Facultad de Ciencias Económicas y Empresariales
Universidad de Navarra

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Expectational Stability in Multivariate Models

Seonghoon Cho
School of Economics
Yonsei University

Antonio Moreno
Facultad de Ciencias Económicas y Empresariales
Universidad de Navarra

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ABSTRACT

This paper shows that the concept of Expectational stability (E-stability) in a multivariate framework is inherently model-dependent. Whereas a Rational Expectations equilibrium (REE) is subject to model-specific parameter restrictions from the economic model at hand, a perceived law of motion (PLM) is postulated without such restrictions because economic agents are not likely to know the restrictions a priori. Therefore, an unrestricted PLM is in general overparameterized relative to an REE of interest in multivariate models even when the functional form is the same as the REE. Since E-stability necessarily involves model-specific extents of overparameterization, it is model-dependent in general. An immediate implication is that E-stability in a multivariate framework is not directly comparable across models and, in particular, across different representations of a given model. This implies that one may draw different conclusions on E-stability of an REE to one model under alternative representations of the model and the REE. We discuss a potential direction to develop a model-independent concept of E-stability.

Seonghoon Cho
School of Economics, Yonsei University
Seoul, Korea
sc719@yonsei.ac.kr

Antonio Moreno
Universidad de Navarra
Pamplona, Spain
antmoreno@unav.es

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Seonghoon Cho† Antonio Moreno ‡

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Abstract

This paper shows that the concept of Expectational stability (E-stability) in a multivariate framework is inherently model-dependent. Whereas a Rational Expectations equilibrium (REE) is subject to model-specific parameter restrictions from the economic model at hand, a perceived law of motion (PLM) is postulated without such restrictions because economic agents are not likely to know the restrictions *a priori*. Therefore, an unrestricted PLM is in general overparameterized relative to an REE of interest in multivariate models even when the functional form is the same as the REE. Since E-stability necessarily involves model-specific extents of overparameterization, it is model-dependent in general. An immediate implication is that E-stability in a multivariate framework is not directly comparable across models and, in particular, across different representations of a given model. This implies that one may draw different conclusions on E-stability of an REE to one model under alternative representations of the model and the REE. We discuss a potential direction to develop a model-independent concept of E-stability.

JEL Classification: C62; D83; D84
Keywords: E-stability; Rational Expectations Equilibrium; Overparameterization; Multivariate model

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†Corresponding author. School of Economics, Yonsei University, 134, Shinchon-Dong, Seodaemun-Gu, Seoul, 120-749, Korea. *E-mail:* `sc719@yonsei.ac.kr`. Tel:+82-2-2123-2470; Fax:+82-2-393-1158

‡Department of Economics, University of Navarra, Edificio de Biblioteca, Entrada Este, 31080 Pamplona, Spain. *E-mail:* `antmoreno@unav.es`. He acknowledges the financial help from the CICYT grant SEJ2005-06302.

1 Introduction

The concept of Expectational stability (E-stability hereafter) proposed and developed by George Evans and Seppo Honkapohja in a series of papers has been one of the major contributions to the literature on convergence to a Rational Expectations equilibrium (REE) under adaptive learning. Based on the results by Marcet and Sargent (1989), Evans and Honkapohja (1998, 1999, 2001) have extensively analyzed the relation between E-stability and least-squares learnability of REEs. It is now well-known that there is a tight relation between them, known as the E-stability Principle. E-stability has been popular in the literature because it is much easier to implement E-stability than to implement least-squares learnability.

Evans and Honkapohja (2001) provide a general treatment of E-stability for multivariate models and several authors have applied E-Stability in this framework.[1] In this paper, we show that the concept of E-stability in a multivariate framework is inherently model-dependent. Consequently, the E-stability property is not directly comparable across models. We show both theoretically and through several examples that one may draw different conclusions on E-stability of the REEs to a model at hand under alternative representations of the model and the REE.

The reason can be understood in terms of overparameterization of the perceived law of motion (PLM) relative to an REE of interest. To build up some intuition, it is instructive to first recall the implications of the well-known overparameterization associated with different PLM classes in a univariate framework. "Weak" E-stability applies when an REE (solution) and the PLM have the same functional form. For each coefficient of a state variable in an REE, an unrestricted PLM parameter is assigned to that variable. This implies that the number of PLM parameters is the same as that of the REE. When a more general functional form of the PLM relative to the REE is postulated, the PLM is overparameterized relative to the REE because the PLM has more state variables, and thus more parameters than the REE. In this case, a different concept, "strong" E-stability, applies. As such, weak and strong E-stability are associated with different learning rules. Intuitively, when economic agents postulate different types of PLMs, their implications on the REE may well be different and it is not surprising that they can lead

[1] A selected list of papers includes Bullard and Mitra (2002), Gauthier (2002), Adam (2003) and Evans and Honkapohja (2003b). Recently, the relation between determinacy, learnability and E-stability has also been explored by Woodford (2003a,b), Giannitsarou (2005), McCallum (2007) and Bullard and Eusepi (2008).

to different conclusions on E-stability for the same solution. For future reference, we define this type of overparameterization as the between-PLM overparameterization.

In this paper, we show that the concept of E-stability in a multivariate framework is in general also subject to a very different type of overparameterization and that the extents of this kind of overparameterization are model-specific. For ease of exposition, let the fundamental (non-fundamental) PLM denote the PLM that has the same functional form as the class of fundamental (non-fundamental) solutions.[2] For instance, consider a fundamental solution to a multivariate model and suppose that the fundamental PLM is postulated. Conceptually, E-stability in this case would be analogous to weak E-stability in a univariate framework because the PLM and the REE are of the same functional form. Indeed, the E-stability conditions described in chapter 10 of Evans and Honkapohja (2001) nest those of the univariate cases so that they are direct generalizations of the weak E-stability conditions from a technical point of view. However, it turns out that the concept of E-stability in multivariate models differs from weak E-stability in univariate models, just as weak and strong E-stability are different.

The reason is that virtually every macroeconomic model imposes model-specific restrictions on the parameters of the REE, and thus not all the coefficients of the state variables in an REE are free in general. In contrast, a PLM is postulated *a priori* without such restrictions and, as Evans and Honkapohja (2001) show, an unrestricted PLM is the most natural benchmark because agents are not likely to know the exact restrictions implied by the model. Hence, the PLM is in general overparameterized relative to the solution even within the same class of PLMs as the REEs. We call this type of overparameterization the within-PLM overparameterization.[3] Since different models impose different restrictions on their REEs, the extents of the within-PLM overparameterization vary across models. Moreover, they also vary across different representations of the same

[2] By fundamental solutions, we mean the REEs that depend on the minimal set of state variables. Non-fundamental (bubble or sunspot) solutions are the REEs that typically depend on additional variables to the minimal set of state variables, plus some other variables outside the model at hand. The fundamental solutions are also known as the minimal state variable (MSV) solutions in the literature. However, the solution obtained via the MSV criterion of McCallum (1983) is also often called the MSV solution. To avoid confusion throughout the paper, we use the fundamental solutions to denote the solutions that depend on the minimal set of state variables and do not use the term MSV solutions.

[3] Evans and Honkapohja (2003a) and Evans and McGough (2005) also examine different representations of sunspot equilibria and show that the stability properties depend on the solution representations. However, they postulate different classes of REEs and PLMs to a given representation of the model, rather than the same PLM to different representations of the model. Therefore, they study the implications of the between-PLM overparameterization, not the within-PLM overparameterization.

model and the same solution. Consequently, the concept of E-stability of the solution depends on each model and its representation.

This is clearly undesirable because E-stability results cannot be comparable across models and various representations of a model on the same ground. Henceforth, a model-independent concept of E-stability in a multivariate framework needs to be developed. In this paper we do not pursue this goal, but briefly discuss the potential avenue for it. Notice that the well-established concepts, weak and strong E-stability, are model-independent in univariate models, because the PLM in a univariate framework is not in general subject to the within-PLM overparameterization. Consequently, one way to derive the model-independent E-stability in a multivariate framework would be to reduce a given multivariate model into a univariate framework and subsequently apply the concepts of weak and strong E-stability. We also discuss some pros and cons of this approach.

For the purpose of this paper, it is sufficient to show that the concept of E-stability is model-dependent in the context of the fundamental class of solutions and the fundamental PLMs. One may show that it is also model-dependent for the non-fundamental class of solutions and the non-fundamental PLMs. When a class of solutions and a broader class of PLMs are considered, analogously to strong E-stability in univariate models, then the PLM would be subject to both the within-PLM and between-PLM overparameterization and hence, E-stability would again be model-dependent. While we do not discuss the issue of underparameterization, E-stability associated with underparameterized PLMs would also be model-dependent in multivariate models.

This paper is organized as follows. In Section 2, we show that a modified version of the Dornbusch model considered by Evans and Honkapohja (1994) and Evans and Honkapohja (2001) can be represented differently and that the E-stability results are different across model representations. Section 3 derives the E-stability conditions in general linear RE models and show that E-stability is subject to the within-PLM overparameterization in a multivariate framework. Section 4 provides several examples where different representations lead to different conclusions on E-stability. We also show that our results are independent of the information structure. Section 5 outlines an avenue to solve the problem of model-dependent E-stability. It also discusses the potential difficulties of doing so. Section 6 concludes.

2 The Dornbusch (1976) Model

Evans and Honkapohja (1994) and Evans and Honkapohja (2001) (EH hereafter) examine E-stability of fundamental solutions to the Dornbusch (1976) model under a univariate representation in terms of the log of the price level. The Dornbusch model considered by EH consists of a Phillips curve, an open economy IS curve, an LM curve and the open-economy parity condition. The model is reproduced as follows:

$$p_t = p_{t-1} + \pi E_{t-1} d_t \tag{1a}$$

$$d_t = -\gamma(r_t - E_{t-1}p_{t+1} + p_t) + \eta(e_t - p_t) \tag{1b}$$

$$r_t = \lambda^{-1}(p_t - \vartheta p_{t-1}) \tag{1c}$$

$$e_t = E_{t-1}e_{t+1} - r_t \tag{1d}$$

where p_t is the (log) price level, d_t is (log) aggregate demand, r_t is the nominal interest rate and e_t is the (log) nominal exchange rate. While EH use contemporaneous expectations in equations (1b) and (1d), we use lagged expectations in order to avoid complications regarding mixed dating of expectations.

The model can be represented in several forms as:

$$x_t = \beta_0 E_{t-1}x_t + \beta_1 E_{t-1}x_{t+1} + \beta_2 E_{t-1}x_{t+2} + \delta x_{t-1} \tag{2}$$

where x_t is defined in table 1 for the 5 representations of the model. For instance, $R4$ is the original model itself and $R1$ is the univariate representation considered by EH. Definitions of $\beta_0,\ \beta_1,\ \beta_2$, and δ for each representation are given in Appendix A.

Table 1: **Five Representations of the Dornbusch Model**

Representation	$R1$	$R2$	$R3$	$R4$	$R2'$
x_t	p_t	$(p_t\ e_t)'$	$(p_t\ d_t\ \ r_t)'$	$(p_t\ d_t\ r_t\ e_t)'$	$(p_t\ d_t)'$

We consider a class of fundamental solutions as:

$$x_t = \bar{b}x_{t-1} \tag{3}$$

where $\bar{b}$ must satisfy the following restriction imposed by the model:

$$\beta_0 \bar{b} + \beta_1 \bar{b}^2 + \beta_2 \bar{b}^3 + \delta = \bar{b}. \tag{4}$$

Since the definitions of β_0, β_1, β_2, and δ are representation-dependent, so is $\bar{b}$. For $R1$, $x_t = p_t$ and the solution to this equation is given by $p_t = \bar{b} p_{t-1}$ where $\bar{b} = \bar{b}_p$ is a scalar. The remaining variables are solved as $d_t = -(1 - \bar{b}_p)/\pi p_{t-1}$, $r_t = [(\bar{b}_p - \vartheta)/\lambda] p_{t-1}$ and $e_t = -[(\bar{b}_p - \vartheta)/(\lambda(1 - \bar{b}_p))] p_{t-1}$. Therefore, they are completely characterized by a single solution parameter, $\bar{b}_p$. For the other representations, $\bar{b}$ can also be defined corresponding to $\bar{b}_p$ as we show in Appendix A. Consequently, while different researchers may analyze different representations of the model and a solution, and there is no "right" or "wrong" representation, they in fact analyze an identical model and solution.

Since we consider a class of fundamental PLMs, this has the same functional form as (3):

$$x_t = b x_{t-1} \tag{5}$$

where b is unrestricted for each representation. Therefore, E-stability of a fundamental REE with respect to the fundamental PLM should be conceptually equivalent across different representations. In $R1$, E-stability of a fundamental solution is defined as "weak" E-stability because the same PLM class is postulated. E-stability in multivariate models shown in chapter 10 of EH may also be analogously interpreted as "weak" E-stability precisely because of the same reason. Furthermore, the conditions of E-stability in multivariate models nest those in univariate models. That is, the conditions in multivariate models are a direct generalization of those in univariate models.

Consequently, it is natural to expect that the E-stability results of the REEs to the model would be the same across different representations of the model and the solutions. However, it turns out that different representations lead to different conclusions on E-stability. The numerical parameter values considered by EH are $\pi = 1.5$, $\gamma = 1.5$, $\lambda = 10$. When $\vartheta = 1.1$ and $\eta = -0.1$, there are three stationary fundamental solutions for $\bar{b}$. When $\vartheta = 0.5$ and $\eta = 0.2$, there is a unique stationary fundamental solution and two non-stationary solutions. All the technical details can be found in the following section where we generalize the E-stability conditions outlined in chapter 10 of EH. Table 2 summarizes the E-stability results. In both cases of indeterminacy and determinacy, the first solution associated with the smallest root is E-stable for all representations.[4]

[4] Even though in this example, the REE associated with the smallest root is E-stable across all

Table 2: **E-stability of REEs to Five Representations of the Dornbusch Model**

	Three stationary solutions			Unique stationary solution		
Representation	b =0.716	b =0.772	b =0.990	b =0.384	b =1.043	b =1.250
$R1$	Yes	No	Yes	Yes	No	Yes
$R2$	Yes	No	No	Yes	No	No
$R3$	Yes	No	Yes	Yes	No	Yes
$R4$	Yes	No	No	Yes	No	No
$R2'$	Yes	No	Yes	Yes	No	Yes

However, the solution associated with the largest stationary root is E-stable only in $R1$, $R3$ and $R2'$.[5] The results for $R1$ are those reported in Evans and Honkapohja (1994) and EH. From the table, it is clear that the concept of E-stability must be in fact different across different model representations of the same model and REE.

What leads to different conclusions on E-stability across different representations of the model and REE? The reason can be understood in terms of the within-PLM overparameterization. Whereas $\bar{b}$ in (3) as a fundamental solution is subject to (4), b in (5) as the fundamental PLM is postulated without restrictions. Specifically, the solution can be completely characterized by a single solution parameter $\bar{b}_p$ as shown in table 3. Across

Table 3: $\bar{\mathbf{b}}$ **in Five Representations of the Dornbusch Model**

Representation	$R1$	$R2$	$R3$	$R4$	$R2'$
$\bar{b}$	$\bar{b}_p$	$\begin{bmatrix} \bar{b}_p & 0 \\ \bar{b}_e & 0 \end{bmatrix}$	$\begin{bmatrix} \bar{b}_p & 0 & 0 \\ \bar{b}_d & 0 & 0 \\ \bar{b}_r & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} \bar{b}_p & 0 & 0 & 0 \\ \bar{b}_d & 0 & 0 & 0 \\ \bar{b}_r & 0 & 0 & 0 \\ \bar{b}_e & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} \bar{b}_p & 0 \\ \bar{b}_d & 0 \end{bmatrix}$

all representations, the solution (3) has only one free parameter while there are 4, 9, 16 and 4 free PLM parameters in $R2$, $R3$, $R4$ and $R2'$, respectively. Consequently, the PLMs

representations, this is not always true, as we show in section 4.

[5]In case of determinacy, the unique stationary solution, $\bar{b} = 0.384$, is E-stable in all representations. However, as Bullard and Mitra (2002) and recently McCallum (2008) show, a determinate but E-unstable REE can exist, so that the REE under determinacy may not be always E-stable across different representations. It is also interesting to see that the non-stationary solution $\bar{b} = 1.250$ can be E-stable or E-unstable depending on representations, although little attention is typically paid to such a solution (an exception is Cochrane (2007)).

are overparameterized relative to respective REEs in multivariate representations, and the concept of E-stability precisely reflects these representation-dependent extents of the within-PLM overparameterization. In addition, the within-PLM overparameterization does not just depend on the dimension of the model representation, but also on the variables with which the model is represented, as E-stability results for $R2$ and $R2'$ are also different. Furthermore, E-stability of the REE in a larger dimensional representation is not "strong" relative to that in a smaller dimensional representation. For instance, E-stability of the solution $\bar{b}$ associated with $\bar{b}_p = 0.99$ in $R3$ does not imply E-stability of the same solution in $R2$. This is clearly undesirable because the same class of PLMs delivers different conclusions on E-stability of a model solution simply because this is represented differently.

More importantly, regardless of the E-stability results, the extents of the PLM overparameterization depend on model representations, implying that the concept of E-stability should be distinguished across model representations, just as we distinguish between weak and strong E-stability across the different PLMs. For instance, while $R2$ and $R4$ yield the same E-stability results, the results in fact reflect different concepts of E-stability.

In the following section, we present E-stability conditions for general multivariate linear macro models and show that the type of E-stability varies not just across different representations of a given model, but also across different models. We also point out critical differences in the economic implications associated with the within and between-PLM overparameterization.

3 Characterizing E-Stability in a General Framework

We present two classes of models under lagged and contemporaneous expectations that nest most of the models considered by EH and their series of papers, and derive E-stability conditions for the fundamental class of REEs. Then we show that the concept of E-stability differs across alternative representations of the same model.[6]

[6]Some models may include mixed dating of expectations as in Adam, Evans, and Honkapohja (2006) and Evans, Honkapohja, and Marimón (2007). While it is straightforward to derive E-stability conditions for such class of models, we do not consider them here for simplicity.

3.1 Lagged Expectations Models

Consider a linear model:

$$y_t = \beta_0 E_{t-1} y_t + \beta_1 E_{t-1} y_{t+1} + \beta_2 E_{t-1} y_{t+2} + \delta y_{t-1} + \epsilon_t, \tag{6}$$

where y_t is an $n \times 1$ vector of variables observed at time t for a natural number n including 1. $\beta_0, \beta_1, \beta_2$ and δ are $n \times n$ matrices of parameters.[7] E_t is the expectation operator conditional on information available at time t. ϵ_t is an error term such that $E_t(\epsilon_{t+1}) = 0$. The class of fundamental RE solutions is given by:

$$y_t = \bar{b} y_{t-1} + \epsilon_t, \tag{7}$$

where the $n \times n$ matrix $\bar{b}$ must solve the following restrictions implied by the model:

$$\beta_2 \bar{b}^3 + \beta_1 \bar{b}^2 + \beta_0 \bar{b} + \delta = \bar{b}. \tag{8}$$

In order to study learnability of the REE of the form (7) in terms of E-stability, a particular functional form of the PLM must be specified. In this paper, we restrict our interest to the fundamental PLM and it is given by:

$$y_t = b y_{t-1} + \epsilon_t, \tag{9}$$

where b is free and not subject to the parameter restrictions in (8). By evaluating the model (6) with the PLM (9), we can derive the actual law of motion (ALM). The mapping from the PLM to the ALM and its derivative with respect to the unrestricted PLM parameters are respectively given by:

$$T(b) = \beta_2 b^3 + \beta_1 b^2 + \beta_0 b + \delta \tag{10}$$

$$DT(b) \equiv \frac{\partial vec(T(b))}{\partial (vec(b))'} = I \otimes (\beta_0 + \beta_1 b + \beta_2 b^2) + b' \otimes (\beta_1 + \beta_2 b) + (b^2)' \otimes \beta_2. \tag{11}$$

Let $DT(\bar{b}) \equiv DT(b)|_{b=\bar{b}}$ be $DT(b)$ when b is evaluated with an REE $\bar{b}$. Following EH, a fundamental solution (7) is said to be E-stable if all the eigenvalues of $DT(\bar{b})$ have real parts less than 1.[8]

[7] Throughout the paper, we ignore constants and persistent exogenous variables for ease of exposition.

[8] It is straightforward to compute $DT(b)$ using the simple formula, $d(XY) = X(dY) + d(X)Y$, where

It is crucial to understand that the PLM coefficient matrix b in (9) is unrestricted whereas the REE coefficient matrix $\bar{b}$ is restricted to satisfy (8). The number of parameters in $\bar{b}$ is at most n^2 for the identification of the model while b has exactly n^2. Throughout this paper, we assume that $\bar{b}$ strictly has less free parameters than n^2 in multivariate models, as virtually every structural macro model has parameter restrictions on its REE. Then, the PLM is overparameterized relative to the REE in multivariate (representations of these) models. In addition, the coefficient matrices $(\beta_0, \beta_1, \beta_2, \delta)$ are model-specific and $\bar{b}$ is restricted by them. Therefore, while the PLM is not model-dependent by itself, the extents of overparameterization of the PLM relative to an REE are model-dependent. Furthermore, as we showed in the previous section, the degrees of overparameterization differ across different representations of a given model and its REE. This type of overparameterization is what we call the within-PLM overparameterization. Therefore, E-stability must be defined with respect to a model, its representation and the class of PLM considered. Consequently, E-stability is not comparable across different models as well as different representations of a given model.

In the literature, however, E-stability is defined with respect to a particular PLM form, without an explicit reference to a model and its representation. For ease of exposition, let us classify RE models depending on the dimension of y_t and the values of β_2 as in table 4: E-stability conditions of fundamental solutions for the $LU1$ and $LM1$ models

Table 4: **Classes of RE Models under Lagged Expectations**

	$\beta_2 = 0$		$\beta_2 \neq 0$	
	$n = 1$	$n > 1$	$n = 1$	$n > 1$
Class	$LU1$	$LM1$	$LU2$	$LM2$

with respect to the fundamental PLM are given in pages 196 and 231 of EH, respectively. E-stability of $LU2$ is also discussed on page 215 of EH. Although E-stability in $LM2$ is not discussed in their book, it is straightforward to derive the E-stability condition as in (11). Since $LM2$ nests $LU2, LM1$ and $LU1$ as special cases, it seems natural to interpret E-stability of a fundamental REE with the corresponding fundamental PLM as the same kind for all classes of models as "weak" E-stability for univariate models. However, the concept of E-stability differs across multivariate models because it is defined

$X = b$ and $Y = b^2$.

with respect to the model-dependent within-PLM overparameterization. An immediate consequence is that one may draw different conclusions on E-stability of REEs to a given model when researchers use different representations of the same model. An example of this kind is given in the previous section: The representations $R1$ through $R4$ and $R2'$ of the Dornbusch model belong to $LU2$, $LM1$, $LM2$, $LM1$ and $LM2$, respectively.

We now compare the implications of the within-PLM and between-PLM overparameterizations on E-stability. When a more general functional form of the PLM relative to the solution of interest is postulated, E-stability is subject to the between-PLM overparameterization, as different classes of PLMs represent different ways in which agents forecast the economic variables at hand. Consequently, it is natural for E-stability to have different economic implications on the REE across different PLMs. An example of this kind is strong E-stability of REEs to univariate models. For a given PLM, strong E-stability is model-independent.[9] In contrast, E-stability associated with the within-PLM overparameterization in multivariate models is model-specific in spite of the fact that the PLM and the solution have the same functional form. Weak E-stability in multivariate models is such an example. Another example is strong E-stability in multivariate models, which is subject to both the within-PLM and between-PLM overparameterizations. As EH argue, unrestricted PLMs are the most natural ones because agents with imperfect information are unlikely to know the existence of these equilibrium restrictions. Unfortunately, the specification of unrestricted PLMs is precisely the source of the E-stability mismatch across representations in multivariate models.

As a result, a concept of model-independent E-stability in a multivariate framework is called for, so that it be comparable across models and yield the same E-stability results independently of the representations of a given model. To do so, one may have to impose the model specific restrictions on the PLM parameters. However, imposing such restrictions directly on the PLM is not so natural as we discussed above. Furthermore, if agents were able to impose such restrictions on the PLM, they would directly compute the RE solution. Instead, note that only E-stability in a univariate framework, such as $LU1$ and $LU2$, is in general comparable across models. Therefore, if a given multivariate model can be reduced into a univariate representation, then E-stability would be model-independent in general. We sketch this idea and discuss the pros and cons of this suggested approach

[9]Strictly speaking, however, strong E-stability must also be defined with a particular PLM because different general PLMs imply different extents of overparameterization, leading to different concepts of E-stability.

in section 5. Now we turn to the class of models with contemporaneous expectations.

3.2 Contemporaneous Expectations Models

Consider a linear model where expectations are taken contemporaneously:

$$y_t = \beta_1 E_t y_{t+1} + \beta_2 E_t y_{t+2} + \delta y_{t-1} + \epsilon_t. \tag{12}$$

The class of fundamental solutions and the restrictions satisfied by the REE are given by:

$$y_t = \bar{b} y_{t-1} + \epsilon_t \tag{13}$$

$$\beta_2 \bar{b}^3 + \beta_1 \bar{b}^2 + \delta = \bar{b}. \tag{14}$$

The fundamental PLM has the same functional form as (13) but without the parameter restriction (14):

$$y_t = b y_{t-1} + \epsilon_t. \tag{15}$$

The T-mapping from the PLM to the ALM and the derivative of the mapping with respect to the unrestricted PLM parameters are respectively given by:

$$T(b) = (I - \beta_1 b - \beta_2 b^2)^{-1} \delta \tag{16}$$

$$DT(b) = [F(b)^{-1}\delta]' \otimes F(b)^{-1}(\beta_1 + \beta_2 b) + (bF(b)^{-1}\delta)' \otimes F(b)^{-1}\beta_2 \tag{17}$$

where $F(b) = (I - \beta_1 b - \beta_2 b^2)$. For ease of exposition, we classify RE models depending on the dimension of y_t and the values of β_2 as in table 5, analogously to table 4: The

Table 5: **Classes of RE Models**

	$\beta_2 = 0$		$\beta_2 \neq 0$	
	$n = 1$	$n > 1$	$n = 1$	$n > 1$
Class	$CU1$	$CM1$	$CU2$	$CM2$

E-stability conditions for $CU1$ and $CM1$ are given in pages 202 and 238 of EH. The E-stability conditions for $CU2$ and $CM2$ are not explicitly discussed. However, once again, it is straightforward to derive the E-stability conditions for $CU2$ and $CM2$. All the arguments laid out in models with lagged expectations are preserved under contemporaneous expectations.

4 Examples

In this section, we present several models that can be represented in two forms and derive the conditions under which a particular REE to a model can be E-stable or E-unstable, depending on the representation. First, we present a bivariate model composed of two independent univariate equations under lagged expectations. Then we show that a solution to the bivariate model consisting of individually E-stable solutions to each univariate model can be E-unstable. We also show that exactly the same results are obtained when a two-variable model has a recursive structure, where the second variable is independent of the first one but the first variable depends on the second one. Second, we present a bivariate model that has no E-stable REE. Then we show that when the model is represented in a univariate form, it has one or more E-stable solutions. By comparing the extents of the PLM overparameterization in the two models, we show that E-stability is not just representation-dependent, but also model-dependent. We perform analogous exercises under the models with contemporaneous expectations.

4.1 Models with Lagged Expectations

4.1.1 Model A: Combination of Independent Univariate Equations

We consider a model that can be represented in $LU1$ and $LM1$ forms.

$LU1$ **Representation**: Consider two completely unrelated univariate equations belonging to $LU1$. The (representation of the) model, the fundamental solutions, the solution restrictions, the fundamental PLM, the T-map and its derivative corresponding to equations (6) through (11) are respectively given by:

$$
\begin{aligned}
y_{i,t} &= \beta_{0,i}E_{t-1}y_{i,t} + \beta_{1,i}E_{t-1}y_{i,t+1} + \delta_i y_{i,t-1} + \epsilon_{i,t} && \text{(18a)}\\
y_{i,t} &= \bar{b}_i y_{i,t-1} + \epsilon_{i,t} && \text{(18b)}\\
\bar{b}_i &= \beta_{1,i}\bar{b}_i^2 + \beta_{0,i}\bar{b}_i + \delta_i && \text{(18c)}\\
y_{i,t} &= b_i y_{i,t-1} + \epsilon_{i,t} && \text{(18d)}\\
T(b_i) &= \beta_{1,i}b_i^2 + \beta_{0,i}b_i + \delta_i && \text{(18e)}\\
DT(b_i) &= \beta_{0,i} + 2\beta_{1,i}b_i && \text{(18f)}
\end{aligned}
$$

for $i = 1, 2$. Suppose that there are two real-valued but not necessarily stationary solutions, with $\bar{b}_i(1) < \bar{b}_i(2)$ (without loss of generality) in each equation.

$LM1$ **Representation**: The $LU1$ representation of the model can be written in a bivariate $LM1$ form with $x_t = (y_{1,t}\ y_{2,t})'$ and $v_t = (\epsilon_{1,t}\ \epsilon_{2,t})'$. The analogous equations to (18) are as follows:

$$
\begin{aligned}
x_t &= \beta_0 E_{t-1}x_t + \beta_1 E_{t-1}x_{t+1} + \delta x_{t-1} + v_t && \text{(19a)}\\
x_t &= \bar{b}x_{t-1} + v_t && \text{(19b)}\\
\bar{b} &= \beta_1\bar{b}^2 + \beta_0\bar{b} + \delta && \text{(19c)}\\
x_t &= bx_{t-1} + v_t && \text{(19d)}\\
T(b) &= \beta_1 b^2 + \beta_0 b + \delta && \text{(19e)}\\
DT(b) &= I \otimes (\beta_0 + \beta_1 b) + b' \otimes \beta_1 && \text{(19f)}
\end{aligned}
$$

where

$$
\beta_0 = \begin{bmatrix} \beta_{0,1} & 0 \\ 0 & \beta_{0,2} \end{bmatrix}, \beta_1 = \begin{bmatrix} \beta_{1,1} & 0 \\ 0 & \beta_{1,2} \end{bmatrix}, \delta = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix}. \tag{20}
$$

(18a)-(18c) and (19a)-(19c) are just different representations of the same model, solution and solution restrictions. Specifically, the solution $\bar{b}$ is given by:

$$
\bar{b} = \begin{bmatrix} \bar{b}_1 & 0 \\ 0 & \bar{b}_2 \end{bmatrix} \tag{21}
$$

where $\bar{b}_i$ is identical to that in (18b) subject to (18c) for $i = 1, 2$. Consequently, it is natural to expect that (18f) and (19f) deliver the same conclusions on E-stability.

(19f) is the condition stated by Proposition 10.1 of EH in a multivariate context, which generalizes the E-stability condition in univariate models. Indeed, when the model is univariate, (19f) is identical to (18f). The latter condition is stated in Proposition 8.2 of EH.

However, it turns out that E-stability defined in (19f) differs from that defined in (18f). When evaluated with $\bar{b}$ in (21), it is straightforward to show that $DT(\bar{b})$ is diagonal (so that its eigenvalues are the diagonal elements) and can be analytically expressed as:

$$diag(DT(\bar{b})) = \left[\begin{array}{cccc} \beta_{0,1} + 2\beta_{1,1}\bar{b}_1 & \beta_{0,2} + \beta_{1,2}(\bar{b}_1 + \bar{b}_2) & \beta_{0,1} + \beta_{1,1}(\bar{b}_1 + \bar{b}_2) & \beta_{0,2} + 2\beta_{1,2}\bar{b}_2 \end{array} \right]'. \tag{22}$$

Here is where the discrepancy between the E-stability conditions in the $LU1$ and $LM1$ representations arises. The off-diagonal elements of $\bar{b}$ are in fact zeros and thus are not free. However, b is postulated without such restrictions and $DT(b)$ produces non-zero second and third diagonal elements. For example, the second diagonal element contains the parameters of the second equation, $\beta_{0,2}$ and $\beta_{1,2}$, and the completely unrelated parameter of the first equation, $\bar{b}_1$. Note that the first and fourth diagonal elements are just the E-stability conditions of each equation in (18f). Hence, the second and third roots are the additional conditions induced by the overparameterized PLM in the $LM1$ representation. Therefore, (18f) and (19f) are conditions for different types of E-stability, implying that the concept of E-stability of an REE to a given model is representation-dependent.

If the E-stability results were the same across different representations, then the fact that the concept of E-stability is model-dependent would not pose a problem in practice. However, the results on E-stability may actually differ across representations. We now derive a condition under which the solution $\bar{b}$ consisting of the E-stable solutions $\bar{b}_1(1)$ and $\bar{b}_2(1)$ in the $LU1$ form is not E-stable in the $LM1$ representation. Suppose that all the parameter values are positive. Then, one such condition is given by:

$$\bar{b}_1(1) > \bar{b}_2(2). \tag{23}$$

That is, whenever the two solutions of the first equation are larger than those of the second equation in $LU1$ form, the solution consisting of individually E-stable solutions to both equations turns out to be E-unstable in the $LM1$ representation.[10]

[10] To see this, note that $\beta_{0,i} = 1 - \beta_{1,i}(\bar{b}_i(1) + \bar{b}_i(2))$, for $i = 1, 2$. Therefore, the second diagonal element of $DT(\bar{b})$ can be written as $\beta_{0,2} + \beta_{1,2}(\bar{b}_1(1) + \bar{b}_2(1)) = 1 + \beta_{1,2}(\bar{b}_1(1) - \bar{b}_2(2))$. Therefore, it is greater than 1 as long as $\bar{b}_1(1) > \bar{b}_2(2)$. By symmetry, the other case is $\bar{b}_2(1) > \bar{b}_1(2)$.

As a numerical example, suppose that $\beta_{1,1} = 0.4$, $\beta_{0,1} = 0.32$, $\delta_1 = 0.288$, $\beta_{1,2} = 0.5$, $\beta_{0,2} = 0.35$ and $\delta_2 = 0.21$. Table 6 shows the two solutions of each equation and the four

Table 6: $DT(\bar{b})$ of the $LU1$ and $LM1$

		$LM1$			
		$LU1$			
$\bar{b}_1$	$\bar{b}_2$	DT_{11}	DT_{44}	DT_{22}	DT_{33}
$\bar{b}_1(1) = 0.8$	$\bar{b}_2(1) = 0.6$	0.96	0.95	1.05	0.88
$\bar{b}_1(1) = 0.8$	$\bar{b}_2(2) = 0.7$	0.96	1.05	1.1	0.92
$\bar{b}_1(2) = 0.9$	$\bar{b}_2(1) = 0.6$	1.04	0.95	1.1	0.92
$\bar{b}_1(2) = 0.9$	$\bar{b}_2(2) = 0.7$	1.04	1.05	1.15	0.96

diagonal elements of $DT(\bar{b})$. As can be seen from the table, while $\bar{b}_1(1)$ and $\bar{b}_2(1)$ are E-stable in $LU1$, the solution $\bar{b}$ corresponding to $\bar{b}_1(1)$ and $\bar{b}_2(1)$ is not E-stable in $LM1$. Note also that the results are independent of the stationarity of the solutions; as long as (23) holds, the same outcome is obtained.

While we provide this example in order to clearly show that the concept of E-stability depends on the representation of a given model, there is no reason why we should put the two independent equations in one bivariate framework. A less trivial example would be a recursive two-equation-two-variable $(y_{1,t}, y_{2,t})$ model where $y_{2,t}$ is an autonomous process and also influences $y_{1,t}$. Thus, consider the following model:

$$\begin{aligned} y_{1,t} &= f(y_{2,t}) + \beta_{0,1} E_{t-1} y_{1,t} + \beta_{1,1} E_{t-1} y_{1,t+1} + \delta_1 y_{1,t-1} + \epsilon_{1,t} \\ y_{2,t} &= \beta_{0,2} E_{t-1} y_{2,t} + \beta_{1,2} E_{t-1} y_{2,t+1} + \delta_2 y_{2,t-1} + \epsilon_{2,t} \end{aligned}$$

where $f(y_{2,t})$ can adopt any form such as $\kappa y_{2,t}$, $\kappa E_{t-1} y_{2,t}$, $\kappa E_{t-1} y_{2,t+1}$ and $\kappa y_{2,t-1}$. Then, it can be analytically shown that none of the previous results is altered.[11] This is because the solution $\bar{b}$ would be upper triangular and $DT(\bar{b})$ would be block-recursive (upper triangular) with the same diagonal elements as those in equation (22). An economic example of this kind would be a two-country model where the home country is a small open economy depending on a foreign country, which is a relatively large closed economy.

[11] There is however, one additional E-stability condition for the first equation. For example, suppose $f(y_{2,t}) = \kappa y_{2,t}$. Then the PLM of the first equation would be $y_{1,t} = b_1 y_{1,t-1} + c y_{2,t}$. Therefore, E-stability must also be examined with respect to c. In our example, the conclusions on E-stability are not affected by this additional condition.

4.1.2 Model B: Bivariate Model and its Univariate Representation

Consider a model that can be represented in $LU2$ and $LM1$ forms:

$$y_t = \beta_{0,y}E_{t-1}y_t + \beta_{1,y}E_{t-1}y_{t+1} + E_{t-1}z_{t+1} + \delta_y y_{t-1} + \epsilon_t \quad (24)$$

$$z_t = \beta_{2,y}E_{t-1}y_{t+1} \quad (25)$$

$LU2$ **Representation**: The model can be represented in a univariate form in terms of y_t by substituting out z_t. This $LU2$ representation of the model, the fundamental solutions, the solution restriction, the fundamental PLM, the T-map and its derivative, corresponding to equations (6) through (11), are respectively given by:[12]

$$y_t = \beta_{0,y}E_{t-1}y_t + \beta_{1,y}E_{t-1}y_{t+1} + \beta_{2,y}E_{t-1}y_{t+2} + \delta_y y_{t-1} + \epsilon_t \quad (26a)$$

$$y_t = \bar{b}_y y_{t-1} + \epsilon_t \quad (26b)$$

$$\bar{b}_y = \beta_{2,y}\bar{b}_y^3 + \beta_{1,y}\bar{b}_y^2 + \beta_{0,y}\bar{b}_y + \delta_y \quad (26c)$$

$$y_t = b_y y_{t-1} + \epsilon_t \quad (26d)$$

$$T(b_y) = \beta_{2,y}b_y^3 + \beta_{1,y}b_y^2 + \beta_{0,y}b_y + \delta_y \quad (26e)$$

$$DT(b_y) = 3\beta_{2,y}b_y^2 + 2\beta_{1,y}b_y + \beta_{0,y} \quad (26f)$$

$LM1$ **Representation**: In matrix form, the model can also be written as:

$$x_t = \beta_0 E_{t-1}x_t + \beta_1 E_{t-1}x_{t+1} + \delta x_{t-1} + v_t \quad (27)$$

where $x_t = (y_t \ z_t)'$, $v_t = (\epsilon_t, 0)'$, β_0, β_1 and δ are given by:

$$\beta_0 = \begin{bmatrix} \beta_{0,y} & 0 \\ 0 & 0 \end{bmatrix}, \beta_1 = \begin{bmatrix} \beta_{1,y} & 1 \\ \beta_{2,y} & 0 \end{bmatrix}, \delta = \begin{bmatrix} \delta_y & 0 \\ 0 & 0 \end{bmatrix}. \quad (28)$$

[13] Since the functional form of the $LM1$ representation of this model is identical to

[12]Once the fundamental REE to the first equation is obtained and E-stability is examined, the fundamental solutions to the z_t equation can be obtained. Since this equation does not involve its own expectational term, we do not need to examine E-stability for the solutions to this equation. We also ignore innovations to this equation for simplicity.

[13]When a model is given in $LU2$ form (equation (26a)), it is sometimes easy to examine determinacy of the model and solve for the REEs by transforming the model into $LM1$ using an auxiliary expectational variable, z_t in equation (25). This kind of model transformation is not uncommon in the literature and in his study of E-stability and determinacy, McCallum (2007) generalizes models by employing such transformation. George Evans and Seppo Honkapohja pointed out to us that representing (24) with (25)

(19a), the fundamental solution, the solution restriction, fundamental PLM, T-map and its derivative with respect to the unrestricted PLM parameters are exactly of the same form as (19b) through (19f). However, the extents of the restrictions on $\bar{b}$ are of course different from those in the LM1 representation of the Model A, shown in the previous subsection. This is simply because the definitions of β_0, β_1, and δ are different, and $\bar{b}$ is given by:

$$\bar{b} = \begin{bmatrix} \bar{b}_y & 0 \\ \beta_{2,y}\bar{b}_y^2 & 0 \end{bmatrix}. \tag{29}$$

In Appendix B, we show that the E-stability conditions are given by:

$$3\beta_{2,y}\bar{b}_y^2 + 2\beta_{1,y}\bar{b}_y + \beta_{0,y} < 1, \;\; \beta_{0,y} + 2\beta_{1,y}\bar{b}_y + \beta_{2,y}\bar{b}_y^2 < 2, \;\; \beta_{0,y} + \beta_{1,y}\bar{b}_y + \beta_{2,y}\bar{b}_y^2 < 1. \tag{30}$$

Note that the first condition is the $LU2$ E-stability condition for $\bar{b}_y$. Therefore, one can reject E-stability of $\bar{b}$ in $LM1$ and accept E-stability of the same solution in $LU2$ if the first condition is met but either the second or the third, or both conditions are violated.

We replicate the example in section 9.5.1 of EH in order to show that the finding of representation-dependent E-stability is independent of the uniqueness of a stationary fundamental solution. With $\beta_{0,y} = -0.4$, $\beta_{1,y} = 1.9$, $\beta_{2,y} = -1$ and $\delta_y = 0.45$, there exists a unique stationary fundamental solution, $\bar{b}_y = 0.9$ and a pair of complex conjugates.[14] With $\bar{b}_y = 0.9$, the three values in (30) are (0.59, 2.21, 0.5). Since the first condition holds, the solution must be E-stable in $LU2$, but not in $LM1$ because the second condition is violated. Indeed, when $\bar{b}_y = 0.9$, $DT(\bar{b}_y) = 0.59$ so that $\bar{b}_y$ is E-stable, but the eigenvalues of $DT(\bar{b})$ are $1.1050 \pm 0.6316i$, 0 and 0.5, implying a rejection of E-stability in $LM1$.

For a comparison with contemporaneous expectations models below, we also consider a numerical example with multiple stationary solutions. EH show that a model in the $LU2$ form with $\beta_{0,y} = -3.53968254$, $\beta_{1,y} = 6.66666667$, $\beta_{2,y} = -3.17460318$ and $\delta_y = 1$ has two E-stable solutions. But when it is represented in $LM1$ form, none of the REEs becomes E-stable, as table 7 shows.

into the form of (27) may not be appropriate for the purpose of examining E-stability because z_t is itself a forecasting variable for agents. However, the opposite directional transformation from the $LM1$ into $LU2$ form would pose no such problem as we do here.

[14]Since the absolute values of the complex roots are less than 1, the model is still indeterminate although the real-valued fundamental solution is unique. We thank Evans and Honkapohja for pointing this out.

Table 7: $DT(\bar{b})$ of the $LU2$ and $LM1$

	$LU2$	$LM1$			
$\bar{b}_y$	$DT(\bar{b}_y)$	Eigenvalues of $DT(\bar{b})$			
0.5	0.75	$1.17-0.48i$	$1.17+0.48i$	0	-1
0.7	1.13	3.29	0.94	0	-0.43
0.9	0.75	4.82	1.07	0.750	-0.11

4.1.3 Comparison between Model A and Model B

We have shown that E-stability of fundamental REEs to models A and B is representation-dependent. Here we show that E-stability is in general model-dependent as well when models are represented in multivariate form. In the $LM1$ representation of both models A and B, the fundamental PLM is given by $b = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$. However, the RE solutions are restricted by $\bar{b}_A = \begin{bmatrix} \bar{b}_1 & 0 \\ 0 & \bar{b}_2 \end{bmatrix}$ and $\bar{b}_B = \begin{bmatrix} \bar{b}_y & 0 \\ \beta_{2,y}\bar{b}_y^2 & 0 \end{bmatrix}$ for models A and B, respectively. That is, while the PLM is model-independent, the REEs differ across models. $\bar{b}_A$ has two independent parameters on its diagonal position. $\bar{b}_B$ has two non-zero elements on the first column, but they are not independent. As such, the way the PLM is overparameterized relative to the respective REE is different. This difference is reflected in the E-stability conditions: The E-stability condition for the $LM1$ representation of Model A is that all the elements of (22) be less than one. In contrast, it is given by (30) for Model B. Therefore, the extents of the within-PLM overparameterizations differ across multivariate models in general and, consequently, the concept of E-stability is model-dependent.

Univariate representations of the models are, however, in general not model-dependent. In both the $LU1$ representation of Model A and the $LU2$ representation of Model B, the PLM has one unrestricted parameter b and the REE has also only one solution parameter. Therefore, E-stability is not subject to the within-PLM overparameterization.[15]

[15] Even in univariate models, the fundamental PLM could potentially be overparameterized relative to the fundamental solutions. For instance, suppose that a univariate model has n state variables so that the REE has n solution parameters. If the number of structural parameters of the model is less than n, then the number of independent solution (reduced-form) parameters would be less than n as well. Then, the PLM would be technically overparameterized as well. We do not investigate this issue in the present paper.

The concept of E-stability applied in these univariate representations is precisely "weak" E-stability. Consequently, E-stability in the $LU1$ or $LU2$ representations can be interpreted as "weak" E-stability conditions in a multivariate framework in the sense that the functional form of the PLM is identical to that of the REE, and the model-specific restriction (19c) is taken into account. We discuss this issue further in section 5.

In models A and B, the E-stability conditions in $LM1$ are "stronger" than those in the univariate representation of each model because the former are sufficient for the latter. However, it is not known whether the concept of E-stability in any arbitrary multivariate representation of a model is stronger than that in the univariate representations in general. Also, E-stability in higher dimensional representations is neither necessary nor sufficient for that in a lower dimensional representation, as the numerical example of the Dornbusch model showed in section 2.

4.2 Models with Contemporaneous Expectations

In this section, we show that all the findings of the previous section are not altered in models with contemporaneous expectations.

4.2.1 Model C: Combination of Independent Univariate Equations

We consider a model that can be represented in $CU1$ and $CM1$ forms.

$CU1$ **Representation**: Consider two completely unrelated univariate equations belonging to $CU1$. The model, solutions, the solution restriction, the PLM, the T-map and its derivatives corresponding to equations (12) through (17) are, respectively, given by:

$$y_{i,t} = \beta_{1,i} E_t y_{i,t+1} + \delta_i y_{i,t-1} + \epsilon_{i,t} \tag{31a}$$

$$y_{i,t} = \bar{b}_i y_{i,t-1} + \bar{c}_i \epsilon_{i,t} \tag{31b}$$

$$\bar{b}_i = (1 - \beta_{1,i}\bar{b}_i)^{-1}\delta_i, \ \bar{c}_i = (1 - \beta_{1,i}\bar{b}_i)^{-1} \tag{31c}$$

$$y_{i,t} = b_i y_{i,t-1} + c_i \epsilon_{i,t} \tag{31d}$$

$$T(b_i) = (1 - \beta_{1,i} b_i)^{-1}\delta_i, \ \ T(c_i) = (1 - \beta_{1,i} b_i)^{-1} \tag{31e}$$

$$DT(b_i) = (1 - \beta_{1,i} b_i)^{-2}\delta_i \beta_{1,i} = \beta_{1,i} b_i^2 / \delta_i, \tag{31f}$$

for $i = 1, 2$. Suppose that there are two real-valued but not necessarily stationary solutions, and $\bar{b}_i(1) < \bar{b}_i(2)$ (without loss of generality) in each equation. Note that the

E-stability condition for c_i is not required as it does not appear in the T-mapping.

$CM1$ **Representation**: The $CU1$ representation of the model can be written in a bivariate $CM1$ form with $x_t = (y_{1,t}\; y_{2,t})'$ and $v_t = (\epsilon_{1,t}\; \epsilon_{2,t})'$. The corresponding equations to (31) are as follows:

$$x_t = \beta_1 E_t x_{t+1} + \delta x_{t-1} + v_t \tag{32a}$$
$$x_t = \bar{b} x_{t-1} + \bar{c} v_t \tag{32b}$$
$$\bar{b} = (I - \beta_1 \bar{b})^{-1}\delta, \; \bar{c} = (I - \beta_1 \bar{b})^{-1} \tag{32c}$$
$$x_t = b x_{t-1} + c v_t \tag{32d}$$
$$T(b) = (I - \beta_1 b)^{-1}\delta, \;\; T(c) = (I - \beta_1 b)^{-1} \tag{32e}$$
$$DT(b) = [(I - \beta_1 b)^{-1}\delta]' \otimes (I - \beta_1 b)^{-1}\beta_1 \tag{32f}$$

where β_1 and δ are identical to those in equation (20). As in Model A under lagged expectations, (31a)-(31c) and (32a)-(32c) are just different representations of the same model, solution and solution restrictions. The solution $\bar{b}$ is given by $\bar{b} = \begin{bmatrix} \bar{b}_1 & 0 \\ 0 & \bar{b}_2 \end{bmatrix}$. It is straightforward to derive the eigenvalues of $DT(b)$ analytically when $b = \bar{b}$:

$$diag[DT(\bar{b})] = [\;\beta_{1,1}\bar{b}_1^2/\delta_1 \quad \beta_{1,2}\bar{b}_1\bar{b}_2/\delta_2 \quad \beta_{1,1}\bar{b}_1\bar{b}_2/\delta_1 \quad \beta_{1,2}\bar{b}_2^2/\delta_2\;]'.$$

Exactly the same problem arises here as in the lagged expectations models. The second and the third diagonal elements of $DT(\bar{b})$ are not zeros. The PLM (32d) is overparameterized relative to the solution (32b). Therefore, E-stability for the fundamental solutions hinges on the model representation and the PLM.

One can show that the condition for the mismatch of the E-stability conditions in $CU1$ and $CM2$ forms is identical to the condition (23) with lagged expectations models:[16]

$$\bar{b}_1(1) > \bar{b}_2(2).$$

A numerical example can be illustrated as follows. The parameter values $\beta_{1,1} = 0.5882$, $\delta_1 = 0.4235$, $\beta_{1,2} = 0.7692$ and $\delta_2 = 0.3231$ yield the same solutions as those to Model A. Table 8, analogously to Table 6, reports the eigenvalues of $DT(\bar{b})$. Again, while $\bar{b}_1(1)$

[16] To see this, note that $\delta_2 = \bar{b}_2(1)\bar{b}_2(2)\beta_{1,2}$. With $\bar{b}_1 = \bar{b}_1(1)$ and $\bar{b}_2 = \bar{b}_2(1)$, the second diagonal element of $DT(\bar{b})$ is then $\bar{b}_1(1)/\bar{b}_2(2)$.

Table 8: $DT(\bar{b})$ of the $CU1$ and $CM1$

		$CM1$			
		$CU1$			
$\bar{b}_1$	$\bar{b}_2$	DT_{11}	DT_{44}	DT_{22}	DT_{33}
$\bar{b}_1(1) = 0.8$	$\bar{b}_2(1) = 0.6$	0.89	0.86	1.14	0.67
$\bar{b}_1(1) = 0.8$	$\bar{b}_2(2) = 0.7$	0.89	1.17	1.33	0.78
$\bar{b}_1(2) = 0.9$	$\bar{b}_2(1) = 0.6$	1.13	0.86	1.29	0.75
$\bar{b}_1(2) = 0.9$	$\bar{b}_2(2) = 0.7$	1.13	1.17	1.50	0.88

and $\bar{b}_2(1)$ are E-stable in $CU1$, the solution $\bar{b}$ corresponding to $\bar{b}_1(1)$ and $\bar{b}_2(1)$ is not E-stable in $CM1$. This example together with the one under lagged expectations shows that the discrepancies of E-stability across model representations do not stem from the information structure. One can also show that the analysis of the recursive models with lagged expectations is isomorphic to those with contemporaneous expectations.

4.2.2 Model D: Bivariate Model and its Univariate Representation

We consider a model that can be represented in $CU2$ and $CM1$ forms:

$$y_t = \beta_{1,y} E_t y_{t+1} + E_t z_{t+1} + \delta_y y_{t-1} + \epsilon_t \tag{33}$$

$$z_t = \beta_{2,y} E_t y_{t+1} \tag{34}$$

$CU2$ **Representation**: The model can be represented in a univariate form in terms of y_t by substituting out z_t. This $CU2$ representation of the model, the fundamental solutions, the solution restriction, the PLM, the T-map and its derivative are given by:

$$y_t = \beta_{1,y} E_t y_{t+1} + \beta_{2,y} E_t y_{t+2} + \delta_y y_{t-1} + \epsilon_t \tag{35a}$$

$$y_t = \bar{b}_y y_{t-1} + \epsilon_t \tag{35b}$$

$$\bar{b}_y = \beta_{2,y} \bar{b}_y^3 + \beta_{1,y} \bar{b}_y^2 + \delta_y \tag{35c}$$

$$y_t = b_y y_{t-1} + \epsilon_t \tag{35d}$$

$$T(b_y) = \beta_{2,y} b_y^3 + \beta_{1,y} b_y^2 + \delta_y \tag{35e}$$

$$DT(b_y) = 3\beta_{2,y} b_y^2 + 2\beta_{1,y} b_y. \tag{35f}$$

$CM1$ **Representation**: In matrix form, the model can also be written as the following bivariate $CM1$ form:

$$x_t = \beta_1 E_t x_{t+1} + \delta x_{t-1} + v_t \tag{36}$$

where $x_t = (y_t \ z_t)'$ and $v_t = (\epsilon_t, 0)'$, and β_1 and δ are identical to those in equation (28). The fundamental solution, the solution restriction, the fundamental PLM, the T-map and its derivative with respect to the unrestricted PLM parameters are exactly of the same form as (32b) through (32f). However, the extents of the restrictions on $\bar{b}$ are again different from those in the $CM1$ representation of Model C in the previous subsection. The reason is simply that the definitions of β_1 and δ are different, and $\bar{b}$ as an REE is given by $\bar{b} = \begin{bmatrix} \bar{b}_y & 0 \\ \beta_{2,y}\bar{b}_y^2 & 0 \end{bmatrix}$.

In Appendix C, we show that the E-stability conditions for $CM1$ are given by:

$$(\delta_y - \beta_{1,y}\bar{b}_y^2 - 2\beta_{2,y}\bar{b}_y^3)/\delta_y > 0, \ (\beta_{1,y}\bar{b}_y^2 + \beta_{2,y}\bar{b}_y^3)/\delta_y < 2 \tag{37}$$

and the first condition is identical to the E-stability condition in $CU2$.

A numerical example analogous to that in Model B can be illustrated as follows. For the $LU2$ representation of Model B, $\bar{b}_y$ must solve $\beta_{2,y}\bar{b}_y^3 + \beta_{1,y}\bar{b}_y^2 + \beta_{0,y}\bar{b}_y + \delta_y = \bar{b}_y$. Note that this equation divided by $(1 - \beta_{0,y})$ becomes the solution restriction for the $CU2$ representation of the present Model D. Using the values in section 4.1.2, we redefine $\beta_{1,y} = 1.4685$, $\beta_{2,y} = -0.6993$ and $\delta_y = 0.2203$. Therefore we obtain the same solutions with $\bar{b}_y = 0.5$, 0.7 and 0.9. Table 9 shows the univariate and multivariate E-stability conditions. Therefore, while the solutions with $\bar{b}_y = 0.5$ and 0.9 are E-stable in $CU2$, the

Table 9: $DT(\bar{b})$ of the $CU2$ and $CM1$

	$CU2$	$CM1$			
$\bar{b}_y$	$DT(\bar{b}_y)$	Eigenvalues of $DT(\bar{b})$			
0.5	0.87	0.71	0.56	0	0
0.7	1.09	1.40	0.78	0	0
0.9	0.77	1.80	1.29	0	0

multivariate solution $\bar{b}$ associated with $\bar{b}_y = 0.9$ is not, implying that E-stability results differ across representations. These results are however, not exactly the same as those in Model B under lagged expectations. In Model B and Model D, the solutions with $\bar{b}_y = 0.5$ and 0.9 are E-stable in univariate representations. However, no solution was E-stable in the multivariate representation of the Model B. This is because the E-stability conditions (37) for models under contemporaneous expectations are not the same as those in (30) for models under lagged expectations.

5 Model-independent E-stability

We have shown that the concept of E-stability varies across multivariate models or multivariate representations of a given model and this finding is independent of the information structure or the stationarity of solutions. Naturally, a model-independent concept of E-stability would be desirable. We argue that the source of model-dependent E-stability is the within-PLM overparameterization. Consequently, it is tempting to directly impose the model-specific restrictions on the PLM in a given multivariate model. However, this approach would not be economically sensible in a learning setting, where the unrestricted PLM is naturally consistent with bounded rationality and imperfect information, which are central concepts in the learning literature. If agents knew the precise restrictions implied by the RE model, then they would directly compute the REE, rather than specify a plausible PLM and examine the learning dynamics over time.

As we hinted from several examples in the previous section, E-stability is not model-dependent in a univariate representation in general. Therefore, a straightforward way to make a PLM not subject to within-PLM overparameterization would be to recursively reduce a multivariate system into a univariate representation for each variable and examine the standard weak E-stability in a univariate framework sequentially. However, such a recursive system reduction comes at the cost of notational and analytical complications. In general, the resulting univariate representation would involve different dates at which expectations are formed and more lagged state variables than the original multivariate representation. In addition, the same procedure must be carried out for the remaining variables recursively.

We illustrate the recursive system reduction into a univariate representation with a simple bivariate model. Consider the following model:

$$x_{1,t} = \beta_{11} E_t x_{1,t+1} + \beta_{12} E_t x_{2,t+1} + \delta_{11} x_{1,t-1} + \delta_{12} x_{2,t-1}, \quad (38)$$

$$x_{2,t} = \beta_{21} E_t x_{1,t+1} + \beta_{22} E_t x_{2,t+1} + \delta_{21} x_{1,t-1} + \delta_{22} x_{2,t-1}. \quad (39)$$

For simplicity, we abstract from exogenous disturbances. In Appendix D, we show that the model can be represented in a univariate form for $x_{1,t}$ as:

$$x_{1,t} = f_1 E_t x_{1,t+1} + f_2 E_t x_{1,t+2} + f_3 E_{t-1} x_{1,t} + f_4 E_{t-1} x_{1,t+1} + f_5 x_{1,t-1} + f_6 x_{1,t-2}$$

where f_1 through f_6 are very complicated functions of the structural parameters in the original model. The fundamental solution is of the following form:

$$x_{1,t} = \bar{b}_1 x_{1,t-1} + \bar{b}_2 x_{1,t-2}. \tag{40}$$

The fundamental PLM is given by $x_{1,t} = b_1 x_{1,t-1} + b_2 x_{1,t-2}$ without parameter restrictions. Then the T-mapping from the fundamental PLM to the ALM is given by:

$$T_1(b_1, b_2) = (F^{-1}((f_1 + f_2 b_1)b_2 + (f_3 + f_4 b_1)b_1 + f_4 b_2 + f_5), \quad F^{-1}((f_3 + f_4 b_1)b_2 + f_6)).$$

where $F = 1 - (f_1 + f_2 b_1)b_1 - f_2 b_2$. A standard weak E-stability condition can be derived by constructing the Jacobian matrix of $T_1(b_1, b_2)$ and obtaining its eigenvalues.

Once (40) is proven to be E-stable, we need to examine the E-stability of a solution in the univariate representation of $x_{2,t}$. Now $x_{1,t}$ becomes an exogenous process and, from equation (39), the fundamental solution would have the following form:

$$x_{2,t} = \bar{b}_3 x_{2,t-1} + \bar{b}_4 x_{1,t} + \bar{b}_5 x_{1,t-1}. \tag{41}$$

An analogous equation without restrictions on the parameters can be used as the fundamental PLM. Then the T-mapping can be easily derived as:

$$T_2(b_3, b_4, b_5) = (\frac{\delta_{22}}{1 - \beta_{22} b_3}, \frac{\beta_{22} b_4 b_1 + \beta_{21} b_1 + \beta_{22} b_5}{1 - \beta_{22} b_3}, \frac{\beta_{22} b_4 b_2 + \beta_{21} b_2 + \delta_{21}}{1 - \beta_{22} b_3})$$

A standard weak E-stability condition can also be derived by constructing the Jacobian matrix of $T_2(b_3, b_4, b_5)$ and obtaining its eigenvalues. Therefore, we may conclude that if (40) and (41) are both weakly E-stable in a univariate framework, the solution to the bivariate model consisting of these two equations is weakly E-stable in a multivariate context.

We remark two more potential difficulties in the proposed recursive system reduction. First, it is not clear whether the system reduction is robust against the order of the variables with which the model is reduced. Second, it must be proved that the coefficients of the state variables are independent of each other. If not, E-stability would still be model-dependent. Therefore, although the system reduction is a plausible way of constructing a model-independent concept of E-stability in a multivariate framework, it may not always be possible to do so.

6 Conclusion

This paper shows that the concept of E-stability in a multivariate framework is model-dependent. We also show that the model-specific nature of E-stability surfaces independently of the uniqueness of the fundamental solution, stability of the REEs and information structure. An immediate consequence of our analysis is that it is hard to compare the results of E-stability across models. Consequently, the development of a model-independent concept of E-stability is called for in a multivariate framework.

We show that the source of model-dependent E-stability lies in the fact that a postulated PLM is in general overparameterized relative to the REE, which is subject to the model-specific restrictions. Therefore, developing model-independent E-stability conditions requires that the PLM at hand be not subject to the within-PLM overparameterization. In this paper, we propose a tentative method of recursive system reduction of a given multivariate model into a univariate representation. However, such a procedure comes at the cost of analytical complication, especially when the model at hand is a large-scale model. Furthermore, the validity of the system reduction approach should be formally examined. We leave a formal treatment of the system reduction method as a future research topic.

Even if model-independent E-stability conditions are developed, the cost of implementing such conditions can be high. As is well-known, E-stability has been extensively used as an indirect way of exploring learnability under the E-stability principle because implementing E-stability is technically much simpler. Therefore, one may have to directly rely on learnability conditions in studies of macroeconomic dynamics if the cost of implementing the model-independent E-stability conditions is high.

According to our results, the concept of E-stability would also be model-dependent in studies on the relation between determinacy, learnability and E-stability. For example, under fairly general conditions, E-stability and learnability are shown to be equivalent. Assuming this, Bullard and Mitra (2002) and Bullard and Eusepi (2008) study the relation between determinacy and learnability. One important finding of Bullard and Mitra

(2002) is that determinacy does not necessarily imply learnability and indeterminacy does not necessarily imply lack of learnability. Alternatively, Heinemann (2000) and Giannitsarou (2005) show that E-stability and learnability may not be identical in some environments. All of these studies may deal with different types of E-stability if their models are multivariate. We leave the study of the interrelation between determinacy, learnability and model-independent E-stability as a future research topic.

Appendix

A. Five Representations of the Dornbusch Model

All representations of the model can be written in the following general form:

$$Ax_t = B_0 E_{t-1} x_t + B_1 E_{t-1} x_{t+1} + B_2 E_{t-1} x_{t+2} + D x_{t-1}. \tag{42}$$

By pre-multiplying this equation by A^{-1}, the model can be written as:

$$x_t = \beta_0 E_{t-1} x_t + \beta_1 E_{t-1} x_{t+1} + \beta_2 E_{t-1} x_{t+2} + \delta x_{t-1} \tag{43}$$

where $\beta_0 = A^{-1}B_0$, $\beta_1 = A^{-1}B_1$, $\beta_2 = A^{-1}B_2$ and $\delta = A^{-1}D$. Since x_{t-1} is the only state vector, the fundamental solution has the following form:

$$x_t = \bar{b} x_{t-1} \tag{44}$$

where $\bar{b}$ is subject to:

$$\beta_0 \bar{b} + \beta_1 \bar{b}^2 + \beta_2 \bar{b}^3 + \delta = \bar{b}. \tag{45}$$

Finally, the fundamental PLM is given by:

$$x_t = b x_{t-1} \tag{46}$$

where b is unrestricted. x_t, the parameter matrices A, B_0, B_1, B_2, D (or β_0, β_1, β_2, δ) and $\bar{b}$ are representation-specific and they are defined as follows:

$R1$. **Univariate Representation with $x_t = p_t$:** The fundamental solution is given by $p_t = \bar{b}_p p_{t-1}$, Let $\alpha_0 = 1 + \pi(\gamma + \eta + \gamma/\lambda + \eta/\lambda + \gamma\vartheta/\lambda)$, $\alpha_1 = 1 + \pi(2\gamma + \eta + \gamma/\lambda)$, $\alpha_2 = -\pi\gamma$ and $\delta_0 = 1 + \pi\vartheta(\gamma + \eta)/\lambda$. Then, β_0, β_1, β_2, δ and $\bar{b}$ are defined as:

$$\beta_0 = \alpha_0,\ \beta_1 = \alpha_1,\ \beta_2 = \alpha_2,\ \delta = \delta_0,\ \bar{b} = \bar{b}_p.$$

We need to solve for the remaining variables sequentially. They can be characterized in terms of $\bar{b}_p$ as $d_t = \bar{b}_d p_{t-1}$, $r_t = \bar{b}_r p_{t-1}$ and $e_t = \bar{b}_e p_{t-1}$ where $\bar{b}_d = -(1 - \bar{b}_p)/\pi$, $\bar{b}_r = (\bar{b}_p - \vartheta)/\lambda$ and $\bar{b}_e = -(\bar{b}_p - \vartheta)/(\lambda(1 - \bar{b}_p))$.

$R2$. **Bi-variate Representation with** $x_t = (p_t\ e_t)'$**:**

$$A = \begin{bmatrix} 1 & 0 \\ 1/\lambda & 1 \end{bmatrix},\ B_0 = \begin{bmatrix} -\pi(\gamma+\eta+\gamma/\lambda) & \pi\eta \\ 0 & 0 \end{bmatrix},\ B_1 = \begin{bmatrix} \pi\gamma & 0 \\ 0 & 1 \end{bmatrix},$$
$$B_2 = 0_{2\times 2},\ D = \begin{bmatrix} 1+\pi\gamma\vartheta/\gamma & 0 \\ -\vartheta/\gamma & 0 \end{bmatrix},\ \bar{b} = \begin{bmatrix} \bar{b}_p & 0 \\ \bar{b}_e & 0 \end{bmatrix}.$$

$R3$. **Tri-variate Representation with** $x_t = (p_t\ d_t\ r_t)'$**:**

$$A = \begin{bmatrix} 1 & 0 & 0 \\ \gamma+\eta & 1 & \gamma+\eta \\ -\lambda^{-1} & 0 & 1 \end{bmatrix},\ B_0 = \begin{bmatrix} 0 & \pi & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ B_1 = \begin{bmatrix} 0 & 0 & 0 \\ 2\gamma+\eta & 1 & \gamma \\ 0 & 0 & 1 \end{bmatrix},$$
$$B_2 = \begin{bmatrix} 0 & 0 & 0 \\ -\gamma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -\vartheta/\lambda & 0 & 0 \end{bmatrix},\ \bar{b} = \begin{bmatrix} \bar{b}_p & 0 & 0 \\ \bar{b}_d & 0 & 0 \\ \bar{b}_r & 0 & 0 \end{bmatrix}.$$

$R4$. **Four-variable Representation with** $x_t = (p_t\ d_t\ \ r_t\ e_t)'$:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \gamma+\eta & 1 & \gamma & -\eta \\ -\lambda^{-1} & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix},\ B_0 = \begin{bmatrix} 0 & \pi & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix},\ B_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$
$$B_2 = 0_{4\times 4},\ D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\theta/\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ \bar{b} = \begin{bmatrix} \bar{b}_p & 0 & 0 & 0 \\ \bar{b}_d & 0 & 0 & 0 \\ \bar{b}_r & 0 & 0 & 0 \\ \bar{b}_e & 0 & 0 & 0 \end{bmatrix}.$$

$R2'$. **Bi-variate Representation with** $x_t = (p_t\ d_t)'$**:**

$$A = \begin{bmatrix} 1 & 0 \\ \frac{(\lambda+1)(\gamma+\eta)}{\lambda} & 1 \end{bmatrix},\ B_0 = \begin{bmatrix} 0 & \pi \\ -\frac{\gamma\vartheta}{\lambda} & 0 \end{bmatrix},\ B_1 = \begin{bmatrix} 0 & 0 \\ 2\gamma+\eta+\frac{\gamma}{\lambda} & 1 \end{bmatrix},$$
$$B_2 = \begin{bmatrix} 0 & 0 \\ -\gamma & 1 \end{bmatrix},\ D = \begin{bmatrix} \delta & 0 \\ -\frac{\vartheta(\gamma+\eta)}{\lambda} & 0 \end{bmatrix},\ \bar{b} = \begin{bmatrix} \bar{b}_p & 0 \\ \bar{b}_d & 0 \end{bmatrix}.$$

Table 10: $DT(\bar{b})$ for Five Representations of the Dornbusch Model

Panel A. $R1$ representation

$\vartheta = 1.1$ and $\eta = -0.1$		$\vartheta = 0.5$ and $\eta = 0.2$	
$\bar{b}$	$DT(\bar{b})$	$\bar{b}$	$DT(\bar{b})$
0.716	0.966	0.384	−0.285
0.772	1.028	1.043	1.307
0.990	0.866	1.250	0.596

Panel B. $R2$ Representation

$\vartheta = 1.1$ and $\eta = -0.1$		$\vartheta = 0.5$ and $\eta = 0.2$	
$\bar{b}$	$DT(\bar{b})$	$\bar{b}$	$DT(\bar{b})$
0.716	0.821 0.807 −0.669 −0.030	0.384	0.380 0.003 −1.944 −1.072
0.772	1.137 0.800 −0.532 −0.041	1.043	2.191 0.742 0.426 −0.884
0.990	1.999 1.134 −0.042 $\pm 0.399i$	1.250	2.854 1.218 0.304 −0.296

Panel C. $R3$ Representation

$\vartheta = 1.1$ and $\eta = -0.1$		$\vartheta = 0.5$ and $\eta = 0.2$	
$\bar{b}$	$DT(\bar{b})$	$\bar{b}$	$DT(\bar{b})$
0.716	−2.58 0.99 0.56(2) −2.87(2) 0(3)	0.384	−3.11 0.68 0.28(2) −3.09(2) 0(3)
0.772	−2.54 1.01 0.59(2) −2.90(2) 0(3)	1.043	−2.84 1.08 0.76(2) −3.57(2) 0(3)
0.990	−2.28 0.96 0.69(2) −3.00(2) 0(3)	1.250	−2.43 0.88 0.80(2) −3.61(2) 0(3)

Panel D. $R4$ Representation

$\vartheta = 1.1$ and $\eta = -0.1$		$\vartheta = 0.5$ and $\eta = 0.2$	
$\bar{b}$	$DT(\bar{b})$	$\bar{b}$	$DT(\bar{b})$
0.716	−3.29 0.97 0.72 −2.87(3) 0.56(3) 0(7)	0.384	−3.32 0.51 0.39 −3.09(3) 0.28(3) 0(7)
0.772	−3.35 1.03 0.78 −2.90(3) 0.59(3) 0(7)	1.043	−4.04 1.42 0.86 −3.57(3) 0.76(3) 0(7)
0.990	−3.53 $1.1 \pm 0.13i$ −3.00(3) 0.69(3) 0(7)	1.250	−4.17 1.43 1.18 −3.61(3) 0.80(3) 0(7)

Panel E. $R2'$ representation

$\vartheta = 1.1$ and $\eta = -0.1$		$\vartheta = 0.5$ and $\eta = 0.2$	
$\bar{b}$	$DT(\bar{b})$	$\bar{b}$	$DT(\bar{b})$
0.716	0.990 −2.584 0.555 −2.865	0.384	0.687 −3.108 0.282 −3.087
0.772	1.008 −2.546 0.591 −2.901	1.043	1.080 −2.842 0.763 −3.568
0.990	0.959 −2.279 0.689 −2.999	1.250	0.883 −2.437 0.804 −3.609

Note: The number of repeated eigenvalues of $DT(\bar{b})$ in Panels C and D is in parentheses.

All the representations are nested in the $LM2$ class of RE models in section 3. E-stability of an REE of each representation can be examined by computing the derivatives of the T-mapping from the fundamental PLM to the ALM evaluated with the REE, $b = \bar{b}$ ($DT(\bar{b})$) in (11). In each representation of the model, there are three fundamental solutions, $\bar{b}$ corresponding to the three values of $\bar{b}_p$. Table 10 shows the derivatives of the T-mapping computed for the three values of b in each representation.

B. E-stability Conditions for the $LM1$ Representation of Model B in Section 4.1.2

Here we derive $DT(\bar{b})$ analytically. Let $P(\xi : \bar{b}_y)$ be the characteristic function of $DT(\bar{b})$. Then $P(\xi : \bar{b}_y) = |DT(\bar{b}) - \xi I_2|$ where

$$DT(\bar{b}) = I_2 \otimes \left(\begin{bmatrix} \beta_{0,y} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \beta_{1,y} & 1 \\ \beta_{2,y} & 0 \end{bmatrix} \begin{bmatrix} \bar{b}_y & 0 \\ \beta_{2,y}\bar{b}_y^2 & 0 \end{bmatrix} \right) + \begin{bmatrix} \bar{b}_y & 0 \\ \beta_{2,y}\bar{b}_y^2 & 0 \end{bmatrix}' \otimes \begin{bmatrix} \beta_{1,y} & 1 \\ \beta_{2,y} & 0 \end{bmatrix}.$$

Direct computation yields:

$$\begin{aligned} P(\xi &: \bar{b}_y) = C(\xi : \bar{b}_y)\xi(\xi - \beta_{0,y} - \beta_{1,y}\bar{b}_y - \beta_{2,y}\bar{b}_y^2) \\ C(\xi &: \bar{b}_y) = \xi^2 - (\beta_{0,y} + 2\beta_{1,y}\bar{b}_y + \beta_{2,y}\bar{b}^2)\xi - 2\beta_{2,y}\bar{b}_y^2. \end{aligned}$$

In this example, the analytical solution of $\bar{b}_y$ is not available in general. However, we can still characterize the E-stability condition, i.e., the condition under which the real part of all roots of $P(\xi : \bar{b}_y)$ is less than 1, as follows:

$$\begin{gathered} C(1 : \bar{b}_y) > 0 \\ \xi_1 + \xi_2 = \beta_{0,y} + 2\beta_{1,y}\bar{b}_y + \beta_{2,y}\bar{b}_y^2 < 2 \\ \xi_3 = \beta_{0,y} + \beta_{1,y}\bar{b}_y + \beta_{2,y}\bar{b}_y^2 < 1 \Leftrightarrow (\bar{b}_y - \delta_y)/\bar{b}_y < 1 \end{gathered}$$

where ξ_1 and ξ_2 are the two roots of $C(\xi : \bar{b}_y)$. $C(1 : \bar{b}_y) > 0$ implies that $3\beta_{2,y}\bar{b}_y^2 + 2\beta_{1,y}\bar{b}_y + \beta_{0,y} < 1$, which is precisely the E-stability condition for the $LU2$ representation of the model. The second and the third conditions are the additional conditions associated with the $LM1$ representation of the model.

C. E-stability Conditions for the $CM1$ Representation of Model D in Section 4.2.2

Let $P(\xi : \bar{b}_y)$ be the characteristic function of $DT(\bar{b})$. Direct computation yields:

$$\begin{aligned} P(\xi &: \bar{b}_y) = C(\xi : \bar{b}_y)\xi^2 \\ C(\xi &: \bar{b}_y) = \xi^2 - \frac{\beta_{1,y}\bar{b}_y^2 + \beta_{2,y}\bar{b}_y^3}{\delta_y}\xi - \frac{\beta_{2,y}\bar{b}_y^3}{\delta_y}. \end{aligned}$$

For the roots of $C(\xi : \bar{b}_y)$ to have real parts less than 1, it must be that $C(1 : \bar{b}_y) > 0$ and $\xi_1 + \xi_2 < 2$ where ξ_1 and ξ_2 are the two roots of $C(\xi : \bar{b}_y)$. These two E-stability conditions are then given by:

$$(\delta_y - \beta_{1,y}\bar{b}_y^2 - 2\beta_{2,y}\bar{b}_y^3)/\delta_y > 0, (\beta_{1,y}\bar{b}_y^2 + \beta_{2,y}\bar{b}_y^3)/\delta_y < 2.$$

From the definition of (17), the E-stability condition for the $CU2$ representation is:

$$\frac{\delta_y(\beta_{1,y} + 2\beta_{2,y}\bar{b}_y)}{(1 - \beta_{1,y}\bar{b}_y - \beta_{2,y}\bar{b}_y^2)^2} < 1.$$

But $(1 - \beta_{1,y}\bar{b}_y - \beta_{2,y}\bar{b}_y^2) = (\delta_y/\bar{b}_y)$ since $\bar{b}_y$ must solve $\beta_{2,y}\bar{b}_y^3 + \beta_{1,y}\bar{b}_y^2 + \delta_y = \bar{b}_y$. Therefore, a rearrangement of this condition becomes the first E-stability condition for $CM1$.

D. System Reduction into a Univariate Framework

Let us reproduce the general bivariate model as:

$$x_{1,t} = \beta_{11}E_t x_{1,t+1} + \beta_{12}E_t x_{2,t+1} + \delta_{11}x_{1,t-1} + \delta_{12}x_{2,t-1}, \quad (47)$$

$$x_{2,t} = \beta_{21}E_t x_{1,t+1} + \beta_{22}E_t x_{2,t+1} + \delta_{21}x_{1,t-1} + \delta_{22}x_{2,t-1}. \quad (48)$$

First, we eliminate $E_t x_{2,t+1}$ in (47) by pre-multiplying (47) and (48) by β_{22} and β_{12}, respectively, and then subtracting the second equation from the first one as:

$$z_t = k_1 x_{2,t} + k_2 x_{2,t-1} \quad (49)$$

where $k_1 = \beta_{12}$, $k_2 = (\beta_{22}\delta_{12} - \beta_{12}\delta_{22})$ and

$$z_t = \beta_{22}x_{1,t} - (\beta_{22}\beta_{11} - \beta_{12}\beta_{21})E_t x_{1,t+1} - (\beta_{22}\delta_{11} - \beta_{12}\delta_{21})x_{1,t-1}.$$

Then (48) can be expressed as:

$$z_t = \beta_{21}(k_1 E_t x_{1,t+1} + k_2 E_{t-1} x_{1,t}) + \beta_{22}(k_1 E_t x_{2,t+1} + k_2 E_{t-1} x_{2,t}) + \delta_{21}(k_1 x_{1,t-1} + k_2 x_{1,t-2}) + \delta_{22} z_{t-1}. \tag{50}$$

From equation (49),

$$(k_1 E_t x_{2,t+1} + k_2 E_{t-1} x_{2,t}) = E_t z_{t+1} + (k_2/k_1) E_{t-1} z_t - (k_2/k_1) z_t.$$

Therefore, (50) becomes:

$$\begin{aligned}(1 + \beta_{22}(k_2/k_1)) z_t &= \beta_{21}(k_1 E_t x_{1,t+1} + k_2 E_{t-1} x_{1,t}) + \delta_{21}(k_1 x_{1,t-1} + k_2 x_{1,t-2}) \\ &\quad + \beta_{22} E_t z_{t+1} + \beta_{22}(k_2/k_1) E_{t-1} z_t + \delta_{22} z_{t-1}. \end{aligned} \tag{51}$$

Note that (51) consists of the variable $x_{1,t}$ only. By rearranging this equation in terms of $x_{1,t}$, we have:

$$x_{1,t} = f_1 E_t x_{1,t+1} + f_2 E_t x_{1,t+2} + f_3 E_{t-1} x_{1,t} + f_4 E_{t-1} x_{1,t+1} + f_5 x_{1,t-1} + f_6 x_{1,t-2},$$

where

$$\begin{aligned}
f_1 &= \Delta^{-1}\left(\beta_{12}\beta_{11} + \beta_{22}^2\delta_{12}\beta_{11} - \beta_{22}\delta_{12}\beta_{12}\beta_{21} - \beta_{22}\beta_{12}\delta_{22}\beta_{11} + \beta_{12}^2\delta_{22}\beta_{21} + \beta_{22}\beta_{12}\right) \\
f_2 &= -\Delta^{-1}\beta_{12}\left(\beta_{22}\beta_{11} - \beta_{12}\beta_{21}\right) \\
f_3 &= \Delta^{-1}\left(-\beta_{12}\delta_{22}\beta_{11} + \delta_{12}\beta_{12}\beta_{21} + \beta_{22}^2\delta_{12} - \beta_{22}\beta_{12}\delta_{22}\right) \\
f_4 &= -\Delta^{-1}\left(\beta_{22}\delta_{12} - \beta_{12}\delta_{22}\right)\left(\beta_{22}\beta_{11} - \beta_{12}\beta_{21}\right) \\
f_5 &= \Delta^{-1}\beta_{12}\left(\delta_{11} + \delta_{22}\right) \\
f_6 &= -\Delta^{-1}\beta_{12}\left(\delta_{22}\delta_{11} - \delta_{21}\delta_{12}\right)
\end{aligned}$$

and $\Delta = \beta_{12} + \beta_{22}^2\delta_{12} - \beta_{22}\beta_{12}\delta_{22} + \beta_{12}\beta_{22}\delta_{11} - \beta_{12}^2\delta_{21}$.

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