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Multi-battle contests*

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Abstract

We study equilibrium in a multistage race in which players compete in a sequence of simultaneous move component contests. Players may win a prize for winning each component contest, as well as a prize for winning the overall race. Each component contest is an all-pay auction with complete information. We characterize the unique equilibrium analytically and demonstrate that it exhibits endogenous uncertainty. Even a large lead by one player does not fully discourage the other player, and each feasible state is reached with positive probability in equilibrium (pervasiveness). Total effort may exceed the value of the prize by a factor that is proportional to the maximum number of stages. Important applications are to war, sports, and R&D contests and the results have empirical counterparts there.

Keywords: all-pay auction, contest, race, conflict, multi-stage, R&D, endogenous uncertainty, preemption, discouragement.

JEL classification numbers: D72, D74

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1 Introduction

Final success or final failure often results not from a single strike, but from a series of battles. Players often compete in a sequence of battles and final victory is often awarded as a function of the numbers of battle victories that the players accumulated. If two players compete in a series of battles, each player may receive one 'victory point' for winning a battle, and the player whose number of victory points first reaches some given minimum number, (which may vary across players), is awarded the prize for final victory. We call this structure a multi-battle contest or a race.

Obvious examples of such races can be found in many contexts, for instance, in sports, politics, warfare, and R&D competition. A tennis match, for instance, consists of a series of single battles. To be victorious, a player needs to win a certain number of sets before his or her competitor does. In a chess tournament between two players, the victorious player is the one who arrives at a certain number of victory points prior to his opponent. Similarly, in Formula I races or team leagues, teams collect winning points and the team that wins the largest number of points becomes the champion of the year. Examples exist in many other areas of sports competition.[1]

In politics, elections are won as the outcome of election campaigns and the candidate wins who wins the majority of votes. In some contexts the decision making of the members of the voting group is sequential. Klumpp and Polborn (2005) address this issue in the context of primaries in the US presidential elections.

In the context of conventional warfare, military conflict generally occurs in series of battles, typically called campaigns. The military adversaries typically start in some status quo in which each is in command of a number of soldiers, guns, armies, cities, countries, fortresses, military production capacity or other resources that can be turned into means of warfare. They fight each other in battles. In the sequence of successes and failures the potential for military conflict is weakened up to the point where one of the adversaries' military power is completely destroyed or sufficiently weakened to make him give up. Final victory becomes therefore a function of the

[1] For a survey on the theory of contests in sports see Szymanski (2003). The fact that many different battles interact in sports is acknowledged in this context. The particular micro structure of annual championships or simple matches has not been studied in this context, however.

sequence of outcomes of these battles.[2]

Another important application in which multi-battle contests have received considerable attention already is the area of R&D competition. Harris and Vickers (1987) described a multi-battle contest for a patent and called it a patent race: two players expend efforts on R&D in a sequence of single component contests. In each component contest one of the players wins, and the winner is determined as a stochastic function of the players' efforts in the respective component contest. The player who is first to win a given number of component contests wins the patent.

Our analysis will relate to the R&D literature on races and will answer some questions that have been left open in this literature, but the results obtained hold more generally to races in the context of sports, warfare or politics as well. When considering the single battles, we will assume that these are described by all-pay auctions with fully informed players. This reduced form description of the patent race traces back to Dasgupta (1986), who modeled such a race as a one-shot all-pay auction. It is natural to consider the patent race as a dynamic process which consists of a sequence of such all-pay auctions. Fudenberg, Gilbert, Stiglitz and Tirole (1983), Harris and Vickers (1985, 1987), Leininger (1991), and Budd, Harris and Vickers (1993) looked at the dynamics of patent races. Fudenberg, Gilbert, Stiglitz and Tirole (1983) consider a race in which the set of possible action choices at each stage is limited to three discrete values, and in which these effort choices add over time and determine the contestants' performance status. Harris and Vickers (1985) allow for a continuum of effort levels at each stage of the race, but make the contestants move sequentially. Harris and Vickers (1987) look both at the race and the tug-of-war with simultaneous effort choices at each stage. Similar to our framework, period efforts translate into probabilities of winning the period battle at the respective stage, and the number of battle wins over time determine the overall outcome. One difference between their analysis and our approach is the type of contest. They consider contest suc-

[2] Alternatively, if an adversary who wins the battle for a fortress or fortified city does not destroy it, but conquers and uses it himself, a ruler wins only once he possesses all $n_1 + n_2$ fortified places. Hence, in this case the lead in victories is decisive for final victory. We call such a conflict a tug-of-war. Starting with n_1 and n_2 fortresses under their respective control, ruler 1 needs a lead of n_2 more battle wins than ruler 2 for final victory, and similarly, ruler 2 needs a lead of n_1 more victories than ruler 2 to reach final success. In this paper we consider a formal model of a race. The analysis of a tug-of-war is addressed in a companion paper (Konrad and Kovenock, 2005).

cess functions with exogenous noise, whereas we consider contest success as the outcome of an all-pay auction without noise. Leininger (1991) considers a framework in which the contestants' expenditures are sunk and period efforts accumulate, and where the prize is allocated to the contestant who makes the largest overall effort. Finite and given total expenditure budgets allow for a recursive structure and solution of the problem. Finally, Budd, Harris and Vickers (1993) turn to a tug-of-war in continuous time that follows a stochastic law of motion. They also need sufficient exogenous noise for the existence of a pure strategy equilibrium.

We address the multi-stage race as in Harris and Vickers (1987), but give a complete analytical characterization of the equilibrium in a context with all-pay auction component contests. The absence of exogenous noise will require the battle equilibria to be in mixed strategies. Uncertainty in the race is then not being assumed exogenously, but being derived endogenously. One further notable difference in our analysis of racing that is particularly important is the role of an intermediate prize that is obtained by the winner of a component contest, apart from the benefit a victory in a component contest has for winning the overall multi-battle game. Empirically, the counterpart of this additional benefit in R&D races could be an information spillover or the cost reduction in other production processes that winning a component contest in the development of a patentable product may have. Such intermediate prizes have important consequences for the equilibrium outcome. In many models of R&D races a player who gains a sufficient lead wins the remaining battles, sometimes without further effort, because the player who is lagging behind gives up. If intermediate prizes are awarded for winning component contests, the player who is lagging far behind may catch up, and does catch up with a considerable probability in the equilibrium. The strategic advantage even of a considerable lead in the R&D race is diminished.

Our results extend to the other examples of races discussed above and have their empirical counterparts there. In sports contests intermediate prizes that are awarded in single matches of a game or in single tournaments in a series of tournaments that determines the annual championship make sure that players who are lagging behind will not simply give up. They expend effort and catch up with a considerable probability in the equilibrium, and may even become the leader again. Suspense is sustained by the existence of intermediate prizes, even if the sequence of battle victories is not very balanced for some rounds. As suspense is one of the desirable features of sports events (see, e.g., Hoehn and Szymanski 1999), this result may explain why such intermediate prizes are frequently observed in races which are carefully designed. In Formula I races, for instance, each Grand Prix generates some benefits to the winner, apart from the championship points that count for the overall championship that is awarded on an annual basis. Similarly, the PGA tour has large purses of prize money in the various tournaments, but each victory also contributes to the grand prize which is awarded at the end of the tour.

Finally, a property of the equilibrium outcome of the race we consider may explain why warfare and other types of multi-battle contests appear to be extremely wasteful in the sense that, overall, the players expend more effort than the value of the prize that is at stake. In particular, political scientists considered the question why so many resources are expended and so much wealth is destroyed in a war, and why war may continue for so long.[3] We show that the resources that are expended in a multi-battle contest may, for some realizations of the sequence of battle victories, sum up to amounts that exceed the value of the prize, even by a factor that is proportional to the maximum number of battles that is consistent with the overall contest.

2 The multi-battle race

Consider two players A and B. The players take part in a race which is comprised of a sequence of one shot simultaneous move component contests ("battles"). The series of component contests awards a prize to the winner of the race, and this prize is valued at Z_A and Z_B by these players. We assume that $Z_A \geq Z_B > 0$. In order to win the prize, player A must win n of these component contests before player B wins m component contests, with $(n, m) \gg 0$. Similarly, B receives the prize Z_B for winning the race if he wins m component contests before A wins n contests. In addition to the final prize Z_A or Z_B, additional intermediate prizes are awarded to the winner of each component contest. We assume here that each of these intermediate prizes is valued at $\Delta \geq 0$ by both contestants.

[3] This literature also came up with a number of good answers, concentrating on issues such as asymmetric information, learning, limits to what can be enforced in a contract, and dynamic games with multiple equilibria. See, for instance, Fearon (1995) for an overview and Powell (2004a, 2004b), Garfinkel and Skaperdas (2000) and Slantchev (2003) for recent contributions that consider different effects. The overdissipation result in this paper adds to these explanations.

Starting from the initial state (n, m) the first contest, C^{nm}, is played. If player A wins the contest, he receives the intermediate prize $\Delta \geq 0$ and the state moves to $(n-1, m)$, indicating that in order to win the race player A then only needs to win $n-1$ component contests before B wins m component contests. If player B wins the first contest, B receives the intermediate prize Δ and the state moves to $(n, m-1)$, indicating that player B only needs to win $m-1$ more contests in order to win the race. The outcome of each component contest becomes public information at its conclusion. In each state, the players simultaneously decide upon their expenditures in the next contest. The solution concept that we will use is subgame perfect equilibrium. In each state (i, j) we shall denote by $v_A(i, j)$ and $v_B(i, j)$ the subgame perfect equilibrium continuation values of players A and B, respectively, which is the value the player attributes to starting the race at this state. We argue below that these continuation values are well-defined.

Since a player wins the race if he is the first to reach a position with no component contests left to win, we may set

$$\begin{array}{llll} v_A(0,j) = Z_A & \text{and} & v_B(0,j) = 0 & \text{for all } j > 0 \\ v_B(i,0) = Z_B & \text{and} & v_A(i,0) = 0 & \text{for all } i > 0 \end{array} \tag{1}$$

At any (i, j) that is not such an end state, a component contest takes place. Each of the component contests, C^{ij}, is described as follows. Players simultaneously choose efforts, denoted $a \geq 0$ and $b \geq 0$, respectively. A player who expends a strictly higher effort than his opponent wins the component contest. If the players expend the same effort, for simplicity we assume that player A wins the contest at (i, j) if $v_A(i-1, j) > v_B(i, j-1)$ and player B wins the contest if the inequality is reversed. In the event players expend the same effort and $v_A(i-1, j) = v_B(i, j-1)$ we assume that the winner of the component contest is chosen by a fair randomizing device.[4]

Figure 1 illustrates a race for $n = 6$ and $m = 4$. Players start at $(6, 4)$, can reach any of the dark states (i, j) with $i \leq 6$ and $j \leq 4$, and will finally end on the upper or right boundary. One of the issues we address is whether the race is *pervasive* in the sense that, starting from any (i', j'), all states (i, j) with $i \leq i'$and $j \leq j'$ are reached with positive probability in the subgame

[4] Our tie breaking rule is chosen to avoid the notational diffculties in having to carry out the analysis for a finite grid and then taking the mesh of the grid to zero. The practical implication of the assumption is to avoid having a player with a positive continuation value outbid by "an epsilon" a player with zero continuation value who bids zero. This tie breaking rule only affects equilibrium behavior when $\Delta = 0$.

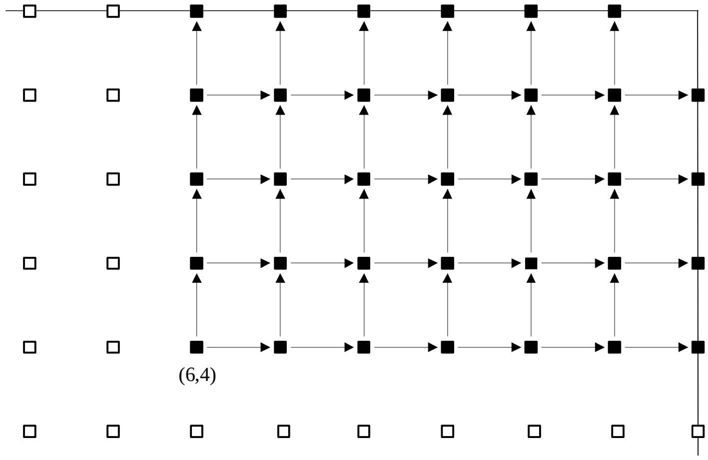


Figure 1:

perfect equilibrium. This is in contrast to some models of patent races where a sufficient lead by one contestant leads the other contestant to give up.

Starting or continuing from the state $(n, m) = (1, 1)$, which we will term the "*decisive state*," this contest is just a standard all-pay auction with complete information with prizes $z_A(1,1) \equiv Z_A + \Delta$ and $z_B(1,1) \equiv Z_B + \Delta$. It is now well known (see Hillman and Riley (1989) and Baye, Kovenock, and de Vries (1993,1996)) that the two player all-pay auction has a unique equilibrium in mixed strategies. The following proposition characterizes the unique equilibrium in the all-pay auction between two contestants with these valuations.

Proposition 1 *(Hillman and Riley, 1989) Let* $\Gamma(z_A, z_B)$ *be a two-player first price all-pay auction with prize values* z_A *and* z_B*, where* $z_A \geq z_B > 0$. $\Gamma(z_A, z_B)$ *has a unique Nash equilibrium in mixed strategies. In this equilibrium players' efforts (bids) are chosen randomly according to the cumulative distribution functions*

$$F_A(a) = \begin{cases} \frac{a}{z_B} & \text{for} \quad a \in [0, z_B] \\ 1 & \text{for} \quad a > z_B \end{cases} \tag{2}$$

and

$$G_B(b) = \begin{cases} \frac{z_A - z_B}{z_A} + \frac{b}{z_A} & \text{for} \quad b \in [0, z_B] \\ 1 & \text{for} \quad b > z_B. \end{cases} \tag{3}$$

Equilibrium payoffs are $u_A^* = z_A - z_B$ *and* $u_B^* = 0$. *In equilibrium, the expected sum of the efforts is* $E(a+b) = \frac{1}{2}z_B[1+\frac{z_B}{z_A}]$ *and the probability of winning the prize is* $p_A = 1 - \frac{z_B}{2z_A}$ *and* $p_B = \frac{z_B}{2z_A}$, *for players A and B respectively.*

According to this proposition, the player who attributes a higher value to winning has a payoff equal to the difference between his own and his rival's value of winning, and the other player has an equilibrium payoff that is equal to zero. This determines the values that the players attribute to being at the state $(1,1)$ as

$$\begin{aligned} v_A(1,1) &= \max[z_A(1,1) - z_B(1,1), 0] \\ &= [(v_A(0,1) - v_A(1,0)) + \Delta] - [(v_B(1,0) - v_B(0,1)) + \Delta] \\ &= Z_A - Z_B, \end{aligned} \tag{4}$$

and similarly,

$$v_B(1,1) = 0. \tag{5}$$

Note that the size of the intermediate prize Δ does not affect the continuation values $v_A(1,1)$ and $v_B(1,1)$. It does, however, affect the equilibrium distribution of efforts and the respective probabilities of winning the contest C^{11}. From Proposition 1 the expected sum of the efforts in C^{11} is $\frac{1}{2}(Z_B+\Delta)[1+\frac{Z_B+\Delta}{Z_A+\Delta}]$ and the respective probabilities of winning are $p_A = 1 - \frac{Z_B+\Delta}{2(Z_A+\Delta)}$ and $p_B = \frac{Z_B+\Delta}{2(Z_A+\Delta)}$.

Define $\Sigma(k) = \{(i,j) >> 0 : i+j = k\}$. Now that the continuation values at $(1,1),(2,0)$ and $(0,2)$ are uniquely defined, we can consider the states in $\Sigma(3), (2,1)$ and $(1,2)$. The contest C^{21} can either lead to $(1,1)$ or to $(2,0)$, and the value of winning this component contest is equal to the intermediate prize Δ plus the absolute value of the difference in the respective contestant's continuation values at $(1,1)$ and $(2,0)$, which are uniquely determined. Hence, using subgame perfection, C^{21} reduces to a problem that is equivalent to a standard all-pay auction with complete information, which again has a unique equilibrium which is determined analogously to the equilibrium for C^{11}, and which uniquely determines the continuation values $v_A(2,1)$ and $v_B(2,1)$. Similar reasoning applies for C^{12}, and the continuation values for the end states $(0,3)$ and $(3,0)$ are also well defined by (1). More generally, in any component contest C^{ij} the value of the "prize" of winning the contest is equal to Δ plus the absolute value of the difference in the respective player's continuation values in states $(i-1,j)$ and $(i,j-1)$ Formally, in C^{ij} players play an all-pay auction with prizes

$$\begin{aligned} z_A(i,j) &= v_A(i-1,j) - v_A(i,j-1) + \Delta \text{ and} \\ z_B(i,j) &= v_B(i,j-1) - v_B(i-1,j) + \Delta. \end{aligned} \tag{6}$$

This illustrates how unique continuation values for all states (n, m) can be calculated recursively.

More generally, in order to characterize the nature of the subgame perfect equilibrium in the multi-battle contest we first define the set of "separating states."

Definition 2 *Suppose $Z_A \geq Z_B > 0$ and consider the set of states $(i, k-i) \in \Sigma(k)$ for $k \geq 2$. We define a state $(i_k, k-i_k)$ to be a separating state if it has the following separating property:*

$$\begin{array}{l} v_A(i,k-i) = Z_A \text{ and } v_B(i,k-i) = 0 \text{ for all } i < i_k \\ v_A(i,k-i) = 0 \text{ and } v_B(i,k-i) = Z_B \text{ for all } i > i_k \end{array} \tag{7}$$

The separating property provides some structure to the state space and has implications for the continuation values of the separating states themselves. One implication is that there can be at most two separating states for each $\Sigma(k)$. This follows by contradiction: suppose $(i_k, k-i_k)$, $(j_k, k-j_k)$ and $(l_k, k-l_k)$ are separating states in $\Sigma(k)$, and let $i_k > j_k > l_k$. Then, by the separating property of i_k, $v_A(j_k, k-j_k) = Z_A$ and by the separating property of l_k, $v_A(j_k, k-j_k) = 0$; hence, a contradiction. When there are two separating states (i,j) and (i',j') with $i > i'$ in $\Sigma(k)$, they must be neighboring in the sense that $i = i'+1$, and the continuation values at these states need to be $v_A(i,j) = 0, v_B(i,j) = Z_B$, $v_A(i',j') = Z_A$, $v_B(i',j') = 0$. This structure and the neighboring property are useful in proving the main proposition of the paper.

Proposition 3 *(i) For every $k \geq 2$ there exist one or two separating states in $\Sigma(k)$. (ii) The state $(i,j) >> 0$ is a separating state if and only if $\frac{j-1}{i} \leq \frac{Z_B}{Z_A} \leq \frac{j}{i-1}$. Hence, if there exists an $(i,\ j) \in \Sigma(k)$ such that $\frac{j-1}{i} < \frac{Z_B}{Z_A} < \frac{j}{i-1}$ then (i,j) is the unique separating state in $\Sigma(k)$. (iii) $(i,j+1)$ and $(i+1,j)$ comprise the set of separating states in $\Sigma(k)$ if and only if $\frac{Z_B}{Z_A} = \frac{j}{i}$ for $(i,j) \in \Sigma(k-1)$. (iv) For any (i,j), $v_A(i,j) = \min(Z_A, \max(0, jZ_A - iZ_B))$ and $v_B(i,j) = \min(Z_B, \max(0, iZ_B - jZ_A))$. If (i_k, j_k) is a separating state then*

$v_A(i_k, j_k) = \max(0, j_k Z_A - i_k Z_B) \leq Z_A$ *and* $v_B(i_k, j_k) = \max(0, i_k Z_B - j_k Z_A) \leq Z_B$. *(v) An immediate consequence is that* $v_A(i, j) > 0$ *if and only if* $\frac{Z_B}{Z_A} < \frac{j}{i}$ *and* $v_B(i, j) > 0$ *if and only if* $\frac{Z_B}{Z_A} > \frac{j}{i}$.

The proof of Proposition 3 is relegated to the Appendix. Figure 2 illustrates the structure of the problem, with $(i_k, k - i_k)$ a unique separating state in $\Sigma(k)$.

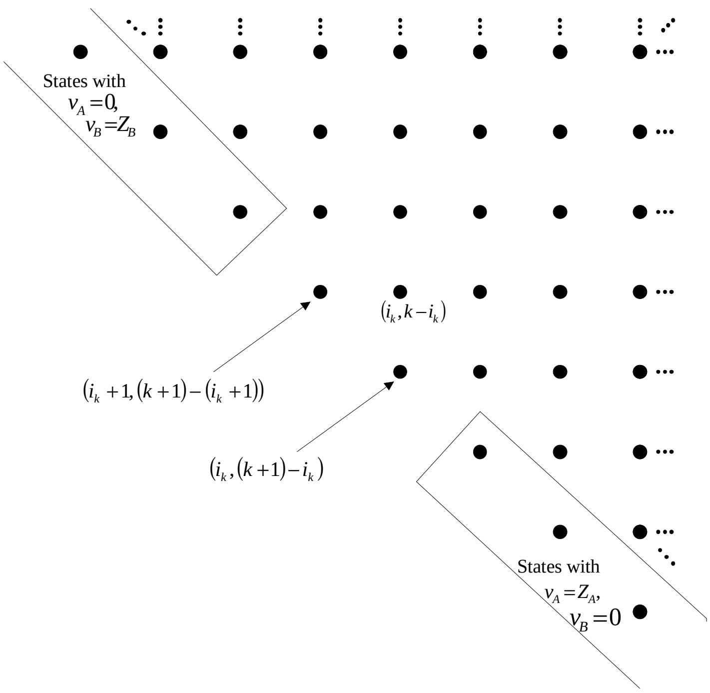


Figure 2:

Most non-separating states in the set $\Sigma(k + 1)$ simply inherit their continuation values from the fact that the component contest in this state leads to two possible states which do not differ in the continuation values for players. Accordingly, the component contest in such a state is symmetric, and is essentially about the intermediate prize Δ only. The situation for separating states and for states next to separating states is more complex. If $(i_k, k - i_k)$ is a unique separating state in $\Sigma(k)$, then the separating state in $\Sigma(k + 1)$ is either the state to the left of $(i_k, k - i_k)$ or the state right below $(i_k, k - i_k)$, or both. The latter happens in the non-generic case in which the separating state $(i_k, k - i_k)$ is located right on the line through $(0,0)$ with slope Z_B/Z_A. If $\Sigma(k)$ has two separating states, then the state in $\Sigma(k+1)$ that is located next to these two states is the (unique) separating state in $\Sigma(k+1)$.

To facilitate our analysis of the equilibrium distributions define a *non-trivial component contest* as a component contest in which both contestants expend positive effort with positive probability, and a *trivial component contest* as a contest which both contestants expend zero contest effort with probability one. [5]

3 No Intermediate Prizes (Δ=0)

The special case with terminal prizes, Z_A and Z_B, but no intermediate prizes, $\Delta = 0$, is of particular interest. All of the results in Proposition 3 hold for this special case. Indeed, for $\Delta = 0$ the equilibrium distributions take a very simple form. The following corollary summarizes the characterization of these distributions in this special case:

Corollary 4 *Suppose $Z_A \geq Z_B > 0$ and $\Delta = 0$. A non-trivial component contest occurs at (i,j) if and only if $v_A(i-1,j) > 0$ and $v_B(i,j-1) > 0$, which holds if and only if $\frac{j-1}{i} < \frac{Z_B}{Z_A} < \frac{j}{i-1}$. If $\frac{Z_B}{Z_A} \leq \frac{j-1}{i}$ then, starting in state (i,j), player A is able to win the remaining contests with no effort. If $\frac{Z_B}{Z_A} \geq \frac{j}{i-1}$ then starting in (i,j), player B is able to win the remaining contests with no effort.*

Note that one implication of the corollary is the following: Since for a given k the sets $\{[\frac{k-i-1}{i}, \frac{k-i}{i-1}), i = 1, ..., k-1\}$ partition the interval $[0, \infty)$ and for each of the states $(i,j) \in \Sigma(k)$, $\frac{j}{i}$ lies in exactly one interval, at most one such state generates a non-trivial contest. That is, the inequalities $\frac{j-1}{i} < \frac{Z_B}{Z_A} < \frac{j}{i-1}$ can jointly hold for at most one such state. However, it is possible that the inequalities do not jointly hold for any such state. This occurs when there are two separating states in $\Sigma(k)$. That is, $\frac{Z_B}{Z_A}$ takes the value of one of the non-zero endpoints of these intervals: $\frac{Z_B}{Z_A} = \frac{k-i-1}{i}$, for

[5] It is straightforward to show that, given the continuous strategy space and the tiebreaking rule that we employ, it cannot be equilibrium behavior for one contestant to expend positive effort with positive probability and the other to expend positive effort with probability zero.

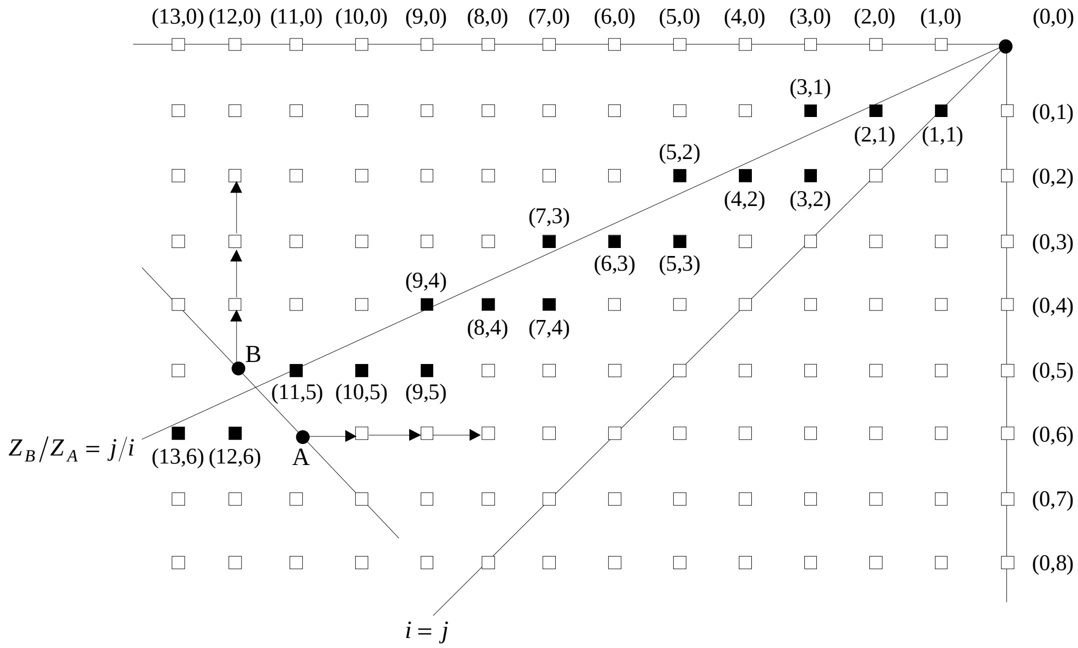


Figure 3:

some $i \in 1, .., k-2$. In this case, for $i' \leq i$, player A wins at $(i', k-i')$ without having to expend any further effort and for $i' > i$, player B wins at $(i', k-i')$ without having to expend any further effort. Both $(i, k-i)$ and $(i+1, k-1-i)$ are separating states in $\Sigma(k)$. Note that this guarantees that the state $(i+1, k-i)$ is a separating state in $\Sigma(k+1)$ at which there is a non-trivial contest.

Figure 3 shows the separating states for the set of initial states $(i, j) \in \Sigma(k), k \leq 19$, and with prize values $Z_A = 11$, $Z_B = 5$, and $\Delta = 0$. The states indicated with black boxes are those with non-trivial component contests. The states labeled A and B are separating states in $\Sigma(17)$, but each have trivial component contests. At state B, player B wins the remaining contests with no effort and at point A player A wins the remaining contests with no effort.

At this point we have yet to specify the precise form of the local strategies employed in states in which non-trivial component contests occur. Corollary 4 states that a non-trivial component contest arises at (i, j) if and only if $\frac{j-1}{i} < \frac{Z_B}{Z_A} < \frac{j}{i-1}$. At (i, j) the contest is an all-pay auction with prizes

$z_A(i,j) = v_A(i-1,j) - v_A(i,j-1)$ and $z_B(i,j) = v_B(i,j-1) - v_B(i-1,j)$. Utilizing the characterization of equilibrium continuation values in Proposition 3 it is straightforward to show that $z_A(i,j) \gtreqless z_B(i,j)$ as $\frac{Z_B}{Z_A} \lesseqgtr \frac{j}{i}$ and, therefore, that $v_A(i,j) > 0$ if and only if $\frac{Z_B}{Z_A} < \frac{j}{i}$ and $v_B(i,j) > 0$ if and only if $\frac{Z_B}{Z_A} > \frac{j}{i}$. Moreover, the functional forms of the equilibrium distributions vary across three cases:

Case 1: Suppose $\frac{j-1}{i} < \frac{Z_B}{Z_A} \leq \frac{j-1}{i-1}$. It immediately follows that $\frac{Z_B}{Z_A} < \frac{j}{i}$. In this case, $z_A(i,j) = Z_A$ and $z_B(i,j) = iZ_B - (j-1)Z_A$, so that $z_A(i,j) - z_B(i,j) = jZ_A - iZ_B > 0$. From Proposition 1, equilibrium distributions in state (i,j) therefore have a common support on the interval $[0,\ iZ_B - (j-1)Z_A]$ and take the form $F_A(a) = \frac{a}{iZ_B-(j-1)Z_A}$ and $G_B(b) = \frac{jZ_A - iZ_B}{Z_A} + \frac{b}{Z_A}$ over that interval.

Case 2: Suppose $\frac{j-1}{i-1} < \frac{Z_B}{Z_A} \leq \frac{j}{i}$. Then $z_A(i,j) = jZ_A - (i-1)Z_B$ and $z_B(i,j) = Z_B$, so that $z_A(i,j) - z_B(i,j) = jZ_A - iZ_B \geq 0$. From Proposition 1, equilibrium distributions in state (i,j) therefore have a common support on the interval $[0,\ Z_B]$ and take the form $F_A(a) = \frac{a}{Z_B}$ and $G_B(b) = \frac{jZ_A - iZ_B}{jZ_A-(i-1)Z_B} + \frac{b}{jZ_A-(i-1)Z_B}$ over that interval.

Case 3: Suppose $\frac{j}{i} \leq \frac{Z_B}{Z_A} < \frac{j}{i-1}$. Then $z_A(i,j) = jZ_A - (i-1)Z_B$ and $z_B(i,j) = Z_B$, which implies that $z_A(i,j) - z_B(i,j) = jZ_A - iZ_B \leq 0$. From Proposition 1, equilibrium distributions in state (i,j) therefore have a common support on the interval $[0,\ jZ_A - (i-1)Z_B]$ and take the form $F_A(a) = \frac{iZ_B - jZ_A}{Z_B} + \frac{a}{Z_B}$ and $G_B(b) = \frac{b}{jZ_A-(i-1)Z_B}$ over that interval.

Note that Proposition 1 also provides simple formulae for calculating the probability that each contestant wins the component contest at (i,j) and the expected sum of efforts in the component contest. In adapting Cases 1 and 2 above to these formulae, one need only insert the expressions for $z_A(i,j)$ and $z_B(i,j)$ above in place of z_A and z_B in the proposition, since in both cases (and in the proposition) contestant A has the larger prize. To apply the formulae in Proposition 1 to Case 3, all indices must be inverted since $z_A(i,j) \leq z_B(i,j)$ in Case 3, but the proposition assumes the reverse inequality.

Before moving on to examine several of the general properties of the race when $\Delta > 0$ a few more remarks are in order on the case where $\Delta = 0$. First, the treatment of asymmetric per unit costs of effort when $\Delta = 0$ is especially straightforward. Since behavior is invariant with respect to positive affine transformations of utility, we may incorporate asymmetric constant per unit effort costs in our model by dividing each contestant's utility by the corresponding contestant's per unit effort cost. Hence, if the prize values for A and B are Z_A and Z_B, respectively, and the corresponding per unit costs of effort are c_A and c_B, then the equilibrium of the game parameterized by (Z_A, Z_B, c_A, c_B) is identical to that of a game with unit cost equal to one for both players and transformed values $\widetilde{Z}_A = \frac{Z_A}{c_A}$ and $\widetilde{Z}_B = \frac{Z_B}{c_B}$. With $\Delta = 0$, all of our previous results then go through with these values inserted in place of Z_A and Z_B.[6] Although the treatment of asymmetric unit costs of effort can be carried out through backward induction in the case where $\Delta > 0$, the analysis requires somewhat more involved calculations, since transforming a contestant's utility by dividing by the unit cost of effort not only changes the terminal prizes, but also changes each component contest prize. If $c_A \neq c_B$, then $\frac{\Delta}{c_A} \neq \frac{\Delta}{c_B}$, and the component contest prizes become asymmetric in the transformed game. This means that the component contest prize values do not net out in calculating continuation values, so that the continuation value at any (i, j) is a complicated function of component contest prize values at states $(i', j') \leq (i, j)$ as well as the terminal prizes.

The game with $\Delta = 0$ is also useful in illustrating the potential for overdissipation of rents in a multi-battle contest. Since only one player wins the terminal prize, the maximum possible rent to be earned in this game is Z_A. Since a player has the right to opt out of the contest by bidding zero at every non-terminal state, for any such multi-battle contest (m, n), in equilibrium there can be no player whose expected effort exceeds the player's value of the prize. Moreover, it is easily demonstrated that the expected sum of efforts cannot exceed Z_A[7]. However, because of the dynamic nature of the model, unlike the one shot first price all-pay auction, there are many parameter specifications for which, in equilibrium, there is a positive probability that an individual contestant will expend a higher cost of effort than the contestant's value of the prize.[8] To illustrate this, suppose that $Z_B = \phi Z_A$ where $1 > \phi > \frac{n-1}{n}$, and

[6] Note , however, that if $\widetilde{Z}_B > \widetilde{Z}_A$ the indices in the analysis will have to be reversed, since the analysis has assumed that $Z_A \geq Z_B$.

[7] These definitions of overdissipation are the asymmetric contest analogues of Expected Individual Overdissipation (EIO)and Expected Aggregate Overdissipation (EAO) introduced by Baye et al. (1999). A symmetric equilibrium exhibits Expected Individual Overdissipation if an individual player's expected bid exceeds the value of the prize. Expected Aggregate Overdissipation arises when the expected sum of payments of the players exceeds the value of the prize.

[8] In the context of symmetric contests, Baye et al. (1999) refer to this as Probabilistic Individual Overdissipation (PIO).

look at the contest that starts at the intial state (n, n). The play of the game will reach the "decisive state" $(1, 1)$ if and only if player B wins every contest starting from the states (m, m), $n \geq m \geq 2$, and player A wins every contest starting from a state $(m, m-1), n \geq m \geq 2$. (See Figure 4). This in some sense represents a situation of maximal dissipation in the game starting at (n, n), because the contest remains non-trivial for the longest possible time. It is straightforward to demonstrate that in a contest at (m, m), $n \geq m \geq 2$, $z_A(m, m) = Z_A$, $z_B(m, m) = mZ_B - (m-1)Z_A = Z_A[1-(1-\phi)m]$, so that $z_A(m, m) > z_B(m, m)$ and, from Proposition 1, the probability that B wins at (m, m) is $p_B(m, m) = \frac{[1-(1-\phi)m]}{2}$. In contests starting from states of the form $(m, m-1), n \geq m \geq 2$, the corresponding prizes are $z_A(m, m-1) = (m-1)(Z_A - Z_B) = (m-1)(1-\phi)Z_A$, and $z_B(m, m-1) = Z_B = \phi Z_A$, so that $z_B(m, m-1) - z_A(m, m-1) = [1-m+\phi m]Z_A$. Since by assumption $\phi > \frac{n-1}{n}$, and $\frac{n-1}{n} > \frac{m-1}{m}$ for $m < n$, it follows that $z_B(m, m-1) > z_A(m, m-1)$ and the probability that A wins at $(m, m-1)$ is $p_A(m, m-1) = \frac{(m-1)(1-\phi)}{2\phi}$. For ϕ strictly less than but close to 1 the win probability $p_A(m, m-1)$ is positive but close to zero and $p_B(m, m)$ is close to $\frac{1}{2}$. Hence, the probability of reaching the decisive state $(1, 1)$ is positive, but can be quite small for large n.[9] However, the dissipation in the event that the contest reaches the decisive state can be quite large. To reach the decisive state $(1, 1)$ from the initial state (n, n), $2n-2$ non-trivial contests must be fought. For each non-trivial component contest (i, j) fought we know from Proposition 1 that the upper bound of the support of both contestants' equilibrium (local) strategies is the value of the smallest prize, $\min(z_A(i, j), z_B(i, j))$. Hence, it is possible for both players to draw realizations of effort arbitrarily close to this upper bound in each component contest. For $C^{m,m}$ this upper bound is $Z_A[1-(1-\phi)m]$ and for $C^{m,m-1}$ it is $Z_A(m-1)(1-\phi)$. Hence, for ϕ very close to 1 the supremum of the support of each player's equilibrium effort distribution is close to zero in $C^{m,m-1}$ but approaches Z_A in $C^{m,m}$. If realizations of the local strategies occur arbitrarily close to this upper bound of the equilibrium support for each player in each component contest on the path from the state (n, n) to state $(1, 1)$, and again in the decisive contest at $(1, 1)$, the total effort expended by a single contestant could reach arbitrarily

[9] We have derived an exact expression for this probability, but it is not very enligthening. A very loose upper bound on the probability, is $2^{-2(n-1)}$, which would arise if the player with the smaller prize value at each stage of the form (m, m) or $(m, m-1)$ won with probability $\frac{1}{2}$.

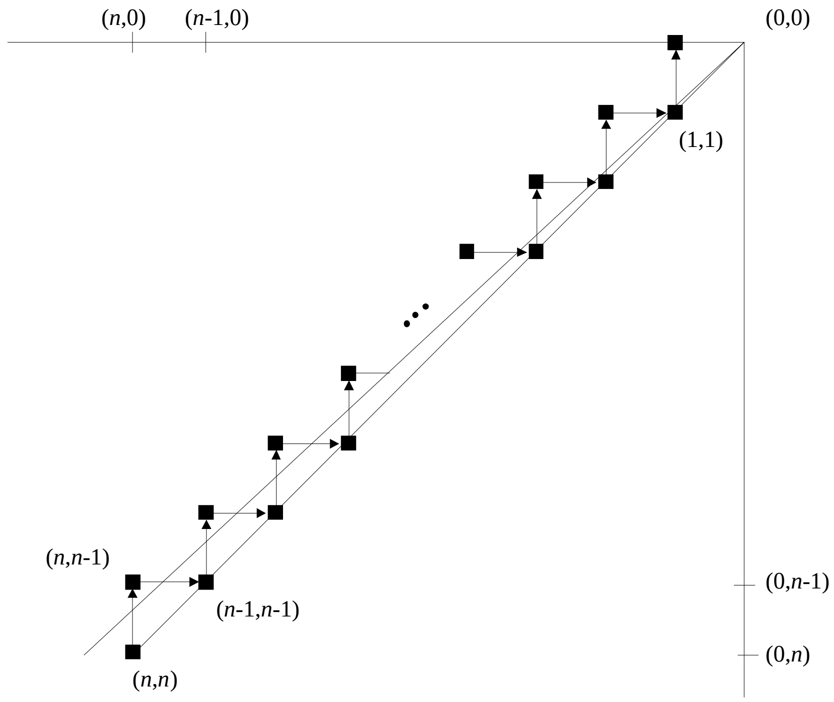


Figure 4:

close to nZ_A. Obviously, realizations of aggregate overdissipation could be double this.

Hence, it turns out that the least upper bound on the degree of possible individual overdissipation in a realization of the subgame perfect equilibrium strategies can be quite large. Note that this cannot arise in either the one-shot all-pay auction or the version of our model with $Z_A = Z_B = 0$ and $\Delta > 0$. The existence of at least one prize of positive value which is captured as a result of a sequence of expenditures is crucial to the result. Overdissipation arises because "sunk costs are sunk costs." Expenditures arising in the past have no effect on a contestant's willingness to expend effort to capture the terminal prize from a given state (i, j). No matter what a contestant's past expenditure, contestant A is still willing to pay up to Z_A to secure the prize rather than earn zero and contestant B is still willing to pay up to Z_B to secure the prize rather than earn zero. The competition that evolves reflects these forces in an all-pay setting.

4 Pervasiveness and the Nonmonotonicity of Effort with $\Delta > 0$

Note that $\Delta > 0$ implies that each contestant has a positive value of winning at any state $(i, k-i) \gg 0$. An immediate consequence is then

Corollary 5 *If $\Delta > 0$ a non-trivial contest occurs at all points $(i, k-i) \gg 0$.*

Corollary 5 reveals that the intermediate prizes are important to obtain a positive contest effort if players are in states that are some distance from the separating states. Intermediate prizes avoid contests becoming trivial. Consider sports contests. Intermediate prizes may consist of purely psychological rewards or ego-rents. For instance, a player or team who already leads by a large margin may enjoy a further increase in his lead, making his victory even more spectacular, or a player who is close to final defeat may enjoy some reward from winning at least another single battle, showing that he or she is at least a serious competitor. Moreover, in many sports contests, monetary prizes are attached to battle victories. The winner of a single Grand Prix Formula I race receives at least a cup and some reward in terms of increased market value and sponsoring contracts, and in tennis or golf tournaments considerable prize money is at stake in each single tournament. A comparison of Corollaries 4 and 5 shows that such intermediate prizes are important to avoid the series of battles becoming rather uninteresting once one of the players has accumulated a sufficient advantage that the other player gives up.

An interesting question in races is whether the current state of the race uniquely determines how the race evolves. Particularly in the literature on patent races, the point has been made that a lead by one contestant can be sufficient to guarantee that this contestant also wins the final prize with probability 1. Intuitively, if contestant A leads by sufficiently many component contest wins, then B gives up, knowing that any effort B might make to catch up with A will be rendered useless if A may react by increasing his effort to keep B at a distance all the way to the finish line. This is not the case in the race we consider. Note that B has a strictly positive probability of winning for any $(i, k-i) \gg 0$. More generally speaking, we define the race as *pervasive* if the equilibrium probability that state $(i', k'-i')$ is reached starting from a given $(i, k-i) \geq (i', k'-i')$ is strictly positive. We conclude from Proposition 1:

Corollary 6 *The multi-battle contest with $\Delta > 0$ is pervasive.*

Indeed, it is possible to characterize completely the nature of the equilibrium local strategies employed in any particular component contest (i, j). The following corollary determines the nature of these distributions in states (i, j) which cannot lead to a separating state after a single component contest.

Corollary 7 *The transition probability from an interior state $(i, k-i) \gg 0$ to $(i, k-i-1)$ and to $(i-1, k-i)$ is equal to 1/2 for all $(i, k-i)$ and $\Delta > 0$ for which $i_{k-1} \notin \{i, i-1\}$.*

Proof. The separating property (7) of i_{k-1} implies that $v_A(i, k-i-1) = v_A(i-1, k-i)$ and $v_B(i, k-i-1) = v_B(i-1, k-i)$ if $i_{k-1} \notin \{i, i-1\}$. Accordingly, $z_A = z_B = \triangle$ at $(i, k-i)$, and this implies that the equilibrium of the component contest is symmetric at $(i, k-i)$. ■

Corollary 7 is somewhat surprising. It suggests that a contestant who is lagging far behind for some time and is only one or two battles away from final defeat may still catch up, may still move back towards the range of separating states, and may even win the final prize with a considerable probability. Hence, intermediate prizes are rather important for producing suspense.

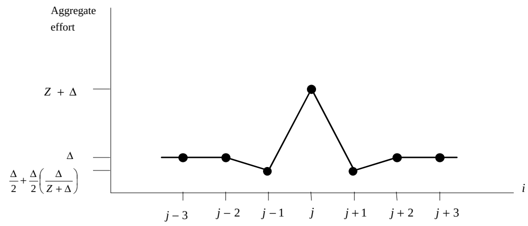
Moreover, this result also shows that the results for R&D races discussed in the introduction, according to which a small lead by one of the contestants may reduce contest effort to zero in subsequent rounds, is sensitive to the assumption that no intermediate prizes are at stake. With such intermediate prizes, as long as a state of final victory has not been reached, each contestant preserves a positive probability of becoming the winner of the final prize.

A complete characterization of equilibrium local strategies employed in separating states and states that are within one component contest outcome of a separating state is easily obtained by inserting the values for $v_A(i, j)$ and $z_B(i, j)$ derived in Proposition 3 into the expression for state (i, j) prizes in equation (6), and then applying Proposition 1. For illustrative purposes, an important special case of our analysis is that in which $Z_A = Z_B$ and $\Delta > 0$, which we now address.

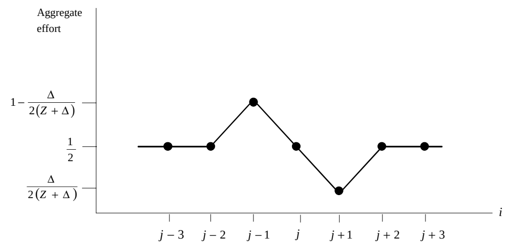
With $Z_A = Z_B \equiv Z$ and $\Delta > 0$, from Proposition 3 we know that (i, j) is a separating state if and only if $\frac{j-1}{i} \leq \frac{Z_B}{Z_A} = 1 \leq \frac{j}{i-1}$. An immediate consequence is that if $(i, j) \in \Sigma(2k)$ for $k \geq 1$, then $(i, j) = (k, k)$ is the unique separating state in $\Sigma(2k)$. If $(i, j) \in \Sigma(2k+1)$ for $k \geq 1$, then

$(i,j) = (k, k+1)$ and $(i,j) = (k+1, k)$ comprise the set of separating states. In the former case it is straightforward to show, again from Proposition 3, that $z_A(k,k) = v_A(k-1,k) - v_A(k,k-1) + \Delta = Z + \Delta$ and $z_B(k,k) = v_B(k,k-1) - v_B(k-1,k) + \Delta = Z + \Delta$, so that, Proposition 1 then implies that a symmetric non-trivial all-pay auction is played at $(i,j) = (k,k)$. This auction dissipates all rents, so that $v_A(k,k) = v_B(k,k) = 0$, $E(a+b) = Z + \Delta$, and, from the symmetry of the component contest, we find that each contestant is equally likely to win: $p_A = p_B = \frac{1}{2}$. This then immediately allows us to derive the expected values of the prizes at the component contests starting in states of the form $(i,j) = (k, k+1)$ and $(i,j) = (k+1, k)$. In the former case, $z_A(k, k+1) = v_A(k-1, k+1) - v_A(k,k) + \Delta = Z + \Delta$ and $z_B(k, k+1) = v_B(k,k) - v_B(k-1, k+1) + \Delta = \Delta$. Hence, from Proposition 1 at $(i,j) = (k, k+1)$ we have an asymmetric component contest with $v_A(k, k+1) = Z$, $v_B(k, k+1) = 0, E(a+b) = \frac{\Delta}{2} + \frac{\Delta}{2}(\frac{\Delta}{Z+\Delta})$, $p_A = 1 - \frac{\Delta}{2(Z+\Delta)}$ and $p_B = \frac{\Delta}{2(Z+\Delta)}$. In a similar fashion, it is straightforward to show that at $(i,j) = (k+1, k)$, $z_A(k+1, k) = \Delta$, $z_B(k+1, k) = Z + \Delta$, $v_A(k+1, k) = 0$, $v_B(k+1, k) = Z, E(a+b) = \frac{\Delta}{2} + \frac{\Delta}{2}(\frac{\Delta}{Z+\Delta})$, $p_B = 1 - \frac{\Delta}{2(Z+\Delta)}$ and $p_A = \frac{\Delta}{2(Z+\Delta)}$.

To calculate expected total effort and component contest win probabilities for states that are not separating, we divide up the analysis. Suppose first that $(i,j) \in \Sigma(2k)$ for $k \geq 1$, but that $i < k$ (for $k = 1$ this set is empty). Then (i,j) is not a separating state and we claim that in the component contest starting in (i,j), $E(a+b) = \Delta$ and $p_A = p_B = \frac{1}{2}$. To see this, we examine the two states $(i-1, j)$ and $(i, j-1)$ that can be immediately reached from the component contest at (i,j). If contestant A wins the component contest, the state moves to $(i-1, j)$. Note that $(i-1, j)$ is either a winning terminal state for player A or $(i-1, j) \in \Sigma(2k-1)$ with $i-1 < k-1$. In either case, $v_A(i-1, j) = Z$ and $v_B(i-1, j) = 0$. If contestant B wins the contest at (i,j) the state moves to $(i, j-1) \in \Sigma(2k-1)$ with $i \leq k-1$, so again $v_A(i, j-1) = Z$ and $v_B(i, j-1) = 0$. (Note that if $i = k-1, (i, j-1) \in \Sigma(2k-1)$ is of the form $(k-1, k)$ and, even though this is a separating state, it still satisfies $v_A(k-1, k) = Z$ and $v_B(k-1, k) = 0$). Hence, the prizes contested in the component contest at (i,j) are $z_A(i,j) = \Delta$ and $z_B(i,j) = \Delta$. From Proposition 1 this yields $E(a+b) = \Delta$ and $p_A = p_B = \frac{1}{2}$. A similar argument applies to the case where $(i,j) \in \Sigma(2k)$ for $k \geq 1$, but that $i > k$. Hence, as is the case for the state (k,k), for any off-diagonal state $(i,j) \in \Sigma(2k)$, $p_A = p_B = \frac{1}{2}$. However, the expected aggregate effort in an off-diagonal state $(i,j) \in \Sigma(2k)$ is $E(a+b) = \Delta$.



Panel A: Expected aggregate effort at *(i, j)* as a function of *i* for given *j* assuming that all relevant points are interior ($Z_A = Z_B = Z, \Delta > 0$)



Panel B: The probability that contestant *A* wins the component contest at *(i, j)*

Figure 5:

A somewhat more straightforward argument shows that the same type of result holds for non-separating states contained in $\Sigma(2k+1)$. Suppose that $(i,j) \in \Sigma(2k+1)$ for $k \geq 1$, and (i,j) is not a separating state. Then, as we have just shown, any state that may be reached immediately from the component contest at (i,j) is either a terminal state or an element of $\Sigma(2k)$ for which the advantaged contestant has continuation value Z and the disadvantaged player has continuation value 0. Hence the prize values for both contestants at (i,j) are equal to Δ and Proposition 1 demonstrates that this rent is completely dissipated and each player is equally likely to win the component contest. That is, at (i,j) the expected aggregate effort is $E(a+b) = \Delta$ and $p_A = p_B = \frac{1}{2}$.

To summarize, with $Z_A = Z_B \equiv Z$ and $\Delta > 0$, across component contests both the expected aggregate effort of the two contestants and the individual contestant win probabilities are non-monotonic in the ratio $\frac{j}{i}$. Panel A of Figure 5 shows the expected aggregate effort as a function of i for a given j, assuming that all of the relevant points are still interior. As can be seen in the Figure, this effort takes a maximum of $Z+\Delta$ on the diagonal where $i=j$, decreases to $\frac{\Delta}{2}+\frac{\Delta}{2}(\frac{\Delta}{Z+\Delta})$ at the two points, $(j-1,j)$ and $(j+1,j)$, just off of the diagonal, and then increases to Δ and remains there for all points satisfying $i \leq j-2$ or $i \geq j+2$. Moreover, as shown in Panel B of Figure 5 the probability that contestant A wins the contest is equal to $\frac{1}{2}$ for any interior state $i \leq j-2$, increases to $1-\frac{\Delta}{2(Z+\Delta)}$ at $i=j-1$, decreases again to $\frac{1}{2}$ at $i=j$, and further decreases to $\frac{\Delta}{2(Z+\Delta)}$ at $i=j+1$, before increasing and remaining at $\frac{1}{2}$ for all interior $i \geq j+2$.

It is also instructive to restrict the parameter values further and examine multi-battle contests in which there are no terminal prizes ($Z_A = Z_B \equiv 0$) but for which the contestants have a common prize $\Delta > 0$ for each component contest. Although this is formally not a special case of the analysis above (since that analysis assumed $Z_A \geq Z_B > 0$) and, in particular, Proposition 3 does not apply, it is straightforward to show that, in this case, for every component contest in an interior state (i,j), in equilibrium $E(a+b) = \Delta$ and $p_A = p_B = \frac{1}{2}$.

In fact, because each contestant wins each component contest with equal probability, the probability that a contest starting at an interior state (m,n) evolves and reaches the interior state (i,j), with $(m,n) \geq (i,j)$, is the probability that out of $(m+n)-(i+j)$ consecutive component contests starting with the component contest at (m,n) precisely $m-i$ of these component contests are won by contestant A and $n-j$ of the component contests are won by contestant B.[10] We label this probability $\Pr[(m,n),(i,j)]$. Since there are $2^{(m+n)-(i+j)}$ possible sequences of outcomes of $(m+n)-(i+j)$ consecutive component contests with $p_A = p_B = \frac{1}{2}$, each equally likely, and we know that any sequence hitting a terminal surface will be absorbed there and will not hit (i,j) thereafter, the probability of hitting (i,j) is the number of ways in which exactly $m-i$ victories for contestant A can be chosen from $(m+n)-(i+j)$ component contests played, divided by the total number of sequences of $(m+n)-(i+j)$ contest outcomes, $2^{(m+n)-(i+j)}$. That is, for $(m,n) \geq (i,j)$, with (i,j) interior,

$$\Pr[(m,n),(i,j)] = \frac{(m+n-i-j)!}{(m-i)!(n-j)!}\left(\frac{1}{2}\right)^{m+n-i-j}$$

[10] Since (i,j) is assumed to be an interior state, any path that reaches a terminal state before $(m+n)-(i+j)$ component contests are played would require that either contestant A win more than $m-i$ times or contestant B win more than $n-j$ times. Such a path would never hit (i,j), so we can ignore the fact that some paths may lead to fewer than $(m+n)-(i+j)$ plays of a componenet contest with $p_A = p_B = \frac{1}{2}$.

Note that this calculation must be modified for terminal states of the form $(0,j)$ or $(i,0)$. Since these states cannot be reached once the terminal surface is hit at another state, the only way in which the evolution of the contest can reach such a state is to go through the corresponding penultimate state, $(1,j)$ in the case of $(0,j)$ and $(i,1)$ in the case of $(i,0)$. Since at state $(1,j)$ the probability that $(0,j)$ is reached is $\frac{1}{2}$, we may calculate $\Pr[(m,n),(0,j)]$ for each terminal state of the form $(0,j)$ from the following identity: $\Pr[(m,n),(0,j)] = \frac{1}{2}\Pr[(m,n),(1,j)]$. A similar argument shows that $\Pr[(m,n),(i,0)] = \frac{1}{2}\Pr[(m,n),(i,1)]$. Hence, we have for $(m,n) \geq (1,j)$

$$\Pr[(m,n),(0,j)] = \frac{1}{2}\frac{(m+n-1-j)!}{(m-1)!(n-j)!}\left(\frac{1}{2}\right)^{m+n-1-j}$$

and for $(m,n) \geq (i,1)$

$$\Pr[(m,n),(i,0)] = \frac{1}{2}\frac{(m+n-i-1)!}{(m-i)!(n-1)!}\left(\frac{1}{2}\right)^{m+n-i-1}$$

A special case that is of particular interest is the case where contestants start at a symmetric point (n,n). As the logic of the formulae above indicate, the calculation of the hitting probabilities for interior states $(i,j) \leq (n,n)$ is $\left(\frac{1}{2}\right)^{2n-i-j}$ times the corresponding entry in a suitably constructed Pascal Triangle, where the top point of the triangle has the entry 1 at (n,n), which makes up *zeroth* entry in the *zeroth* row[11], the first row contains the nodes $(n-1,n)$ and $(n,n-1)$, both with entries in the triangle equal to 1, the second row contains the nodes, $(n-2,n),(n-1,n-1)$ and $(n,n-2)$ with corresponding entries, $1,2,$ and 1, and so forth. In this fashion the number in the triangle corresponding to the interior state (i,j) is ${}_{2n-i-j}C_{n-j} = \frac{(2n-i-j)!}{(n-i)!(n-j)!}$ where $2n-i-j$ is the number of the row (again treating, as is standard, the tip of the triangle as the *zeroth* row) and $n-j$ is the element in that row (with convention that the initial element is the *zeroth* entry).

For states (i,j) that are terminal, $\left(\frac{1}{2}\right)^{2n-i-j}$ times the corresponding elements of the Pascal Triangle constructed above do not provide the correct hitting probabilities, since a state of the form $(i,0)$ (resp. $(0,j)$) cannot

[11] This is a convention common in the discussion of Pascal's Triangle, presumably derived from the interpretation of the triangle as a representation of the coefficients of the expansion of $(1+x)^n$, where the power n corresponds to the row of the triangle and the m^{th} entry within each row is the coefficient of x^m in the expansion.

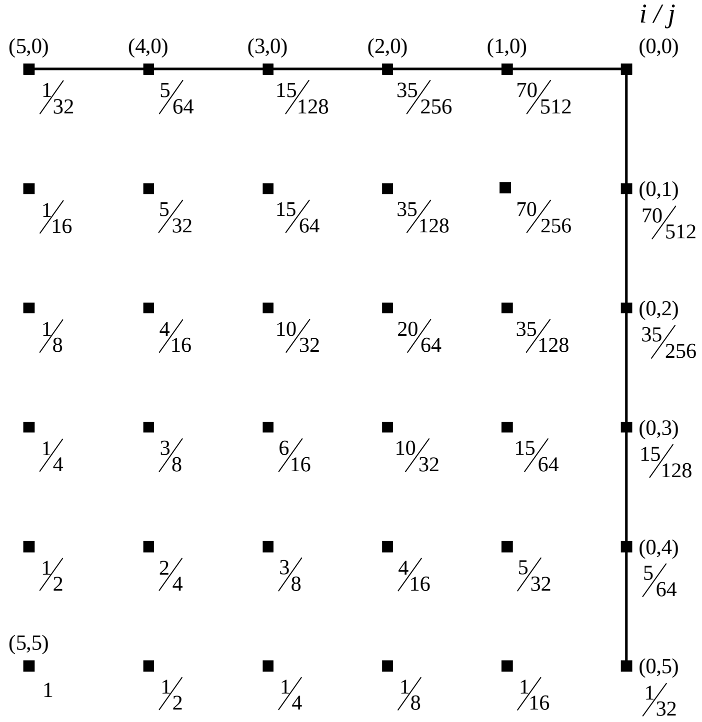


Figure 6:

be reached with positive probability on a path that hits $(i+1,0)$ (resp. $(0,j+1)$). The expressions for $\Pr[(m,n),(i,0)]$ and $\Pr[(m,n),(0,j)]$ above show how these probabilities are determined from the corresponding Pascal Triangle entry from the adjacent interior state. Figure 6 illustrates these hitting probabilities for the case where $(n,n)=(5,5)$.

Note that our calculation above provides a simple way in which to calculate the probability that a contest starting at (n,n) reaches the decisive state $(1,1)$. In this case, $\Pr[(n,n),(1,1)] = \frac{(2n-2)!}{(n-1)!(n-1)!}\left(\frac{1}{2}\right)^{2(n-1)}$. Note that this is exactly the probability that $n-1$ heads come up in $2n-2$ tosses of a fair coin. That is, it is the *central binomial coefficient* times $\left(\frac{1}{2}\right)^{2(n-1)}$. For large n, this may be approximated with Stirling's formula to obtain $\Pr[(n,n),(1,1)] \approx [\pi(n-1)]^{-\frac{1}{2}}$. Hence, for $Z_A = Z_B = 0$ and $\Delta > 0$, the probability that a symmetric contest starting at (n,n) leads to the maximum duration of active conflict is of the order $(n-1)^{\frac{-1}{2}}$.

5 Conclusions

We studied a general type of multi-battle contest in which players interact repeatedly in battles. In each interaction they fight a battle, each of which follows the rules of an all-pay auction with complete information and simultaneous effort choices. Players accumulate battle victories until one of them has accumulated a sufficiently large number of such victories. This player then wins and an overall prize is allocated to him. However, players may also receive an intermediate prize for winning each battle. This structure is called a race and emerges in many different contexts. In some applications the multi-battle set-up is the outcome of the technology of conflict or of what is technologically feasible, and changes over time, like in warfare. Sometimes the structure may have evolved in an evolutionary process, for instance, as when fighting between animals has a multi-stage structure. Sometimes elements of the multi-battle structure can be chosen by a contest designer. Sports championships are natural examples. In tennis matches, for instance, to win the match, a player has to win two or three sets. Annual championships are awarded on the basis of the outcomes of a series of tournaments. The number of battle victories, but also the size of the intermediate prizes and of the overall prize are chosen by the organizer.

We have provided a complete analytical solution for the race and a characterization of the unique equilibrium. We found that there are states in a race that have a separating property: winning the battle at such states shifts a considerable rent to the winner of the battle, even though a long series of battle victories may be required for each of the players to reach final victory from this state. This result is in line with the insights from the R&D literature that suggest that a sufficient lead by one contestant is important. However, unlike in this literature, the conflict does not slack off completely outside these separating states, and the final winner of the overall contest is not readily determined as an outcome of the battle in a separating state. The player who is lagging far behind may catch up, and does catch up with a considerable probability in the equilibrium, if there are intermediate prizes that are allocated to the winners of the component contests. Introducing such intermediate prizes sustains suspense in sports tournaments and may explain why they are chosen in these contests. Intermediate prizes are also important in warfare and make the overall outcome of a multi-battle war less predictable, even if one of the enemies has gained a considerable advantage, and even if there is no exogenous uncertainty in the environment.

Intermediate prizes also make the race pervasive in the sense that, from any combination of past battle victories that does not terminate the game, any other combination of numbers of battle victories with weakly larger numbers of battle victories on each side can be reached with strictly positive probability. Moreover, with strictly positive probability the game moves along trajectories of battle victories over which the sum of the players' efforts can considerably exceed the value of the prizes that are allocated between them. This may explain why, in some instances of sports contests, or in warfare, the aggregate, total amount of effort that is expended by the players may by far exceed the value that is at stake.

6 Appendix

To prove Proposition 3 we first prove by induction that for each $k = 1, 2, 3, ...$ there exists at least one $i_k \in \{1, ..., k-1\}$ that has the separating property described in Definition 2 and possesses the property that

$$\begin{array}{c} (k - i_k)Z_A - i_k Z_B \leq Z_A \\ i_k Z_B - (k - i_k)Z_A \leq Z_B. \end{array} \tag{8}$$

We also demonstrate that these states have continuation values

$$\begin{array}{rcl} v_A(i_k, k - i_k) & = & \max(0, (k - i_k)Z_A - i_k Z_B) \\ v_B(i_k, k - i_k) & = & \max(0, i_k Z_B - (k - i_k)Z_A). \end{array} \tag{9}$$

The properties of these states are then used to demonstrate claims (i) through (v) of Proposition 3.

Note that the property holds for $k = 2$: $v_A(0, 2) = Z_A$, $v_B(0, 2) = 0$, $v_A(2, 0) = 0$ and $v_B(2, 0) = Z_B$. Moreover, by Proposition 1, $v_A(1, 1) = Z_A - Z_B = \max(0, 1 \cdot Z_A - 1 \cdot Z_B)$, with $1 \cdot Z_A - 1 \cdot Z_B \leq Z_A$, and $v_B(1, 1) = 0 = \max(0, Z_B - Z_A))$, and $Z_B - Z_A \leq Z_B$.

Assume now that a separating state $(i_k, k - i_k)$ exists in $\Sigma(k)$ with $(k - i_k)Z_A - i_k Z_B \leq Z_A$ and $i_k Z_B - (k - i_k)Z_A \leq Z_B$ and continuation values as in (9). Let this separating state be depicted in Figure 2. Turn to states $(i, j) \in \Sigma(k + 1)$, which are the points at the south-west frontier of the set of points in Figure 2. We show that, then, a separating state $(i_{k+1}, k + 1 - i_{k+1})$ exists such that this state has the separating property (7) and fulfills

$$\begin{array}{c} ((k + 1) - i_{k+1})Z_A - i_{k+1} Z_B \leq Z_A \\ i_{k+1} Z_B - ((k + 1) - i_{k+1})Z_A \leq Z_B \end{array} \tag{10}$$

and

$$\begin{array}{lcl} v_A(i_{k+1},(k+1)-i_{k+1}) & = & \max(0,(k+1-i_{k+1})Z_A - i_{k+1}Z_B) \\ v_B(i_{k+1},(k+1)-i_{k+1}) & = & \max(0,i_{k+1}Z_B-(k+1-i_{k+1})Z_A). \end{array} \quad (11)$$

For $i < i_k$, the component contest at $(i,(k+1)-i)$ leads to $(i,k-i)$ or $(i-1,k-(i-1))$. As $(i_k,k-i_k)$ is a separating state, by Definition 1, the continuation values are $v_A = Z_A$ and $v_B = 0$ for both these states. This makes the prize of winning the component contest at $(i,(k+1)-i)$ the same for both contestants and equal to $z_A = z_B = \Delta$. Invoking Proposition 1, each contestant wins this component contest with $p_A = p_B = 1/2$ and chooses expected effort $Ea = Eb = \Delta/2$. Accordingly, the continuation values for both contestants at state $(i,(k+1)-i)$ are the same as in $(i,k-i)$ or in $(i-1,k-(i-1))$:

$$v_A(i,(k+1)-i) = Z_A \text{ and } v_B(i,(k+1)-i) = 0 \text{ for all } i < i_k. \quad (12)$$

For $i > i_k+1$, the component contest at $(i,(k+1)-i)$ leads to $(i,k-i)$ if B wins and to $(i-1,(k+1)-i)$ if A wins, with $v_A(i-1,(k+1)-i) = v_A(i,k-i) = 0$ and $v_B(i-1,(k+1)-i) = v_B(i,k-i) = Z_B$. Hence, using subgame perfection, the component contest at $(i,(k+1)-i)$ is a symmetric all-pay auction with complete information with equilibrium win probabilities $p_A = p_B = 1/2$ and equilibrium expected efforts $Ea = Eb = \Delta/2$. This yields

$$v_A(i,(k+1)-i) = 0 \text{ and } v_B(i,(k+1)-i) = Z_B \text{ for all } i > i_k+1. \quad (13)$$

Two states in $\Sigma(k+1)$ remain to be considered: $(i_k,k+1-i_k)$ and $(i_k+1,(k+1)-(i_k+1))$. Which of them is a separating state i_{k+1} will depend on the size of $(k-i_k)Z_A - i_kZ_B$.

Let $(k-i_k)Z_A - i_kZ_B \geq 0$. As depicted in Figure 7, by (9) this implies

$$v_A(i_k,k-i_k) = (k-i_k)Z_A - i_kZ_B \geq 0 \text{ and } v_B(i_k,k-i_k) = 0. \quad (14)$$

From $(i_k,(k+1)-i_k)$, the state moves to $(i_k,k-i_k)$, with continuation values given in (14), or to $(i_k-1,k-(i_k-1))$ at which $v_A = Z_A$ and $v_B = 0$. A attributes a prize to winning at $(i_k,(k+1)-i_k)$ that is equal to $Z_A - [(k-i_k)Z_A - i_kZ_B] + \Delta$ and is at least as large as Δ by the first line of (8), and B attributes a prize to winning that is equal to Δ. Applying Proposition 1, both contestants randomize on the interval $[0,\Delta]$, and B does not have a

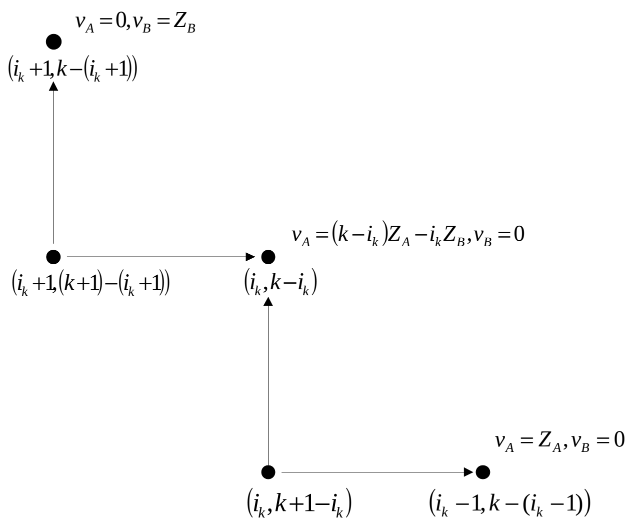


Figure 7:

mass point at Δ. As $a = \Delta$ is in A's equilibrium support, A's expected payoff equals the payoff from choosing $a = \Delta$, by which A wins with probability 1 the intermediate prize Δ and enters into state $(i_k - 1, k - (i_k - 1))$ at which A has a continuation value $v_A = Z_A$. Equilibrium effort Δ and intermediate prize Δ cancel out in the payoff, and, therefore, A's continuation value at $(i_k, (k+1) - i_k)$ is $v_A(i_k, (k+1) - i_k) = v_A(i_k - 1, k - (i_k - 1)) = Z_A$. Contestant B moves from $(i_k, (k+1) - i_k)$ to a state in which B's continuation value is zero. Among B's equilibrium effort choices is $b = 0$, and, as A has no mass point at $a = 0$, B loses with probability 1 when choosing $b = 0$ and moves to $(i_k - 1, k - (i_k - 1))$ with $v_B(i_k - 1, k - (i_k - 1)) = 0$. Hence, also $v_B(i_k, (k+1) - i_k) = 0$.

From $(i_k + 1, (k+1) - (i_k + 1))$, if A wins, the players move to $(i_k, k - i_k)$ with continuation values as in (14). Otherwise, they move to $(i_k + 1, k - (i_k + 1))$, with the continuation values $v_A = 0$ and $v_B = Z_B$ by the separation property of i_k. We need to distinguish between two subcases. *Subcase 1*: Let $Z_B > (k - i_k)Z_A - i_k Z_B$. Then B has a higher prize of winning than A. Making use of Proposition 1, $v_A(i_k + 1, (k+1) - (i_k + 1)) = 0$ and $v_B(i_k + 1, (k+1) - (i_k + 1)) = [Z_B + \Delta] - [(k - i_k)Z_A - i_k Z_B + \Delta] = (i_k + 1)Z_B - ((k+1) - (i_k + 1))Z_A$. Note further that this value is positive, but smaller than Z_B. Hence, this v_B fulfills the conditions in the second line of (10) and of (11). *Subcase 2*:

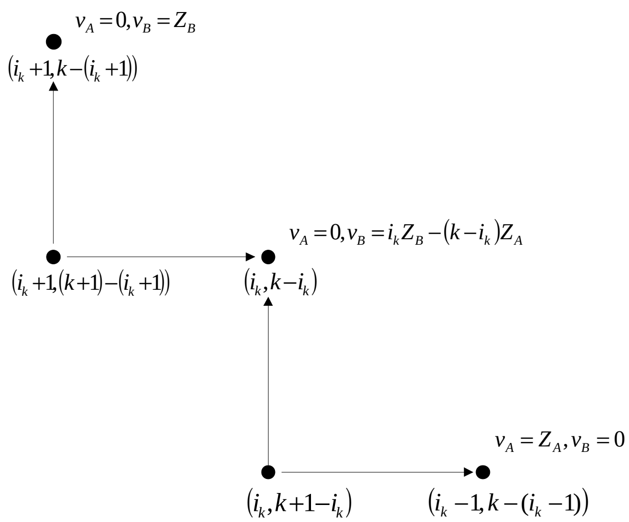


Figure 8:

Let $Z_B \leq (k - i_k)Z_A - i_k Z_B$. A has the higher prize of winning than B. This yields continuation values at $(i_k + 1, (k + 1) - (i_k + 1))$ of $v_B = 0$ and $v_A = [(k - i_k)Z_A - i_k Z_B + \Delta] - [Z_B + \Delta] = ((k+1) - (i_k+1))Z_A - (i_k+1)Z_B$. Moreover, this $v_A \leq Z_A$. Hence, the conditions in the first lines of (10) and (11) are fulfilled.

Together with the properties of states in $\Sigma(k+1)$ with $i < i_k$ and $i > i_k+1$ this shows that, for the case $(k - i_k)Z_A - i_k Z_B \geq 0$ at i_k, the state with $i_{k+1} = i_k + 1$ is a separating state.

Turn now to the case $i_k Z_B - (k - i_k)Z_A \geq 0$. This case implies that

$$v_A(i_k, k - i_k) = 0 \text{ and } v_B(i_k, k - i_k) = i_k Z_B - (k - i_k)Z_A. \qquad (15)$$

As shown in Figure 8, starting in $(i_k, (k + 1) - i_k)$, the state moves either to $(i_k, k - i_k)$, with continuation values given in (15), or to $(i_k - 1, k - (i_k - 1))$ at which $v_A = Z_A$ and $v_B = 0$. A attributes a prize to winning at $(i_k, (k+1) - i_k)$ that is equal to $Z_A + \Delta$, and B attributes a prize to winning that is equal to $i_k Z_B - (k - i_k)Z_A + \Delta$. Using property (8) and $Z_B \leq Z_A$, we get $i_k Z_B - (k - i_k)Z_A \leq Z_A$. Applying Proposition 1, the equilibrium payoff is 0 for B and $(Z_A + \Delta) - [i_k Z_B - (k - i_k)Z_A + \Delta] = ((k+1) - i_k)Z_A - i_k Z_B$ for A with $Z_A \geq ((k + 1) - i_k)Z_A - i_k Z_B \geq 0$. Moreover, starting in $(i_k + 1, (k + 1) - (i_k + 1))$, the state moves either to $(i_k, k - i_k)$, with v_A and v_B given in (15), or to

$(i_k+1, k-(i_k+1))$ at which $v_A = 0$ and $v_B = Z_B$ by the separating property of i_k. Hence, $v_A(i_k+1,(k+1)-(i_k+1)) = 0$. As $b = \Delta$ is in B's equilibrium support and makes B win with probability 1 and leads to state $(i_k+1, k-(i_k+1))$, B's continuation value at $(i_k+1,(k+1)-(i_k+1))$ is $v_B = Z_B$.

Together with the properties of states $i < i_k$ and $i > i_k+1$ in $\Sigma(k)$ this shows that, if $i_k Z_B - (k-i_k)Z_A \geq 0$ holds, a separating state $(i_{k+1},(k+1)-i_{k+1})$ has $i_{k+1} = i_k$. Players have continuation values $v_B = 0$ and $v_A = ((k+1)-i_{k+1})Z_A - i_{k+1}Z_B < Z_A$ at this state, in line with (10) and (11).[12]

Overall we have shown: if $\Sigma(k)$ has a separating state then does $\Sigma(k+1)$, and, together with the existence of a separating state in $\Sigma(2)$ this concludes the induction proof. We now turn to the properties in Proposition 3.

For (i) recall that there cannot be more than two separating states in $\Sigma(k)$. The Lemma establishes existence of at least one separating state, and together these results establish (i).

For (ii) note that (8) is equivalent to

$$\frac{j-1}{i} \leq \frac{Z_B}{Z_A} \leq \frac{j}{i-1} \tag{16}$$

for $i + j = k$. Hence, all separating states $(i_k, k-i_k)$ that have been constructed in the induction proof fulfill (8) and, hence fulfill (16). Suppose there is some other separating state (i,j) in $\Sigma(k)$ that does not fulfill (16). Then, by (i), there is another separating state $(i_k, k-i_k)$ in $\Sigma(k)$ that fulfills (16). If this $(i_k, k-i_k)$ fulfills both inequalities in (16) strictly, then, by (9), either $v_A(i_k, k-i_k) \notin \{0, Z_A\}$, or $v_B(i_k, k-i_k) \notin \{0, Z_B\}$. This rules out that, in addition to $(i_k, k-i_k)$ a second separating state can exist. Suppose then that one of the weak inequalities holds with equality, for instance, $\frac{k-i_k-1}{i_k} = \frac{Z_B}{Z_A}$. Using this in (9) yields $v_A(i_k, k-i_k) = Z_A$ and $v_B(i_k, k-i_k) = 0$. In turn, this implies that the only possible further separating state that may exist is $(i_k+1, k-(i_k+1))$. However, for this state the two inequalities in condition (16) hold: the left-hand side in (16) becomes $\frac{k-(i_k+1)-1}{i_k+1} \leq \frac{Z_B}{Z_A}$, or $(k-(i_k+1)-1)Z_A - Z_B(i_k+1) \leq 0$, or, using $\frac{k-i_k-1}{i_k} = \frac{Z_B}{Z_A}$, it can be written

[12] For completeness note that we treated the case $(k-i_k)Z_A - i_k Z_B = 0$ twice. Indeed, for $(k-i_k)Z_A - i_k Z_B = 0$, both the states $(i_k+1,(k+1)-(i_k+1))$ and $(i_k,(k+1)-i_k)$ have the separating property and the induction argument from $k+1$ to $k+2$ works for any of these two separating states and leads to $(i_k+1,(k+2)-(i_k+1))$ as a unique separating state for $k+2$.

as $-Z_A + Z_B \leq 0$, which is true, and similarly, the right-hand side in (16) becomes $\frac{Z_B}{Z_A} \leq \frac{k-(i_k+1)}{i_k+1-1}$, or $(k-(i_k+1))Z_A \geq i_k Z_B$, which, by $\frac{k-i_k-1}{i_k} = \frac{Z_B}{Z_A}$ is just fulfilled with equality. This shows that all separating states must fulfill the condition (16).

For the if-part of (ii), note that, for each k, the condition (16) determines either one state $(i_k, k-i_k)$ for which it holds with strict inequalities or two states such that left-hand side inequality holds with strict equality and the right-hand side inequality holds with strict inequality, and similarly for replacing left and right. A separating state which fulfills (16) exists by (i). If there is only one state that fulfills (16), then the separating state must be this state. If the condition determines two states, one of them must be a separating state, for which also (9) applies. But then, by the continuation values at this separating state, the other state that fulfills the condition (16) also becomes a separating state.

Turn now to (iv). The claim for separating states in (iv) holds by (8). The continuation values in non-separating states are also determined by the separating property:

$$\begin{cases} v_A(i,j) = \begin{cases} Z_A & \text{for} \quad i < \min[i_k : i_k \in S(k)] \\ 0 & \text{for} \quad i > \max[i_k : i_k \in S(k)] \end{cases} \\ v_B(i,j) = \begin{cases} 0 & \text{for} \quad i < \min[i_k : i_k \in S(k)] \\ Z_B & \text{for} \quad i > \max[i_k : i_k \in S(k)] \end{cases} \end{cases} \quad \text{and}$$

with $S(k)$ the set of separating states $(i, k-i)$ in $\Sigma(k)$. The general representation of continuation values of non-separating states is then confirmed by $\min[Z_A, \max(0, (k-i)Z_A - iZ_B] = Z_A$ as $\frac{k-i-1}{i} > \frac{Z_B}{Z_A}$, $\min[Z_B, \max(0, iZ_B - (k-i)Z_B] = 0$ as $\frac{k-i-1}{i} > \frac{Z_B}{Z_A}$, $\min[Z_B, \max(0, iZ_B - (k-i)Z_B] = Z_B$ as $\frac{Z_B}{Z_A} > \frac{k-i}{i-1}$, and $\min[Z_A, \max(0, (k-i)Z_A - iZ_B] = 0$ as $\frac{Z_B}{Z_A} > \frac{k-i}{i-1}$. This completes (iv). Property (v) follows immediately from (iv).

For part (iii), consider three states, $(i,j) \in \Sigma(k-1)$ and $(i+1, j)$, $(i, j+1) \in \Sigma(k)$. Let $\frac{Z_B}{Z_A} = \frac{j}{i}$. Then by (iv), $v_A(i,j) = v_B(i,j) = 0$. By (ii), this state (i,j) has to be a separating state. In turn, using the separating property for (i,j), it must hold that $v_A(i-1, j+1) = Z_A$, $v_A(i+1, j-1) = 0$, $v_B(i-1, j+1) = 0$ and $v_B(i+1, j-1) = Z_B$. Using the results from Proposition 1 for states $(i+1, j)$ and $(i, j+1)$, this implies that $v_A(i+1, j) = 0$, $v_B(i+1, j) = Z_B$, $v_A(i, j+1) = Z_A$, $v_B(i, j+1) = 0$. Accordingly, the only candidates for separating states in $\Sigma(k)$ are $(i+1, j)$ and $(i, j+1)$, and by existence of such a state, both must be separating states. Conversely, let

$(i+1,j)$ and $(i,j+1)$ be separating states in $\Sigma(k)$. Then, by (8) this implies that $(j+1)Z_A - iZ_B \leq Z_A$ and $(i+1)Z_B - jZ_A \leq Z_B$. Both together imply that $jZ_A - iZ_B = 0$, or $\frac{Z_B}{Z_A} = \frac{j}{i}$. But by (ii) this implies that (i,j) is a unique separating state in $\Sigma(k-1)$.

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