, we easily obtain that seeding A:1-4,2-3 clearly dominates seeding B:1-3,2-4 with respect to the top player's win probability.

The comparison between seedings C:1-2,3-4 and A:1-4,2-3 with respect to the top player's win probability is more subtle: player 1 is more likely to win the final in seeding C:1-2,3-4 (where he meets either player 3 or player 4) than in seeding A:1-4,2-3 (where he meets either 2 or 3) . But, in seeding A:1-4,2-3 player 1 makes it to the final for sure, while in seeding C :1-2,3-4 only with some probability less than one (since he first has to win the semifinal against player 2). It turns out that this last handicap is significant, and it is always the case that seeding A:1-4,2-3 yields a higher overall win probability for player 1.

3.1.1 The Two-Type Case

We briefly consider here the case where $v_1 = v_2 = v_H > v_L = v_3 = v_4$. Obviously, seedings A:1-4,2-3 and B:1-3,2-4 are here equivalent.

Seedings A:1-4,2-3 and B:1-3,2-4. Let $q_{ij}^S(k)$ denote the probability that i beats j in a semifinal among i and j. Based on these probabilities we can compute expected values for the final, conditional on winning a semifinal. By Lemma 1, the winning probabilities q_{ij}^S are themselves determined by these conditional expected values. Thus, the equilibrium is found by solving for a fixed-point.

Conditional on winning the semifinal, player 1 faces player 2 in the final with probability $q_{23}^S(k)$. This results in a payoff of zero for both finalists since they are of equal strength. Player 1 meets player 3 in the final with probability $1 - q_{23}^S(k)$. Since 3 has valuation $v_L < v_H$, player 1 expects a payoff of $v_H - v_L$ in that case. In any case, there is the additional payoff k for making it to the final. Thus, player 1's expected value from winning the semifinal is given by

$$q_{23}^S(k) \cdot 0 + (1 - q_{23}^S(k))(v_H - v_L) + k = (1 - q_{23}^S(k))(v_H - v_L) + k. \tag{6}$$

Analogously, the expected value for player 2 is given by

$$(1 - q_{14}^S(k))(v_H - v_L) + k, \tag{7}$$

In the final Player 4 faces player 2 with probability $q_{23}^S(k)$ and player 3 with probability $1 - q_{23}^S(k)$. Player 4's expected payoff is k in both cases, and analogously for player 3.

Given the above computed values, Lemma 1 tells us that the winning probabilities $q_{14}^S(k)$ and $q_{23}^S(k)$ are determined by the following equations:

$$q_{14}^S(k) = 1 - \frac{k}{2[(1 - q_{23}^S(k))(v_H - v_L) + k]} \tag{8}$$

and

$$q_{23}^S(k) = 1 - \frac{k}{2[(1 - q_{14}^S(k))(v_H - v_L) + k]}. \tag{9}$$

Solving the above system (under the restriction $q \in [0, 1]$) yields the symmetric solution

$$q_{14}^S(k) = q_{23}^S(k) = 1 + \frac{k}{2(v_H - v_L)} - \frac{1}{2(v_H - v_L)}\sqrt{(2(v_H - v_L) + k)k} \tag{10}$$

By Lemma 1, in each of the two semifinals the expected effort is given by

$$\frac{1}{2}k+\frac{1}{2}\frac{k^2}{(1-q^S(k))(v_H-v_L)+k} \tag{11}$$

where $q(k)\in\{q^S_{14}(k),q^S_{23}(k)\}$. Note that

$$\lim_{k\to 0} q^S_{14}(k)=\lim_{k\to 0} q^S_{23}(k)=1 \tag{12}$$

Intuitively, the weak players have only a small chance to win the final, and hence exert almost no effort. This implies that the strong players do not have to exert a lot of effort in the semifinals either. Moreover, each of the strong players knows that he is going to meet the other strong player in the final (and thus that the payoff from the final will be low). This reduces the strong players' valuation for winning the semifinals.

Players 1 and 2 meet in the final with probability 1 (as k tends to zero). Since both 1 and 2 have the same valuation v_H, total expected effort in the final is v_H.

Seeding C:1-2,3-4. The final will be between a player with valuation v_H and a player with valuation v_L. Hence, by Lemma 1, expected effort in the final is $\frac{v_L}{2}\left(1+\frac{v_L}{v_H}\right)$.

Consider first the semifinal between the strong players 1 and 2. Since the winner of this semifinal will meet a weak player in the final, both 1 and 2 expect a payoff v_H-v_L+k in the final. By Lemma 1 , total expected effort in this semifinal is v_H-v_L+k (note that, for small k, this is less than total effort in a final among two strong players, which yields a total effort of v_H).

Consider now the semifinal between the weak players. Both have an expected payoff of k in the final since this is the payoff in a final against a strong competitor with valuation v_H. Hence, total expected effort in this semifinal is also k.

Total limit effort in seeding C:1-2,3-4 is thus given by:

$$TE_C=\lim_{k\to 0}[\frac{1}{2}v_L\left(1+\frac{v_L}{v_H}\right)+v_H-v_L+2k]=v_H-\frac{1}{2}v_L\left(1-\frac{v_L}{v_H}\right) \tag{13}$$

We can conclude that

$$TE_A=TE_B=v_H>v_H-\frac{1}{2}v_L\left(1-\frac{v_L}{v_H}\right)=TE_C. \tag{14}$$

3.2 Total Effort

We now return to the general four-player case, and we assume first that the designer chooses the seeding s in order to maximize total expected tournament effort TE_s . Let R be the set of all players, and let $F(s)$ be the set of players reaching the final for a given seeding s. The designer solves

$$\max_{s\in\{A,B,C\}}\{\sum_{i\in R} e_i^S + \sum_{i\in F(s)} e_i^F\} \tag{15}$$

Proposition 1. *For any valuations, the limit total tournament effort (as k goes to zero) is maximized in seeding B:1-3,2-4, where it equals* $\frac{1}{2}(v_2 + \frac{v_2^2}{v_1})$ *. This also equals the total effort in a one-shot simultaneous contest among all players*[15].

Proof: See Appendix.

3.3 Probability of a Final among the Two Top Players

We now assume that the designer chooses the seeding s in order to maximize the probability of a final among the two top players. As already indicated in the previous section, seeding B:1-3,2-4 is again optimal.

Proposition 2. *For any valuations, a final among players 1 and 2 occurs with limit probability one in seeding B:1-3,2-4, and with limit probability of one-half in seeding* A *:1-4, 2-3.*[16]

Proof: See Lemmas 2, 3, 4 in the Appendix.

Interestingly, our model predicts that the probability of a final among the two top players under random seeding (which roughly corresponds to the method now employed by the NCAA for the Final Four) is $\frac{1}{3}(1 + \frac{1}{2} + 0) = \frac{1}{2}$, which equals the probability of such a final under seeding A:1-4,2-3. Thus, reseeding according to A:1-4,2-3 is not likely

[15] This "revenue-equivalence" result is interesting from a theoretical point of view; at least in sports it is not feasible to organize a simultaneous contest among all participants.

[16] The probability for seeding C :1-2,3-4 is obviously zero.

to increase the probability of a final among the two top players, but, as we show in the next section, it will increase the probability that the top team wins the tournament.

Let us briefly discuss the relevance of the above finding for elimination tournaments among 2^N players where $N > 2$ is the number of rounds needed to produce a winner.

Order the agents by their valuations $v_1 \geq v_2 ... \geq v_{2^N}$ Let M_{ij} denote a match among players i and j, and let $M^w_{(ij)(hl)}$ denote a match among the winners in the matches M_{ij} and M_{hl}.

Definition 1. *We say that seeding s eliminates player i in round $l < N$ if, as k tends to zero, the probability that i reaches stages $l + 1$ (given that she reached stage l) tends to zero. Obviously, at most 2^{N-l} players can be eliminated at stage l.*

For example, recall our results for round $l = 1$ of a tournament with 4 players (see Lemmas 2, 3 and 4 in the Appendix): Seeding C:1-2,3-4 does not eliminate any player, and all four possible finals have positive probability. Seeding A :1-4,2-3 eliminates only player 4 and the finals 1-2 and 1-3 have both positive probability. Finally, the optimal seeding B :1-3,2-4 eliminates both players 3 and 4, and only the final 1-2 has positive probability (one).

It turns out that it is always possible to seed the players such that the two strongest players participate in the final with probability one (as k tends to zero). Consider for example a tournament among 8 players with the following structure of matches: Round 1: $M_{18}, M_{27}, M_{36}, M_{45}$;Round 2: $M^w_{(18)(36)}, M^w_{(27)(45)}$; Round 3: final among winners in semifinals. It is easy to see that players $6, 7$ and 8 are eliminated at stage 1. This seeding does not eliminate players 4 and 5 at stage 1 since they are in symmetric positions given their possible future opponents. Thus we obtain either the semifinals 1-3,2-4 or 1-3,2-5. By the logic of seeding B:1-3,2-4, the two respective weaker players get eliminated in stage 2, and we again obtain the desired final among the two best players.

It is important to note that, whereas in the four-player case seeding B:1-3,2-4 was the unique one with the property that it ensures a final among the two best players, there is more design freedom if there $2^N > 4$ players.

Finally, we conjecture that any seeding where the two top players meet in the final for sure is also maximizing total expected effort.

3.4 The Top Player's Win Probability

Seeding B:1-3,2-4 was found to be optimal for the previous two criteria. But recall that seeding A:1-4,2-3 is the one most often observed in real tournaments. It is reassuring to find that seeding A:1-4,2-3 is optimal with respect to the important criterion of maximizing the top player's win probability.

Proposition 3. *For any valuations, player 1's limit win probability (as k goes to zero) is maximized in seeding A:1-4,2-3.*

Proof: See Appendix.

3.5 Fairness

We now study the win probabilities of all players, and check which seedings have the property that the probabilities to win the tournament are naturally ordered according to the players' ranking.

Proposition 4. *Let $p_i(s)$ denote the probability that player i wins the tournament for a given seeding $s \in \{A, B, C\}$. We have:*

1. $p_1(A) > p_2(A) > p_3(A) > p_4(A)$;
2. $p_1(B) > p_2(B) > p_3(B) = p_4(B) = 0$.
3. In seeding C :1-2,3-4 it may happen that $v_i > v_j$ but $p_i(s) < p_j(s)$.

Proof: See Appendix.

4 Some Empirical Observations

Do these theoretical results bear any empirical relevance? To investigate the empirical content of the theory presented above, we confront the theoretical propositions with the data from 100 regional NCAA college basketball tournament already mentioned in the introduction. The data contain results for semi-finals and finals of the four regional tournaments (East, West, Midwest, South) over the period 1979 to 2003. All 16 participating teams in the respective regional elimination tournaments are initially ranked according to their previous performance during the selection process. This allows to identify seedings A:1-4, 2-3, B:1-3,2-4 or C:1-2,3-4 at the stage of semifinals using the rankings as proxy for the teams' valuations.[17] The availability of a ranking of the teams in each region *at the beginning* of the respective tournament allows to construct a relative ordering of the four teams left in the semi-finals, and to reveal the seeding of the tournament at the level of semi-finals. Because already two rounds have been played prior to the semi-final, and only those teams who won elimination tournaments on the previous levels are left, the seedings are presumably random and unaffected by the initial seeding of the tournament.[18] As shown in Table 1, we observe all seedings, although A:1-4,2-3 (41 observations) and B:1-3,2-4 (44 observations) are almost three times more common than C:1-2,3-4 (15 observations).

[17]Data and relevant links can be found on the internet under http://old.sportsline.com/u/madness/2002/history/index.html. During the selection process, the basketball selection committee elects the 64 "best" college teams of the respective season and allocates them to the regional conferences. This selection is based on several measures reflecting the teams' recent performance, including a "rating percentage index". In principle, the teams are then split into groups of four according to their ranking. The four highest ranked teams are allocated to the four regions, always only one team to one region, followed by the split of the next group of four teams, etc., until all teams are allocated to regions. The aim of this procedure is to balance the brackets in all regions. The regional brackets are initially seedings of type A of the respective 16 teams. Teams' regional allocation may change from one year to another. Details about rules and working of the college basketball championship are contained in the respective Division I Men's Basketball Championship Handbook, available on the internet under http://www.ncaa.org/library/handbooks/basketball/2003.

[18]This is only true if ordinal seedings are concerned, as is the case here. Particular combinations regarding the cardinal ranking of teams, e.g. semifinals among the two strongest regional teams, are not possible as consequence of the seeding A of regional brackets. We make the identifying assumption that during the first two rounds of the tournament there is sufficient noise in the match outcomes to generate a random selection of teams, and thus (ordinal) seedings. We also analyzed a modified sample to test the robustness of the results and to rule out any potential biases due to selection effects stemming from initial seedings. In particular, we excluded teams that could only achieve a certain relative rank in the semi-final in a particular seeding as consequence of the initial seeding, see the appendix 6.4 for details. The results are qualitatively identical to those obtained from the full sample.

The first testable implication of the theory is a consequence of Lemma 1 and states that, everything else equal, the weaker the underdog compared to the favorite, the higher the lower the total effort they exert in an all-pay auction, i.e. during a match (regardless whether semi-final or final). The main problem with testing theoretical predictions about effort exertion, however, is to find a good and observable proxy for effort. This is particularly difficult for such a complex team sport as basketball.[19] Taking the number of scored points as indicator for effort, one implicitly assumes that effort mostly reflects offensive effort, but neglects that higher effort might mean better defense, that is fewer points scored by the rival team. Comparing the number of scored points across matches teams with different relative rankings, displayed in Table 2, effort is indeed higher if more equal teams play against each other, or, respectively, better ranked underdogs play against a particular favorite, as predicted by theory. While the differences in means are not significant at conventional levels, the null that the number of points scored is the same regardless of the pairing is rejected in favor of the order suggested by the theory at the 1 percent level using a Jonckheere-Terpstra-test.[20]

Testing the result of Proposition 1 that seeding B:1-3,2-4 maximizes total tournament effort causes similar problems regarding the measuring of effort. Again, when looking at the total sum of points as proxy of effort as depicted in Table 1, the data support the theory, but the differences are not significant.[21]

Proposition 2 before claimed that the probability of a final between the two relatively strongest teams of the semi-finalists is higher in seeding B:1-3,2-4 than in A:1-4, 2-3 (while it is zero by definition in C:1-2,3-4). A look at the data provides little support for this claim: Under seeding B:1-3,2-4, 22 out of 44 observed finals are between the two highest ranked teams, while under seeding A:1-4,2-3 23 out of 41 finals involve the two strongest teams. However, closer inspections suggest that this difference is not significant: A dummy for seed A:1-4,2-3 turns out to be positive but insignificant in logit estimates of the event of a final between teams 1 and 2, and remains so when a

[19]In a well known study about the incentive effects of prizes in tournaments, Ehrenberg and Bognanno (1990) looked at golf tournaments. Their proxy for effort was the number of hits required for a round. In contrast to their application, basketball is more complex, since offensive and defensive are important, tactics plays a role, partly influenced by the composition of a team and its rival team, etc.

[20]The null of irrelevance of the pairings is also rejected when different groups of pairings are compared to each other.

[21]Experiments with alternative proxies for effort yield similar results.

full set of year or regional dummies are included as explanatory variables.[22]

A further testable result of the theory is that the probability of the strongest team winning the tournament is maximized by seeding A:1-4,2-3, see Proposition 3. Raw data mildly support this hypothesis: Under seeding A:1-4,2-3, the best team wins the tournament in 22 out of 41 cases, under seeding B:1-3,2-4 only in 18 out of 44 cases. However, in 9 out of 15 cases, the strongest team wins the tournament under seeding C:1-2,3-4 . To investigate this result in more depth, we adopt a modified version of the framework suggested by Klaassen and Magnus (2003) to forecast the winner of a tennis match. To test the influence of seedings on the winning probability of the best team in the tournament, we estimate a logit model of this event on a modified measure of ranks based on the expected round of elimination of a team as well as the respective seed.[23] Compared to seeding A:1-4, 2-3, both seeding B:1-3,2-4 and C:1-2,3-4 entail a significantly smaller probability of the strongest team winning the tournament. While the respective coefficient for seeding B:1-3,2-4 is significant only on the 10 percent level, the coefficient for seeding C:1-4,2-3 is significant at the 5 percent level. This result corroborates the theoretical claim.

As final result, Proposition 4 implies that the probability of winning the tournament is positively correlated with the ranking of the teams. The raw frequencies of winner types by seeding displayed in Table 1 illustrate that this prediction is approximately corroborated by the data. Higher ranked teams are more likely to win a tournament. The only seeding for which this relationship is not strictly monotonous, is seeding C:1-2,3-4 , which is in line with the theoretical prediction.

To sum up, data from college basketball tournaments are broadly in line with the predictions from the game theoretic model about optimal seeding presented above.

[22] The control group is seed B, the number of observations is 85.

[23] The modified measure of ranks is based on the round in which a team is expected to lose and drop out of the tournament. The rank of a team i, r_i, is given by $r_i = 2 - log_2(rank_i)$. The probability of winning the final is a function of the own rank of a finalist team as well as of the rank of the competing finalist team, r_j. The estimated specification is a logit model of the probability that the team ranked 1 prior to the tournament wins the final. The number of observations is 100. In addition to the difference in modified ranks, $d = r_i - r_j$ used by Klaassen and Magnus (2003), we use dummies for seedings B and C as explanatory variables. As in the Klaassen and Magnus paper, the effect of the rank difference d is positive and significant.

However, not all implications are reflected in the data. This has to do with practical difficulties to come up with a satisfactory empirical proxy for effort. Moreover, other factors than the respective seeding shape the tournament outcomes, so that it is difficult to identify any direct causal effects of seedings on teams' performance or the observed tournament results. Finally, due to data limitations, it is difficult to perform more detailed tests, leading to statistically more significant results. Therefore, the empirical observations reported here are to be taken as corroborating evidence for the relevance of the problem and the applicability of the theoretical results, rather than a concise test of the theory.

Table 1: **Seedings: Tournament winner and Total Effort** as a function of seeding in 100 regional conference tournaments (1979-2003)

Seeding (Observations)	All (100)	A (41)	B (44)	C (15)
Team ranked 1	48	53.66	38.64	60
Team ranked 2	25	19.51	31.82	20
Team ranked 3	17	14.63	18.18	20
Team ranked 4	10	12.20	11.36	0
Mean Total Effort (Points)	435.08	435.68	435.80	431.33
(Std.Dev.)	(46.82)	(55.08)	(42.13)	(36.91)

Note: Entries are frequencies in percentage points. Mean Total Effort (Points) is the mean of the sum of all points scored in semi-finals and final of a conference tournament.

Table 2: **Relative Strength of Teams: Composition of Finals and Total Effort** in 100 regional conference tournaments (1979-2003)

		Composition of Finals			Total Points per Match	
Match	All	Seeding A	Seeding B	Seeding C	Mean	Std.Dev.
1 vs 2	45	56.10	50.00	0	149.06	24.23
1 vs 3	15	24.39	0	33.33	148.54	22.81
1 vs 4	15	0	20.45	40.00	144.30	20.63
2 vs 3	10	0	15.91	20.00	142.16	22.39
2 vs 4	3	4.88	0	6.67	141.74	21.95
3 vs 4	12	14.63	13.64	0	141	21.11

Note: Entries for composition of finals are in percentage points.

5 Concluding Remarks

We have analyzed optimal seedings in an elimination tournament where players have to exert effort in order to advance to the next stage. We established that seedings involving a delayed encounter among the top players are optimal for a variety of criteria. We have also exhibited the effects of switching the ranks of the opponents that play against the top players in the semifinals. In principle, it is possible to generalize the analysis conducted here to tournaments with more players (and possibly more prizes). But, the exponentially growing number of seedings, and the complexity of the fixed-point arguments suggest that analytic solutions are difficult to come by.

Our model and results offer a wealth of testable hypothesis. We have compared its predictions to the results of the NCAA basketball tournaments. In a companion paper we look at a data set containing 150 ATP tennis tournaments

6 Appendix

All results are based on the three basic lemmas that determine equilibrium behavior for each seeding. Let $q_{ij}^S(k)$ denote the limit probability that i beats j if they meet in a semifinal, and let $q_{ij}^S = \lim_{k\to 0} q_{ij}^S(k)$. $TE_{ij}^S(k)$ denotes total equilibrium effort in a semifinal among i and j; $TE_s^F(k)$ denotes total equilibrium effort in a final resulting from seeding s; $TE_s(k)$ denotes total equilibrium effort in all three matches of seeding s, and define $TE_s = \lim_{k\to 0} TE_s(k)$;

Lemma 2. *Consider seeding A :1-4,2-3 . In the limit, as $k \to 0$, player 1 reaches the final with probability one, while players 2 and 3 reach the final with probability one-half each. In addition the following hold:*

$$\lim_{k\to 0} (TE_{14}^S(k) + TE_{23}^S(k)) = 0 \tag{16}$$

and

$$TE_A = \lim_{k\to 0} TE_A(k) = \lim_{k\to 0} TE_A^F(k) = \frac{1}{4}\left(v_2 + \frac{v_2^2}{v_1} + v_3 + \frac{v_3^2}{v_1}\right) \tag{17}$$

Proof: By equations (4) and (5), player 1's valuation for the semifinal is $(1 - q_{32}^S(k))(v_1 - v_2 + k) + q_{32}^S(k)(v_1 - v_3 + k)$ and player 4's valuation is k. By equation (3), we know that the total expected effort in this semifinal is given by

$$TE_{14}^S(k) = \frac{1}{2}k + \frac{1}{2}\frac{k^2}{(1 - q_{32}^S(k))(v_1 - v_2 + k) + q_{32}^S(k)(v_1 - v_3 + k) + k}. \tag{18}$$

Player's 4 probability of winning is given by

$$q_{41}^S(k) = \frac{1}{2}\frac{k}{(1 - q_{32}^S(k))(v_1 - v_2 + k) + q_{32}^S(k)(v_1 - v_3 + k) + k} \tag{19}$$

Players 2 and 3 play in the other semifinal. Their valuations for the semifinal are $q_{41}^S(k)(v_j - v_4 + k) + (1 - q_{41}^S(k))k,\ j = 2, 3$, and expected total efforts in this semifinal is given by

$$\begin{aligned} TE_{23}^S(k) \quad &= \quad \frac{1}{2}[q_{41}^S(k)(v_3 - v_4 + k) + (1 - q_{41}^S(k))k] + \\ &\quad\quad \frac{1}{2}\frac{(q_{41}^S(k)(v_3 - v_4 + k) + (1 - q_{41}^S(k))k)^2}{q_{41}^S(k)(v_2 - v_4 + k) + (1 - q_{41}^S(k))k} \end{aligned} \tag{20}$$

Player's 3 probability of winning is given by

$$q_{32}^S(k) = \frac{1}{2}\frac{q_{41}^S(k)(v_3 - v_4 + k) + (1 - q_{41}^S(k))k}{q_{41}^S(k)(v_2 - v_4 + k) + (1 - q_{41}^S(k))k}. \tag{21}$$

In the limit, as $k \to 0$, the unique fixed point is $q^S_{41} = 0$ and $q^S_{32} = 1/2$. We have then

$$TE^F_A = \frac{1}{4}\left(v_2 + \frac{v_2^2}{v_1} + v_3 + \frac{v_3^2}{v_1}\right)$$

Q.E.D.

Lemma 3. *Consider seeding B:1-3,2-4. In the limit, as $k \to 0$, the final takes place among players 1 and 2 with probability one. In addition, the following hold:*

$$\lim_{k\to 0}\left(TE^S_{13}(k) + TE^S_{24}(k)\right) = 0 \tag{22}$$

and

$$TE_B = \lim_{k\to 0} TE_B(k) = \lim_{k\to 0} TE^F_B(k) = \frac{1}{2}(v_2 + \frac{v_2^2}{v_1}) \tag{23}$$

Proof: Player 1's valuation for the semifinal is $(1-q^S_{42}(k))(v_1-v_2+k)+q^S_{42}(k)(v_1-v_4+k)$ and player 3's valuation is $(1-q^S_{42}(k))k + q^S_{42}(k)(v_3 - v_4 + k)$. Total expected effort in this semifinal is given by

$$\begin{aligned} TE^S_{13}(k) &= \frac{1}{2}((1-q^S_{42}(k))k + q^S_{42}(k)(v_3 - v_4 + k)) + \\ & \frac{1}{2}\frac{((1-q^S_{42}(k))k + q^S_{42}(k)(v_3 - v_4 + k))^2}{(1-q^S_{42}(k))(v_1 - v_2 + k) + q^S_{42}(k)(v_1 - v_4 + k)} \end{aligned} \tag{24}$$

Player's 3 probability of winning is given by

$$q^S_{31}(k) = \frac{1}{2}\frac{(1-q^S_{42}(k))k + q^S_{42}(k)(v_3 - v_4 + k)}{(1-q^S_{42}(k))(v_1 - v_2 + k) + q^S_{42}(k)(v_1 - v_4 + k)} \tag{25}$$

Players 2 and 4 play in the other semifinal. Their valuations for the semifinal are $q^S_{31}(k)(v_2 - v_3 + k) + (1 - q^S_{31}(k))k$ for player 2 and k for player 4. Expected total effort in this semifinal is given by

$$TE^S_{24}(k) = \frac{1}{2}k + \frac{1}{2}\frac{k^2}{q^S_{31}(k)(v_2 - v_3 + k) + (1 - q^S_{31}(k))k}. \tag{26}$$

Player's 4 probability of winning the semifinal is given by

$$q_{42}(k) = \frac{1}{2}\frac{k}{q_{31}(k)(v_2 - v_3 + k) + (1 - q_{31}(k))k}. \tag{27}$$

Solving for q^S_{31} by combining equations (25) and (27) yields

$$
\begin{aligned}
q^S_{31}(k) \quad &= \quad \frac{-k[2(v_1 - v_2 + k) + (v_3 - v_4)]}{4(v_2 - v_3)(v_1 - v_2 + k)} + \\
&\sqrt{\left(\frac{k[2(v_1 - v_2 + k) + (v_3 - v_4)]}{2(v_2 - v_3)(v_1 - v_2 + k)}\right)^2 + k(2k + (v_2 - v_4))}. \qquad (28)
\end{aligned}
$$

Taking the limit in equation (28), we obtain $\lim_{k\to 0} q^S_{31}(k) = 0$.

By equation (27) we also have

$$
q^S_{42}(k) = \frac{1}{2} \frac{k}{q^S_{31}(k)(v_2 - v_3) + k}. \qquad (29)
$$

Note that $q_{31}(k)$ converges faster to zero than k, due to the square root. But then, $q^S_{42} \to 0$ as $k \to 0$. To see this, we apply l'Hospital's rule. Denote:

$$
g(k) = \frac{-k[2(v_1 - v_2 + k) + (v_3 - v_4)]}{4(v_2 - v_3)(v_1 - v_2 + k)},
$$

$$
z(k) = \left(\frac{k[2(v_1 - v_2 + k) + (v_3 - v_4)]}{2(v_2 - v_3)(v_1 - v_2 + k)}\right)^2
$$

and

$$
f(k) = k(2k + (v_2 - v_4)).
$$

Taking the derivatives of the numerator and denominator in the expression for $q^S_{31}(k)$ yields the expression

$$
\frac{2\sqrt{z(k) + f(k)}}{2\sqrt{z(k) + f(k)} + g'(k)2\sqrt{z(k) + f(k)} + z'(k) + f'(k)}. \qquad (30)
$$

Note that $z(0) = f(0) = z'(0) = 0$, but that $f'(0) > 0$. Hence $\lim_{k\to 0} q^S_{42}(k) = 0$.

Q.E.D

Lemma 4. *Consider seeding C:1-2,3-4. In the limit, as $k \to 0$, players 3 and 4 reach the final with probability one-half each. Player 2 reaches the final with probability $\frac{1}{2}\frac{2v_2 - v_3 - v_4}{2v_1 - v_3 - v_4}$. In addition the following hold:*

$$
\lim_{k\to 0} TE^S_{34}(k) = 0 \qquad (31)
$$

$$\lim_{k\to 0} TE_{12}^S(k) = \frac{1}{4}[(2v_2 - v_3 - v_4) + \frac{(2v_2 - v_3 - v_4)^2}{2v_1 - v_3 - v_4}] \tag{32}$$

$$\begin{aligned} TE_C &= \lim_{k\to 0} TE_C(k) = \frac{1}{2}v_2 + \frac{1}{4}\left(\frac{v_4^2 + v_3^2}{v_1}\right) + \\ & \frac{1}{4}\frac{(2v_2 - v_3 - v_4)}{(2v_1 - v_3 - v_4)}\left(2v_2 - v_3 - v_4 + \frac{v_4^2 + v_3^2}{v_2} - \frac{v_4^2 + v_3^2}{v_1}\right) \end{aligned} \tag{33}$$

Proof: Player 1's valuation for the semifinal is $q_{43}^S(k)(v_1 - v_4 + k) + (1 - q_{43}^S(k))(v_1 - v_3 + k)$ and player 2's valuation is $q_{43}^S(k)(v_2 - v_4 + k) + (1 - q_{43}^S(k))(v_2 - v_3 + k)$. The probability of winning for player 2 is

$$q_{21}^S(k) = \frac{1}{2}\frac{q_{43}^S(k)(v_2 - v_4 + k) + (1 - q_{43}^S(k))(v_2 - v_3 + k)}{q_{43}^S(k)(v_1 - v_4 + k) + (1 - q_{43}^S(k))(v_1 - v_3 + k)} \tag{34}$$

By equation (3) we know that total expected effort in this semifinal is

$$\begin{aligned} TE_{12}^S(k) &= \frac{1}{2}(q_{43}^S(k)(v_2 - v_4 + k) + (1 - q_{43}^S(k))(v_2 - v_3 + k)) + \\ &+ \frac{1}{2}\frac{(q_{43}^S(k)(v_2 - v_4 + k) + (1 - q_{43}^S(k))(v_2 - v_3 + k))^2}{q_{43}^S(k)(v_1 - v_4 + k) + (1 - q_{43}^S(k))(v_1 - v_3 + k)}. \end{aligned} \tag{35}$$

Players 3 and 4 play in the other semifinal. Their valuations for the semifinal are k and expected total effort is also $TE_{34}^S(k) = k$. The respective probabilities of winning are $q_{43}^S = 1 - q_{43}^S = \frac{1}{2}$

The expected effort in the final is given by

$$TE_C^F(k) = \frac{1}{2}[q_{43}^S q_{21}^S\left(v_4 + \frac{v_4^2}{v_2}\right) + (1 - q_{43}^S)q_{21}^S\left(v_3 + \frac{v_3^2}{v_2}\right) + \tag{36}$$

$$+ q_{43}^S(1 - q_{21}^S)\left(v_4 + \frac{v_4^2}{v_1}\right) + (1 - q_{43}^S)(1 - q_{21}^S)\left(v_3 + \frac{v_3^2}{v_1}\right)] \tag{37}$$

Total expected effort is given by

$$TE_C(k) = TE_{12}^S(k) + TE_{34}^S(k) + TE_C^F(k). \tag{38}$$

Note that

$$\lim_{k\to 0} TE_{12}^S(k) = \frac{1}{4}[(2v_2 - v_3 - v_4)) + \frac{(2v_2 - v_3 - v_4)^2}{2v_1 - v_3 - v_4}]$$

Since $q_{43}^S = 1/2$ we also obtain

$$\lim_{k\to 0} q_{21}(k) = \frac{1}{2}\frac{2v_2 - v_3 - v_4}{2v_1 - v_3 - v_4}. \tag{39}$$

Combining all pieces gives the desired formula for TE_C. *Q.E.D.*

6.1 Proof of Proposition 1

The equivalence between seeding B:1-3,2-4 and the one-shot simultaneous contest among the four players is immediate by Lemma 3 above, and by Theorem 1 in Baye et al. (1996) where total effort in the one-shot model is computed. We now compare the total efforts in each seeding.

Since $v_3 \leq v_2$ we immediately obtain that

$TE_A = \frac{1}{4}\left(v_2 + \frac{v_2^2}{v_1} + v_3 + \frac{v_3^2}{v_1}\right) \leq TE_B = \frac{1}{2}(v_2 + \frac{v_2^2}{v_1})$, with strict inequality for $v_3 < v_2$.

Hence, in order to find the optimal seeding, it remains to compare

$$TE_B = \frac{1}{2}v_2 + \frac{v_2^2}{2v_1} \tag{40}$$

with

$$\begin{aligned} TE_C &= \frac{1}{2}v_2 + \frac{1}{4}\left(\frac{v_4^2 + v_3^2}{v_1}\right) + \\ &\quad \frac{1}{2}\frac{(2v_2 - v_3 - v_4)}{(2v_1 - v_3 - v_4)}\left(v_2 - \frac{v_3}{2} - \frac{v_4}{2} + \frac{1}{2}\frac{v_4^2 + v_3^2}{v_2} - \frac{1}{2}\frac{v_4^2 + v_3^2}{v_1}\right) \end{aligned} \tag{41}$$

The idea is to look for values $v_i^*, i = 3, 4$ which maximize $TE_C(v_1, v_2, v_3, v_4)$ under the restriction $0 \leq v_4 \leq v_3 \leq v_2 \leq v_1$, while treating v_1 and v_2 as exogenous parameters.

We write $TE_C(v_3, v_4)$ for fixed v_2 and v_1. Note that $TE_C(v_3, v_4)$ is symmetric. It can be shown that its maximizers must be symmetric, that is, $v_3^* = v_4^*$. Hence we set $v_4 = v_3$ and look at total effort as a function of v_3 only:

$$TE_C(v_3) = \frac{1}{2}\left(v_2 + \frac{(v_2 - v_3)^2}{(v_1 - v_3)} + \frac{1}{2}\frac{(v_2 - v_3)}{(v_1 - v_3)}\frac{v_3^2}{v_2} + \frac{1}{2}\frac{(v_1 - v_2)}{(v_1 - v_3)}\frac{v_3^2}{v_1}\right). \tag{42}$$

In the remaining part of the proof we show that the function TE_C is strictly convex for all $v_3 \in [v_2, v_4]$, and that it achieves maxima at $v_3^* = v_2$ and at $v_3^* = v_4$ only if $v_4 = 0$.

The following facts are used:

(i)

$$TE_C'(v_3)_{v_3=v_2} = \frac{v_2}{v_1} > 0 \tag{43}$$

(ii)

$$TE_C'(v_3)_{v_3=0} = \frac{v_2(2v_2 - 4v_1)}{(-v_1)^2} < 0. \tag{44}$$

(iii) The numerator of $TE_C(v_3)$ is a polynomial of third degree in v_3 . Hence it can have at most two roots between 0 and v_1, where its first derivative is equal to zero.

(iv) $TE_C(0) = TE_C(v_2) = \frac{1}{2}v_2 + \frac{1}{2}\frac{v_2^2}{v_1}$.

Combining facts(i), (ii) and (iv), it must be that TE_C' has exactly one root in $[0, v_2]$ and it must be a minimum. Therefore TE_C is strictly convex on the interval $[v_2, v_4]$ and it satisfies

$$TE_C(v_3) \leq TE_C(0) = TE_C(v_2) = \frac{1}{2}v_2 + \frac{1}{2}\frac{v_2^2}{v_1} = TE_B$$

Q.E.D.

6.2 Proof of Proposition 3

Let q_{ij}^Fdenote the limit probability (as k goes to zero) that player i beats player j if these players meet in the final. Recall that q_{ij}^S denote the limit probability that i beats j if they meet in a semifinal. Moreover, let $p_1(s), s \in \{A, B, C\}$ denote the limit probability that the strongest player, player 1, wins the tournament with a given seeding. We have

$$p_1(A) = q_{14}^S(q_{12}^F q_{23}^F + q_{13}^F q_{32}^F) \tag{45}$$

$$p_1(B) = q_{13}^S(q_{12}^F q_{24}^F + q_{14}^F q_{42}^F) \tag{46}$$

$$p_1(C) = q_{12}^S(q_{13}^F q_{34}^F + q_{14}^F q_{43}^F) \tag{47}$$

By Lemmas 1, 2, 3 and 4 we obtain that:

$$p_1(A) = \frac{1}{2}(1 - \frac{1}{2}\frac{v_2}{v_1} + 1 - \frac{1}{2}\frac{v_3}{v_1}) \tag{48}$$

$$p_1(B) = 1 - \frac{1}{2}\frac{v_2}{v_1} \tag{49}$$

$$p_1(C) = \frac{1}{2}\left(1 - \frac{1}{2}\frac{v_2 - \frac{1}{2}(v_3 + v_4)}{v_1 - \frac{1}{2}(v_3 + v_4)}\right)\left(1 - \frac{1}{2}\frac{v_3}{v_1} + 1 - \frac{1}{2}\frac{v_4}{v_1}\right) \tag{50}$$

We clearly have $p_1(A) > p_1(B)$. For the other inequality, note that $p_1(C)$ can be written as an expression of the sum $t \equiv v_3 + v_4$:

$$p_1(C) = p_1(C, t) = \frac{1}{2}\left(1 - \frac{1}{2}\frac{v_2 - \frac{t}{2}}{v_1 - \frac{t}{2}}\right)\left(2 - \frac{1}{2}\frac{t}{v_1}\right). \tag{51}$$

We first show that, on the interval of definition $[0, 2v_2]$, $p_1(C, t)$ attains a maximum at $t = 2v_2$. Note that for all $t \in [0, 2v_2]$

$$\frac{\partial^2 p_1(C, t)}{\partial t^2} = -\frac{v_1 - v_2}{(t - 2v_1)^3} > 0 \tag{52}$$

Hence, $p_1(C, t)$ is strictly convex in t and its maximum can not be interior. We also get

$$p_1(C, 2v_2) = p_1(C, 0) = 1 - \frac{1}{2}\frac{v_2}{v_1} \tag{53}$$

Thus, for any v_3 and v_4, we obtain that

$$p_1(C) \leq 1 - \frac{1}{2}\frac{v_2}{v_1}. \tag{54}$$

On the other hand, since $p_1(A)$ is strictly decreasing in v_3 , we obtain for all $v_3 \leq v_2$ that

$$p_1(A) \geq 1 - \frac{1}{2}\frac{v_2}{v_1}. \tag{55}$$

For all v_3 and v_4 such that $v_4 < v_3 < v_2$ we obtain $p_1(A) > p_1(C)$. *Q.E.D.*

6.3 Proof of Proposition 4

The results for seedings A:1-4,2-3 and B:1-3,2-4 follow by Lemmas 2 , 3 in the Appendix. We now display an example where $p_3(C) > p_4(C) > p_2(C)$.

By Lemma 4, we have:

$$p_2(C) = \frac{1}{2}\left(\frac{1}{2}\frac{v_2 - \frac{1}{2}(v_3+v_4)}{v_1 - \frac{1}{2}(v_3+v_4)}\right)\left(1 - \frac{1}{2}\frac{v_3}{v_2} + 1 - \frac{1}{2}\frac{v_4}{v_2}\right). \tag{56}$$

and for $j = 3, 4$,

$$p_j(C) = \frac{1}{2}\left[\left(1 - \frac{1}{2}\frac{v_2 - \frac{1}{2}(v_3+v_4)}{v_1 - \frac{1}{2}(v_3+v_4)}\right)\left(\frac{1}{2}\frac{v_j}{v_1}\right) + \left(\frac{1}{2}\frac{v_2 - \frac{1}{2}(v_3+v_4)}{v_1 - \frac{1}{2}(v_3+v_4)}\right)\left(\frac{1}{2}\frac{v_j}{v_2}\right)\right] \tag{57}$$

For $v_1 = 15, v_2 = 13, v_3 = 11, v_4 = 10$, we obtain $p_2(C) = 0.166 < 0.174 = p_4(C) < 0.324 = p_3(C)$. *Q.E.D.*

6.4 Robustness of Empirical Results

As mentioned in the text, the empirical results rest on the assumption that all seedings can be identified by ordinal rankings of teams reaching the semi-finals, and that the occurrence of a particular seeding is random. In particular, the (ability) distribution of teams should be the same regardless of the seeding to identify the effects of seedings on outcomes. If the teams competing in particular seedings are in some way predetermined by the way the 16 teams were seeded initially, the results might be biased: the teams meeting in semi-finals are selected in a particular way, and hence might differ systematically with respect to their ability, effort exertion and winning probabilities. The initial seeding of teams in NCAA basketball tournaments is of type A, i.e. the team ranked 1 plays team 16 in the first round, and the winner meets the winner of a match of teams 8 and 9, while in the second bracket the winner of the match between team 2 and team 15 encounters the winner of the match 7-10 etc. This particular seeding has potential consequences on the possible pairings in semi-finals, since, for example, teams ranked 1 and 2 can only meet in a regional final, not before. In order to rule out any potential selection effects resulting from this, we exclude semi-finals between particular teams that could only happen under certain seedings since teams initially were seeded according to A. Thus, for example, we exclude any tournament with semi-finals, in which a team that was absolutely ranked 2 prior to the tournament is the second best remaining team. Given the initial seeding, this can only happen in semi-final seedings A and B, but is impossible in seeding C, which requires the highest ranked two teams meeting in a semi-final. This is impossible due to the initial seeding, so we drop such a tournament from the sample. On the other hand, a semi-final with team 3 as second best remaining team is possible under all three seedings given the initial brackets, so we keep it in the sample.[24] This sample selection leads us to drop 30 observations and leaves us with 70 tournaments with teams in semifinals that have absolute and relative ranks, which are compatible with any of the three possible seedings, given that the initial seeding was A. The statistics corresponding to Tables 1 and 2 are depicted in Tables 3 and 4. Tests of the theoretical predictions using the selected sample deliver virtually identical, if not somewhat stronger, results than those in the main text obtained from the unselected sample. In particular, the differences in effort as measured by all points scored are more pronounced and as expected, with seeding B eliciting the highest effort. However, as with the full sample, the differences are not significant. The same holds for coefficient estimates of seeding dummies in regressions of all points

[24] One can easily verify using this argument that the set of teams that have the relatively highest ability or valuation in the semi-final, which can be obtained under all three seedings is $\{1, 2, ..., 10\}$, where the numbers are the absolute ranks of teams prior to the tournament; likewise, the set of second-best teams is $\{3, ..., 13\}$, the set of teams relatively ranked 3 is $\{4, ..., 14\}$, and the set of weakest teams is $\{7, ..., 16\}$.

scored. Logit estimates to test Proposition 2 reveal a positive but insignificant effect of seeding A on the probability of a final between the two relatively strongest teams of the semi-finalists compared to seeding B. Logit estimates for the probability of the best team winning the tournament deliver a positive effect of seeding A if seedings B and C are the reference groups. For seeding A as reference group, the coefficients for seedings B and C are negative. All effects are significant on the 5 percent level. This provides even stronger evidence in favor of Proposition 3 than those obtained with the unselected sample presented in the text. However, the data contradict Proposition 4 to a certain extent, since under seeding B the monotonicity of winning probabilities in terms of relative rankings seems not to hold.

Table 3: **Seedings: Tournament winner and Total Effort** as a function of seeding in 70 selected regional conference tournaments (1979-2003)

Seeding	All	A	B	C
(Observations)	(70)	(25)	(30)	(15)
Team ranked 1	48	22	17	9
Team ranked 2	7	1	3	3
Team ranked 3	12	1	8	3
Team ranked 4	3	1	2	0
Mean Total Effort (Points)	431.39	427.88	434.33	431.33
(Std.Dev.)	(50.38)	(61.91)	(46.80)	(36.91)

Note: Entries are frequencies. Mean Total Effort (Points) is the mean of the sum of all points scored in semi-finals and final of a conference tournament.

In a companion paper, we test the theoretical predictions using data from ATP tennis tournaments, and data from national soccer cup competitions in Germany, England, France, Italy and Spain. Tennis tournaments involve 64 to 128 players and a much more randomized initial seeding (including wildcards etc.), so the selection issue is much less of a problem, since already four or five rounds have been played before the relative strength of semi-finalists is used to determine the respective seeding (instead of only two rounds in basketball). In soccer, the encounters on any stage, and in particular the matches between the semi-finalists, are obtained by drawing lots. Hence,

Table 4: **Relative Strength of Teams: Composition of Finals and Total Effort** in 70 selected regional conference tournaments (1979-2003)

		Composition of Finals			Total Points per Match	
Match	All	Seeding A	Seeding B	Seeding C	Mean	Std.Dev.
1 vs 2	30	16	14	0	437.63	54.01
1 vs 3	13	8	0	5	427.23	43.14
1 vs 4	13	0	7	6	424.46	59.31
2 vs 3	7	0	4	3	424	38.88
2 vs 4	1	0	0	1	414	.
3 vs 4	6	1	5	0	435.67	53.15

Note: Entries for composition of finals are frequencies.

seedings are completely random, allowing to identify the causal effect of seedings on effort or winning probabilities without potential selection problems.

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