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Abstract

We study the alternating-offer bargaining problem of sharing a common value pie under incomplete information on both sides and no depreciation between two identical players. We characterise the essentially unique perfect Bayesian equilibrium of this game which turns out to be in gradually increasing offers. (JEL *C73, C78, D44, D82, J12.* Keywords: *Gradual bargaining*, *Common values*, *Incomplete information*, *Repeated games.*)

Introduction

We study a bargaining situation where the mutual offers made by two privately informed players signal, in the course of bargaining, their private information to the opponent. The players bargain on an indivisible object that is of either a high or low value. Both players know these possible values. Before bargaining commences, each player is sent one private signal of publicly known precision refining his prior on the object's value. These signals can be either high or low and their precision (accuracy) is the probability with which this signal equals the true value of the object. We call the player with the higher signal accuracy player one (P1) and the other player two (P2). The main rule of bargaining is that once a player has made an offer and this was rejected, she must subsequently offer a strictly higher payment to the opponent. We define a minimal admissible offer-increment kept constant at 1 (currency) unit. As the high value of the object is increased, the relative size of this increment gets arbitrarily small.

Many economic applications lend themselves to our interpretation of bargaining. Our study of non-depreciating common values—the non-depreciation part of which we share with

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Zeuthen (1930)—complements the analysis of depreciating private values by Ståhl (1972) or Rubinstein (1982). Their assumption of depreciation typically leads to immediate or temporally very finely tuned agreement in subgame perfect equilibrium. This phenomenon, however, is often not observed in bargaining situations in which signalling of common values matters. Some examples captured within our framework are: (1) A partnership dissolution problem where two asymmetrically informed players jointly own a firm. (2) Agreeing on a profit sharing rule between two firms involved in a joint venture. (3) The 'buying out' of parties holding dispersed property rights (or patents) needed for the production of some good or service. (4) Deciding whether to spin-off some yet-to-be-proven innovation ('selling the project to the manager') or developing it inside the firm. (5) Splitting an inheritance (eg. an Amish farm or company) under the provision of maintaining it as a unit.[1]

Incomplete information bargaining models are typically plagued by a plethora of equilibria. One might expect that the signalling aspect introduced by the common value nature of the object further accentuates this problem. This is, however, incorrect. Indeed our analysis shows that we can identify pairs of signal accuracies for which our game has the following *essentially unique* perfect Bayesian equilibrium:[2]

- Players who observe a high signal, ie. high-type players, always continue bidding up in minimal increments and with probability 1 until some highest equilibrium offer is reached.
- Low-type players start by using the same probability 1 continuation actions as above until one player's certain quitting payoff becomes lower than their prior-based continuation payoffs. This is first the case with the less-precisely informed P2. In order to avoid quitting by P2, P1 needs to play a mixed stage action in order to *i*) directly change P2's continuation payoff and *ii*) change P2's beliefs on P1's signal given that P1's continuation action is observed. Whether the low-type P1 or P2 starts mixing depends on whether P1's or P2's bid first exceeds the object's expectation given that both players receive high signals (ie. the highest possible value of the object). After a low-type player starts mixing, all subsequent stage moves by both players are mixed until the same highest equilibrium offer is reached as above.

Depending on the individual signal accuracies, there are other equilibria in our bargaining game. Disregarding knife-edge cases, however, these are all (essentially) unique for their parameter region. We provide an example in the appendix which illustrates this for *any* combination of signal accuracies. We focus on the equilibrium outlined above because, disregarding knife-edge cases, it puts the highest requirements on the agents' patience and is hardest to implement. Once this equilibrium is characterised, other equilibria for different signal precisions

[1] In the Roman Republic, a proconsul was a promagistrate who, after serving as consul, spent a year as a governor of a province. Each province had two consuls. In certain provinces negotiations between the two consuls, each of whom had veto power, decided who was to become proconsul.

[2] We call an equilibrium 'essentially' unique if all stage actions but (for certain parameter-values) the final one are unique and all final actions lead to the same outcome.

can be found relatively easily by dropping the mixture or continuation requirements on the final stages of the game.

The reason for the essential uniqueness of our proposed equilibrium is that players do not allow their beliefs to be 'wilfully manipulated' by the opponent. That is, Pi cannot plan an equilibrium action which induces P-i to respond using a belief-triggered action (eg. immediate quitting after a jump-bid) which is beneficial only to a certain type of Pi. Thus observing an on- or off-equilibrium-path action by Pi, the low-type P-i's *equilibrium* response must be based on the belief which makes her indifferent between accepting the current offer by quitting (beneficial to a high-type Pi) and continuing to make a (minimally) higher own offer (beneficial to a low-type Pi). This suffices to force unique on and off-equilibrium path beliefs. Furthermore, the efficient agreement is reached gradually and stochastically over a stretch of multiple rounds of offers and counteroffers—not immediately as implied by depreciation in other bargaining models.

There is a rich literature on bargaining with incomplete information. Extensive surveys on bargaining under incomplete information are presented by Kennan and Wilson (1993) and Ausubel, Cramton, and Deneckere (2002) but, to present, the only full analysis of incomplete information bargaining over an object's pure common value allowing for bids by both players is Schweinzer (2003). The present study extends this model to asymmetric incomplete information on both sides. The main difference to the setting with incomplete information on one side is the introduction of imperfectly informative signals to *both* players. Hence in contrast to the one sided incomplete information scenario, the better informed player need not necessarily know everything the less-well informed player knows but gains additional information by learning the lower-precision signal. We model this signal accuracy as a discrete version of ideas developed by Athey and Levin (1998) and Persico (2000) as an extension of their concept of signal sufficiency.

Our model of incomplete information on both sides encompasses all situations where the asymmetrically informed players can form more precise conditional expectations on the object's value if they learn the opponent's signal. The special assumptions on priors, possible bids and preferences extend easily. Generalising the type space to a larger set of possible values retains the result in the sense that a low-signal player always mimics the high-signal player. Nothing but the technical difficulty stops us from introducing depreciation. If common values are replaced by private values and the incomplete information is about the players' willingness to pay our results remain applicable in a different realm.

1 The model

We consider two risk-neutral players $\{\mathrm{P1}, \mathrm{P2}\}$ and the simplest case of an indivisible object taking only two possible common values $\theta \in \{\underline{\theta}, \bar{\theta}\}$ with $\underline{\theta} = 0$, $\bar{\theta} \in \mathbb{R}$, $\bar{\theta} \geq 3$. Nature chooses θ with the publicly known probability $\varphi^0 = \mathrm{pr}(\theta = \bar{\theta}) = {}^1/_2$. Subsequently, Nature sends a private signal $s \in \{\underline{s}, \bar{s}\}$ to each player. These signals are of publicly known accuracy $p_i = \text{pr}(s_i = \bar{s}_i|\bar{\theta}) = \text{pr}(s_i = \underline{s}_i|\underline{\theta}) \in [1/2, 1]$, $i = 1, 2$, where $p = 1/2$ is uninformative and $p = 1$ is fully revealing.[3] We assume p_1, p_2 to be i.i.d. conditional on the realised value θ and denote (p_1, p_2) by $\mathbf{p}$. Hence the unconditional ex-ante probability of receiving a high signal is $\text{pr}(s_i = \bar{s}) = \varphi^0 = 1/2$ for both players. We call the more accurately informed player, ie. the player with the higher p, P1. We define Pi's beliefs as the conditional probability with which he believes the other player to have received a high signal, ie. $\varphi_i = \text{pr}(s_{-i} = \bar{s}|s_i)$. On the basis of his own signal, player Pi updates these beliefs through his observation of his opponent's bidding behaviour.

By assumption, the game starts with P1 offering a payment o_1^1 (subscripts are players, superscripts time periods) to P2 for sole ownership of the object. As a convention, we sign P1's offers positive and P2's negative. If P2 accepts the offer, P1 pays the offered amount to P2, P1 gets the object and the game is over. If P2 does not accept P1's offer, nothing is paid, and P2 makes an own offer. Players go on making alternating offers until one player quits. We denote by $g_i^t = |o_i^t - o_i^{t-2}|$ the *current increment* over the last own offer, set $o_2^0 = o_1^{-1} = 0$ and introduce the notion of player i's *bid* as the running sum of both players' offer-increments $b_i^t = \sum_{\hat{t}=1}^{t} g^{\hat{t}} = |o_{-i}^{t-1}| + o_i^t$. (Thus if P1's initial offer of, say, $o_1^1 = 1$ is followed by $o_1^3 = 3$ then $g_1^3 = 2$, if $o_2^2 = -1$, $b_1^3 = 4$.) It turns out to simplify the formal analysis to use this bid as stage action.[4] Pure bids b_i^t are restricted to the set of possible bids $B = \{0, 1, \ldots\} \subset \mathbb{N}$ bounded above by some $\bar{B} \in B$ ('all the money in the world') with $\bar{B} > \bar{\theta}$. We require bids to be strictly increasing in t, ie. all continuation increments are $g_i^t \geq 1$ while $g_i^t < 1$ is interpreted as quitting ('q'). This implicitly defines the minimal admissible bidding increment as 1 'currency unit.' Hence by increasing the above value $\bar{\theta}$, one decreases the relative admissible minimal bidding increment. Mixed bids attach probability α_i^t to the pure continuation bid b_i^t and the complementary probability to quitting, ie. accepting the current offer and ending the game. We denote such mixed actions as $[\alpha_i^t : b_i^t, q]$ where the continuation action b_i^t is played with probability α_i^t and q with the complementary probability $(1 - \alpha_i^t)$.[5] Finally we define a *jump bid* as $j_i^t = g_i^t - 1 \geq 0$ and keep a running sum of player i's jump bids as $J_i^t = \sum_{\hat{t}=1}^{t} j_i^{\hat{t}}$.

Pi's (repeated game) strategy β_i consists of the sequence of (mixed) stage actions $[\alpha_i^t : b_i^t, q]$ for each possible plan of the opponent. Players observe the opponents' pure offers and enjoy perfect recall. The players' final expected payoffs are written $u_i(\beta|s)$ and consist of the object's value minus payments made for the winner of the object and the payments received for the looser. Player i's quitting payoff when accepting an offer at t is written $u_i^t(q)$. The history of play h^t consists of all observed continuation actions not including time t.

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[3] The case of $\mathbf{p} = (1, 1/2)$ is a game with incomplete information on one side as analysed in Schweinzer (2003).

[4] Our dynamic game can be alternatively understood as a finitely repeated game of incomplete information as defined by Aumann and Maschler (1966) and subsequently developed by Mertens, Sorin, and Zamir (1994, chp. IV). That literature, however, typically derives average payoffs from long interactions which do not arise naturally in our context. We will, however, use the repeated game terminology whenever convenient.

[5] We refrain from a more general definition of a mixed stage action (over a larger support of pure actions) because we will not need anything more complicated than the above.

For ease of exposition, we introduce some notation. $\overline{\text{P}i}$ is the high-type Pi, $\underline{\text{P}i}$ the low type. Since $\bar{s}_i$ is player i's high signal and $\underline{s}_i$ his low signal, we write the possible signal profiles as $\underline{\underline{s}}$, $\bar{\underline{s}}_1^2$, $\bar{\underline{s}}_2^1$, and $\bar{\bar{s}}$. Similarly, we write the object's expected value given the possible signal combinations as $\underline{\underline{\text{E}}}$, $\bar{\underline{\text{E}}}_1^2$, $\bar{\underline{\text{E}}}_2^1$, $\bar{\bar{\text{E}}}$. Finally, we denote the low type Pi's, time-t beliefs by $\underline{\varphi}_i^t$ and the high type's beliefs by $\bar{\varphi}_i^t$. In the appendix we also occasionally discriminate Pi's mixed continuation probability given a low signal $\underline{\alpha}_i$ from the probability i chooses when having received a high signal $\bar{\alpha}_i$.

To sum up, our model—called a 'queto' game $\mathcal{Q}_B$—is a standard alternating-offer bargaining game with incomplete information over common values on both sides and no discounting.[6] We end this section by stating the definitions required to formulate our result.

Definition 1. *An equilibrium is called* essentially unique *if it is unique unless $\lfloor \bar{\bar{E}} \rfloor = \bar{\bar{E}}$, in which case the final equilibrium mixture of the game (all of which leading to the same outcome) is arbitrary.*[7]

Definition 2. *A privately informed player's strategy is called* separating *if it deterministically reveals the object's value prior to the final equilibrium stage of the game.*

Definition 3. *A strategy is called* minimal-increment *strategy if all actions it contains increase the previous own bid by a mixture $\alpha \in [0, 1]$ between quitting and the minimum of one.*

2 Discussion

The idea of the equilibrium candidate β^* outlined in the introduction is that both low types always mix between periods t_s and $t_f - 1$ (which we call the 'main game') while both high types always continue increasing their offers minimally with probability 1 until bidding reaches $\lfloor \bar{\bar{\text{E}}} \rfloor$. Both low-type players mix in equilibrium in order to make their low-type opponents mix in turn. Compared to the game with incomplete information on one side this creates the complication that these mixtures change both players' beliefs which are crucial the calculation of the continuation payoff expectations (and thus the own next-period mixture condition). Hence the low-type's mixture conditions at each stage of the main game are harder to enforce than in the game with incomplete information on one side. The next paragraphs try to convey the intuition of what precisely is going.

It is helpful for understanding the equilibrium dynamics to see that, on the one hand, $\underline{\text{P2}}$ is made indifferent between quitting and minimally increasing through her beliefs $\underline{\varphi}_2^t$ set by the previous period's α_1^{t-1}. $\underline{\text{P1}}$, on the other hand, is made indifferent between quitting and minimally increasing through $\underline{\text{P2}}$'s next period's mixture probability α_2^{t+1}. Thus the mixing dynamics of the game are, for odd $\bar{\bar{\text{E}}}$, as follows: $\underline{\text{P1}}$ starts mixing at period t_s (defined as

[6] The name derives from the player's stage actions of either *qu*itting or v*eto*ing the current proposal. The idea of our game is similar to the *quitting games* introduced by Vieille and Solan (2001) in the context of complete information stochastic games. They define quitting games as sequential games in which, at any stage, each player has the choice between a single continuation bid and quitting.

[7] The notation $\lfloor x \rfloor$ denotes the greatest integer not exceeding some real x; $\lceil x \rceil$ is the integer directly above x.

the period before the less well informed P2's continuation payoffs from the prior-based β^* are below her quitting payoffs at t_s+1). P1's mixing with *any* $\alpha_1^{t_s}$ (based on his prior beliefs $\underline{\varphi}_1^{t_s}$) determines a unique mixture probability $\alpha_2^{t_s+1}$ at the subsequent stage. This mixture probability, in turn, determines a unique belief $\underline{\varphi}_1^{t_s+2}$ which allows P1's mixing and thus in turn determines a unique $\alpha_2^{t_s+4}$ and so on until the final period t_f-1 where P2 mixes. We refer to this chain of reasoning which determines a unique α_2^t at all even t in the main game as the *forward chain*.

Conversely, P2's final belief $\underline{\varphi}_2^{t_f-1}$ (before P1 quits at t_f) is determined from her mixture condition over terminal payoffs. This indifference belief, in turn, determines a unique $\alpha_1^{t_f-2}$ which generates this belief. This $\alpha_1^{t_f-2}$, however, also determines P2's payoffs at t_f-3 and thus requires a unique $\underline{\varphi}_2^{t_f-3}$ in order to ensure P2's indifference. This belief again determines uniquely its generating $\alpha_1^{t_f-4}$ and so on until P1's first mixture period is reached at t_s. Thus all α_1^t for odd t are uniquely determined through this *backward chain* from the terminal beliefs.

The dynamics for even $\bar{\bar{\text{E}}}$ are similar: P2 quits at the continuation bid $\bar{\bar{\text{E}}}$ which determines P1's previous period's beliefs. These, in turn, determine P2's mixture one more period ahead and so on until the prior-based equilibrium continuation payoff exceeds the quitting payoff and P2 stops mixing but continues minimally with probability 1. Since this leaves P1's terminal mixture probability undefined, it determines P2's mixture probability one period *backwards*. Hence there is no need for P1 to mix *before* P2's first mixture because his following period mixture directly manipulates her continuation payoff.

The basic requirement from β^* is that, at each stage of the main game, a low-type player must be indifferent between all pure actions contained in the support of his mixed action

1. P1 mixes at odd t iff $u_1^t(q) = u_1^{t+1}(\beta^*|s)$ or

$$\frac{t-1}{2} + J_2^{t-1} = (1-\underline{\varphi}_1^t)\left[(1-\alpha_2^{t+1})\left(\underline{\underline{\text{E}}} - \frac{t+1}{2} - J_1^t\right) + \alpha_2^{t+1}\left(\frac{t+1}{2} + J_2^{t+1}\right)\right] + \underline{\varphi}_1^t\left[\frac{t+1}{2} + J_2^{t+1}\right]$$

resulting in a the mixture probability

$$\alpha_2^{t+1} = \frac{(1-\underline{\varphi}_1^t)(t + J_1^t + J_2^{t+1} - \underline{\underline{\text{E}}} + 1) - j_2^{t-1} - 1}{(1-\underline{\varphi}_1^t)(t + J_1^t + J_2^{t+1} - \underline{\underline{\text{E}}} + 1)} \tag{2.1}$$

which equals in equilibrium

$$^*\alpha_2^{t+1} = \frac{(1-\underline{\varphi}_1^t)(t - \underline{\underline{\text{E}}} + 1) - 1}{(1-\underline{\varphi}_1^t)(t - \underline{\underline{\text{E}}} + 1)}.$$

In addition, P1's beliefs $\underline{\varphi}_1^t$ must stem from the application of Bayes' rule

$$\begin{aligned} \underline{\varphi}_1^t = \mathrm{pr}(\bar{s}_2|h^{t-1}, \underline{s}_1) &= \frac{\mathrm{pr}(b_2^{t-1}|\bar{s}_2)\,\mathrm{pr}(\bar{s}_2)}{\mathrm{pr}(b_2^{t-1}|\bar{s}_2)\,\mathrm{pr}(\bar{s}_2) + \mathrm{pr}(b_2^{t-1}|\underline{s}_2)\,\mathrm{pr}(\underline{s}_2)} \\ &= \frac{\underline{\varphi}_1^{t-2}}{\underline{\varphi}_1^{t-2} + (1-\underline{\varphi}_1^{t-2})\alpha_2^{t-1}}. \end{aligned} \tag{2.2}$$

2. Similarly, P2 mixes at even t iff $u_2^t(q) = u_2^{t+1}(\beta^*|s)$ or

$$\begin{aligned} \frac{t}{2} + J_1^{t-1} = (1-\underline{\varphi}_2^t) & \left[(1-\alpha_1^{t+1})\left(\underline{\underline{\mathrm{E}}} - \frac{t}{2} - J_2^t\right) + \alpha_1^{t+1}\left(\frac{t+2}{2} + J_1^{t+1}\right)\right] + \\ \underline{\varphi}_2^t & \left[\frac{t+2}{2} + J_1^{t+1}\right] \end{aligned}$$

resulting in

$$\alpha_1^{t+1} = \frac{(1-\underline{\varphi}_2^t)(t + J_1^{t+1} + J_2^t - \underline{\underline{\mathrm{E}}} + 1) - j_1^{t-1} - 1}{(1-\underline{\varphi}_2^t)(t + J_1^{t+1} + J_2^t - \underline{\underline{\mathrm{E}}} + 1)} \tag{2.3}$$

which equals in equilibrium

$$^*\alpha_1^{t+1} = \frac{(1-\underline{\varphi}_2^t)(t - \underline{\underline{\mathrm{E}}} + 1) - 1}{(1-\underline{\varphi}_2^t)(t - \underline{\underline{\mathrm{E}}} + 1)}.$$

As above, P2's beliefs $\underline{\varphi}_2^t$ must stem from the application of Bayes' rule

$$\begin{aligned} \underline{\varphi}_2^t = \mathrm{pr}(\bar{s}_1|h^{t-1}, \underline{s}_2) &= \frac{\mathrm{pr}(b_1^{t-1}|\bar{s}_1)\,\mathrm{pr}(\bar{s}_1)}{\mathrm{pr}(b_1^{t-1}|\bar{s}_1)\,\mathrm{pr}(\bar{s}_1) + \mathrm{pr}(b_1^{t-1}|\underline{s}_1)\,\mathrm{pr}(\underline{s}_1)} \\ &= \frac{\underline{\varphi}_2^{t-2}}{\underline{\varphi}_2^{t-2} + (1-\underline{\varphi}_2^{t-2})\alpha_1^{t-1}}. \end{aligned} \tag{2.4}$$

The high-type posteriors $\bar{\varphi}_i^t$ are formed accordingly as a by-product of the low types' mixing. The candidate equilibrium β^* further prescribes

- Both high types always increase their offers minimally with probability 1 as long as the minimum continuation bid is below $\lceil \bar{\bar{\mathrm{E}}} \rceil$. They quit with probability 1 if the minimum continuation bid is higher.
- Both low types increase minimally with probability 1 until the less-precisely informed player's prior-based continuation expectation is below her quitting payoff (at period t_s). Depending on $\lceil \bar{\bar{\mathrm{E}}} \rceil$, players start to mix at either this or the following period.
- Any deviation from β^* is countered using minimum increase strategies.

These prescriptions and the above conditions (2.1) and (2.3) apply at each stage of the main game and, together with Bayes' rule (2.2) and (2.4), are sufficient to fully define the equilibrium strategy profile β^*.

The equilibrium β^* only exists for certain combinations of signal-accuracies. Sufficient and necessary conditions for this existence are stated in assumption 2. These conditions depend on the equilibrium start of mixing t_s^* which we can only determine through an iterative procedure in proposition 1. We cannot eliminate this inconvenience because we are unaware of a closed form representation of the ratio of Euler Γ-functions (of t_s^*) we need to describe the players' belief processes.[8] Assumption 1 allows us to avoid the duplication of our efforts for the case of the less accurately informed player moving first. It is apparent that for small bidding increments compared to the object's value, the difference in terms of payoffs is negligible.

3 Results

Assumption 1. *P1 has more accurate information than P2:* $^1/_2 \leq p_2 < p_1 \leq 1$.

Assumption 2. *The influence of the bidding grid is low in the sense that* $\lfloor \bar{\bar{E}} \rfloor > \underline{\bar{E}}_2^1$. *Moreover, in equilibrium, i) the high type moving at period* $t = \lfloor \bar{\bar{E}} \rfloor$ *bids* $\lfloor \bar{\bar{E}} \rfloor$ *with probability 1 and ii) the low type moving at* $t = \lfloor \bar{\bar{E}} \rfloor - 1$ *mixes between bidding* $\lfloor \bar{\bar{E}} \rfloor$ *and quitting.*[9]

The following theorem summarises our main result which is proved in the remainder of this section. All proofs of lemmata and propositions can be found in the appendix.

Theorem 1. *For* (p_1, p_2) *satisfying assumptions 1 and 2,* β^* *is the essentially unique perfect Bayesian equilibrium of* $\mathcal{Q}_B$. *This equilibrium involves gradually increasing offers.*

Proof. The first two lemmata establish a unique belief-structure on and off any equilibrium path. Lemmata 4, 5, and 6 provide a backward induction chain from the highest possible bid forward to the first period. Thus these three lemmata establish β^* as the essentially unique equilibrium of $\mathcal{Q}_B$. Since the β^* is explicitly constructed, this also ensures existence. Finally, proposition 1 gives an exact procedure for the calculation of the period where P1 starts mixing. From that, expected payoffs can be calculated for any $(\bar{\theta}, p)$. □

[8] To alleviate this nuisance, computational procedures pinning down t_s^* are available from the author. We additionally provide very simple sufficient conditions (A.4) & (A.5) which make it easy to find a profile (p_1, p_2) for which the equilibrium β^* exits.

[9] As shown in lemma 4, for odd $\lfloor \bar{\bar{\mathrm{E}}} \rfloor$ and start of mixing at period t_s, requirement *i*) amounts to the following condition on $\overline{\mathrm{P1}}$'s final period equilibrium beliefs

$$\bar{\varphi}_1^{\lfloor \bar{\bar{\mathrm{E}}} \rfloor} = \frac{\underline{\varphi}_1^{\lfloor \bar{\bar{\mathrm{E}}} \rfloor} \operatorname{pr}(\bar{s}_1, \bar{s}_2) \operatorname{pr}(\underline{s}_1, \underline{s}_2)}{\underline{\varphi}_1^{\lfloor \bar{\bar{\mathrm{E}}} \rfloor} \operatorname{pr}(\bar{s}_1, \bar{s}_2) \operatorname{pr}(\underline{s}_1, \underline{s}_2) + (1 - \underline{\varphi}_1^{\lfloor \bar{\bar{\mathrm{E}}} \rfloor}) \operatorname{pr}(\bar{s}_1, \underline{s}_2) \operatorname{pr}(\underline{s}_1, \bar{s}_2)} > \frac{\lfloor \bar{\bar{\mathrm{E}}} \rfloor - \underline{\bar{\mathrm{E}}}_2^1}{\bar{\bar{\mathrm{E}}} - \underline{\bar{\mathrm{E}}}_2^1} \quad \text{where}$$

$$\underline{\varphi}_1^{\lfloor \bar{\bar{\mathrm{E}}} \rfloor} = \underline{\varphi}_1^1 \prod_{\tau = \frac{t_s - 1}{2}}^{\frac{\lfloor \bar{\bar{\mathrm{E}}} \rfloor - t_s}{2}} \frac{(2\tau + 1) - \underline{\underline{\mathrm{E}}} - 1}{(2\tau + 1) - \underline{\underline{\mathrm{E}}}} = \underline{\varphi}_1^1 \frac{\Gamma(\frac{\lfloor \bar{\bar{\mathrm{E}}} \rfloor - t_s - \underline{\underline{\mathrm{E}}} + 4}{2})}{\Gamma(\frac{\lfloor \bar{\bar{\mathrm{E}}} \rfloor - t_s - \underline{\underline{\mathrm{E}}} + 3}{2})} \frac{\Gamma(\frac{t_s - \underline{\underline{\mathrm{E}}}}{2})}{\Gamma(\frac{t_s - \underline{\underline{\mathrm{E}}} + 1}{2})} \in [\underline{\varphi}_1^1, 1]. \tag{3.1}$$

Notice that this is a condition on P1's *prior* information. In addition we require that all involved mixture probabilities and beliefs must be well-formed, ie. elements of the unit interval. (There is a similar condition for even $\lfloor \bar{\bar{\mathrm{E}}} \rfloor$.)

The first two lemmata establish a unique belief-structure accompanying β^*.

Lemma 1. *There is no separating equilibrium.*

Lemma 2. *After observing the opponent deviate from the prescribed equilibrium β^*, the observing player believes that a low-type opponent has mixed with the unique probability which makes her indifferent between quitting and continuing at the stage directly following the observed deviation. If the mixture-probability required for indifference is greater than 1, the responding player quits and if it is smaller than zero, she continues with her equilibrium action with probability 1.*

Lemma 3 is an accounting argument used in the successive reasoning.

Lemma 3. *Under assumption 1, we have $0 < \underline{\underline{E}} < \underline{E}_1 < \underline{E}_2 < \bar{\underline{E}}_1^2 < \bar{\underline{E}}_2^1 < \bar{\bar{E}} < \bar{\theta}$.*

Our main argument is a backward induction proof from the highest possible bid ('all the money in the world') back to the first period of the game. It comprises of the following three lemmata. Lemma 4 shows that there is a highest equilibrium bid of $\lfloor \bar{\bar{E}} \rfloor$ at a period which we call t_f. Lemma 5 establishes optimality and uniqueness of β^* in the main game.

Lemma 4. *In any equilibrium, Pi quits with probability 1 if her minimal admissible continuation bid $b_i^{t_f}$ exceeds the object's highest possible expected value $\bar{\bar{E}}$.*

The next lemma addresses main-game stages—ie. between the period where mixing starts (called t_s) and the period of the last equilibrium continuation bid of the game (called t_f)—and argues that bids are increased minimally. We first fix the terminal payoffs and then proceed by backward induction. The argument against jump bids is made by reducing all deviation payoffs to modified (and smaller) versions of equilibrium payoffs for later periods. The lemma is lengthy and uses much notation but its nature is straightforward backward induction.

Lemma 5. *Given the final period t_f defined by the previous lemma, we proceed by backward induction until a low-type player stops mixing. We call the period of the last low-type mixture t_s. The resulting backward induction path coincides with the equilibrium path β^*.*

Lemma 6. *β^* prescribes optimal actions for the 'preplay'-phase where $1 \leq t \leq t_s$.*

Combined, the above three lemmata establish existence and essential uniqueness of the equilibrium β^*. It does not involve jump bids. In particular they show that there is a period t_s, where low types start to play mixed actions. The next step determines this period t_s and calculates the corresponding payoffs.

Proposition 1. *In the equilibrium β^*, the first mixing period t_s^* is determined through the final mixing belief $\underline{\varphi}^{t_f-1}$. This t_s^* pins down all expected payoffs.*

The above proposition fully and uniquely determines t_s^* using a trial & error procedure but fails to find a closed form representation of t_s^*. Hence the resulting payoff characterisations are unwieldy and we prefer to present the results of simulations to analyse the payoff implications of varying the player's signal precisions. For very low $\bar{\theta}$ and therefore very short games, the influence of the rule that P1 starts the game blur the general picture. This effects vanishes when larger values of $\bar{\theta}$ are considered. The simulations show that t_s^* is decreasing when the sum of available information $p_1 + p_2$ is going down and increasing when it is going up. This is intuitive as $\underline{\underline{\mathrm{E}}}$ and $\bar{\bar{\mathrm{E}}}$ move closer together with less information and further apart with more available information. As to be expected, Pi's payoff expectation from β^* moves in the same direction as p_i when holding p_{-i} fixed. The simulation packages are available from the author.

Conclusion

We present the essentially unique solution to an alternating-offers bargaining problem where two players are asymmetrically informed about an object's common value. Extending the existing literature, we study the effects of the players mutually signalling this private information during the bargaining process. We find that a privately informed player cannot deterministically mislead his opponent through actions (such as jump-bids) which can only be beneficial to a certain type. Thus the opponent—after observing such an action—will not make inferences about the player's type after such an observation which differ from those made in equilibrium. Indeed if she would, she could be made to believe anything the deviating player wishes. Hence the player observing a deviation has no option but to keep using her equilibrium strategy which only allows for unique beliefs on and off the equilibrium path. This restricts the players to using partially revealing, semi-separating strategies which gradually reveal their information to the opponent through the use of type-dependent lotteries. This, in turn, necessitates the mixing by both players which grants an information rent to the more accurately informed player.

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Appendix

A.1 Proofs

Proof of lemma 1. Suppose the contrary is true and let $\hat{\beta}_1(\underline{s}_1) = (q, \ldots)$ and $\hat{\beta}_1(\bar{s}_1) = (1, \ldots)$ be P1's type dependent equilibrium separating strategies with full information revelation at $t = 1$. Notice that $\underline{\text{P1}}$'s strategy $\hat{\beta}_1(\underline{s}_1)$ gives him a payoff of zero because it prescribes immediate quitting while $\hat{\beta}_1(\bar{s}_1)$ ensures him a positive payoff. Hence $\underline{\text{P1}}$ will optimally deviate from his separating equilibrium action q to mimicking his high-type by bidding 1 and thereby securing a payoff strictly higher than zero. But this contradicts the separating strategies $\hat{\beta}_1$ being part of an equilibrium. The same argument holds at any period of the game where P2 can condition on the information revealed by P1. The argument for P2 is symmetric. □

Proof of lemma 2. Suppose the observed deviation from β^* is by P1. P2's equilibrium strategy β_2^* is a complete contingent plan containing an equilibrium reaction to each possible action by P1 at each of his information sets—including any possible deviation. In particular, β_2^* prescribes her equilibrium reaction $[{}^*\alpha_2^{t+1} : {}^*b_2^{t+1}, q]$ to an observed deviation $\tilde{g}_1^t > 1$. This equilibrium response is determined through the above lemma 1 which says that no equilibrium action by P1 can reveal the object's true value to P2. Since P2's pure continuation would be beneficial to $\underline{\text{P1}}$ and pure quitting would be beneficial to $\overline{\text{P1}}$, the only non-revealing action open to P1 is to make $\underline{\text{P2}}$ precisely indifferent between quitting and continuing with a higher offer. This means, in turn, that P2's *equilibrium* beliefs on the mixture probabilities involved with observing the deviation $\tilde{g}_1^t$ are uniquely determined. (P2 cannot be made indifferent as long as her quitting payoffs are smaller than her prior-based expected continuation payoff, ie. in the so-called 'preplay'-phase of the game.) Again, the argument for a deviation by P2 is symmetric. □

Proof of lemma 3. The players' beliefs after observing their own signals $\varphi_i = \mathrm{pr}(s_{-i} = \bar{s}|s_i)$, $i = 1, 2$, are given by

$$\underline{\varphi}_1^1 = p_1 + p_2 - 2p_1p_2, \quad \underline{\varphi}_2^1 = (1-p_1)p_2 + p_1(1-p_2),$$
$$\bar{\varphi}_1^1 = p_1p_2 + (1-p_1)(1-p_2), \quad \bar{\varphi}_2^1 = (1-p_1)(1-p_2) + p_1p_2.$$

Then the above claim follows from the definitions of the object's expectations

$$\left.\begin{array}{l} \bar{\underline{\mathrm{E}}}_1^2 = \dfrac{\bar{\theta}(1-p_1)p_2}{p_1+p_2-2p_1p_2}, \ \underline{\underline{\mathrm{E}}} = \dfrac{\bar{\theta}(1-p_1)(1-p_2)}{(1-p_1)(1-p_2)+p_1p_2)} \\ \bar{\bar{\mathrm{E}}} = \dfrac{\bar{\theta}p_1p_2}{p_1p_2+(1-p_1)(1-p_2)}, \ \bar{\underline{\mathrm{E}}}_2^1 = \dfrac{\bar{\theta}p_1(1-p_2)}{p_1+p_2-2p_1p_2} \end{array}\right\} \begin{array}{l} \underline{\mathrm{E}}_2 = (1-\underline{\varphi}_2^1)\underline{\underline{\mathrm{E}}} + \underline{\varphi}_2^1\bar{\underline{\mathrm{E}}}_2^1 = (1-p_2)\bar{\theta}, \\ \underline{\mathrm{E}}_1 = (1-\underline{\varphi}_1^1)\underline{\underline{\mathrm{E}}} + \underline{\varphi}_1^1\bar{\underline{\mathrm{E}}}_1^2 = (1-p_1)\bar{\theta} \end{array}$$

since

$$\underline{\underline{\mathrm{E}}} > \bar{\bar{\mathrm{E}}} \Leftrightarrow p_1p_2 > (1-p_1)(1-p_2), \quad \bar{\underline{\mathrm{E}}}_2^1 > \bar{\underline{\mathrm{E}}}_1^2 \Leftrightarrow p_1 > p_2,$$

$$\bar{\underline{\mathrm{E}}}_1^2 > \underline{\underline{\mathrm{E}}} \text{ because, if not, } 2p_2 < \frac{p_1}{1+p_1} \text{ which is a contradiction, and}$$

$$\bar{\bar{\mathrm{E}}} > \bar{\underline{\mathrm{E}}}_2^1 \text{ because, if not, } 2p_1 < \frac{p_2}{1+p_2} \text{ which is a contradiction.}$$ □

Proof of lemma 4. Because we are interested in finding the highest possible continuation bid, we assume that players' beliefs are $\varphi_i = 1$, ie. they believe in high-type opponents with probability 1. Fig. 1 shows the terminal (quitting-)payoffs for odd and even $t_f = \lfloor \bar{\bar{\mathrm{E}}} \rfloor - J_1 - J_2$. These time-$t$ quitting stage payoffs are obtained through summation as[10]

$$\begin{array}{ll} {}_1u^t(q) = \left(\dfrac{t-1}{2} + J_2^{t-1}, \mathrm{E}(s) - \dfrac{t-1}{2} - J_2^{t-1}\right) & \text{if P1 quits (odd } t\text{)}, \\ {}_2u^t(q) = \left(\mathrm{E}(s) - \dfrac{t}{2} - J_1^{t-1}, \dfrac{t}{2} + J_1^{t-1}\right) & \text{if P2 quits (even } t\text{)} \end{array} \tag{A.1}$$

where $J_i^t = \sum_t j_i^t$ is the sum of player i's jump bids over the minimum increment before t. If no player quits, one bidder must eventually bid the highest possible bid $\bar{B}$. Suppose that Pi makes this last admissible continuation bid $b_i^{\tilde{t}} = \bar{B} \geq \bar{\theta} > \mathrm{E}(s)$ at some period $\tilde{t}$. Then, at $\tilde{t}+1$, P$-i$ must accept Pi's offer, quit with probability 1 and obtain

$${}_1u^{\tilde{t}+1}(q) = \left(\frac{\tilde{t}}{2} + J_2^{\tilde{t}}, \mathrm{E}(s) - \frac{\tilde{t}}{2} - J_2^{\tilde{t}}\right), \quad {}_2u^{\tilde{t}+1}(q) = \left(\mathrm{E}(s) - \frac{\tilde{t}+1}{2} - J_1^{\tilde{t}}, \frac{\tilde{t}+1}{2} + J_1^{\tilde{t}}\right).$$

$$\text{Knowing this, } \begin{array}{ll} \text{P1 quits at } \tilde{t} \text{ if} & \dfrac{\tilde{t}-1}{2} + J_2^{\tilde{t}-1} > \mathrm{E}(s) - \dfrac{\tilde{t}-1}{2} - J_1^{\tilde{t}} \\ \text{P2 quits at } \tilde{t} \text{ if} & \dfrac{\tilde{t}}{2} + J_1^{\tilde{t}-1} > \mathrm{E}(s) - \dfrac{\tilde{t}}{2} - J_2^{\tilde{t}} \end{array} \tag{A.2}$$

both resulting in the quitting condition $\tilde{t} > \mathrm{E}(s) - J_1^{\tilde{t}} - J_2^{\tilde{t}}$. Plugging $b_i^{\tilde{t}} = \bar{B} = \sum_{\tilde{t}} g_i^t = \tilde{t} + J_1^{\tilde{t}} + J_2^{\tilde{t}}$

[10] To facilitate readability of this and the following two lemmata, we occasionally prefix payoffs by the player index to indicate the player who is about to move, eg. ${}_1u_2^t(q)$ denotes P2's payoff from P1 quitting at t.

back into (A.2), we obtain $\bar{B} - J_1^{\tilde{t}} - J_2^{\tilde{t}} > \mathrm{E}(s) - J_1^{\tilde{t}} - J_2^{\tilde{t}}$ or $\bar{B} > \mathrm{E}(s)$ which is true by assumption. Folding back $t = \tilde{t} - 1$ yields the desired result. We consequently define

$$t_f = \lfloor \bar{\bar{\mathrm{E}}} \rfloor - J_1^{t-1} - J_2^{t-1} \tag{A.3}$$

as the period of the last (ie. highest) equilibrium continuation bid of the game. Any player moving after t_f quits with probability 1. $\square$

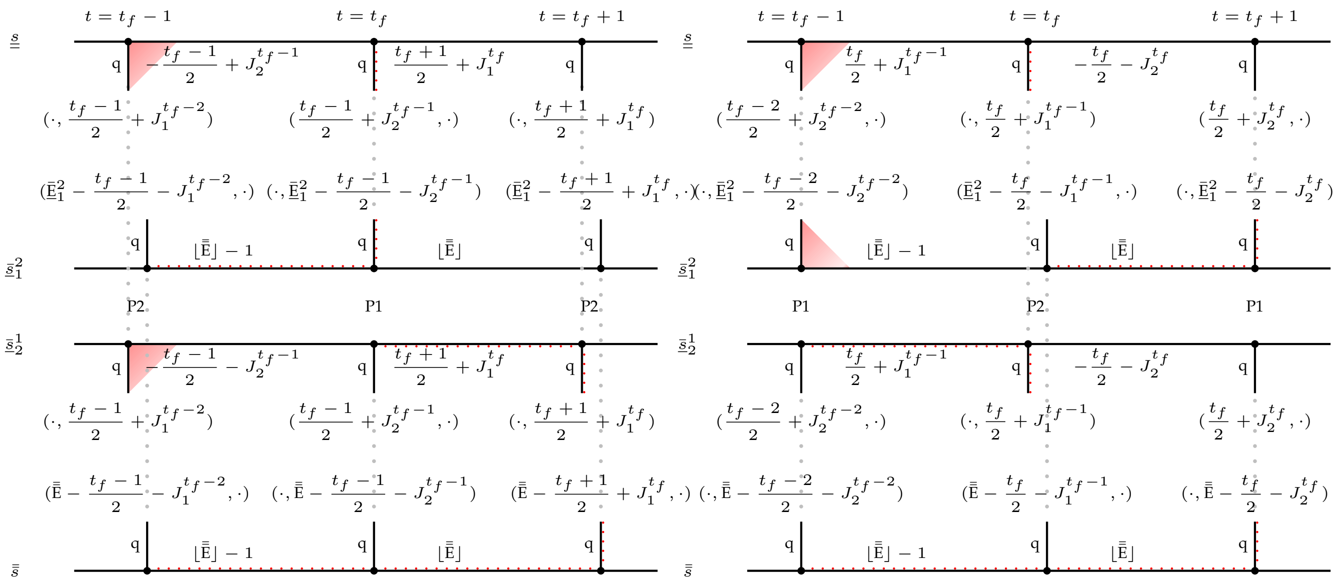

Figure 1: The endgame for odd t_f (left) and the 'dual' endgame for even t_f (right).

Proof of lemma 5. We define $A^t = \{b_i^t \in B | b_{-i}^{t-1} < b_i^t \leq \bar{\bar{\mathrm{E}}}\} \cup \{q\}$, the set of feasible bids at t not exceeding the highest equilibrium bid.

1. $A^{t_f+1} = \{q\}$. At $t = t_f + 1$ and all later periods, the moving player quits with payoffs

$$_1u^{t_f+1}(q) = \left(\frac{t_f}{2} + J_2^{t_f}, \mathrm{E}(s) - \frac{t_f}{2} - J_2^{t_f}\right), {}_2u^{t_f+1}(q) = \left(\mathrm{E}(s) - \frac{t_f+1}{2} - J_1^{t_f}, \frac{t_f+1}{2} + J_1^{t_f}\right).$$

2. $A^{t_f} = \{q, \lfloor \bar{\bar{\mathrm{E}}} \rfloor\}$. At $t = t_f$, players have the following options (from fig. 1)

$$\text{P1:} \quad \begin{aligned} {}_1u^{t_f}(q) &= \left(\frac{t_f-1}{2} + J_2^{t_f-1}, \mathrm{E}(s) - \frac{t_f-1}{2} - J_2^{t_f-1}\right), \\ {}_1u^{t_f}(\lfloor \bar{\bar{\mathrm{E}}} \rfloor) &= {}_2u^{t_f+1}(q) = \left(\mathrm{E}(s) - \frac{t_f+1}{2} - J_1^{t_f}, \frac{t_f+1}{2} + J_1^{t_f}\right), \end{aligned}$$

$$\text{P2:} \quad \begin{aligned} {}_2u^{t_f}(q) &= \left(\mathrm{E}(s) - \frac{t_f}{2} - J_1^{t_f-1}, \frac{t_f}{2} + J_1^{t_f-1}\right), \\ {}_2u^{t_f}(\lfloor \bar{\bar{\mathrm{E}}} \rfloor) &= {}_1u^{t_f+1}(q) = \left(\frac{t_f}{2} + J_2^{t_f}, \mathrm{E}(s) - \frac{t_f}{2} - J_2^{t_f}\right). \end{aligned}$$

(a) $\underline{\text{P1}}$ chooses q over $\lfloor\bar{\bar{\mathrm{E}}}\rfloor$ if

$$
\begin{aligned}
{}_1u_1^{t_f}(q|\underline{s}_1) = \frac{t_f-1}{2} + J_2^{t_f-1} &> \underline{\mathrm{E}}_1 - \frac{t_f-1}{2} - J_1^{t_f} = {}_2u_1^{t_f+1}(q|\underline{s}_1) \\
\lfloor\bar{\bar{\mathrm{E}}}\rfloor = t_f + J_1^{t_f} + J_2^{t_f-1} &> \underline{\mathrm{E}}_1 = \underline{\varphi}_1^{t_f}\underline{\bar{\mathrm{E}}}_1^2 + (1-\underline{\varphi}_1^{t_f})\underline{\underline{\mathrm{E}}} \\
\underline{\varphi}_1^{t_f} &< \frac{\lfloor\bar{\bar{\mathrm{E}}}\rfloor - \underline{\underline{\mathrm{E}}}}{\underline{\bar{\mathrm{E}}}_1^2 - \underline{\underline{\mathrm{E}}}} \quad \text{which is true whenever } \lfloor\bar{\bar{\mathrm{E}}}\rfloor > \underline{\bar{\mathrm{E}}}_1^2.
\end{aligned}
$$

(b) $\overline{\text{P1}}$ chooses $\lfloor\bar{\bar{\mathrm{E}}}\rfloor$ over q if

$$
\begin{aligned}
{}_1u_1^{t_f}(q|\bar{s}_1) = \frac{t_f-1}{2} + J_2^{t_f-1} &< \bar{\mathrm{E}}^1 - \frac{t_f-1}{2} - J_1^{t_f} = {}_2u_1^{t_f+1}(q|\bar{s}_1) \\
\lfloor\bar{\bar{\mathrm{E}}}\rfloor = t_f + J_1^{t_f} + J_2^{t_f-1} &< \bar{\mathrm{E}}^1 = \bar{\varphi}_1^{t_f}\bar{\bar{\mathrm{E}}} + (1-\bar{\varphi}_1^{t_f})\underline{\bar{\mathrm{E}}}_2^1 \\
\bar{\varphi}_1^{t_f} &> \frac{\lfloor\bar{\bar{\mathrm{E}}}\rfloor - \underline{\bar{\mathrm{E}}}_2^1}{\bar{\bar{\mathrm{E}}} - \underline{\bar{\mathrm{E}}}_2^1}.
\end{aligned}
$$

A simple sufficient condition for this to hold is to require the same already of $\overline{\text{P1}}$'s prior. Then, by inserting the definitions of the object's expectations and beliefs from the proof of lemma 3, we explore the knife-edge case of

$$
\begin{aligned}
\bar{\varphi}_1^1 &= \frac{\lfloor\bar{\bar{\mathrm{E}}}\rfloor - \underline{\bar{\mathrm{E}}}_2^1}{\bar{\bar{\mathrm{E}}} - \underline{\bar{\mathrm{E}}}_2^1} \\
p_1p_2 + (1-p_1)(1-p_2) &= \frac{\lfloor\bar{\bar{\mathrm{E}}}\rfloor - \frac{\bar{\theta}p_1(1-p_2)}{p_1+p_2-2p_1p_2}}{\frac{\bar{\theta}p_1p_2}{(1-p_1)(1-p_2)+p_1p_2} - \frac{\bar{\theta}p_1(1-p_2)}{p_1+p_2-2p_1p_2}} \qquad \text{(A.4)} \\
\lfloor\bar{\bar{\mathrm{E}}}\rfloor &= \left\lfloor \frac{\bar{\theta}p_1p_2}{(1-p_1)(1-p_2)+p_1p_2} \right\rfloor = p_1\bar{\theta}
\end{aligned}
$$

which is solved by $p_1 = 1, p_2 = {}^1/_2$. Any mixing at all along the equilibrium path will increase this solution set to two intervals and a qualifying $\mathbf{p}$ can *always* be found if there is at least one mixture period in the game which strictly increases the belief above the prior $\bar{\varphi}_1^1$. Precisely this is assumed by assumption 2.

(c) $\underline{\text{P2}}$ chooses q over $\lfloor\bar{\bar{\mathrm{E}}}\rfloor$ if

$$
\begin{aligned}
{}_2u_2^{t_f}(q|\underline{s}_2) = \frac{t_f}{2} + J_1^{t_f-1} &> \underline{\mathrm{E}}_2 - \frac{t_f}{2} - J_2^{t_f} = {}_1u_2^{t_f+1}(q|\underline{s}_2) \\
\lfloor\bar{\bar{\mathrm{E}}}\rfloor = t_f + J_1^{t_f-1} + J_2^{t_f} &> \underline{\mathrm{E}}_2 = \underline{\varphi}_2^{t_f}\underline{\bar{\mathrm{E}}}_2^1 + (1-\underline{\varphi}_2^{t_f})\underline{\underline{\mathrm{E}}} \\
\underline{\varphi}_2^{t_f} &< \frac{\lfloor\bar{\bar{\mathrm{E}}}\rfloor - \underline{\underline{\mathrm{E}}}}{\underline{\bar{\mathrm{E}}}_2^1 - \underline{\underline{\mathrm{E}}}} \quad \text{which is true whenever } \lfloor\bar{\bar{\mathrm{E}}}\rfloor > \underline{\bar{\mathrm{E}}}_2^1.
\end{aligned}
$$

(d) $\overline{\mathrm{P2}}$ chooses $\lfloor \bar{\bar{\mathrm{E}}} \rfloor$ over q if

$$\begin{aligned} {}_2u_2^{t_f}(q|\bar{s}_2) = \frac{t_f}{2} + J_1^{t_f-1} &< \bar{\mathrm{E}}^2 - \frac{t_f}{2} - J_2^{t_f} = {}_2u_1^{t_f+1}(q|\bar{s}_2) \\ \lfloor \bar{\bar{\mathrm{E}}} \rfloor = t_f + J_1^{t_f-1} + J_2^{t_f} &< \bar{\mathrm{E}}^2 = \bar{\varphi}_2^{t_f} \bar{\bar{\mathrm{E}}} + (1 - \bar{\varphi}_2^{t_f}) \underline{\mathrm{E}}_1^2 \\ \bar{\varphi}_2^{t_f} &> \frac{\lfloor \bar{\bar{\mathrm{E}}} \rfloor - \underline{\mathrm{E}}_1^2}{\bar{\bar{\mathrm{E}}} - \underline{\mathrm{E}}_1^2} \end{aligned} \tag{A.5}$$

which can be ensured through mixing for the same reason as in (b).

Thus low types quit for certain from lemma 3 and the assumption on the bidding grid that $\lfloor \bar{\bar{\mathrm{E}}} \rfloor > \underline{\mathrm{E}}_2^1 > \underline{\mathrm{E}}_1^2$. The above sufficiency conditions under which high types continue with probability 1 guarantee that, for $\lfloor \bar{\bar{\mathrm{E}}} \rfloor < \bar{\bar{\mathrm{E}}}$, there always exists an equilibrium β^*. These conditions are, however, far from necessary. The necessary and sufficient conditions which are stated in assumption 2 are arrived at as follows: For odd $\lfloor \bar{\bar{\mathrm{E}}} \rfloor$ we first calculate $\underline{\varphi}_1^{t_f}$ from the equilibrium continuation probabilities α_1^t in (2.3) and Bayes' rule (2.4). (This is done explicitly in proposition 1.) Since it does not matter when the signal s_1 arrives, we can transform this low type's belief into the high type's belief by applying

$$\bar{\varphi}_1^t = \bar{\varphi}_1(\underline{\varphi}_1^t) = \frac{\underline{\varphi}_1^t \operatorname{pr}(\bar{s}_1, \bar{s}_2) \operatorname{pr}(\underline{s}_1, \underline{s}_2)}{\underline{\varphi}_1^t \operatorname{pr}(\bar{s}_1, \bar{s}_2) \operatorname{pr}(\underline{s}_1, \underline{s}_2) + (1 - \underline{\varphi}_1^t) \operatorname{pr}(\bar{s}_1, \underline{s}_2) \operatorname{pr}(\underline{s}_1, \bar{s}_2)}$$

and then assuming that the final $\bar{\varphi}_1^{t_f}$ is sufficiently large. This results in condition (3.1) which indeed ensures that $\overline{\mathrm{P1}}$ continues. For even $\lfloor \bar{\bar{\mathrm{E}}} \rfloor$ we similarly calculate $\underline{\varphi}_2^{t_f}$ from the equilibrium α_2^t in (2.1) and Bayes' rule (2.2), convert the resulting belief into the high-type's and ensure continuation by requiring $\bar{\varphi}_2(\underline{\varphi}_2^{\lfloor \bar{\bar{\mathrm{E}}} \rfloor}) > \frac{\lfloor \bar{\bar{\mathrm{E}}} \rfloor - \underline{\mathrm{E}}_1^2}{\bar{\bar{\mathrm{E}}} - \underline{\mathrm{E}}_1^2}$.

Thus under assumption 2, at the final equilibrium period of the game, high types continue and low types exit with probability 1.

3. $A^{t_f-1} = \{q, \lfloor \bar{\bar{\mathrm{E}}} \rfloor - 1, \lfloor \bar{\bar{\mathrm{E}}} \rfloor\}$. In this step at $t = t_f - 1$ we formulate the payoffs from a jump deviation j_i^t in terms of $\tilde{t}_f = t_f - j_i^t$, the shortening of the game relative to the equilibrium duration due to the jump and $\tilde{J}_i = J_i + j_i^t$, the (stage) payoff consequence of the jump. The idea is that the same can be done with all previous-period deviations, thereby transforming payoffs already on the backward induction path into their $(\tilde{t}_f, \tilde{J})$-versions. As apparent from the formulation of quitting payoffs (A.1) and their equivalents below, the $(\tilde{t}_f, \tilde{J})$-versions cannot exceed the backward induction payoffs.

Players have the following pure options at period $t_f - 1$ (from fig. 1)

$$\text{P1:}\quad \begin{aligned} {}_1u^{t_f-1}(q) &= \left(\frac{t_f-2}{2} + J_2^{t_f-2}, \mathrm{E}(s) - \frac{t_f-2}{2} - J_2^{t_f-2}\right), \\ {}_1u^{t_f-1}(\lfloor\bar{\bar{\mathrm{E}}}\rfloor - 1) &= {}_2u^{t_f}(\beta^*), \\ {}_1u^{t_f-1}(\lfloor\bar{\bar{\mathrm{E}}}\rfloor) &= {}_2u^{\tilde{t}_f+1}(q) = \left(\mathrm{E}(s) - \frac{\tilde{t}_f+1}{2} - \tilde{J}_1^{\tilde{t}_f-1}, \frac{\tilde{t}_f+1}{2} + \tilde{J}_1^{\tilde{t}_f-1}\right), \\ &= \left(\mathrm{E}(s) - \frac{t_f+2}{2} - J_1^{t_f-2}, \frac{t_f+2}{2} + J_1^{t_f-2}\right) \end{aligned}$$

$$\text{P2:}\quad \begin{aligned} {}_2u^{t_f-1}(q) &= \left(\mathrm{E}(s) - \frac{t_f-1}{2} - J_1^{t_f-2}, \frac{t_f-1}{2} + J_1^{t_f-2}\right), \\ {}_2u_2^{t_f-1}(\lfloor\bar{\bar{\mathrm{E}}}\rfloor - 1) &= {}_1u^{t_f}(\beta^*), \\ {}_2u_2^{t_f-1}(\lfloor\bar{\bar{\mathrm{E}}}\rfloor) &= {}_1u^{\tilde{t}_f+1}(\beta^* = q) = \left(\frac{\tilde{t}_f}{2} + \tilde{J}_2^{\tilde{t}_f-1}, \mathrm{E}(s) - \frac{\tilde{t}_f}{2} - \tilde{J}_2^{\tilde{t}_f-1}\right) \\ &= \left(\frac{t_f+1}{2} + J_2^{t_f-2}, \mathrm{E}(s) - \frac{t_f+1}{2} - J_2^{t_f-2}\right) \end{aligned}$$

(a) $\underline{\text{P1}}$ mixes between q and $\lfloor\bar{\bar{\mathrm{E}}}\rfloor - 1$ if ${}_1u_1^{t_f-1}(q) = {}_1u_1^{t_f-1}(\lfloor\bar{\bar{\mathrm{E}}}\rfloor - 1)$ or

$$\frac{t_f-2}{2} + J_2^{t_f-2} = (1 - \underline{\varphi}_1^{t_f-1})\left[\underline{\underline{\mathrm{E}}} - \frac{t_f}{2} - J_1^{t_f-1}\right] + \underline{\varphi}_1^{t_f-1}\left[\frac{t_f}{2} + J_2^{t_f}\right]$$

resulting in the terminal low-type belief condition fixing the backward-chain for even $\lfloor\bar{\bar{\mathrm{E}}}\rfloor$

$$\underline{\varphi}_1^{t_f-1} = \frac{\underline{\underline{\mathrm{E}}} - t_f - J_1^{t_f-1} - J_2^{t_f-2} + 1}{\underline{\underline{\mathrm{E}}} - t_f - J_1^{t_f-1} - J_2^{t_f}} = \frac{\lfloor\bar{\bar{\mathrm{E}}}\rfloor - \underline{\underline{\mathrm{E}}} - 1}{\lfloor\bar{\bar{\mathrm{E}}}\rfloor - \underline{\underline{\mathrm{E}}}}. \tag{A.6}$$

$\underline{\text{P1}}$ jumps $j_1^{t_f-1} = 1$ if ${}_1u_1^{t_f-1}(\lfloor\bar{\bar{\mathrm{E}}}\rfloor) > {}_1u_1^{t_f-1}(q)$ or

$$\begin{aligned} \mathrm{E}(\underline{s}_1) - \frac{\tilde{t}_f+1}{2} - \tilde{J}_1^{\tilde{t}_f} = \mathrm{E}(\underline{s}_1) - \frac{t_f}{2} - J_1^{t_f-3} - j_1^{t_f-1} &> \frac{t_f-2}{2} + J_2^{t_f-2} \\ \lfloor\bar{\bar{\mathrm{E}}}\rfloor = t_f + J_1^{t_f-3} + J_2^{t_f-2} &< \mathrm{E}(\underline{s}_1) \end{aligned}$$

which is impossible. Hence for suitable beliefs $\underline{\text{P1}}$ is willing to mix.

(b) $\overline{\text{P1}}$ quits if ${}_1u_1^{t_f-1}(q) > {}_1u_1^{t_f-1}(\lfloor\bar{\bar{\mathrm{E}}}\rfloor - 1)$ or

$$\begin{aligned} \frac{t_f-2}{2} + J_2^{t_f-2} &> (1 - \bar{\varphi}_1^{t_f-1})\left[\bar{\underline{\mathrm{E}}}_2^1 - \frac{t_f}{2} - J_1^{t_f-1}\right] + \bar{\varphi}_1^{t_f-1}\left[\frac{t_f}{2} + J_2^{t_f}\right] \\ \bar{\varphi}_1^{t_f-1} &< \frac{\bar{\bar{\mathrm{E}}} - \bar{\underline{\mathrm{E}}}_2^1 - 1}{\bar{\bar{\mathrm{E}}} - \bar{\underline{\mathrm{E}}}_2^1} \end{aligned}$$

contradicting (A.6) because $\bar{\varphi}_i^t > \underline{\varphi}_i^t$. So if $\underline{\text{P1}}$ mixes at $t_f - 1$ (as assumed), then $\overline{\text{P1}}$

cannot quit with positive probability. $\overline{\text{P1}}$ jumps if ${}_1u_1^{t_f-1}(\lfloor\bar{\bar{\text{E}}}\rfloor) > {}_1u_1^{t_f-1}(\lfloor\bar{\bar{\text{E}}}\rfloor - 1)$ or

$$\begin{gathered}
\text{E}(\bar{s}_1) - \frac{\tilde{t}_f + 1}{2} - \tilde{J}_1^{\tilde{t}_f} = \text{E}(\bar{s}_1) - \frac{t_f}{2} - J_1^{t_f-3} - j_1^{t_f-1} > \\
(1 - \bar{\varphi}_1^{t_f-1})\left[\underline{\bar{\text{E}}}_2^1 - \frac{t_f}{2} - J_1^{t_f-1}\right] + \bar{\varphi}_1^{t_f-1}\left[\frac{t_f}{2} + J_2^{t_f}\right]
\end{gathered}$$

resulting in the two conditions

$$\begin{gathered}
(1 - \bar{\varphi}_1^{t_f-1})\left[\underline{\bar{\text{E}}}_2^1 - \frac{t_f}{2} - J_1^{t_f-3} - 1\right] > (1 - \bar{\varphi}_1^{t_f-1})\left[\underline{\bar{\text{E}}}_2^1 - \frac{t_f}{2} - J_1^{t_f-1}\right], \\
\bar{\varphi}_1^{t_f-1}\left[\bar{\bar{\text{E}}} - \frac{t_f}{2} - J_1^{t_f-3} - 1\right] > \bar{\varphi}_1^{t_f-1}\left[\frac{t_f}{2} + J_2^{t_f}\right] \Leftrightarrow \bar{\bar{\text{E}}} > \lfloor\bar{\bar{\text{E}}}\rfloor + 1
\end{gathered}$$

which are both impossible to satisfy. Hence $\overline{\text{P1}}$ continues with probability 1.

(c) $\underline{\text{P2}}$ mixes between q and $\lfloor\bar{\bar{\text{E}}}\rfloor - 1$ if ${}_2u_2^{t_f-1}(q) = {}_2u_2^{t_f-1}(\lfloor\bar{\bar{\text{E}}}\rfloor - 1)$ or

$$\frac{t_f - 2}{2} + J_1^{t_f-2} = (1 - \underline{\varphi}_2^{t_f-1})\left[\underline{\underline{\text{E}}} - \frac{t_f - 1}{2} - J_2^{t_f-1}\right] + \underline{\varphi}_2^{t_f-1}\left[\frac{t_f + 1}{2} + J_1^{t_f}\right]$$

resulting in the familiar terminal low-type belief condition fixing the backward-chain for odd $\lfloor\bar{\bar{\text{E}}}\rfloor$

$$\underline{\varphi}_2^{t_f-1} = \frac{\underline{\underline{\text{E}}} - t_f - J_1^{t_f-2} - J_2^{t_f-1} + 1}{\underline{\underline{\text{E}}} - t_f - J_1^{t_f} - J_2^{t_f-1}} = \frac{\lfloor\bar{\bar{\text{E}}}\rfloor - \underline{\underline{\text{E}}} - 1}{\lfloor\bar{\bar{\text{E}}}\rfloor - \underline{\underline{\text{E}}}}. \tag{A.7}$$

$\underline{\text{P2}}$ jumps $j_2^{t_f-1} = 1$ if ${}_2u_2^{t_f-1}(\lfloor\bar{\bar{\text{E}}}\rfloor) > {}_2u_2^{t_f-1}(q)$ or

$$\begin{gathered}
\text{E}(\underline{s}_2) - \frac{\tilde{t}_f}{2} - \tilde{J}_2^{\tilde{t}_f} = \text{E}(\underline{s}_2) - \frac{t_f - 1}{2} - J_2^{t_f-3} - j_2^{t_f-1} > \frac{t_f - 1}{2} + J_1^{t_f-2} \\
\lfloor\bar{\bar{\text{E}}}\rfloor = t_f + J_1^{t_f-2} + J_2^{t_f-3} < \text{E}(\underline{s}_2)
\end{gathered}$$

which is impossible. Hence for suitable beliefs $\underline{\text{P2}}$ mixes.

(d) $\overline{\text{P2}}$ quits if ${}_2u_2^{t_f-1}(q) > {}_2u_2^{t_f-1}(\lfloor\bar{\bar{\text{E}}}\rfloor - 1)$ or

$$\begin{gathered}
\frac{t_f - 1}{2} + J_1^{t_f-2} > (1 - \bar{\varphi}_2^{t_f-1})\left[\underline{\underline{\text{E}}} - \frac{t_f - 1}{2} - J_2^{t_f-1}\right] + \bar{\varphi}_2^{t_f-1}\left[\frac{t_f + 1}{2} + J_1^{t_f}\right] \\
\bar{\varphi}_2^{t_f-1} < \frac{\lfloor\bar{\bar{\text{E}}}\rfloor - \underline{\bar{\text{E}}}_1^2 - 1}{\lfloor\bar{\bar{\text{E}}}\rfloor - \underline{\bar{\text{E}}}_1^2}
\end{gathered}$$

contradicting (A.7) because $\bar{\varphi}_i^t > \underline{\varphi}_i^t$ and $\overline{\text{P2}}$ will not quit with positive probability.

$\overline{\text{P2}}$ jumps if ${}_2u_2^{t_f-1}(\lfloor\bar{\bar{\text{E}}}\rfloor) > {}_2u_2^{t_f-1}(\lfloor\bar{\bar{\text{E}}}\rfloor - 1)$ or

$$\begin{aligned} \text{E}(\bar{s}_2) - \frac{\tilde{t}_f}{2} - \tilde{J}_2^{\tilde{t}_f} = \text{E}(\bar{s}_2) - \frac{t_f-1}{2} - J_2^{t_f-3} - j_2^{t_f-1} > \\ (1-\bar{\varphi}_2^{t_f-1})\left[\underline{\bar{\text{E}}}_1^2 - \frac{t_f-1}{2} - J_2^{t_f-1}\right] + \bar{\varphi}_2^{t_f-1}\left[\frac{t_f+1}{2} + J_1^{t_f}\right] \end{aligned}$$

resulting in the two contradictions

$$\begin{aligned} (1-\bar{\varphi}_2^{t_f-1})\left[\underline{\bar{\text{E}}}_1^2 - \frac{t_f-1}{2} - J_2^{t_f-3} - 1\right] > (1-\bar{\varphi}_2^{t_f-1})\left[\underline{\bar{\text{E}}}_1^2 - \frac{t_f-1}{2} - J_2^{t_f-1}\right], \\ \bar{\varphi}_2^{t_f-1}\left[\bar{\bar{\text{E}}} - \frac{t_f-1}{2} - J_2^{t_f-3} - 1\right] > \bar{\varphi}_2^{t_f-1}\left[\frac{t_f+1}{2} + J_1^{t_f}\right] \Leftrightarrow \bar{\bar{\text{E}}} > \lfloor\bar{\bar{\text{E}}}\rfloor + 1. \end{aligned}$$

Hence $\overline{\text{P2}}$ continues with probability 1.

4. $A^{t_f-2} = \{q, \lfloor\bar{\bar{\text{E}}}\rfloor - 2, \lfloor\bar{\bar{\text{E}}}\rfloor - 1, \lfloor\bar{\bar{\text{E}}}\rfloor\}$. We avoid to list the pure continuation actions for the high types at $t = t_f - 2$ which are virtually identical to those in the previous step. We do, however, state the mixture and deviation conditions for the low types which confirm the equilibrium stage mixture conditions (2.1) and (2.3).

   (a) $\underline{\text{P1}}$ mixes between q and $\lfloor\bar{\bar{\text{E}}}\rfloor - 2$ if ${}_1\underline{u}_1^{t_f-2}(q) = {}_1\underline{u}_1^{t_f-2}(\lfloor\bar{\bar{\text{E}}}\rfloor - 2)$ or

$$\begin{aligned} \frac{t_f-3}{2} + J_2^{t_f-3} = \; & (1-\underline{\varphi}_1^{t_f-2})\left[(1-\alpha_2^{t_f-1})\left(\underline{\underline{\text{E}}} - \frac{t_f-1}{2} - J_1^{t_f-2}\right)\right. \\ & + \left.\alpha_2^{t_f-1}\left(\frac{t_f-1}{2} + J_2^{t_f-1}\right)\right] + \underline{\varphi}_1^{t_f-2}\left[\frac{t_f-1}{2} + J_2^{t_f-1}\right] \end{aligned}$$

   giving the stage mixing condition (2.1)

$$\begin{aligned} \alpha_2^{t_f-1} = \; & \frac{(1-\underline{\varphi}_1^{t_f-2})\left(t_f + J_1^{t_f-2} + J_2^{t_f-1} - \underline{\underline{\text{E}}} - 1\right) - j_2^{t_f-3} - 1}{(1-\underline{\varphi}_1^{t_f-2})\left(t_f + J_1^{t_f-2} + J_2^{t_f-1} - \underline{\underline{\text{E}}} - 1\right)} \\ = \; & \frac{(1-\underline{\varphi}_1^{t_f-2})(\lfloor\bar{\bar{\text{E}}}\rfloor - \underline{\underline{\text{E}}} - 1) - 1}{(1-\underline{\varphi}_1^{t_f-2})(\lfloor\bar{\bar{\text{E}}}\rfloor - \underline{\underline{\text{E}}} - 1)} \text{ in equilibrium.} \end{aligned}$$

   (b) $\underline{\text{P1}}$ jumps $j_1^{t_f-2} = 2$ (hence $\tilde{t}_f = t_f - 2$ and $\tilde{J}_1^{\tilde{t}_f-2} = J_1^{\tilde{t}_f-2} + 2$) if ${}_1\underline{u}_1^{t_f-2}(\lfloor\bar{\bar{\text{E}}}\rfloor) > {}_1\underline{u}_1^{t_f-2}(\lfloor\bar{\bar{\text{E}}}\rfloor - 2)$ or, for ${}_1\underline{u}_1^{t_f-1}(\beta^*) = \underline{\bar{\text{E}}}_2^1 - \frac{t_f+1}{2} - J_1^{t_f}$,

$$\begin{aligned} & {}_1\underline{u}_1^{t_f-1}(\lfloor\bar{\bar{\text{E}}}\rfloor) = {}_2\underline{u}_1^{\tilde{t}_f+1}(q) = \text{E}(\underline{s}_1) - \frac{\tilde{t}_f+1}{2} - \tilde{J}_1^{\tilde{t}_f} = \text{E}(\underline{s}_1) - \frac{t_f+3}{2} - J_1^{t_f-4} = \\ & (1-\underline{\varphi}_1^{t_f-2})\left[\underline{\underline{\text{E}}} - \frac{t_f+3}{2} - J_1^{t_f-4}\right] + \underline{\varphi}_1^{t_f-2}\left[\underline{\bar{\text{E}}}_1^2 - \frac{t_f+3}{2} - J_1^{t_f-4}\right] > \\ & (1-\underline{\varphi}_1^{t_f-2})\left[(1-\underline{\alpha}_2^{t_f-1})\left(\underline{\underline{\text{E}}} - \frac{t_f-1}{2} - J_1^{t_f-2}\right) + \underline{\alpha}_2^{t_f-1}\left(\frac{t_f-1}{2} + J_2^{t_f-1}\right)\right] \\ & + \underline{\varphi}_1^{t_f-2}\left[\bar{\bar{\text{E}}} - \frac{t_f+1}{2} - J_1^{t_f}\right] = {}_1\underline{u}_1^{t_f-2}(\lfloor\bar{\bar{\text{E}}}\rfloor - 2) \end{aligned}$$

leading to the low-signal condition

$$\underline{\underline{\mathrm{E}}} - \frac{t_f+3}{2} - J_1^{t_f-4} > (1-\underline{\alpha}_2^{t_f-1})\left(\underline{\underline{\mathrm{E}}} - \frac{t_f-1}{2} - J_1^{t_f-2}\right) + \underline{\alpha}_2^{t_f-1}\left(\frac{t_f-1}{2} + J_2^{t_f-1}\right)$$

which cannot be satisfied. Neither can the second, high-signal condition hold

$$\underline{\bar{\mathrm{E}}}_1^2 - \frac{t_f+3}{2} - J_1^{t_f-4} > \frac{t_f-1}{2} - J_1^{t_f-1} \Leftrightarrow \underline{\bar{\mathrm{E}}}_1^2 - 1 > \lfloor \bar{\bar{\mathrm{E}}} \rfloor = t_f + J_1^{t_f-4} + J_2^{t_f-1}.$$

Thus a jump of $j_1^{t_f-2} = 2$ is not profitable for $\underline{\mathrm{P1}}$.

(c) $\underline{\mathrm{P1}}$ jumps $j_1^{t_f-2} = 1$ (hence $\tilde{t}_f = t_f - 1$ and $\tilde{J}_1^{\tilde{t}_f-2} = J_1^{\tilde{t}_f-2} + 1$) if ${}_1\underline{u}_1^{t_f-2}(\lfloor \bar{\bar{\mathrm{E}}} \rfloor - 1) > {}_1\underline{u}_1^{t_f-2}(\lfloor \bar{\bar{\mathrm{E}}} \rfloor - 2)$ or

$$\begin{aligned}&{}_1\underline{u}_1^{t_f-1}(\bar{\bar{\mathrm{E}}} - 1) = {}_2\underline{u}_1^{\tilde{t}_f}(\beta^*) = (1-\underline{\varphi}_1^{t_f-2})\left[\underline{\underline{\mathrm{E}}} - \frac{t_f-1}{2} - J_1^{t_f-4} - 1\right] + \underline{\varphi}_1^{t_f-2}\left[\frac{t_f-1}{2} + J_2^{t_f-1}\right] > \\ &(1-\underline{\varphi}_1^{t_f-2})\left[(1-\underline{\alpha}_2^{t_f-1})\left(\underline{\underline{\mathrm{E}}} - \frac{t_f-1}{2} - J_1^{t_f-2}\right) + \underline{\alpha}_2^{t_f-1}\left(\frac{t_f-1}{2} + J_2^{t_f-1}\right)\right] \\ &+\underline{\varphi}_1^{t_f-2}\left[\frac{t_f-1}{2} + J_2^{t_f-1}\right] = {}_1\underline{u}_1^{t_f-2}(\lfloor \bar{\bar{\mathrm{E}}} \rfloor - 2)\end{aligned}$$

resulting in the low-signal condition

$$\underline{\underline{\mathrm{E}}} - \frac{t_f+1}{2} - J_1^{t_f-4} > (1-\underline{\alpha}_2^{t_f-1})\left(\underline{\underline{\mathrm{E}}} - \frac{t_f-1}{2} - J_1^{t_f-2}\right) + \underline{\alpha}_2^{t_f-1}\left(\frac{t_f-1}{2} + J_2^{t_f-1}\right)$$

which cannot be satisfied. For the second, high-signal condition is $\frac{t_f-1}{2} + J_2^{t_f-1} > \frac{t_f-1}{2} + J_2^{t_f-1}$ for which again a jump of $j_1^{t_f-2} = 1$ cannot be profitable.

(d) $\overline{\mathrm{P1}}$ prefers minimally increasing with probability 1 over quitting as in period $t_f - 1$.

(e) $\overline{\mathrm{P1}}$ jumps $j_1^{t_f-2} = 2$ implying $\tilde{t}_f = t_f - 2$ and $\tilde{J}_1^{\tilde{t}_f-2} = J_1^{\tilde{t}_f-2} + 2$ if ${}_1u_1^{t_f-2}(\lfloor \bar{\bar{\mathrm{E}}} \rfloor) > {}_1u_1^{t_f-2}(\lfloor \bar{\bar{\mathrm{E}}} \rfloor - 2)$ or, for

$$\begin{aligned}&{}_1\bar{u}_1^{t_f-1}(\beta^*) = \underline{\bar{\mathrm{E}}}_2^1 - \frac{t_f+1}{2} - J_1^{t_f}, \\ &{}_1\bar{u}_1^{t_f-1}(\lfloor \bar{\bar{\mathrm{E}}} \rfloor) = {}_2u_1^{\tilde{t}_f+1}(q) = \mathrm{E}(\bar{s}_1) - \frac{\tilde{t}_f+1}{2} - \tilde{J}_1^{\tilde{t}_f} = \mathrm{E}(\bar{s}_1) - \frac{t_f+3}{2} - J_1^{t_f-4} = \\ &(1-\bar{\varphi}_1^{t_f-2})\left[\underline{\bar{\mathrm{E}}}_2^1 - \frac{t_f+3}{2} - J_1^{t_f-4}\right] + \varphi_1^{t_f-2}\left[\bar{\bar{\mathrm{E}}} - \frac{t_f+3}{2} - J_1^{t_f-4}\right] > \\ &(1-\bar{\varphi}_1^{t_f-2})\left[(1-\underline{\alpha}_2^{t_f-1})\left(\underline{\bar{\mathrm{E}}}_2^1 - \frac{t_f-1}{2} - J_1^{t_f-2}\right) + \underline{\alpha}_2^{t_f-1}\left(\underline{\bar{\mathrm{E}}}_2^1 - \frac{t_f+1}{2} - J_1^{t_f}\right)\right] \\ &+\varphi_1^{t_f-2}\left[\bar{\bar{\mathrm{E}}} - \frac{t_f+1}{2} - J_1^{t_f}\right] = {}_1u_1^{t_f-2}(\lfloor \bar{\bar{\mathrm{E}}} \rfloor - 2)\end{aligned}$$

leading to both

$$\underline{\bar{\mathrm{E}}}_2^1 - \frac{t_f+3}{2} - J_1^{t_f-4} > (1-\underline{\alpha}_2^{t_f-1})\left(\underline{\bar{\mathrm{E}}}_2^1 - \frac{t_f-1}{2} - J_1^{t_f-2}\right) + \underline{\alpha}_2^{t_f-1}\left(\underline{\bar{\mathrm{E}}}_2^1 - \frac{t_f+1}{2} - J_1^{t_f}\right),$$
$$\text{and } \bar{\bar{\mathrm{E}}} - \frac{t_f+3}{2} - J_1^{t_f-4} > \bar{\bar{\mathrm{E}}} - \frac{t_f+1}{2} - J_1^{t_f}$$

which cannot hold. Hence a jump of $j_1^{t_f-2} = 2$ is not profitable for $\overline{\text{P1}}$.

(f) $\overline{\text{P1}}$ jumps $j_1^{t_f-2} = 1$ implying $\tilde{t}_f = t_f - 1$ and $\tilde{J}_1^{\tilde{t}_f-2} = J_1^{\tilde{t}_f-2} + 1$ if ${}_1u_1^{t_f-2}(\lfloor\bar{\bar{\mathrm{E}}}\rfloor - 1) > {}_1u_1^{t_f-2}(\lfloor\bar{\bar{\mathrm{E}}}\rfloor - 2)$ or, for

$$\begin{aligned}
&{}_1\bar{u}_1^{t_f-1}(\bar{\bar{\mathrm{E}}}-1) = {}_2u_1^{\tilde{t}_f}(\beta^*) = (1-\bar{\varphi}_1^{t_f-2})\left[\underline{\bar{\mathrm{E}}}_2^1 - \frac{\tilde{t}_f}{2} - \tilde{J}_1^{\tilde{t}_f-1}\right] + \bar{\varphi}_1^{t_f-2}\left[\frac{\tilde{t}_f}{2} + \tilde{J}_2^{\tilde{t}_f}\right] \\
&= (1-\bar{\varphi}_1^{t_f-2})\left[\underline{\bar{\mathrm{E}}}_2^1 - \frac{t_f-1}{2} - J_1^{t_f-4} - 1\right] + \bar{\varphi}_1^{t_f-2}\left[\frac{t_f-1}{2} + J_2^{t_f-1}\right] > \\
&(1-\bar{\varphi}_1^{t_f-2})\left[(1-\underline{\alpha}_2^{t_f-1})\left(\underline{\bar{\mathrm{E}}}_2^1 - \frac{t_f-1}{2} - J_1^{t_f-2}\right) + \underline{\alpha}_2^{t_f-1}\left(\underline{\bar{\mathrm{E}}}_2^1 - \frac{t_f+1}{2} - J_1^{t_f}\right)\right] \\
&+\varphi_1^{t_f-2}\left[\bar{\bar{\mathrm{E}}} - \frac{t_f+1}{2} - J_1^{t_f}\right] = {}_1u_1^{t_f-2}(\lfloor\bar{\bar{\mathrm{E}}}\rfloor - 2)
\end{aligned}$$

leading again to the two unfulfillable conditions

$$\underline{\bar{\mathrm{E}}}_2^1 - \frac{t_f}{2} - \frac{1}{2} - J_1^{t_f-4} > (1-\underline{\alpha}_2^{t_f-1})\left(\underline{\bar{\mathrm{E}}}_2^1 - \frac{t_f-1}{2} - J_1^{t_f-2}\right) + \underline{\alpha}_2^{t_f-1}\left(\underline{\bar{\mathrm{E}}}_2^1 - \frac{t_f+1}{2} - J_1^{t_f}\right),$$
$$\text{and } \frac{t_f-1}{2} + J_2^{t_f-1} > \bar{\bar{\mathrm{E}}} - \frac{t_f+1}{2} - J_1^{t_f} \Leftrightarrow \lfloor\bar{\bar{\mathrm{E}}}\rfloor > \bar{\bar{\mathrm{E}}}$$

rendering a jump of $j_1^{t_f-2} = 1$ unprofitable for $\overline{\text{P1}}$.

(g)–(h) The stage-$t_f - 2$ conditions for P2 are very similar to those for P1. Thus we only derive $\underline{\text{P2}}$'s mixing condition between q and $\lfloor\bar{\bar{\mathrm{E}}}\rfloor - 2$.
$\underline{\text{P2}}$ mixes if ${}_2u_2^{t_f-2}(q) = {}_2u_2^{t_f-2}(\lfloor\bar{\bar{\mathrm{E}}}\rfloor - 2)$ or

$$\begin{aligned}
\frac{t_f-2}{2} + J_1^{t_f-3} = \; & (1-\underline{\varphi}_2^{t_f-2})\left[(1-\alpha_1^{t_f-1})\left(\underline{\underline{\mathrm{E}}} - \frac{t_f-2}{2} - J_2^{t_f-2}\right)\right. \\
+ \; & \left.\alpha_1^{t_f-1}\left(\frac{t_f}{2} + J_1^{t_f-1}\right)\right] + \underline{\varphi}_2^{t_f-2}\left[\frac{t_f}{2} + J_1^{t_f-1}\right]
\end{aligned}$$

resulting in the stage mixing condition (2.3)

$$\begin{aligned}
\alpha_1^{t_f-1} &= \frac{(1-\underline{\varphi}_2^{t_f-2})\left(t_f + J_1^{t_f-1} + J_2^{t_f-2} - \underline{\underline{\mathrm{E}}} - 1\right) - j_1^{t_f-3} - 1}{(1-\underline{\varphi}_1^{t_f-2})\left(t_f + J_1^{t_f-1} + J_2^{t_f-2} - \underline{\underline{\mathrm{E}}} - 1\right)} \\
&= \frac{(1-\underline{\varphi}_2^{t_f-2})(\lfloor\bar{\bar{\mathrm{E}}}\rfloor - \underline{\underline{\mathrm{E}}} - 1) - 1}{(1-\underline{\varphi}_2^{t_f-2})(\lfloor\bar{\bar{\mathrm{E}}}\rfloor - \underline{\underline{\mathrm{E}}} - 1)} \text{ in equilibrium.}
\end{aligned}$$

Thus given suitable beliefs, all stage actions are as specified by β^*.

5. $A^t = \{q, \lfloor\bar{\bar{\text{E}}}\rfloor - t, \dots, \lfloor\bar{\bar{\text{E}}}\rfloor\}$. At general $t_s < t < t_f - 1$ we show that a mixed or pure minimal increase action is optimal. As in the previous backward induction step, we formulate the payoffs after a jump j_i^t in terms of $\tilde{t}_f = t_f - j_i^t$, the shortening of the game relative to the equilibrium duration due to the jump, and $\tilde{J}_i = J_i + j_i^t$, the (stage) payoff consequence of the jump. Any continuation action at t leads to a situation where the opponent faces a feasible set A^{t+1} which is smaller than the set A^t. Following a jump with even j_i^t, the same feasible choice set $A^{\tilde{t}+1}$ is already on the equilibrium backward induction path leading to the node at t. For odd-valued jumps j_i^t, the feasible choice set $A^{\tilde{t}+1}$ is on the equilibrium backward induction path of a *dual* game which is identical to $\mathcal{Q}_B$ but has P2 start the game. These payoffs, however, are modified by $t_f = \tilde{t}_f - j_i^t$ and $\tilde{J}_i^{\tilde{t}_f}$. (If the action at $t+1$ is mixed, we need to consider both the $(1-\alpha^{t+1})$-weighed $t+1$-quitting payoff and the the α^{t+1}-weighed $t+2$-continuation payoff. For a pure action, the $t+2$-continuation payoff suffices.) Relative to the quitting payoffs obtained on the equilibrium path, $\tilde{t}_f$ increases $_{-i}u_i(q)$ and $\tilde{J}_i$ reduces the deviation $_{-i}\tilde{u}(q)$ while leaving all $_iu(q)$ unchanged for all periods following the deviation. Since $\tilde{t}_f$ carries only half the payoff-weight of $\tilde{J}_i$, the overall effect of a jump on future quitting payoffs cannot be positive.

   (a) $\overline{\text{P1}}$: The continuation payoffs $\bar{u}_1^{t+2}(\beta)$ from increasing minimally and from jumping j_1^t are given by backward induction through the previous steps. The $(\tilde{t}, \tilde{J}_i)$-version is lower than the minimal increase version. We consider a general period t, x minimal bids ahead of $\lfloor\bar{\bar{\text{E}}}\rfloor$ and look at $\overline{\text{P1}}$'s stage decision. $\overline{\text{P1}}$ expects the following payoff

$$\bar{u}_1^t(\beta) = (1-\bar{\varphi}_i^t)\left[(1-\alpha_2^{t+1})\left(\bar{\theta} - \frac{t+1}{2} - J_1^t\right) + \alpha_2^{t+1} u_1^{t+2}(\beta)\right] + \bar{\varphi}_i^t \bar{u}_1^{t+2} \tag{A.8}$$

which, for α_2^{t+1} as defined in (2.1) as

$$\alpha_2^{t+1} = \frac{(1-\underline{\varphi}_1^t)(t + J_1^t + J_2^{t+1} - \underline{\underline{\text{E}}} + 1) - j_2^{t-1} - 1}{(1-\underline{\varphi}_1^t)(t + J_1^t + J_2^{t+1} - \underline{\underline{\text{E}}} + 1)},$$

equals

$$(1-\bar{\varphi}_1^t)\left[\underbrace{1 - \frac{(1-\underline{\varphi}_1^t)(t + J_1^t + J_2^{t+1} - \underline{\underline{\text{E}}} + 1) - j_2^{t-1} - 1}{(1-\underline{\varphi}_1^t)(t + J_1^t + J_2^{t+1} - \underline{\underline{\text{E}}} + 1)}}_{\text{decreasing in } j_1^t}\underbrace{\left(\bar{\underline{\text{E}}}_2^1 - \frac{t+1}{2} - J_1^t\right)}_{\text{decreasing in } j_1^t}\right.$$
$$\left. + \underbrace{\frac{(1-\underline{\varphi}_1^t)(t + J_1^t + J_2^{t+1} - \underline{\underline{\text{E}}} + 1) - j_2^{t-1} - 1}{(1-\underline{\varphi}_1^t)(t + J_1^t + J_2^{t+1} - \underline{\underline{\text{E}}} + 1)}}_{\text{increasing in } j_1^t}\bar{u}_1^{t+2}(\beta)\right] + \bar{\varphi}_1^t \bar{u}_1^{t+2}(\beta).$$

where $J_1^t = j_1^1 + j_1^3 + \cdots + j_1^t$ is increasing in any jump j_1^t at t. Since the continuation payoff $u_1^{t+2}(\beta)$ is pinned down by a previous equilibrium backward induction step with all future quitting payoffs and continuation probabilities α_2 replaced by their $(\tilde{t}_f, \tilde{J}_i)$-formulations exhibiting the same monotonicity as (A.8), we know that $u_1^{t+2}(\beta)$ is decreased by a jump $j_1^t > 0$ by P1. Thus $u_1^t(\cdot)$ is strictly decreasing in j_1^t and $\overline{\text{P1}}$ finds it optimal to increase his bids minimally by setting $j_1^t = 0$ for all $t_s < t < t_f - 1$.

The intuition is that for $\bar{\underline{s}}_2^1$, $\overline{\text{P1}}$ wants $\underline{\text{P2}}$ to quit as early as possible in order to minimise his payments for the object worth $\bar{\underline{\text{E}}}_2^1$. In case of $\bar{\bar{s}}$, $\overline{\text{P2}}$ only quits at the final stage and thus the players share the object's value half-half.

(b) $\underline{\text{P1}}$ is made indifferent at each stage, whatever the choice of continuation action, because $\underline{\text{P2}}$ ensures this through mixing appropriately at the following stage. She is willing to mix because, from lemma 1, she must be indifferent between continuing and quitting herself. Since $\underline{\text{P1}}$ is thus made indifferent between any continuation action and quitting, he cannot profit from jumping. The same holds for $\underline{\text{P2}}$.

(c) In an argument which fully parallels (a), $\overline{\text{P2}}$ gets

$$\bar{u}_2^t(\beta) = (1-\bar{\varphi}_2^t)\left[(1-\alpha_1^{t+1})\left(\bar{\underline{\text{E}}}_1^2 - \frac{t}{2} - J_2^t\right) + \alpha_1^{t+1}u_2^{t+2}(\beta)\right] + \bar{\varphi}_2^t\bar{u}_2^{t+2}(\beta)$$

which equals for α_1^{t+1} defined in (2.3)

$$(1-\bar{\varphi}_2^t)\left[\underbrace{1 - \frac{(1-\underline{\varphi}_2^t)(t+J_1^{t+1}+J_2^t-\underline{\underline{\text{E}}}+1) - j_1^{t-1} - 1}{(1-\underline{\varphi}_2^t)(t+J_1^{t+1}+J_2^t-\underline{\underline{\text{E}}}+1)}}_{\text{decreasing in } j_2^t}\underbrace{\left(\bar{\underline{\text{E}}}_1^2 - \frac{t}{2} - J_2^t\right)}_{\text{decreasing in } j_1^t}\right.$$
$$\left. + \underbrace{\frac{(1-\underline{\varphi}_2^t)(t+J_1^{t+1}+J_2^t-\underline{\underline{\text{E}}}+1) - j_1^{t-1} - 1}{(1-\underline{\varphi}_2^t)(t+J_1^{t+1}+J_2^t-\underline{\underline{\text{E}}}+1)}}_{\text{increasing in } j_1^t}\bar{u}_2^{t+2}(\beta)\right] + \bar{\varphi}_2^t\bar{u}_2^{t+2}(\beta).$$

Thus again $u_2^t(\cdot)$ is strictly decreasing in j_2^t and $\overline{\text{P2}}$ finds it optimal to increase her bids minimally by setting $j_2^t = 0$ for all $t_s < t < t_f - 1$. □

Proof of lemma 6. The previous lemma pins down continuation payoffs for periods from the last possible equilibrium continuation action at t_f backward to t_s. Again we have to distinguish the cases of odd and even $\lfloor\bar{\bar{\text{E}}}\rfloor$. In equilibrium for odd $\lfloor\bar{\bar{\text{E}}}\rfloor$, we know that $\underline{\text{P2}}$ is the last player to mix at period $t_f - 1$. Therefore we need a first mixture by $\underline{\text{P1}}$ at period t_s in order to uniquely define all α_i^t and φ_i^t of the main game. Conversely, for even $\lfloor\bar{\bar{\text{E}}}\rfloor$, $\underline{\text{P1}}$ is the last player to mix in β^* and the first player to mix at period t_s must be $\underline{\text{P2}}$. Apart from this role reversal, the two cases are identical. Thus we only look at the case of odd $\lfloor\bar{\bar{\text{E}}}\rfloor$—illustrated in fig. 2—in detail.

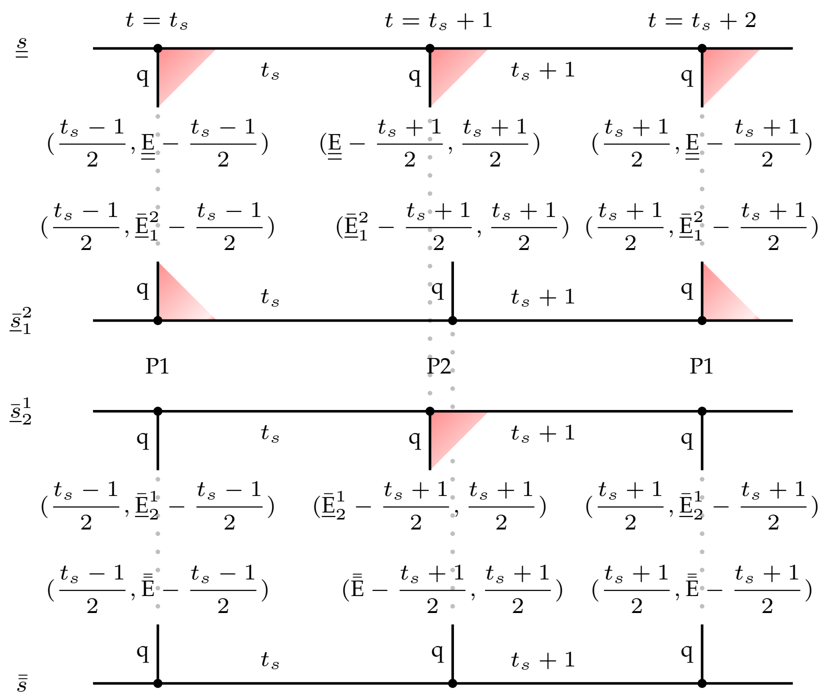

Figure 2: The end of the equilibrium 'preplay'-phase at t_s for odd $\lfloor \bar{\bar{\mathrm{E}}} \rfloor$.

We defined $t_s + 1$ as the first period where $\underline{\mathrm{P2}}$'s quitting payoff is higher than her prior-based continuation payoff from the above lemma 5. Thus, if her beliefs are not manipulated through $\underline{\mathrm{P1}}$'s mixing, $\underline{\mathrm{P2}}$ will quit at $t_s + 1$. Conversely, for $t < t_s$, there is no need for $\underline{\mathrm{P1}}$ to mix and thereby induce $\underline{\mathrm{P2}}$ to continue bidding up because her prior-based continuation payoff is already higher than her quitting payoffs. Thus mixing by $\underline{\mathrm{P1}}$ before t_s would lower his expected payoff because her quitting payoffs are increasing. Thus $\underline{\mathrm{P1}}$ will not quit with positive probability before t_s and $\overline{\mathrm{P1}}$ will do the same because $\bar{\varphi}_1 > \underline{\varphi_1}$. Moreover, since no $\underline{\mathrm{P}i}$ can gain from mixing, beliefs cannot be modified through preplay jumps. Therefore only the negative influence of a jump on own future quitting payoffs ${}_{-i}u_i^t(q)$ for $t \geq t_s$ remains and hence preplay-phase jumps decrease continuation payoffs.

Thus both players and both types increase own bids minimally and with probability one in the preplay-phase. Potential equilibria of (jump, jump-back)-type, where the original equilibrium payoffs are re-instated by a pair of symmetric preplay-jumps, are excluded because each player can increase his payoff by unilaterally deviating from such a candidate to a minimal-increase strategy. □

Proof of proposition 1. For odd $\lfloor \bar{\bar{\mathrm{E}}} \rfloor$, the evolution of $\underline{\mathrm{P2}}$'s equilibrium belief $\underline{\varphi}_2^t$ determines the first mixing period of the game. To see why this is the case, notice that—as argued in (A.7) of the previous lemma—$\underline{\mathrm{P2}}$'s terminal equilibrium mixing belief must be

$$\underline{\varphi}_2^{\lfloor \bar{\bar{\mathrm{E}}} \rfloor - 1} = \frac{\lfloor \bar{\bar{\mathrm{E}}} \rfloor - \underline{\underline{\mathrm{E}}} - 1}{\lfloor \bar{\bar{\mathrm{E}}} \rfloor - \underline{\underline{\mathrm{E}}}}.$$

Inserting this into Bayes' rule and substituting $\underline{\mathrm{P1}}$'s mixture probability for the equilibrium (2.1), one finds $\underline{\underline{\mathrm{P2}}}$'s even time-$t$ belief by calculating backwards as

$$
{}^{*}\underline{\varphi}_2^t = \left(\left(\left(\frac{\lfloor\bar{\bar{\mathrm{E}}}\rfloor - \underline{\underline{\mathrm{E}}} - 1}{\lfloor\bar{\bar{\mathrm{E}}}\rfloor - \underline{\underline{\mathrm{E}}}}\right)\frac{\lfloor\bar{\bar{\mathrm{E}}}\rfloor - \underline{\underline{\mathrm{E}}} - 3}{\lfloor\bar{\bar{\mathrm{E}}}\rfloor - \underline{\underline{\mathrm{E}}} - 2}\right)\frac{\lfloor\bar{\bar{\mathrm{E}}}\rfloor - \underline{\underline{\mathrm{E}}} - 5}{\lfloor\bar{\bar{\mathrm{E}}}\rfloor - \underline{\underline{\mathrm{E}}} - 4}\right)_{\frac{\lfloor\bar{\bar{\mathrm{E}}}\rfloor - t}{2}\ \text{times}} \cdots = \prod_{\tau=1}^{\frac{\lfloor\bar{\bar{\mathrm{E}}}\rfloor - t+1}{2}} \frac{\lfloor\bar{\bar{\mathrm{E}}}\rfloor - \underline{\underline{\mathrm{E}}} - 2\tau + 1}{\lfloor\bar{\bar{\mathrm{E}}}\rfloor - \underline{\underline{\mathrm{E}}} - 2\tau + 2}
$$

which, using the Pochhammer notation $Pochhammer(a,n) = \frac{\Gamma(a+n)}{\Gamma(a)}$, simplifies to

$$
{}^{*}\underline{\varphi}_2^t = \underline{\varphi}_2^{t+2}\frac{t - \underline{\underline{\mathrm{E}}}}{t - \underline{\underline{\mathrm{E}}} + 1} = \frac{Pochhammer\left(\frac{1-\lfloor\bar{\bar{\mathrm{E}}}\rfloor + \underline{\underline{\mathrm{E}}}}{2}, \frac{1-\lfloor\bar{\bar{\mathrm{E}}}\rfloor - t}{2}\right)}{Pochhammer\left(\frac{-\lfloor\bar{\bar{\mathrm{E}}}\rfloor + \underline{\underline{\mathrm{E}}}}{2}, \frac{1-\lfloor\bar{\bar{\mathrm{E}}}\rfloor - t}{2}\right)}. \tag{A.9}
$$

We have found t_s^* as soon as this calculated ${}^{*}\underline{\varphi}_2^t$ becomes smaller than the prior $\underline{\varphi}_2^{t_s}$. Since a closed form solution to (A.9) for t_s is—to the author's knowledge—unavailable, we must use trial & error simulation techniques. This, however, does not impinge on our analytic existence argument and the characterisation of the equilibrium strategies. Now calculating from this t_s^* forward, $\underline{\mathrm{P1}}$'s final period belief is found from ${}^{*}\underline{\varphi}_1^t = \underline{\varphi}_1^{t-2}\frac{t-\underline{\underline{\mathrm{E}}}}{t-\underline{\underline{\mathrm{E}}}+1}$ as

$$
\underline{\varphi}_1^{\lfloor\bar{\bar{\mathrm{E}}}\rfloor} = \underline{\varphi}_1^1 \frac{\Gamma\left(\frac{\lfloor\bar{\bar{\mathrm{E}}}\rfloor - \underline{\underline{\mathrm{E}}} - t_s + 4}{2}\right)}{\Gamma\left(\frac{\lfloor\bar{\bar{\mathrm{E}}}\rfloor - \underline{\underline{\mathrm{E}}} - t_s + 3}{2}\right)} \frac{\Gamma\left(\frac{t_s - \underline{\underline{\mathrm{E}}}}{2}\right)}{\Gamma\left(\frac{t_s - \underline{\underline{\mathrm{E}}}+1}{2}\right)}. \tag{A.10}
$$

The two resulting conditions (A.9) and (A.10) are the sufficient conditions for the existence of the equilibrium β^* as stated by assumption 2. The conditions for even $\lfloor\bar{\bar{\mathrm{E}}}\rfloor$ are identical for reversed roles of the players. Given the first mixture period t_s^*, the player's odd $\lfloor\bar{\bar{\mathrm{E}}}\rfloor$ expected payoffs are determined as

$$
\begin{aligned}
u_1(\beta^*|\bar{\bar{\mathrm{s}}}) &= \bar{\bar{\mathrm{E}}} - \tfrac{\lfloor\bar{\bar{\mathrm{E}}}\rfloor+1}{2}, \quad u_2(\beta^*|\bar{\bar{\mathrm{s}}}) = \tfrac{\lfloor\bar{\bar{\mathrm{E}}}\rfloor+1}{2}, \\
u_1(\beta^*|\bar{\underline{\mathrm{s}}}_2^1) &= \bar{\underline{\mathrm{E}}}_2^1 - u_2(\beta^*|\bar{\underline{\mathrm{s}}}_2^1), \\
u_2(\beta^*|\bar{\underline{\mathrm{s}}}_2^1) &= \sum_{\tau=\frac{t_s+1}{2}}^{\frac{\lfloor\bar{\bar{\mathrm{E}}}\rfloor-1}{2}} \left(\prod_{t=\frac{t_s+1}{2}}^{\tau-1} \alpha_2^{2t}\right) u_2^{2\tau}(q)(1-\alpha_2^{2\tau}) + \left(\prod_{t=1}^{\frac{\lfloor\bar{\bar{\mathrm{E}}}\rfloor-1}{2}} \alpha_2^{2t}\right) u_2^{\lfloor\bar{\bar{\mathrm{E}}}\rfloor}(q), \\
u_1(\beta^*|\bar{\underline{\mathrm{s}}}_1^2) &= \sum_{\tau=\frac{t_s+1}{2}}^{\frac{\lfloor\bar{\bar{\mathrm{E}}}\rfloor+1}{2}} \left(\prod_{t=1}^{\tau-1} \alpha_1^{2t-1}\right) u_1^{2\tau-1}(q)(1-\alpha_1^{2\tau-1}) + \left(\prod_{t=\frac{t_s+1}{2}}^{\frac{\lfloor\bar{\bar{\mathrm{E}}}\rfloor+1}{2}} \alpha_1^{2t-1}\right) u_1^{\lfloor\bar{\bar{\mathrm{E}}}\rfloor}(q), \\
u_2(\beta^*|\bar{\underline{\mathrm{s}}}_1^2) &= \bar{\underline{\mathrm{E}}}_1^2 - u_1(\beta^*|\bar{\underline{\mathrm{s}}}_1^2), \quad u_1(\beta^*|\underline{\underline{\mathrm{s}}}) = \underline{\underline{\mathrm{E}}} - u_2(\beta^*|\underline{\underline{\mathrm{s}}}), \\
u_2(\beta^*|\underline{\underline{\mathrm{s}}} &= \sum_{\tau=\frac{t_s+1}{2}}^{\frac{\lfloor\bar{\bar{\mathrm{E}}}\rfloor-1}{2}} \left(\prod_{t=\frac{t_s+1}{2}}^{\tau-1} \alpha_2^{2t-1}\alpha_2^{2t}\right)\left((1-\alpha_1^{2\tau-1})u_2^{2\tau-1}(q) + \alpha_1^{2\tau-1}(1-\alpha_2^{2\tau})u_2^{2\tau}(q)\right).
\end{aligned}
$$

Obviously, $u_1(\beta^*|\underline{s}_1) = \frac{t_s - 1}{2}$. These payoffs can be directly computed by supplying the equilibrium continuation probabilities α_i^t and the corresponding beliefs φ_i^t from (2.1)–(2.4). For the case of even-valued $\lfloor \bar{\bar{\mathrm{E}}} \rfloor$, the expected payoffs are found in the same way. □

A.2 Examples

A.2.1 Particular p

We look at the simple example of $\theta \in \{0, 5\}$ for the particular, arbitrarily chosen pair of signal accuracies $\mathbf{p} = (.8, .75)$. The set of possible bids is $\{0, 1, 2, 3, 4, 5, \ldots, \bar{B}\}$. The true value of the object is unknown to a player with signal precision $p < 1$ and thus there is some generic uncertainty on the object's value underlying the incomplete information on the opponent's signal. Therefore the true realisation of θ is generally not known to a player even if he were to know the opponent's signal: Learning the opponent's signal is the best a player can hope for. Thus we use signal profiles as states in fig. 3 and not the realisation of θ.

The chosen signal accuracies give rise to the initial beliefs conditional on the own signal $\varphi_i^t = \mathrm{pr}(s_{-i} = \bar{s}|s_i)$. These initial conditional beliefs are calculated from the common priors $\varphi_i^0 = {}^1\!/_2$ using Bayes' rule as, for instance, for P2

$$\begin{aligned}\underline{\varphi}_2^1 = \quad & \mathrm{pr}(\underline{s}_1|\underline{s}_2) = \frac{\mathrm{pr}(\underline{s}_1, \underline{s}_2)}{\mathrm{pr}(\underline{s}_2)} = \frac{\mathrm{pr}(\underline{s}_1, \underline{s}_2|\underline{\theta})\,\mathrm{pr}(\underline{\theta}) + \mathrm{pr}(\underline{s}_1, \underline{s}_2|\bar{\theta})\,\mathrm{pr}(\bar{\theta})}{\mathrm{pr}(\underline{s}_2|\underline{\theta})\,\mathrm{pr}(\underline{\theta}) + \mathrm{pr}(\underline{s}_2|\bar{\theta})\,\mathrm{pr}(\bar{\theta})} = \\ & \frac{\mathrm{pr}(\underline{s}_1, \underline{s}_2|\underline{\theta}) + \mathrm{pr}(\underline{s}_1, \underline{s}_2|\bar{\theta})}{\mathrm{pr}(\underline{s}_2|\underline{\theta}) + \mathrm{pr}(\underline{s}_2|\bar{\theta})} = \frac{p_1p_2 + (1-p_1)(1-p_2)}{p_2 + (1-p_2)}.\end{aligned}$$

Filling in the example values of $\mathbf{p} = (.8, .75)$, the above give

$$\begin{aligned}\underline{\varphi}_1^1 &= p_1 + p_2 - 2p_1p_2 = .35, & \underline{\varphi}_2^1 &= (1-p_1)p_2 + p_1(1-p_2) = .35, \\ \bar{\varphi}_1^1 &= p_1p_2 + (1-p_1)(1-p_2) = .65, & \bar{\varphi}_2^1 &= (1-p_1)(1-p_2) + p_1p_2 = .65.\end{aligned}$$

Since the information on the signal accuracies is symmetric and public, the initial $\underline{\varphi}_1^1 = \underline{\varphi}_2^1$ and $\bar{\varphi}_1^1 = \bar{\varphi}_2^1$ must be identical given the same signal. Next we calculate the 'objective' expectation of the object's value given the different signal combinations. These are

$$\begin{aligned}\underline{\underline{\mathrm{E}}} &= \bar{\theta}\frac{(1-p_1)(1-p_2)}{(1-p_1)(1-p_2) + p_1p_2} = 0.38, & \underline{\bar{\mathrm{E}}}_1^2 &= \bar{\theta}\frac{(1-p_1)p_2}{p_1 + p_2 - 2p_1p_2} = 2.14, \\ \underline{\bar{\mathrm{E}}}_2^1 &= \bar{\theta}\frac{p_1(1-p_2)}{p_1 + p_2 - 2p_1p_2} = 2.86, & \bar{\bar{\mathrm{E}}} &= \bar{\theta}\frac{p_1p_2}{p_1p_2 + (1-p_1)(1-p_2)} = 4.62\end{aligned}$$

giving P2's 'subjective' ex-ante expectation of the object's value as $\underline{\mathrm{E}}_2 = \underline{\varphi}_2^1\underline{\bar{\mathrm{E}}}_2^1 + (1 - \underline{\varphi}_2^1)\underline{\underline{\mathrm{E}}} = (1-p_2)\bar{\theta} = 1.25$.

Following the definition of the equilibrium β^* in section 2 we impose the low-type mixture conditions at each stage of the main game and obtain the following continuation probabilities from solving the resulting system of inequalities (which is derived below)

$$\beta_b^* = \begin{pmatrix} \underline{\alpha}_1^1 = 1 & \underline{\alpha}_2^2 = 0.21 & \underline{\alpha}_1^3 = 0.41 & \underline{\alpha}_2^4 = 0 & \underline{\alpha}_1^5 = 0 \\ \bar{\alpha}_1^1 = 1 & \bar{\alpha}_2^2 = 1 & \bar{\alpha}_1^3 = 1 & \bar{\alpha}_2^4 = 1 & \bar{\alpha}_1^5 = 0 \end{pmatrix} \tag{A.11}$$

together with the belief system

$$\begin{pmatrix} \underline{\varphi}_1^1 = .35 & \underline{\varphi}_2^2 = .35 & \underline{\varphi}_1^3 = .72 & \underline{\varphi}_2^4 = .57 \\ \bar{\varphi}_1^1 = .65 & \bar{\varphi}_2^2 = .65 & \bar{\varphi}_1^3 = .90 & \bar{\varphi}_2^4 = .82 \end{pmatrix}.$$

This candidate equilibrium path is marked red in the extensive form of fig 3 where shaded triangles symbolise mixed actions. To compute the above equilibrium profile, we weigh the low-type player's expected continuation payoffs from a profile β at $t \in \{1, 2, 3, 4\}$

$$u_i^t(\beta|\underline{s}_i) = (1-\underline{\varphi}_i^t)\left[(1-\alpha_{-i}^{t+1})u_i^{t+1}(q|\underline{s}_i, \underline{s}_{-i}) + \alpha_{-i}^{t+1}u_i^{t+2}(\beta|\underline{s}_i)\right] + \underline{\varphi}_i^t\left[u_i^{t+2}(\beta|\underline{s}_i)\right]$$

against the same player's *certain* quitting payoff $u_i^t(q) = \sum^t g_{-i}$ as defined in lemma 4. These quitting payoffs are independent of the object's expected value because they consist solely of the sum of the opponent's bidding increases. Notice that it is crucial for the easy solvability of the game that in the above continuation payoff $u_i^{t+2}(\beta|\underline{s}_i) = u_i^{t+2}(q|\underline{s}_i)$ because $\underline{\text{P}i}$ mixes at $t+2$. Of course, Pi is willing to mix only if the above quitting and continuation payoffs are equal.

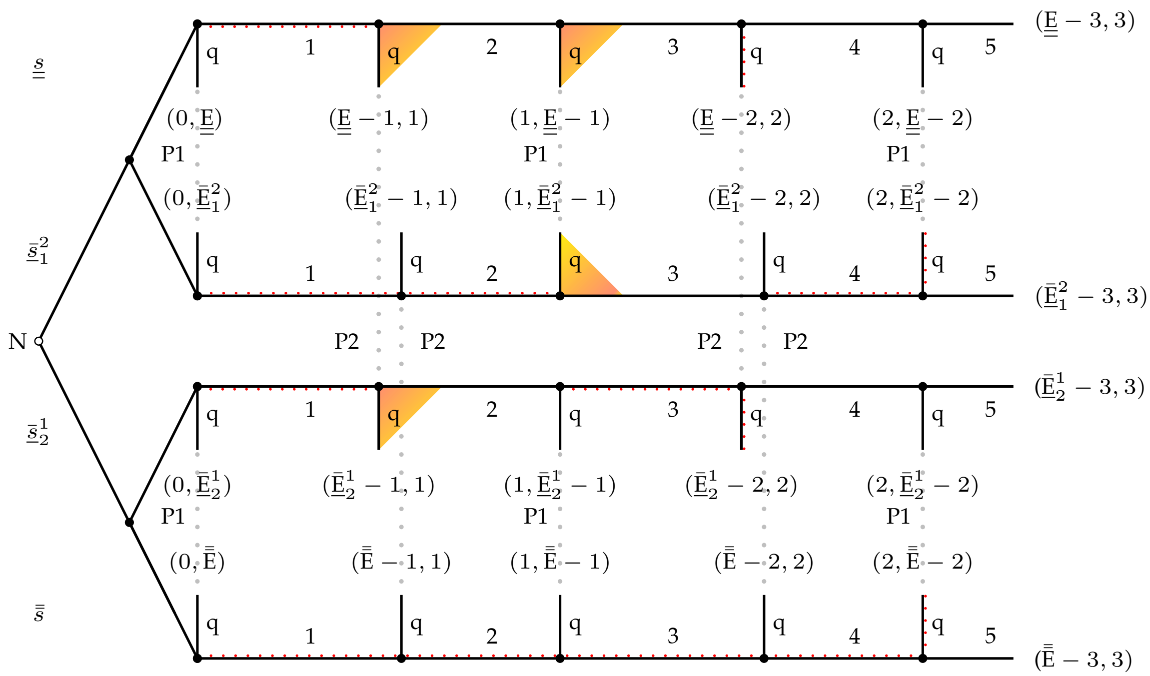

Figure 3: Partial extensive form for $\theta \in \{0, 5\}$ and $\mathbf{p} = (.8, .75)$.

The equilibrium dynamics are as follows; first the *backward chain*: Since $\underline{\text{P2}}$ quits with probability one rather than bidding 4, and $\overline{\text{P2}}$ bids 4 followed by $\overline{\text{P1}}$ quitting, the terminal belief $\underline{\varphi}_1^3$

which makes P1 mix at $t = 3$ is uniquely determined. There is only a single mixture probability α_2^2 which brings about these beliefs through Bayes' rule and thus this mixture probability is determined uniquely as well. The *forward chain* consists just of the indifference condition given the prior beliefs by P2 at $t = 2$ generating P1's mixture probability at $t = 3$: P2's indifference between quitting and minimal continuation uniquely defines the α_1^3 which makes her mix. Notice that this does not include the mixing of P1 at $t_f = 1$ (as in the case of odd $\bar{\bar{\mathrm{E}}}$) because all probabilities are already uniquely determined. As prescribed by β^*, all high types continue with probability 1 until the minimal admissible continuation bids exceed $\lceil\bar{\bar{\mathrm{E}}}\rceil$; then they quit.

Checking the player's equilibrium conditions amounts to setting up the system

$$\begin{array}{lll}
\text{low-type P1:} & t=1, & 0 < (1-\underline{\varphi}_1^1)\left[(1-\underline{\alpha}_2^2)(\underline{\underline{\mathrm{E}}}-1)+\underline{\alpha}_2^2(1)\right]+\underline{\varphi}_1^1(1)\\
\text{high-type P1:} & t=1, & 0 < (1-\bar{\varphi}_1^1)\left[(1-\underline{\alpha}_2^2)(\underline{\bar{\mathrm{E}}}_2^1-1)+\underline{\alpha}_2^2(1)\right]+\bar{\varphi}_1^1(2)\\
\text{low-type P2:} & t=2, & 1 = (1-\underline{\varphi}_2^2)\left[(1-\underline{\alpha}_1^3)(\underline{\underline{\mathrm{E}}}-1)+\underline{\alpha}_1^3(2)\right]+\underline{\varphi}_2^2(2)\\
\text{high-type P2:} & t=2, & 1 < (1-\bar{\varphi}_2^2)\left[(1-\underline{\alpha}_1^3)(\underline{\bar{\mathrm{E}}}_1^2-1)+\underline{\alpha}_1^3(2)\right]+\underline{\varphi}_2^2(\bar{\bar{\mathrm{E}}}-2)\\
\text{BR:} & t=3, & \underline{\varphi}_1^3=\mathrm{pr}(\bar{s}_2|\underline{s}_1,b_2^2=2)=\dfrac{\underline{\varphi}_1^1}{\underline{\varphi}_1^1+(1-\underline{\varphi}_1^1)\underline{\alpha}_2^2},\bar{\varphi}_1^3=\dfrac{\bar{\varphi}_1^1}{\bar{\varphi}_1^1+(1-\bar{\varphi}_1^1)\underline{\alpha}_1^1}
\end{array}$$

$$\begin{array}{lll}
\text{low-type P1:} & t=3, & 1=(1-\underline{\varphi}_1^3)(\underline{\underline{\mathrm{E}}}-2)+\underline{\varphi}_1^3(2)\\
\text{high-type P1:} & t=3, & 1<(1-\bar{\varphi}_1^3)(\underline{\bar{\mathrm{E}}}_2^1-2)+\bar{\varphi}_1^3(2)\\
\text{BR:} & t=4, & \underline{\varphi}_2^4=\mathrm{pr}(\bar{s}_1|\underline{s}_2,b_1^3=3)=\dfrac{\underline{\varphi}_2^2}{\underline{\varphi}_2^2+(1-\underline{\varphi}_2^2)\underline{\alpha}_1^3},\bar{\varphi}_2^4=\dfrac{\bar{\varphi}_2^2}{\bar{\varphi}_2^2+(1-\bar{\varphi}_2^2)\underline{\alpha}_1^3}\\
\text{low-type P2:} & t=4, & 2>(1-\underline{\varphi}_2^4)(\underline{\underline{\mathrm{E}}}-2)+\bar{\varphi}_2^4(\underline{\bar{\mathrm{E}}}_2^1-2)\\
\text{high-type P2:} & t=4, & 2<(1-\bar{\varphi}_2^4)(\underline{\bar{\mathrm{E}}}_1^2-2)+\bar{\varphi}_2^4(\bar{\bar{\mathrm{E}}}-2)\\
\text{BR:} & t=5, & \underline{\varphi}_1^5=\mathrm{pr}(\bar{s}_2|\underline{s}_1,b_2^4=4)=\dfrac{\underline{\varphi}_1^3}{\underline{\varphi}_1^3+(1-\underline{\varphi}_1^3)\underline{\alpha}_2^4}=1,\bar{\varphi}_1^5=\dfrac{\bar{\varphi}_1^3}{\bar{\varphi}_1^3+(1-\bar{\varphi}_1^3)\underline{\alpha}_2^4}=1\\
\text{low-type P1:} & t=5, & 2>(\underline{\bar{\mathrm{E}}}_1^2-3)\\
\text{high-type P1:} & t=5, & 2>(\bar{\bar{\mathrm{E}}}-3)
\end{array}$$

which is solved by (A.11). This, together with an unsuccessful search for deviations,[11] confirms β^* with probabilities (A.11) as equilibrium of our example. Its expected payoffs are

	$(\underline{s}_1,\underline{s}_2)$	$(\underline{s}_1,\bar{s}_2)$	$(\bar{s}_1,\underline{s}_2)$	$(\bar{s}_1,\bar{s}_2)$	$(\underline{s}_1,\cdot)$	$(\bar{s}_1,\cdot)$	$(\cdot,\underline{s}_2)$	$(\cdot,\bar{s}_2)$	$\mathbb{E}[\cdot]$
$u_1(\beta^*\|s)$	−.50	1.41	1.65	2.00	0.17	1.88	·	·	1.02
$u_2(\beta^*\|s)$	0.89	0.73	1.21	2.62	·	·	1.00	1.96	1.48

As pointed out previously, it is not possible to implement β^* for all pairs of signal accuracies $\mathbf{p}$.

[11] In order to confirm that there are no profitable deviations we need to work out the players' on- and off-equilibrium path beliefs. This is done in accord with lemmata 1 and 2. What these lemmata say is that both low-type players must be indifferent between quitting and continuing after each feasible deviation (ie. after each deviation which does not only allow for subsequent quitting). At the same time, the players' equilibrium strategies must state a unique mixture probability in response to each previous period observed action. Since only a single belief is compatible with actually using the equilibrium response, beliefs are fully determined.

As sufficiency condition for existence of β^*, assumption 2 demands that

$$0.8185 = \bar{\varphi}_2^4 > \frac{\lfloor \bar{\bar{\bar{\mathrm{E}}}} \rfloor - \underline{\bar{\mathrm{E}}}_1^2}{\bar{\bar{\bar{\mathrm{E}}}} - \underline{\bar{\mathrm{E}}}_1^2} = 0.7511$$

which ensures that the $\overline{\mathrm{P2}}$ continues at period $\lfloor \bar{\bar{\bar{\mathrm{E}}}} \rfloor = 4$. It is indeed fulfilled. Likewise, $\underline{\mathrm{P1}}$'s period $t_f - 1$ belief allows for his mixing and the grid condition $\lfloor \bar{\bar{\bar{\mathrm{E}}}} \rfloor > \underline{\bar{\mathrm{E}}}_2^1$ holds.

To illustrate a deviation, suppose P2 observes $\hat{b}_1^1 = 2$ instead of the equilibrium-prescribed $b_1^1 = 1$; then her *equilibrium* mixture condition at $t = 2$ turns into

$$2 = (1 - \underline{\hat{\varphi}}_2^2)(\underline{\underline{\underline{\mathrm{E}}}} - 1) + \underline{\hat{\varphi}}_2^2(3) \quad \Leftrightarrow \quad \underline{\hat{\varphi}}_2^2 = \frac{3 - \underline{\underline{\underline{\mathrm{E}}}}}{4 - \underline{\underline{\underline{\mathrm{E}}}}} \approx 0.72 = \frac{\underline{\varphi}_2^1(1)}{\underline{\varphi}_2^1(1) + (1 - \underline{\varphi}_2^1)\hat{\alpha}_1^1}$$

resulting in the requirement of P2 believing that the low-type P1's deviation occurred with probability $\hat{\alpha}_1^1 \approx 0.21$. *Any* other belief will result in P2 either continuing or quitting for certain meaning that P1 could manipulate P2 into doing what is optimal for him. Since P1 would do this *whatever his type*, this cannot be equilibrium behaviour.

The lesson from our example is threefold: (*i*) Players cannot 'lie' to their opponent by playing supposedly fully revealing actions because the opponent would not believe such dubious information. (*ii*) This renders jump-bidding unprofitable because it ties the deviating player to offering a higher-than-equilibrium share to the equilibrium player until agreement is reached. (*iii*) The only way of using private information is to gradually release it by playing partially revealing, type-dependent mixed actions until all information is transferred.

A.2.2 General p

In this subsection we present a fully worked example for the case of $\theta \in \{0, 5\}$. In principle, our problem is to find areas (ie. parameterised equilibria) in $p_1 \times p_2$ demarcated by our equilibrium conditions (which are polynomial inequalities). These conditions are simply the preference of the continuation payoff over the quitting payoff or vice versa and indifference between the two for mixed actions. Imposing these conditions at each stage of the game gives a region of the corresponding information requirements in $p_1 \times p_2$. As $\bar{\theta}$ becomes large, however, these conditions become numerous and of increasingly high order and hence solving for the resulting systems of equilibrium conditions becomes difficult even for state of the art computer math packages.[12]

Since we are looking for *all* full-dimension equilibria now,[13] we have to consider both low- and high-signal mixed actions the continuation probability of which we denote by $\underline{\alpha}$ and $\bar{\alpha}$. To

[12] The field concerned with the study of such problems in general is that of *algebraic geometry*. However, even using specialised computer software developed for the study of algebraic geometry problems we were unable to compute results for cases where $\bar{\theta}$ is significantly larger than in the present example. For a survey of the methods and techniques involved see Baxter and Iserles (2003).

[13] We call solutions 'full-dimension' equilibria if they have an interior in the map $p_1 \times p_2$. The complementary 'measure zero' equilibria are knife-edge parameter cases which we disregard in the present analysis.

denote strategy profiles we use matrices where, for instance

$$\begin{pmatrix} 1 & 1 & 1 & m \\ 1 & m & m & 0 \end{pmatrix} \text{ represents the profile } \begin{pmatrix} \bar{\alpha}_1^1 = 1 & \bar{\alpha}_2^2 = 1 & \bar{\alpha}_1^3 = 1 & \bar{\alpha}_2^4 \in (0,1) \\ \underline{\alpha}_1^1 = 1 & \underline{\alpha}_2^2 = \in (0,1) & \underline{\alpha}_1^3 = \in (0,1) & \underline{\alpha}_2^4 = 0 \end{pmatrix}.$$

The time-5 continuation probability α_1^5 is zero for any $\mathbf{p}$ and any signal and is therefore omitted. In accord with our model specification, the signal accuracy $\mathbf{p} = (p_1, p_2)$ specifies the probability with which the received signal is correctly indicating the true value of the object. The possible range for the publicly known (asymmetric) idiosyncratic signal precision p_i is $[{}^1\!/_2, 1],\ i = \{1, 2\}$ where $p_i = {}^1\!/_2$ means that Pi gets no additional information on top of her priors and $p_i = 1$ means that her signal is fully revealing. Hence the case with incomplete information on one side is described by $\mathbf{p} = (1, 0)$. Matrices such as the one above represent systems of polynomial inequalities solved by a system of restrictions on the constants α and $\mathbf{p}$. These results are summarised in fig. 4. In the following we list the strategies for which solutions in $\mathbf{p}$ (ie. parameterised equilibrium candidates) can be found. We sort them according to the number of pure low-type continuation actions the strategies contain. These parameterised solutions involve recurring conditions $f_1(\cdot)$–$f_3(\cdot)$ for mixed actions which are defined for convenience in the legend of fig. 4.

The remarkable result is that there is a unique map of full-dimension parameterised equilibria which covers $p_1 \times p_2$. Hence the essential uniqueness result of the analysis of the case of incomplete information on one side is preserved in this particular example of incomplete information on both sides for *any* parameter combination.

i) No pure low-type continuation

$$\begin{pmatrix} 1 & m & 1 & m \\ m & 0 & m & 0 \end{pmatrix} \Rightarrow {}^4\!/_5 \le p_1 \le 1 \wedge 0 \le p_2 \le \frac{3p_1 - 4}{p_1 - 3} \text{ for} \tag{A.12}$$

$$\underline{\alpha}_1^1 = \tfrac{7 - 25p_1 + 25p_1^2}{3 - 20p_1 + 25p_1^2},\ \bar{\alpha}_2^2 = \tfrac{5p_1 - 4}{5p_1 - 3},\ \underline{\alpha}_1^3 = \tfrac{5p_1 - 4}{5\underline{\alpha}_1^1 p_1 - \underline{\alpha}_1^1}, \text{ and } \bar{\alpha}_2^4 = \tfrac{5p_1 - 2}{5p_1 - 1}.$$

$$\begin{pmatrix} 1 & m & 1 & m \\ m & m & m & m \end{pmatrix} \Rightarrow {}^4\!/_5 \le p_1 \le 1 \wedge p_2 = {}^1\!/_2 \tag{A.13}$$

This is the essentially unique equilibrium of the case of incomplete information on one side. It is not directly comparable to the other equilibria because of the implicit constraint that $\alpha_2^t = \bar{\alpha}_2^t = \underline{\alpha}_2^t$ for all t stemming from the fact that P1 has perfect information in that model. It is a measure zero equilibrium with $\underline{\alpha}_1^1 = \frac{7 - 25p_1 + 25p_1^2}{3 - 20p_1 + 25p_1^2}$, $\alpha_2^2 = \frac{5p_1 - 4}{5p_1 - 3}$, $\underline{\alpha}_1^3 = \frac{5p_1 - 4}{5\underline{\alpha}_1^1 p_1 - \underline{\alpha}_1^1}$, and $\alpha_2^4 = \frac{5p_1 - 2}{5p_1 - 1}$.

$$\begin{pmatrix} 1 & 1 & 1 & m \\ m & m & m & 0 \end{pmatrix} \Rightarrow \begin{cases} (i) & 0.8453 \le p_1 \le 0.9515 \wedge \frac{3p_1 - 4}{p_1 - 3} \le p_2 \le f_2(p_1) \\ (ii) & 0.9515 < p_1 \le 1 \wedge \frac{3p_1 - 4}{p_1 - 3} \le p_2 \le f_1(p_1) \end{cases} \text{ for} \tag{A.14}$$

$\underline{\alpha}_2^2 = \frac{4-3p_1-3p_2+p_1p_2}{3-3p_1-3p_2+p_1p_2}$, $\underline{\alpha}_1^3 = \frac{-3\underline{\alpha}_1^1-p_1+3\underline{\alpha}_1^1p_1-p_2+3\underline{\alpha}_1^1p_2+2p_1p_2-\underline{\alpha}_1^1p_1p_2}{-2\underline{\alpha}_1^1+2\underline{\alpha}_1^1p_1+2\underline{\alpha}_1^1p_2+\underline{\alpha}_1^1p_1p_2}$, and $\bar{\alpha}_2^4 = \frac{2\underline{\alpha}_2^2-3p_1-2\underline{\alpha}_2^2p_1+2p_2-2\underline{\alpha}_2^2p_2+p_1p_2-\underline{\alpha}_2^2p_1p_2}{-4p_1+p_2+3p_1p_2}$.

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ m & m & 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} (i) & {}^5/_8 \leq p_1 \leq .9078 \wedge f_1(p_1) \leq p_2 \leq 1 \\ (ii) & 0.9078 < p_1 < 1 \wedge \frac{p_1}{4-3p_1} \leq p_2 \leq 1 \end{cases} \text{for} \tag{A.15}$$

$\underline{\alpha}_2^2 = \frac{4-3p_1-3p_2+p_1p_2}{3-3p_1-3p_2+p_1p_2}$ and $\underline{\alpha}_1^1 = \frac{-p_1-p_2+2p_1p_2}{3-3p_1-3p_2+p_1p_2}$.

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ m & m & m & m \end{pmatrix} \Rightarrow f_1(p_1) \leq p_2 \leq \frac{p_1}{4-3p_1} \wedge 0.9078 \leq p_1 \leq 1 \tag{A.16}$$

ii) One pure low-type continuation

$$\begin{pmatrix} 1 & 1 & m & 0 \\ 1 & m & 0 & 0 \end{pmatrix} \Rightarrow {}^1/_2 \leq p_1 \leq {}^3/_5 \wedge {}^3/_5 \leq p_2 \leq \frac{3-2p_1}{2+p_1} \text{ for} \tag{A.17}$$

$\underline{\alpha}_2^2 = \frac{3-3p_1-3p_2+p_1p_2}{2p_1-3p_2+p_1p_2}$ and $\bar{\alpha}_1^3 = \frac{3-5p_2}{2p_1-3p_2+p_1p_2}$.

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} (i) & {}^3/_5 \leq p_1 \leq 0.6202 \wedge \frac{3-2p_1}{2+p_1} \leq p_2 \leq \frac{3p_1}{2+p_1} \\ (ii) & 0.6202 < p_1 \leq {}^4/_5 \wedge \frac{3-2p_1}{2+p_1} \leq p_2 \leq \frac{-4+4p_1}{-4+3p_1} \end{cases}. \tag{A.18}$$

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & m & m & 0 \end{pmatrix} \Rightarrow \begin{cases} (i) & {}^1/_2 \leq p_1 \leq {}^3/_5 \wedge \frac{3-2p_1}{2+p_1} \leq p_2 \leq f_2(p_1) \\ (ii) & {}^3/_5 < p_1 \leq {}^{16}/_{25} \wedge \frac{3p_1}{2+p_1} \leq p_2 \leq f_2(p_1) \end{cases} \text{for} \tag{A.19}$$

$\underline{\alpha}_2^2 = \frac{-3p_1+2p_2+p_1p_2}{-2+2p_1+2p_2+p_1p_2}$ and $\underline{\alpha}_1^3 = \frac{-3+2p_1+2p_2+p_1p_2}{-2+2p_1+2p_2+p_1p_2}$.

$$\begin{pmatrix} 1 & 1 & 1 & m \\ 1 & 0 & m & 0 \end{pmatrix} \Rightarrow \begin{cases} (i) & 0.6202 \leq p_1 \leq 0.64 \wedge \frac{4p_1-4}{3p_1-4} \leq p_2 \leq {}^{3p_1}/_{2p_1} \\ (ii) & 0.64 < p_1 \leq {}^4/_5 \wedge \frac{4p_1-4}{3p_1-4} \leq p_2 \leq f_2(p_1) \\ (iii) & {}^4/_5 < p_1 \leq .8453 \wedge f_3(p_1) \leq p_2 \leq f_2(p_1) \\ (iv) & .8453 < p_1 < 0.9756 \wedge f_3(p_1) \leq p_2 \leq \frac{3p_1-4}{p_1-3} \end{cases} \text{for} \tag{A.20}$$

$\underline{\alpha}_2^4 = \frac{-3p_1+2p_2+p_1p_2}{4p_1+2p_2+3p_1p_2}$ and $\underline{\alpha}_1^3 = \frac{4-4p_1-4p_2+3p_1p_2}{-4p_1+p_2+3p_1p_2}$.

$$\begin{pmatrix} 1 & 1 & 1 & m \\ 1 & m & m & 0 \end{pmatrix} \Rightarrow \begin{cases} (i) & p_1 = {}^1\!/_2 \wedge p_2 = {}^4\!/_5 \wedge \bar{\alpha}_2^4, \underline{\alpha}_2^2, \underline{\alpha}_1^3 \\ (ii) & p_1 = 0.55 \wedge p_2 = 0.77 \wedge \bar{\alpha}_2^4, \underline{\alpha}_2^2, \underline{\alpha}_1^3 \\ (iii) & p_1 = 0.60 \wedge p_2 = 0.7444 \wedge \bar{\alpha}_2^4, \underline{\alpha}_2^2, \underline{\alpha}_1^3 \\ (iv) & p_1 = 0.95 \wedge p_2 = 0.6698 \wedge \bar{\alpha}_2^4, \underline{\alpha}_2^2, \underline{\alpha}_1^3 \\ (v) & p_1 > 0.95 \Rightarrow \emptyset \end{cases} . \tag{A.21}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & m & m & 0 \end{pmatrix} \Rightarrow \begin{cases} (i) & {}^1\!/_2 \leq p_1 <= {}^5\!/_8 \wedge f_2(p_1) \leq p_2 \leq 1 \\ (ii) & {}^5\!/_8 < p_1 < {}^4\!/_5 \wedge f_2(p_1) \leq p_2 \leq f_1(p_1) \\ (iii) & {}^5\!/_8 < p_1 \leq 0.9515 \wedge f_2(p_1) \leq p_2 \leq f_1(p_1) \end{cases} \quad \text{for} \tag{A.22}$$

$$\underline{\alpha}_2^2 = \tfrac{p_1+p_2-2p_1p_2}{-2+2p_1+2p_2+p_1p_2} \text{ and } \underline{\alpha}_1^3 = \tfrac{-3+2p_1+2p_2+p_1p_2}{-2+2p_1+2p_2+p_1p_2}.$$

This is the equilibrium β^* discussed in the main text.

iii) Two pure low-type continuations

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \Rightarrow {}^1\!/_2 \leq p_1 \leq {}^3\!/_5 \wedge {}^1\!/_2 \leq p_2 \leq {}^3\!/_5. \tag{A.23}$$

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \Rightarrow {}^3\!/_5 \leq p_1 \leq {}^4\!/_5 \wedge {}^1\!/_2 \leq p_2 \leq \frac{3-2p_1}{2+p_1}. \tag{A.24}$$

iv) Discussion

The map in fig. 4 has two striking features: (1) The strategy profiles (A.12)–(A.24) fully cover our parameter space $p_1 \times p_2$. (2) There is an *unique* equilibrium for any $\mathbf{p}$ in full dimension.[14] The equilibria are intuitively appealing. For instance in the lower-left-corner equilibrium region (A.23), the players have very little information and cannot effectively discriminate between the high and low signal states. Hence they bid up to the expectation of the object and quit as soon as the required bid exceeds this expectation in a (near-)pooling strategy. As expected from the analysis of the case of incomplete information on one side, (A.13), the essentially unique equilibrium of that case can be retrieved in the more general setting of incomplete information

[14] There are more parameterised equilibria of measure zero but we disregard them in the present discussion. There are no other equilibria in full dimension $\mathbf{p}$.

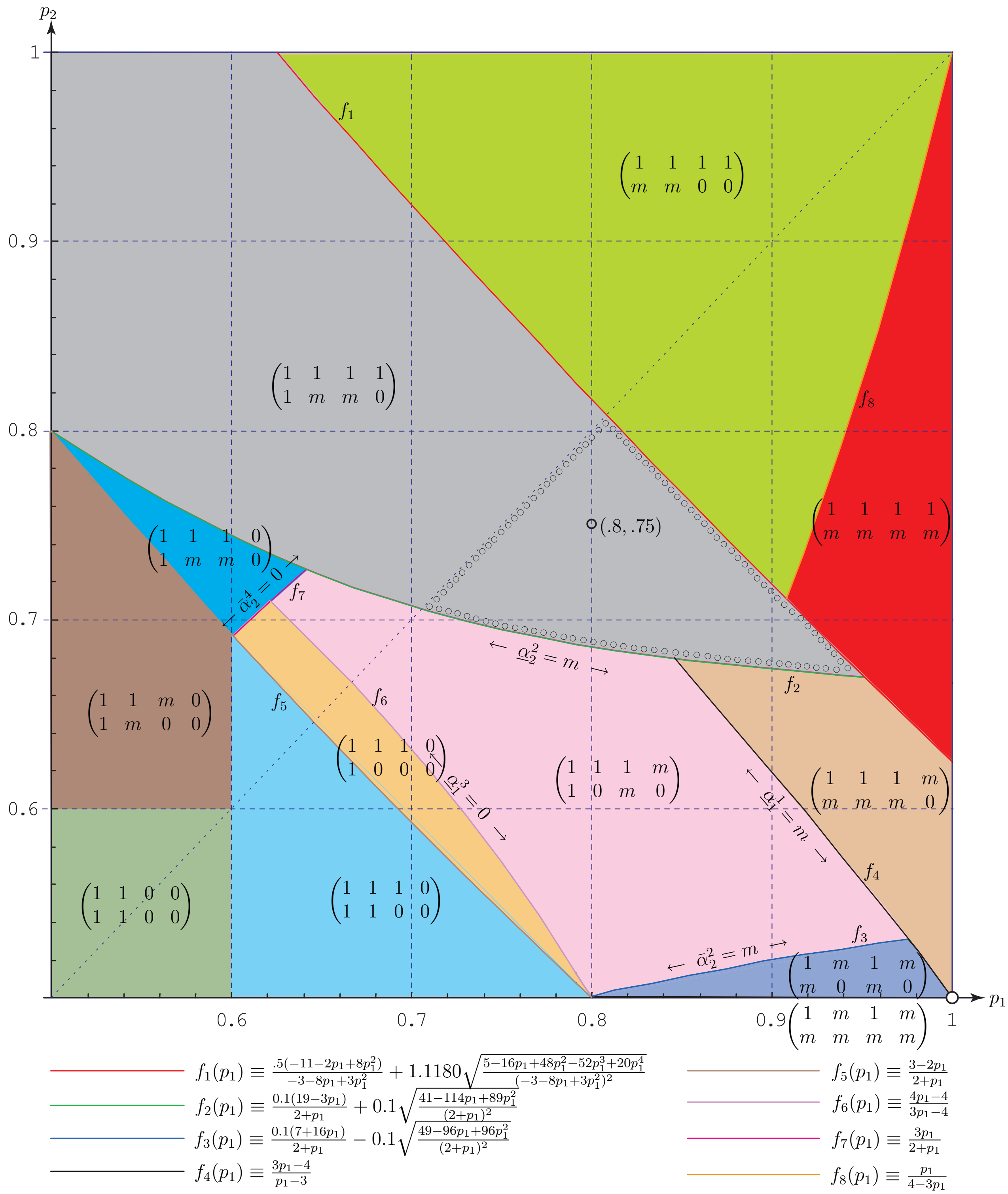

Figure 4: The equilibrium map for $\theta \in \{0, 5\}$ in $p_1 \times p_2$ signal accuracy space; the area β^* is circled.

on both sides. For $\alpha_1^5 = 0$, it occupies the line segment $p_1 \in [^4/_5, 1]$ for $p_2 = {}^1/_2$. The equilibrium β^* discussed in the main section and in last subsection's example for $\mathbf{p} = (.8, .75)$ is confirmed by (A.19). The map shows both equilibria in fully revealing (separating) and non-revealing strategies: In (A.18), for instance, P2 reveals her type at t=2 by quitting. Our general analysis shows that β^* cannot vanish as the object's high value is increased.