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Random Dynamics and Finance: Constructing Implied Binomial Trees from a Predetermined Stationary Density

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ABSTRACT

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Random Dynamics and Finance: Constructing Implied Binomial Trees from a Predetermined Stationary Density *

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Abstract

We introduce a general binomial model for asset prices based on the concept of random maps. The asymptotic stationary distribution for such model is studied using techniques from dynamical systems. In particular, we present a technique to construct a general binomial model with a predetermined stationary distribution. This technique is independent of the chosen distribution making our model potentially useful in financial applications. We briefly explore the suitability of our construction as an implied binomial tree.

KEYWORDS: Implied Binomial Trees, Random map, Binomial model, Perron-Frobenius operator

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1 Introduction

Random dynamical systems are believed to be a useful framework for modeling and analyzing of economic phenomena with stochastic components Schenk-Hoppe (2001). We are interested in developing techniques in theory of dynamical systems which can be implemented in finance. Here we present one possible application that uses the concept of random maps.

A position dependent random map is a discrete time random dynamical system consisting of a collection of transformations that, at each iteration, are selected and applied randomly by means of position dependent probabilities. Such a structure could be used as a generalized binomial model as those discussed in Cox, Ross and Rubinstein (1979), Rubinstein (1994) and Derman and Kani (1994).

In the early years of modern financial mathematics, Cox, Ross and Rubinstein (1979) proposed a binomial approximation to option pricing. This binomial approach has become a favorite among practitioners since it yields approximations for a wide variety of options that cannot be approached otherwise [see for instance Cakici and Foster (2002) or Hui (2005)]. Binomial option pricing models and their convergence to continuous time models have been extensively studied eversince. For instance, Rachev and Ruschendorf (1997) explores the question of which continuous time models arise as limits of generalized binomial models. Hubalek and Schachermayer (1998) explores the conditions needed to ensure that convergence of a general binomial prices imply convergence of option prices for general binomial models. And Diener and Diener (2004) explores the nature of the convergence of binomial models.

Here we use dynamical systems techniques to construct a generalized binomial model as those previously studied in the financial literature [see Hubalek and Schachermayer (1998), Nelson and Ramaswamy (1990), Jackwerth (1999) and references therein]. One appealing feature of this model is the existence of an invariant asymptotic density. This mimics the motivation behind the use of stationary diffusion models in finance. Continuous time diffusion models with stationary densities have been proposed in the literature as appealing asset returns models, for instance Rydberg (1999) introduces a family of an ergodic diffusion with a predetermined stationary distribution. Binomial models as approximations to diffusions have been studied in Nelson and Ramaswamy (1990), in view of which, our model can be naturally interpreted as the discretization of a diffusion.The novelty of our model lays in the fact that Theorem 6.9 allows us to construct a binomial tree that has any given desired distribution as its invariant density. Our techniques are based on the position dependent random map model and its Perron-Frobenius operator. We prove our results theoretically and we produce a program which computes the components of a binomial model whose stationary density is the desired one. A pedagogical version of this program, that constructs a stationary binomial model from any desired lognormal distribution, is available at http://www.mathstat.concordia.ca/pg/Economics200s.zip

Our construction might be of interest to practitioners as well since our method can be used to extract an implied binomial tree from a given risk-neutral density previously inferred from option prices, i.e. our model can produce a stochastic model from a set of option prices (provided we know how to extract a risk-neutral density from these prices). Implied binomial trees were introduced in Rubinstein (1994), Derman and Kani (1994) and Dupire (1994) as a mean to empirically study option prices and to price less-traded options in a market-consistent way. Eversince, there has been a substantial amount of articles discussing further the construction of binomial trees from observed option prices. Relevant articles in the late 90's are Jackwerth (1997), Derman and Kani (1998), Barle and Cakici (1998), Dumas, Fleming and Whaley (1998), Brown and Toft (1999) and Britten-Jones and Neuberger (2000). More recent articles are Jackwerth and Rubinstein (2001) and Li (2001). They all analyze the problem of extracting, from observed option prices, information on the stochastic process behind the underlying asset. Such implied trees were consistent with the volatility smile and were risk-neutral at each step. Since our construction starts from any given density, our model can be very well used to construct an implied binomial tree.

The main purpose of this note is to introduce a novel construction of a binomial tree using random maps. The key feature of this model is that it can be built to have any desired stationary density. As an application, we argue that this construction could be use to produce implied binomial trees. This paper is organized as follows. In Section 2 we present some preliminaries from stochastic analysis. In Section 3 we formulate the definition of a position dependent random map and introduce its Perron-Frobenius operator. In Section 4 we prove an ergodic theorem that will be needed in our application. In Section 5, we build a binomial model from a position dependent random map and present an illustration. In Section 6, we first address the inverse problem of the Perron-Frobenius operator of position dependent random maps. Then we present a method and a computer program to construct binomial models from any density suitable for financial applications. In Section 7, we discuss some interesting features of our construction. We argue that our binomial tree can be considered as an implied binomial tree since this can be built up from a risk-neutral distribution potentially extracted from option opices. This would produce a binomial tree that is consistent with the market in an stationary way. Section 8 presents a method of approximating the invariant density of any given binomial model. In Section 9, we present an interesting modification to our model that causes random arbitrage opportunities but still allows it to accept a stationary density. We construct such model using a perturbed random map.

2 Preliminaries

In this section we present some definitions and notions that will be needed to construct our model. In particular, we state the definition of stationary probability for a Markov process.

Definition 2.1 *Let $(X, \mathcal{B}, \lambda)$ be a probability space. A function $\mathbb{P} : X \times \mathcal{B} \to [0,1]$ is called a stochastic transition function if it has the following properties:*
(i) for any $A \in \mathfrak{B}$, $\mathbb{P}(\cdot, A) : X \to [0,1]$ is a $\mathfrak{B}$-measurable function;
(ii) for any $x \in X$, $\mathbb{P}(x, \cdot) : \mathfrak{B} \to [0,1]$ is a measure.

We can define a Markov process by a transition function $\mathbb{P}$. Let λ be a probabilistic measure on $\mathfrak{B}$ called initial probability. Then we define all probabilities related to the Markov process $\{\mathcal{X}_n\}_{n\geq 0}$ using λ and $\mathbb{P}$:

$$\begin{aligned} \mathbf{P}(\mathcal{X}_0 \in A) &= \lambda(A); \\ \mathbf{P}(\mathcal{X}_1 \in A | \mathcal{X}_0 = x) &= \mathbb{P}(x, A); \\ \mathbf{P}(\mathcal{X}_1 \in A) &= \int_X \mathbb{P}(x, A) d\lambda(x); \end{aligned} \tag{2.1}$$

and in general:

$$\begin{aligned} &\mathbf{P}(\mathcal{X}_{n+1} \in A | \mathcal{X}_n = x) = \mathbb{P}(x, A); \\ &\mathbf{P}(\mathcal{X}_{n+1} \in A) = \underbrace{\int_X \cdots \int_X}_{(n+1)-\text{times}} d\lambda(x_0)\mathbb{P}(x_0, dx_1)\mathbb{P}(x_1, dx_2) \ldots \mathbb{P}(x_{n-1}, dx_n)\mathbb{P}(x_n, A). \end{aligned} \tag{2.2}$$

The n-step transition probability function $\mathbb{P}^n$ is

$$\begin{aligned}\mathbb{P}^n(x,A) &= \mathbf{P}\{\mathcal{X}_n \in A | \mathcal{X}_0 = x\}\\ &= \underbrace{\int_X \cdots \int_X}_{(n-1)-\text{times}} \mathbb{P}(x_0, dx_1)\mathbb{P}(x_1, dx_2)\dots\mathbb{P}(x_{n-2}, dx_{n-1})\mathbb{P}(x_{n-1}, A).\end{aligned} \tag{2.3}$$

Equivalently, Markov process can be understood as a measure on the product space $X_+ = X^{\mathbb{N}\cup\{0\}}$ given by:

$$\mathbf{P}(A_0 \times A_1 \times \cdots \times A_n) = \mathbf{P}(\mathcal{X}_0 \in A_0, \mathcal{X}_1 \in A_1, \dots, \mathcal{X}_n \in A_n),$$

for all $n \geq 1$ and $A_0, A_1, \dots, A_n \in \mathfrak{B}$.

The following two notions will be crucial in our application.

Definition 2.2 *A measure μ on $\mathfrak{B}$ is called a stationary (or invariant) probability measure for a Markov process with transition probability function $\mathbb{P}$ if*

$$\mu(A) = \int_X d\mu(x)\mathbb{P}(x, A), \tag{2.4}$$

for all $A \in \mathcal{B}$. Then, obviously $\mu(A) = \int_X d\mu(x)\mathbb{P}^n(x, A)$.

Definition 2.3 *A Markov process with stationary density μ is ergodic if and only if, when $\mathbb{P}(x, A) = 1$ for all $x \in A$ implies $\mu(A) = 0$ or $\mu(A^c) = 0$.*

Definition 2.4 *A set $A \in \mathcal{B}$ is ergodic with respect to a stationary measure μ of a Markov process if and only if $\mathbb{P}(x, A) = 1$ for all $x \in A$ implies $\mu(A) = 0$ or $\mu(A^c) = 0$.*

Now, we state a theorem for Markov processes concerning its stationary distribution. It is taken from Doob (1953) and it requires the following hypothesis.

Hypothesis: There is a finite measure ϕ on $\mathcal{B}$ with $\phi(X) > 0$, an integer $j \geq 1$, and a positive ε such that

$$\phi(A) \leq \varepsilon \quad \Longrightarrow \mathbb{P}^{(j)}(x, A) \leq 1 - \varepsilon,$$

for all $x \in X$.

Theorem 2.5 *If the hypothesis holds, then the limit*

$$\mu(A) = \lim_{n\to\infty} \mathbb{P}^n(x, A),$$

exists for any $A \in \mathcal{B}$ and is independent of the initial point x. The measure μ is a stationary measure for the Markov process.

Moreover, for any $\mathcal{B}$-measurable function f with

$$\boldsymbol{E}\{|f(\mathcal{X}_1)|\} = \int_\Omega |f(\xi)|\mu(d\xi) < \infty,$$

the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} f(\mathcal{X}_m)$$

exists with probability one. In particular, under the above hypothesis, if there is only one ergodic set,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} f(\mathcal{X}_m) = \int_X f(\xi)\mu(d\xi)$$

with probability one.

3 Position Dependent Random Maps

In this section we define a position dependent random map as a Markov process and discuss the existence of its stationary density. This is formulated via the Perron-Frobenius operator of the random map.

Let $\tau_k : X \to X$, $k = 1, ..., K$, be piecewise one-to-one, non-singular transformations on a common partition $\mathcal{P}$ of X : $\mathcal{P} = \{I_1, ..., I_q\}$ and let $\tau_{k_i} = \tau_k \mid_{I_i}$, $i = 1, ..., q$, $k = 1, ..., K$ be their restrictions to the sets $I_1, ..., I_q$.

Definition 3.1 *We define the position dependent random map*

$$T = \{\tau_1, \dots \tau_k; p_1(x), \dots p_k(x)\}$$

as Markov process $\{T_n\}_{n\geq 0}$ with transition function

$$\mathbb{P}(x, A) = \sum_{k=1}^{K} p_k(x)\chi_A(\tau_k(x)), \tag{3.1}$$

where A is any measurable set and $\{p_k(x)\}_{k=1}^K$ is a set of position dependent measurable probabilities, i.e., $\sum_{k=1}^K p_k(x) = 1$, $p_k(x) \geq 0$, for any $x \in X$ and χ_A denotes the characteristic function of the set A.

The structure of such defined Markov process $\{T_n\}_{n\geq 0}$ can be understood from its state probabilities.

$$\begin{aligned} \mathbf{P}(T_0 \in A) &= \lambda(A); \\ \mathbf{P}(T_1 = \tau_k(x) |\, T_0 = x) &= \mathbb{P}\left(x, \tau_k(x)\right) = p_k(x)\ , \quad k = 1, ..., K\ ; \end{aligned} \tag{3.2}$$

and in general:

$$\mathbf{P}(T_{n+1} = \tau_k(x) |\, T_n = x) = \mathbb{P}\left(x, \tau_k(x)\right) = p_k(x)\ , \quad k = 1, ..., K\ . \tag{3.3}$$

At each step $n+1$ and given a previous state $T_n = x$, the process T_{n+1} can take K different possible states $\tau_1(x), \ldots \tau_K(x)$ with probabilities $p_1(x), \ldots, p_K(x)$ respectively. Notice that the states as well as the probabilities depend on the previous state x through the functions τ_k and p_k. In other words, they are position dependent.

After n steps and given an initial value $T_0 = x$, the Marvov process can take one of the K^n possible different ordered ways in which K transformations can be iterated n times. We denote by $T^n(x) = \tau_{k_n} \circ \tau_{k_{n-1}} \circ \cdots \circ \tau_{k_1}(x)$ one of such n-order iterations. Then, the n-step transition probability function $\mathbb{P}^n$ of the random map is

$$\begin{aligned} \mathbb{P}^n\left(x, T^n(x)\right) &= \mathbf{P}\{T_n = T^n(x) |\, T_0 = x\} \\ &= p_{k_n}(T^{n-1}(x)) \cdot p_{k_{n-1}}(T^{n-2}(x)) \ldots p_{k_2}(T^1(x)) \cdot p_{k_1}(x). \end{aligned} \tag{3.4}$$

In other words, at each step we are defining $T(x) = \tau_k(x)$ with probability $p_k(x)$. The steps of size n are defined as $T^n(x) = \tau_{k_n} \circ \tau_{k_{n-1}} \circ \cdots \circ \tau_{k_1}(x)$ with probability $p_{k_n}(\tau_{k_{n-1}} \circ \cdots \circ \tau_{k_1}(x)) \cdot p_{k_{n-1}}(\tau_{k_{n-2}} \circ \cdots \circ \tau_{k_1}(x)) \cdots p_{k_1}(x)$.

The transition function $\mathbb{P}$ induces an operator $\mathbb{P}_*$ on measures on $(X, \mathcal{B})$ defined by

$$\begin{aligned} \mathbb{P}_*\mu(A) &= \int \mathbb{P}(x, A) d\mu(x) \\ &= \sum_{k=1}^{K} \int p_k(x) \chi_A(\tau_k(x)) d\mu(x) \\ &= \sum_{k=1}^{K} \sum_{i=1}^{q} \int_{\tau_{k_i}^{-1}(A)} p_k(x) d\mu(x). \end{aligned} \tag{3.5}$$

The standard notion of a measure invariant (stationary) for a Markov process gives the following definition of the T-invariant measure:

Definition 3.2 *T preserves a measure μ if and only if*

$$\mu(A) = \sum_{k=1}^{K} \int_{\tau_k^{-1}(A)} p_k(x) d\mu,$$

for any $A \in \mathcal{B}$.

If μ has density f with respect to λ, the $\mathbb{P}_*\mu$ has also a density which we denote by $P_T f$. By change of variables, we obtain

$$\begin{aligned}\int_A P_T f(x) d\lambda(x) &= \sum_{k=1}^{K} \sum_{i=1}^{q} \int_{\tau_{k_i}^{-1}(A)} p_k(x) f(x) d\lambda(x) \\ &= \sum_{k=1}^{K} \sum_{i=1}^{q} \int_A p_k(\tau_{k_i}^{-1} x) f(\tau_{k_i}^{-1} x) \frac{1}{J_{k_i}(\tau_{k_i}^{-1})} d\lambda(x),\end{aligned} \tag{3.6}$$

where J_{k_i} is the Jacobian of τ_{k_i} with respect to λ. Since this holds for any measurable set A we obtain an a.e. equality:

$$(P_T f)(x) = \sum_{k=1}^{K} \sum_{i=1}^{q} p_k(\tau_{k_i}^{-1} x) f(\tau_{k_i}^{-1} x) \frac{1}{J_{k_i}(\tau_{k_i}^{-1})} \chi_{\tau_k(I_i)}(x) \tag{3.7}$$

or

$$(P_T f)(x) = \sum_{k=1}^{K} P_{\tau_k} (p_k f)(x), \tag{3.8}$$

where P_{τ_k} is the Perron-Frobenius operator corresponding to the transformation τ_k Boyarsky and Góra (1997). We call P_T the Perron-Frobenius operator of the random map T. The properties of P_T resemble the properties of the traditional Perron-Frobenius operator. Obviously, $P_T f^* = f^*$ if and only if $f^*\lambda$ is T-invariant. In particular, $\mu = f^*\lambda$ is unique if and only if f^* is the unique fixed point of P_T.

4 Ergodic Theorem

In this section $X = [a, b]$. To prove that T admits a finite number (at least one) of ergodic absolutely continuous invariant measure on $[a, b]$, it is enough to prove that for any $f \in BV(X)$ there exist an $n \in \mathbb{N}$, and real numbers A, B such that

$$||P_T^n f||_{BV} \leq A||f||_{BV} + B||f||_1, \tag{4.1}$$

where $0 < A < 1$ and $0 < B < \infty$ (See Boyarsky and Góra (1997) for (4.1) and Dunford and Schwartz (1964) for BV).

Following Elton (1978), let

$$\Omega = K^\infty = \{(k_1, k_2, \dots) : 1 \leq k_j \leq K \text{ and } k_j \text{ is an integer for each } j\} .$$

Let $\mathcal{A}$ be the σ-algebra generated by the cylinders in Ω. For each $x \in X$, let P_x be the probability measure on $\mathcal{A}$ defined on cylinders by

$$P_x((k_1, k_2, \dots, k_n)) = p_{k_n}(\tau_{k_{n-1}} \circ \dots \circ \tau_{k_1}(x)) \cdot p_{k_{n-1}}(\tau_{k_{n-2}} \circ \dots \circ \tau_{k_1}(x)) \dots p_{k_1}(x) .$$

This is the probability measure for realizations of the Markov process starting at x. For instance, if we consider a Markov process $\{Z_n, n = 0, 1, \dots\}$ with state space X and transition probability $\mathbb{P}$ as defined above, then

$$P(Z_0, Z_1, \dots) \in B | Z_0 = x) = P_x\{(k_1, k_2, \dots) : (x, \tau_{k_1}(x), \tau_{k_2}(\tau_{k_1}(x)), \dots) \in B\}$$

for any $B \in \mathcal{B}$.

Theorem 4.1 *If μ is T-invariant, μ is absolutely continuous and unique among absolutely continuous invariant measures, P_T satisfies (4.1), then for almost every μ point x with probability* 1*:*

$$\frac{1}{n} \sum_{i=0}^{n-1} f(T^i(x)) \to \mu(f)$$

for any $f \in L^1(X, \mu)$.

Proof

Let $\{Z_n\}$ be the Markov process with transition probability $\mathbb{P}$ such that Z_0 has distribution μ. Then the process is stationary since μ is an invariant measure and it is ergodic by (4.1). Let $f \in L^1(X, \lambda)$. Define

$$\Lambda = \{(x_0, x_1, \dots) \in X^\infty : \frac{1}{n} \sum_{j=0}^{n-1} f(x_j) \to \int f d\mu\}.$$

By Theorem 2.5,

$$P((Z_0, Z_1, \dots) \in \Lambda) = 1. \tag{4.2}$$

Observe that

$$\begin{aligned} P((Z_0, Z_1, \dots) \in \Lambda) &= \int P((Z_0, Z_1, \dots) \in \Lambda | Z_0 = x) d\mu(x) \\ &= \int P_x((k_1, k_2, \dots) : (x, \tau_{k_1}(x), \tau_{k_2}(\tau_{k_1}(x)), \dots) \in \Lambda) d\mu(x). \end{aligned} \tag{4.3}$$

Then by (4.2) and (4.3) we have

$$P_{x_0}((k_1, k_2, \dots) : (x_0, \tau_{k_1}(x_0), \tau_{k_2}(\tau_{k_1}(x_0)), \dots) \in \Lambda)) = 1$$

for some $x_0 \in X$.

Let $H = \{(k_1, k_2, \dots) : x_0, \tau_{k_2}(\tau_{k_1}(x_0)), \dots) \in \Lambda)\}$. Thus, $P_{x_0}(H) = 1$ and for $(k_1, k_2, \dots) \in H$,

$$\frac{1}{n} \sum_{j=0}^{n-1} f(\tau_{k_j} \circ \dots \circ \tau_{k_1}(x_0)) \to \int f d\mu.$$

Thus, for almost every μ point x with probability 1:

$$\frac{1}{n} \sum_{i=0}^{n-1} f(T^i(x)) \to \mu(f)$$

for any $f \in L^1(X, \mu)$.

■

5 Generalized Binomial Models

In a now classical paper, Cox, Ross and Rubinstein (1979) proposed a binomial model for asset prices that has played an important role in modern mathematical finance. The binomial model is a simple yet very important model for the price of a single risky security, it is easy to implement and it can be use to price options in a very straigth-forward manner. It has been largely studied in the context of option pricing [He (1990), Hubalek and Schachermayer (1998), Rachev and Ruschendorf (1997), Jackwerth (1999) and references therein]. Its simple structure makes it suitable to approximate option prices when other methods are not available. In this section, we present a generalized binomial model that has not been explored before in the literature. We use the concept of random maps to introduce position dependent jumps and probabilities. The generalization discussed in this section stems from our previous discussion on random maps. At each step, our binomial model will branch out to new states, that depend on the current position, with probabilities that also depend on the current position. In other words, the probability of our price going down (or up) in the next period, is price dependent. This also applies to the price changes, prices decrease (or increase) at each period at different rates that depend on the current price. This is an interesting feature, one would expect that as information becomes available the sizes and chances of certain up-or down-movements change from one instant to another. As an extra feature, this model has asymptotic properties that could be of interest in financial applications.

The classical binomial model studies one risky security price s_1. At each period there are two possibilities: the security price may go up by a factor u or it may go down by a factor d; i.e., $s_1(n) = u \cdot s_1(n-1)$ or $s_1(n) = d \cdot s_1(n-1)$; $n = 1, 2, \dots$ is the time. The probability of an up move during a period is equal to the parameter $\overline{p}_u$, and the probability of going down is $\overline{p}_d = 1 - \overline{p}_u$. The random maps discussed in previous sections lead to a natural extension of this model.

We define a multiperiod multinomial model as follows:

1. $t+1$ trading dates: $n = 0, 1, \dots, t$, $\mathbb{T} = \{0, 1, \dots, t\}$, where the trading horizon t is treated as the terminal date of the economic activity being modeled.

2. A finite probability space Ω with $K < \infty$ elements:
$$\Omega = \{\omega_1, \omega_2, \dots, \omega_K\}.$$

3. A probability measure $\overline{P}$ on Ω with a $\overline{P}(\omega) > 0$ for all $\omega \in \Omega$.

4. A bank account (or riskless asset) process $B = \{B_n; n = 0, \dots, t\}$, where B is a stochastic process with $B_0 = 1$, $B_n(\omega) > 0$ for all n and B_n is the value of the bank account at time n. The quantity $r_n \equiv \frac{B_n - B_{n-1}}{B_{n-1}} \geq 0$, $n = 1, \dots, t$ is the interest rate in the interval $(n-1, n)$. We suppose that the interest rate is constant over time and in some situations, without loss of generality, we suppose it is equal to 1.

5. A risky security process $s = (s_1(n), \ldots, s_L(n))$, $n = 0, 1, \ldots, t$, where s_l is a Markov process for $l = 1, \ldots, L$. $s_l(n)$ is the price of the risky security l at time n. For example, s_l is the price of one share of common stock of a particular corporation. In our discussion, we deal with $L = 1$.

6. Let $\mathbb{F} = \{\mathcal{F}_n; n = 0, \ldots, t\}$ be a filtration defined on $[0, 1]$, with the Lebesgue measurable sets $\mathcal{B}$, where $\mathcal{F}_n$ is the smallest sub-σ-algebra generated by
$$(s_1(0), \ldots, s_1(n)).$$
The measure we consider is the invariant measure for the transition function of Markov process s_1.

We assume that the price of the s_l risky security is an adapted, i.e. $s_1(n)$ is $\mathcal{F}_n$ measurable, stochastic process. Thus, the investors will have full knowledge of the past and present prices. For instance , at time n $s_l(n)$ will be known.

The prices of the securities are assumed to be smaller than a finite number (see Remark 5.1); i.e., the prices have an upper bound $M \in \mathbb{R}$, $0 < M < \infty$, such that $0 < s_l(n) < M$. We normalize the prices over M so that
$$0 < s_l(n) < 1$$
for $1 \leq l \leq L$ and $n = 0, \ldots, t$.

Remark 5.1 *Discrete time models are used to estimate continuous times models over a finite period of time. Thus, the above assumption is natural.*

Without loss of generality, we focus our attention on the one-asset binomial model. Now, we start to depart from the classical approach, we assume that the factors u and d are functions of the prices, $u(x) : (0, 1) \to (1, \infty)$ and $d(x) : (0, 1) \to (0, 1)$; i.e., at time n, u and d depend on the price of the risky security s_1 at time $n - 1$. The examples of u and d are: u and d are constant over subsets of $(0, 1)$; u and d are piecewise linear or piecewise non-linear over $(0, 1)$. Similarly, the probabilities $\overline{p}_u$ and $\overline{p}_d$ can be constant or price dependent. Price dependent probabilities are more general and, perhaps, more realistic. One can argue that the probability of an actual asset price going up or down in a trading market is not constant in time and may depend on current price. This could be explain by the fact that, as market prices unfold, certain up- or down-movements become more likely than others.

Another interesting feature of this model is that the functions u, d, and the probabilities $\overline{p}_u$ and $\overline{p}_d$ can be explicitely obtained from any stationary density we specify for the model. In other words, if we know (or assume) the stationary density that our asset price should have, then we can recuperate the correct functions and probabilities. This feature makes our model somehow similar to those ergodic diffusion processes proposed in the literature as asset models [see Rydberg (1999)]. This is discussed in Section 7.

Given the functions $u(x)$, $d(x)$ and the probabilities $\overline{p}_u$ and $\overline{p}_d$ at time $n = 0$, we can construct the random map T which consists of the transformations τ_u, τ_d and the position dependent probabilities p_u and p_d. The subscript u for τ_u illustrates that the transformation τ_u is the law which moves the price up and the subscript d for τ_d illustrates that the transformation τ_d is the law which moves the price down. The construction of the random map T is straight forward. At time $n+1$, consider the up price to be $\tau_u(s_1(n))$ and the down price to be $\tau_d(s_1(n))$. Also $s_1(n+1) = u(s_1(n)) \cdot s_1(n)$ or $s_1(n) = d(s_1(n)) \cdot s_1(n)$. Therefore, the transformations τ_u and τ_d are given by the following formulas:

$$\tau_u(x) = u(x) \cdot x \qquad \text{and} \qquad \tau_d(x) = d(x) \cdot x. \tag{5.1}$$

Moreover, we extend τ_u and τ_d from (0,1) to the closed interval $[0, 1]$ continuously. For the probabilities, we assume $\overline{p}_u = p_u$ and $\overline{p}_d = p_d$.

We now give a first example to help illustrate the structure of our model. One interesting property of our construction is that the functions u and d are defined piece-wise. This could be used to mimic features actually observed in asset prices like the negative correlation between stock returns and volatility. For instance, in our toy example we choose the functions u and d, which are the derivatives of τ_u and τ_d, in such a way that when the price increases the variability decreases and viceversa. The full consequence of this choice can be observed in Figure 1 where the stationary density is depicted. Recall that prices are set to lie in the interval $[0, 1]$. In Figure 2 we show one possible trajectory of the binomial tree in this example.

Example 5.2 *Suppose $u(x), d(x), \overline{p}_u$ and $\overline{p}_d$ are given:*

$$u(x) = \begin{cases} 2, & 0 < x < \frac{1}{2} \\ \frac{5}{4} + \frac{1}{10x}, & \frac{1}{2} \le x \le \frac{2}{3} \\ \frac{3}{4} + \frac{1}{4x}, & \frac{2}{3} < x < 1 \end{cases}, \tag{5.2}$$

$$d(x) = \begin{cases} \frac{1}{2}, & 0 < x < \frac{1}{2} \\ \frac{3}{4} - \frac{1}{8x}, & \frac{1}{2} \leq x \leq \frac{2}{3} \\ \frac{3}{2} - \frac{1}{2x}, & \frac{2}{3} < x < 1 \end{cases}, \tag{5.3}$$

and

$$\overline{p}_u(x) = \begin{cases} 0.8, & 0 \leq x < \frac{1}{2} \\ 0.725, & \frac{1}{2} \leq x \leq \frac{2}{3} \\ 0.4, & \frac{2}{3} < x \leq 1 \end{cases}, \tag{5.4}$$

$$\overline{p}_d(x) = 1 - p_u(x).$$

Observe that $u(x) \geq 1$ *and* $d(x) \leq 1$*. From* $u(x), d(x), \overline{p}_u(x)$ *and* $\overline{p}_d(x)$*, we construct a random map* $T = \{\tau_u(x), \tau_d(x); p_u(x), p_d(x)\}$*,*

$$\tau_u(x) = \begin{cases} 2x, & 0 \leq x < \frac{1}{2} \\ \frac{5}{4}x + \frac{1}{10}, & \frac{1}{2} \leq x \leq \frac{2}{3} \\ \frac{3}{4}x + \frac{1}{4}, & \frac{2}{3} < x \leq 1 \end{cases}, \tag{5.5}$$

$$\tau_d(x) = \begin{cases} \frac{1}{2}x, & 0 \leq x < \frac{1}{2} \\ \frac{3}{4}x - \frac{1}{8}, & \frac{1}{2} \leq x \leq \frac{2}{3} \\ \frac{3}{2}x - \frac{1}{2}, & \frac{2}{3} < x \leq 1 \end{cases}, \tag{5.6}$$

$p_u(x) = \overline{p}_u(x)$ *and* $p_d(x) = \overline{p}_d(x)$*. For example, if the price of the risky security at time* $n = 0$ *is* 0.25*, then the orbit of the price at times* $n = 1, 2$ *is given by:*

$$\begin{array}{cccccc} & & & & \nearrow & \xrightarrow{p_u=0.725, u=1.45} \quad \tau_u(0.5) = 0.725 \\ & \nearrow & \xrightarrow{p_u=0.8, u=2} \quad \tau_u(0.25) = 0.5 & & & \\ & & & & \searrow & \xrightarrow{p_d=0.275, d=0.5} \quad \tau_d(0.5) = 0.25 \\ s_1(0) = 0.25 & & & & & \\ & & & & \nearrow & \xrightarrow{p_u=0.8, u=2} \quad \tau_u(0.125) = 0.25 \\ & \searrow & \xrightarrow{p_d=0.2, d=0.5} \quad \tau_d(0.25) = 0.125 & & & \\ & & & & \searrow & \xrightarrow{p_d=0.2, d=0.5} \quad \tau_d(0.125) = 0.0625 \end{array} \tag{5.7}$$

Notice that for each starting value we have one binomial tree as the one showed in (5.7). Given a starting value x, this tree describes all the

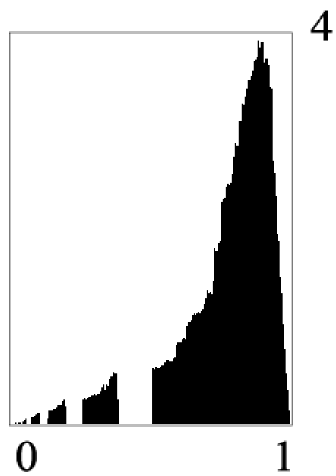


Figure 1

The invariant density of T in Example 5.2, histogram after 2 000 000 iterations.

possible paths the asset price might take in very much the same way as the classical binomial model does. Unlike the classical model, these trees expand or contract as the starting value changes (the structure of the tree remains unchanged for all starting values in the classical binomial model). The classical model can be obtained by setting all functions u, d and p to be constant.

What makes this model interesting is the fact that it accepts a stationary density. The following theorem is the existence theorem of Bahsoun and Góra (2005):

Theorem 5.3 *Let T be as above. If $\sum_{k=1}^{K} \frac{p_k(x)}{|\tau_k'|} \leq \alpha < 1$, and $\frac{p_k(x)}{|\tau_k'|} \in BV(I)$ ($BV(I)$ is the space of functions of bounded variation, see Dunford and Schwartz (1964) for detalis) then T admits a finite number (at least one) of ergodic absolutely continuous invariant measure.*

Remark 5.4 *The condition $\sum_{k=1}^{K} \frac{p_k(x)}{|\tau_k'|} \leq \alpha < 1$ in Theorem 5.3 simply requires that the tree expands in average as it branches out and it does not*

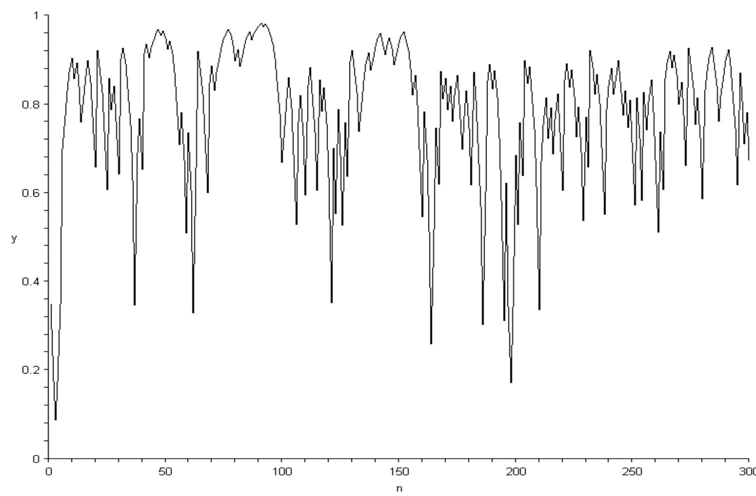


Figure 2

One trajectory of the binomial tree in Example 5.2.

concentrate in a single point. This is a natural condition to ask from a financial binomial tree.

Now, observe that

$$\sup_x \frac{p_u(x)}{|\tau_u'(x)|} + \sup_x \frac{p_d(x)}{|\tau_d'(x)|} = 0.58 + 0.4 = 0.98 < 1.$$

Remark 5.5 *The random map T of Example 5.2 satisfies the assumptions of Theorem 5.3. Thus, it admits an absolutely continuous invariant measure. In Figure 1, the histogram approximating the invariant density of T is shown after 2,000,000 iterations of random map T. The invariant density allows us to find the following probability: $\mu\{x : T(x) \in (\delta_1, \delta_2)\} = \mu(\delta_1, \delta_2)$, where $\mu = f^*\lambda$, f^* is the invariant density. Notice that it is concentrated toward relatively large values but small values are still fairly possible. In other words, our price process will more likely have an upward trend but if and when it goes down, it can do so by a significant percentage.*

One key feature of our model is that it can be constructed from any given discretized density as we will see in the next section.

6 Constructing Binomial Models with a Predetermined Stationary Density

Binomial trees have always played an important role in financial modeling. In the early days, Cox *et al.* (1979) showed how binomial trees provided a simple way of undertsanding the Black-Scholes option pricing model. Later, Rubinstein (1994), Derman and Kani (1994) and Dupire (1994) showed how an implied binomial tree could be extracted from actual option prices. Under certain assumptions, a large set of option prices would contain all information on the stochastic process driving the underlying. Implied binomial trees can be seen as a discrete version of the stochastic process behind the behavior of the underlying asset price. Recovering such a random process from actual option prices seem to be a recurrent topic in the financial literature. Jackwerth and Rubinstein (2001) and Li (2001) are some recent examples.

One important problem in the literature is the search for a model that can explain the relationship among option prices of different strikes and maturities as described by the implied volatility surface. Among all existing models capable of doing that, binomial trees seem to occupy an important place because of their simplicity [empirical studies can be found in Dumas, Fleming and Whaley (1998) and Jackwerth and Rubinstein (2001)]. In this context, the search of an algorithm for constructing an implied binomial is highly desirable. Since our model can be easily constructed from any given density, we believe that it can be of interest.

In the following we assume that we have a suitable density for our asset price and from which we can recover one binomial tree having that density as its stationary law. As we will see, this density could be chosen to be a suitable distribution within a parametric model [Rydberg (1999) for instance]. It can also be chosen to be a risk-neutral probability density obtained from option prices. The problem of extracting risk-neutral densities from option prices has also been a subject of study in the recent years. Jackwerth and Rubinstein (1996), Bahra (1997), Ait-Sahalia and Lo (1998) and Constantinides, Jackwerth and Perrakis (2005) are some examples.

We first address the inverse problem of the Perron-Frobenius operator of position dependent random maps. Then we introduce the notion of $\aleph$-band random map and explain how these are related to our problem. Using these matrices we produce a computer program which approximates the binomial model of a given probability density function. As discussed before, this is relevant for our model since it will allow us to construct a binomial model (through the specification of the functions τ and p) with a predetermined stationary density.

Let $\mathcal{P} = \{I_1, \ldots, I_N\}$ be a partition of $I = [a, b] = [0, 1]$ into intervals. We put $[a, b] = [0, 1]$ to unify the notation in our exposition.

Definition 6.1 *A transformation $\tau : I \to I$ is called $\mathcal{P}$-Markov if, for any $i = 1, \ldots, N$, $\tau|_{I_i}$ is monotonic and $\tau(I_i)$ is a union of intervals of $\mathcal{P}$.*

In the following result we characterize the shape of all possible invariant densities of our random map binomial model. It turns out that the invariant density for our model are piecewise constant functions. This is a nice feature since it will allow us to easily approximate any continuous density with the invariant density of a random map binomial model.

Theorem 6.2 *Let $T = \{\tau_1, \ldots, \tau_K; p_1(x), \ldots, p_K(x)\}$. Suppose that τ_k is $\mathcal{P}$-Markov and piecewise linear, p_k is piecewise constant over $\mathcal{P}$; $k = 1 \ldots K$. We also assume that*

$$\sum_{k=1}^{K} \sup_x \frac{p_k(x)}{|\tau_k'(x)|} \leq \alpha < 1.$$

Then any T-invariant density is constant on intervals of $\mathcal{P}$.

Proof

By the existence result of Bahsoun and Góra (2005), there exists a T-invariant density f, i.e., $P_T f = f$, and it is of bounded variation on I. Moreover,

$$P_T f(x) = \sum_{k_1=1}^{K} \sum_{i=1}^{N} f(\tau_{k_1,i}^{-1}(x)) \frac{p_k(\tau_{k_1,i}^{-1}(x))}{\mid \tau_{k_1,i}{}' \mid} \chi_{\tau_{k_1,i}(I_i)}(x) = f(x).$$

Note that $|\tau_{k_1,i}'|$ and p_{k_1} are constants on I_i, $p_{k_1,i} = p_{k_1|I_i}$, and that f is identically zero outside the range of τ_{k_1}, $k_1 = 1, \ldots, K$. Let $I_q \in \mathcal{P}$. Then $I_q \subset \tau_{k_1}(I_i)$, with i depending on k_1. Let $x, y \in I_q$ be distinct points. Then $\chi_{\tau_{k_1,i}(I_i)}(x) = \chi_{\tau_{k_1,i}(I_i)}(y)$ for all i. Thus,

$$
\begin{aligned}
f(x)-f(y) &= P_T f(x) - P_T f(y) \\
&= \sum_{k_1=1}^{K}\sum_{i=1}^{N}\frac{p_{k_1,i}}{\mid \tau'_{k_1,i}\mid}[f(\tau^{-1}_{k_1,i}(x)) - f(\tau^{-1}_{k_1,i}(y))]\chi_{\tau_{k_1,i}(I_i)}(x) \\
&= \sum_{k_1=1}^{K}\sum_{i_1(k_1)}\frac{p_{k_1,i_1(k_1)}}{\mid \tau'_{k_1,i_1(k_1)}\mid}[f(\tau^{-1}_{k_1,i_1(k_1)}(x)) - f(\tau^{-1}_{k_1,i_1(k_1)}(y))] \\
&\le \sum_{k_1=1}^{K}\sup_x \frac{p_{k_1}}{\mid \tau'_{k_1}\mid}\sum_{i_1(k_1)}[f(\tau^{-1}_{k_1,i_1(k_1)}(x)) - f(\tau^{-1}_{k_1,i_1(k_1)}(y))],
\end{aligned}
\tag{6.1}
$$

where, for notational convenience, we let the index $i_1(k_1)$ run through all the integers $i \in \{1,\dots,N\}$ such that $x \in \tau_{k_1,i}(I_i)$. Similarly, for each $i_1(k_1)$,

$$
\begin{aligned}
f(\tau^{-1}_{k_1,i_1(k_1)}(x)) - f(\tau^{-1}_{k_1,i_1(k_1)}(y)) = \sum_{k_2=1}^{K}\sum_{i_2(k_2)}\frac{p_{k_2,i_2(k_2)}}{\mid \tau'_{k_2,i_2(k_2)}\mid}\cdot \\
[f(\tau^{-1}_{k_2,i_2(k_2)}\tau^{-1}_{k_1,i_1(k_1)}(x)) - f(\tau^{-1}_{k_2,i_2(k_2)}\tau^{-1}_{k_1,i_1(k_1)}(y))],
\end{aligned}
\tag{6.2}
$$

and so on. Therefore,

$$
\begin{aligned}
\mid f(x)-f(y)\mid &\le \sum_{k_1=1}^{K}\sup_x \frac{p_{k_1}}{\mid \tau'_{k_1}\mid}\sum_{i_1(k_1)}[f(\tau^{-1}_{k_1,i_1(k_1)}(x)) - f(\tau^{-1}_{k_1,i_1(k_1)}(y))] \\
&\le \sum_{k_1=1}^{K}\sum_{k_2=1}^{K}\sup_x \frac{p_{k_1}}{\mid \tau'_{k_1}\mid}\sup_x \frac{p_{k_2}}{\mid \tau'_{k_2}\mid}\sum_{i_1(k_1)}\sum_{i_2(k_2)}[f(\tau^{-1}_{k_2,i_2(k_2)}\tau^{-1}_{k_1,i_1(k_1)}(x)) \\
&\qquad - f(\tau^{-1}_{k_2,i_2(k_2)}\tau^{-1}_{k_1,i_1(k_1)}(y))] \\
&\vdots \\
&\le \sum_{k_1=1}^{K}\cdots\sum_{k_n=1}^{K}\sup_x \frac{p_{k_1}}{\mid \tau'_{k_1}\mid}\cdots\sup_x \frac{p_{k_n}}{\mid \tau'_{k_n}\mid}\sum_{i_1(k_1)}\cdots\sum_{i_n(k_n)}[f(\tau^{-1}_{k_n,i_n(k_n)}\cdots\tau^{-1}_{k_1,i_1(k_1)}(x)) \\
&\qquad - f(\tau^{-1}_{k_n,i_n(k_n)}\cdots\tau^{-1}_{k_1,i_1(k_1)}(y))].
\end{aligned}
\tag{6.3}
$$

Now, since all τ_k's are piecewise monotonic on the same partition, for each fixed sequence $(k_1, \ldots, k_n)$

$$\{\tau^{-1}_{k_n,i_n(k_n)} \cdots \tau^{-1}_{k_1,i_1(k_1)}(x), \tau^{-1}_{k_n,i_n(k_n)} \cdots \tau^{-1}_{k_1,i_1(k_1)}(y)\}$$

is a finite collection of at most N^n non-overlapping intervals for each n. Therefore, each multiple sum over $i_n(k_n), \ldots, i_1(k_1)$ in (6.3) is bounded above by variation of f, $V_I f$. Thus,

$$| f(x) - f(y) | \leq \sum_{k_1=1}^{K} \cdots \sum_{k_n=1}^{K} \sup_x \frac{p_{k_1}}{| \tau'_{k_1} |} \cdots \sup_x \frac{p_{k_n}}{| \tau'_{k_n} |} V_I f \leq \alpha^n \cdot V_I f < \varepsilon \tag{6.4}$$

for n large enough. Therefore, $f(x) = f(y)$, and f is constant on I_q. ■

Now we define the class of $\mathcal{P}$–semi-Markov transformations.

Definition 6.3 *A transformation $\tau : I \to I$ is called $\mathcal{P}$–semi-Markov if there exist disjoint intervals $Q_j^{(i)}$, such that for any $i = 1, \ldots, N$, we have $I_i = \cup_{j=1}^{q(i)} Q_j^{(i)}$, $\tau|_{Q_j^{(i)}}$ is monotonic and $\tau(Q_j^{(i)}) \in \mathcal{P}$.*

It is easy to see that any $\mathcal{P}$-Markov transformation is $\mathcal{P}$-semi-Markov and that there exist $\mathcal{P}$-semi-Markov transformations that are not $\mathcal{P}$-Markov.

The following theorem generalizes Theorem 6.2

Theorem 6.4 *Let $T = \{\tau_1, \ldots, \tau_K; p_1(x), \ldots, p_K(x)\}$. Suppose that τ_k is $\mathcal{P}$-semi-Markov and piecewise linear on $Q_j^{(i)}$, such that for any $i = 1, \ldots, N$, p_k is piecewise constant over $\mathcal{P}$; $k = 1 \ldots K$. We also assume that*

$$\sum_{k=1}^{K} \sup_x \frac{p_k(x)}{|\tau'_k(x)|} \leq \alpha < 1.$$

Then any T-invariant density is constant on intervals of $\mathcal{P}$.

Proof

It is easy to see that τ_k's are Q-Markov, where $Q = \{Q_j^{(i)} : 1 \leq j \leq q(i),\ 1 \leq i \leq N\}$. Let f be a T-invariant density. By Theorem 6.2 , f is constant on intervals $Q_j^{(i)}$. Let $f_j^{(i)}$ be the value of f on $Q_j^{(i)}$.

Let us fix $1 \leq i_0 \leq N$, and let $1 \leq j_1, j_2 \leq r(i_0)$.

The Frobenius–Perron equations for a τ–invariant density yield

$$f_{j_1}^{(i_0)} = \sum_{k=1}^{K} \sum_{(i,j)} p_k((\tau_{k,j}^{(i)})^{-1}) |(\tau_{k,j}^{(i)})'|^{-1} f_j^{(i)},$$

$$f_{j_2}^{(i_0)} = \sum_{k=1}^{K} \sum_{(i,j)} p_k((\tau_{k,j}^{(i)})^{-1}) |(\tau_{k,j}^{(i)})'|^{-1} f_j^{(i)},$$

where $\tau_{k,j}^{(i)} = \tau_k|_{Q_j^{(i)}}$, and the sums are over all pairs (i,j) such that $\tau_k(Q_j^{(i)}) = P_{i_0}$. Since both sums on the right-hand side of the equations are equal, $f_{j_1}^{(i_0)} = f_{j_2}^{(i_0)}$.
■

Now we define a Frobenius–Perron matrix associated with a random map. As we mentioned before, this will allow us to estimate the stationary density for our binomial model and later to solve the inverse problem of finding a random map with a predetermined stationary density.

Definition 6.5 *Let τ_k be a $\mathcal{P}$-semi-Markov piecewise linear transformation. We define the Frobenius–Perron matrix associated with τ_k by $M_{\tau_k} = (a_{ij}^k)_{1\le i,j\le N}$, where*

$$a_{ij}^k = \begin{cases} |(\tau_{k,q}^{(i)})'|^{-1} & \text{if } \tau(Q_q^{(i)}) = I_j \\ 0 & \text{otherwise} \end{cases} . \tag{6.5}$$

M_{τ_k} can be identified with the Frobenius–Perron operator P_{τ_k} of τ_k, restricted to the space of functions constant on intervals of $\mathcal{P}$. The Frobenius–Perron operator P_T of T is then represented by

$$M_T = \sum_{k=1}^{K} \Pi_k M_{\tau_k}, \tag{6.6}$$

where Π_k is the diagonal matrix of $p_k(x)$.

Proposition 6.6 *Let $\mathbb{P}$ be an $N\times N$ stochastic matrix. Let $\mathcal{R}$ be a partition of $I=[a,b]$ into N equal intervals. Then there exists a random map T whose transformations are $\mathcal{R}$-semi-Markov transformations and their associated probabilities are piecewise constant over $\mathcal{R}$, and such that $M_T = \mathbb{P}$.*

Proof

Let $\mathbb{P} = (\pi_{ij})_{1\le i,j\le N}$. Let $e_0^{(i)} = a + \frac{i-1}{N}(b-a)$, and let $R_i = [e_0^{(i)}, e_0^{(i+1)}]$, $i = 1, \dots, N$. Fix $1 \le i \le N$. We will construct $\tau_k|_{R_i}$. Let $\pi_{ij_1}, \dots, \pi_{ij_q} > 0$, $\pi_{ij_1} + \cdots + \pi_{ij_q} = 1$, with

$$\pi_{ij_1} + \cdots + \pi_{ij_q} = p_{1,i}(a^1_{ij_1} + \cdots + a^1_{ij_q}) + \cdots + p_{K,i}(a^K_{ij_1} + \cdots + a^K_{ij_q}),$$

where a^k_{ij} and $p_{k,i}$ are as defined above. Let

$$e_s^{(i)} = a + \frac{(i-1) + \pi_{ij_1} + \cdots + \pi_{ij_s}}{N}(b-a)$$

for $s = 1, \dots, q$. We define $Q_s^{(i)} = [e_{s-1}^{(i)}, e_s^{(i)}]$ and $\tau_k|_{Q_s^{(i)}(x)} = \frac{1}{a^k_{ij_s}}(x - e_{s-1}^{(i)}) + e_0^{(j_s)}$. It easy to see that τ_k is an $\mathcal{R}$-semi-Markov, piecewise linear, the random map T is expanding on average, and that $M_T = \mathbb{P}$.

■

6.1 $\aleph$-Band Matrices

Definition 6.7 *An $\mathcal{R}$-semi-Markov piecewise linear transformation is said to be a $\aleph$-band transformation, $\aleph = 2s+1$, $s \le N-1$, if its Frobenius–Perron matrix $M_\tau = (p_{ij})$ satisfies the condition: $p_{ij} = 0$ if $|i-j| > s$, $1 \le i,j \le N$. We call a position dependent random map $\aleph$-band random map if its transformations are $\aleph$-band and the probabilities associated with them are piecewise constant on $\mathcal{R}$.*

Having characterized random maps in terms of its associated Frobenius–Perron matrix we now need the concept of $\aleph$-band transformation in order to construct a random map with a stationary density described by a vector f. The following theorem and the algorithm immediately below describe how this construction can be achieved.

Theorem 6.8 *Let T be a $\aleph$-band transformation on an N-element uniform partition $\mathcal{R}$, $\aleph = 2s+1$, with Frobenius–Perron matrix $M_\tau = (\pi_{ij})$.*

Let $f = (f_1, \dots, f_N)$ be any probabilistic vector with $f_i > 0$, $i = 1, \dots, N$. If

$$f_i \pi_{i,j} = f_j \pi_{j,i}, \tag{6.7}$$

for any $1 \le i,j \le N$, then the density corresponding to the vector f is T- invariant.

Proof

It is enough to show that $fM = f$, or

$$f_1\pi_{1,j} + f_2\pi_{2,j} + \cdots + f_N\pi_{N,j} = f_j,$$

$1 \leq j \leq N$. By equalities (6.7) this is equivalent to

$$f_j\pi_{j,1} + f_j\pi_{j,2} + \cdots + f_j\pi_{j,N} = f_j,$$

which holds by stochasticity of matrix M.

■

Below we present one of many possible constructions of an $\aleph - band$ matrix, $\aleph = 2s + 1$, $s \leq N - 1$, preserving a given vector $f = (f_1, \ldots, f_N)$ with $f_i > 0$, $i = 1, \ldots, N$.

Let $\aleph = 2s+1$, $s \leq N-1$. Let us fix s nonnegative constants $c_1, c_2, \ldots, c_s$ such that $c_1 + c_2 + \cdots + c_s \leq 1$ and other s constants $d_1, d_2, \ldots, d_s$ such that $0 < d_i < 1$, $1 \leq i \leq s$. In the whole construction, all elements with indices larger than N should be ignored.

We start by the construction of the first row and the first column of the matrix.

- If $c_1/(f_2/f_1) < 1$, then we set $\pi_{1,2} = c_1$, else we set $\pi_{1,2} = d_1(f_2/f_1)$. Note that in either case $\pi_{1,2} \leq c_1$.
- For each $1 \leq i \leq s$, if $c_i/(f_{1+i}/f_1) < 1$, then we set $\pi_{1,1+i} = c_i$, else we set $\pi_{1,1+i} = d_i(f_{1+i}/f_1)$. Note that we always have that $\pi_{1,1+i} \leq c_i$.
- Now, we define $\pi_{1,1} = 1-(\pi_{1,2}+\cdots+\pi_{1,1+s})$ and $\pi_{1+i,1} = \pi_{1,1+i}/(f_{1+i}/f_1)$, $i = 1, \ldots, s$. Note that $0 \leq \pi_{1+i,1} \leq 1$, $i = 0, \ldots, s$.
- We set $\pi_{1,j} = 0$ and $\pi_{j,1} = 0$ for $j > 1 + s$.

Now, we construct the second row and the second column of the matrix.

- The element $\pi_{2,1}$ has already been defined.
- As for the elements $\pi_{2,2+i}$ for $i = 1, 2, \ldots, s - 1$:

  If $c_i(1 - \pi_{2,1})/(f_{2+i}/f_2) < 1 - \pi_{2+i,1}$, then we set $\pi_{2,2+i} = c_i(1 - \pi_{2,1})$, else we set $\pi_{2,2+i} = d_i(f_{2+i}/f_2)(1 - \pi_{2+i,1})$. Note that $\pi_{2,2+i} \leq c_i(1 - \pi_{2,1})$ for considered i's.
- Now, we define $\pi_{2,2+s}$: If $c_s(1 - \pi_{2,1})/(f_{2+s}/f_2) < 1$, then we set $\pi_{2,2+s} = c_s(1 - \pi_{2,1})$, else $\pi_{2,2+s} = d_s(f_{2+s}/f_2)$. Again, $\pi_{2,2+s} \leq c_s(1 - \pi_{2,1})$.

- Now, we define $\pi_{2,2} = 1 - (\pi_{2,1} + \pi_{2,3} + \cdots + \pi_{2,2+s})$ and $\pi_{2+i,2} = \pi_{2,2+i}/(f_{2+i}/f_2)$, $i = 1, \ldots, s$. Note that again $0 \leq \pi_{2+i,2} \leq 1$, $i = 0, \ldots, s$.

- We set $\pi_{2,j} = 0$ and $\pi_{j,2} = 0$ for $j > 2 + s$.

Let us assume that the rows and the columns with indices $\leq k-1$ have been defined.

Now we construct the kth row and the kth column of the matrix.

- The elements $\pi_{k,j}$ has already been defined for $j < k$.

- As for the elements $\pi_{k,k+i}$ for $i = 1, 2, \ldots, s$:

  If $c_i(1 - \sum_{j=1}^{k-1} \pi_{k,j})/(f_{k+i}/f_k) < 1 - \sum_{j=1}^{k-1} \pi_{k+i,j}$, then we set $\pi_{k,k+i} = c_i(1 - \sum_{j=1}^{k-1} \pi_{k,j})$. Else, we set $\pi_{k,k+i} = d_i(f_{k+i}/f_k)(1 - \sum_{j=1}^{k-1} \pi_{k+i,j})$. Note that $\pi_{k,k+i} \leq c_i(1 - \sum_{j=1}^{k-1} \pi_{k,j})$ for $i = 1, 2, \ldots, s$.

- Now, we define $\pi_{k,k} = 1 - \sum_{\substack{j=1 \\ j \neq k}}^{k+s} \pi_{k,j}$ and $\pi_{k+i,k} = \pi_{k,k+i}/(f_{k+i}/f_k)$, $i = 1, \ldots, s$. Note that again $0 \leq \pi_{k+i,k} \leq 1$, $i = 0, \ldots, s$.

- We set $\pi_{k,j} = 0$ and $\pi_{j,k} = 0$ for $j > k + s$.

This construction in N steps creates an $\aleph$-band probabilistic matrix satisfying conditions (6.7). In other words, given a piecewise density f, this algorithm gives a way to construct an $\aleph$-band probabilistic matrix that preserves f. In the following theorem we link this up to a random map binomial model.

Theorem 6.9 *Let $f = (f_1, \ldots, f_N)$ be a piecewise constant density on a partition $\mathcal{R}$ of $I = [a, b]$ into N equal intervals. Then there exists an $\aleph$-band random map $T = \{\tau_u, \tau_d; p_u(x); p_d(x)\}$ such that*

$$\sup_x \frac{p_u(x)}{|\tau_u'(x)|} + \sup_x \frac{p_d(x)}{|\tau_d'(x)|} \leq \alpha < 1 \tag{6.8}$$

with $\tau_u(x) \geq x$, $\tau_d(x) \leq x$ and f being T-invariant.

Proof

Let $M = (\pi_{i,j})_{1 \leq i,j \leq N}$ be an $\aleph$-band matrix preserving the vector f.

It is enough to construct nonnegative vectors p_u, p_d, $p_d(i) = 1 - p_u(i)$, $i = 1, \dots, N$, upper triangular $\aleph$-band probabilistic matrix M_u and lower triangular $\aleph$-band probabilistic matrix M_d such that

$$M = \mathrm{Diag(p_u)M_u + Diag(p_d)M_d}, \tag{6.9}$$

where Diag(v) is a diagonal matrix with the elements of the vector v on the diagonal.

Let us introduce numbers $m^+(i) = \sum_{j=i+1}^{i+s} \pi_{i,j}$ and $m^-(i) = \sum_{j=1}^{i-1} \pi_{i,j}$, $i = 1, \dots, N$ (the sum over an empty set is equal to 0). Since M is a probabilistic matrix the intervals $[m^+(i), 1 - m^-(i)]$ are all nonempty.

Let us define vector p_u in such a way that

$$p_u(i) \in [m^+(i), 1 - m^-(i)] \tag{6.10}$$

and set $p_d(i) = 1 - p_u(i)$, $i = 1, \dots, N$. Note that $p_d(i) \geq m^-(i)$, for all i's. Now, for any $1 \leq i \leq N$ we define

$$M_u(i,j) = \begin{cases} \pi_{i,j}/p_u(i), & \text{for} \quad j > i\ , \\ \\ 0, & \text{for} \quad j < i\ , \end{cases}$$

and $M_u(i,i) = 1 - \sum_{j \neq i} M_u(i,j)$. Similarly

$$M_d(i,j) = \begin{cases} \pi_{i,j}/p_d(i), & \text{for} \quad j > i\ , \\ \\ 0, & \text{for} \quad j < i\ , \end{cases}$$

and $M_d(i,i) = 1 - \sum_{j \neq i} M_d(i,j)$. In view of conditions (6.10) both matrices are probabilistic. The condition (6.9) holds by construction. The condition (6.8) is satisfied since the matrices M_u and M_d are probabilistic, thus the piecewise linear maps corresponding to them are piecewise expanding on average.

■

Now, Theorem 6.9 allows us to construct a random map with a predetermined stationary density f. Recall that stationary densities for these random map binomial models are piecewise constant so they can be described by a vector f. Using our construction we can produce matrices M, M_u, M_d and the vectors p_u, p_d for any given vector f. In order to produce a random map, we need to extract the functions τ_u and τ_d.

Let $M_u = (a_{ij}),\ i = 1, \dots q;\ j = 1, \dots, q$ be a stochastic matrix representing the Frobenius-Perron operator of a piecewise linear Markov transformation $\tau_u : [0,1] \to [0,1]$ with respect to a partition of disjoint open intervals $I_1, \dots, I_q$. We construct τ_u by using the nonzero entries of M_u. If $a_{ij} \neq 0$ then

$$a_{ij} = |(\tau_{u,i})'|^{-1} \text{and } \tau_u(I_i) = I_j.$$

Of course the above construction does not produce a unique τ_u. However, our algorithm provides a matrix M_u which produces a particular class of transformations

$$\tau_u(x) \geq x \ \ \forall x \in [0,1].$$

Moreover, our algorithm may produce a continuous τ_u.

The construction of τ_d from M_d is done in the same way with the algorithm providing us with a matrix M_d which produces a transformation

$$\tau_d(x) \leq x \ \ \forall x \in [0,1].$$

We provide a Maple program which produces matrices M, M_u, M_d and the vectors p_u, p_d for any given vector f. The program is posted at: http://www.mathstat.concordia.ca/pg/nband.zip

Note that in the above described algorithm we can freely choose the constants $s \leq N-1, c_1, c_2, \dots, c_s$ such that $c_1+c_2+\cdots+c_s \leq 1$ and $d_1, d_2, \dots, d_s$ such that $0 < d_i < 1$, $1 \leq i \leq s$. These constants have a direct effect on the resulting tree. Constant s is the number of adjacent intervals that can be reached from any given position in either direction in one step. The larger it is, the larger the jumps that can be obtained in one step. Constants c's and d's are related to the probabilities of jumping accross intervals from any given position. So, small values of c's and d's imply lower probabilities of reaching a faraway interval in one step. Then, small value s along with small values of c's and d's make for less variable trajectories whereas large values s, c's and d's make for more fluctuation in the trajectories. We have produced a program that allows to empirically see these effects. It is posted at http://www.mathstat.concordia.ca/pg/Economics200s.zip

6.2 The Starting Density

In this section we have assumed as given the initial density from which we build up our model. The choice of such density could be done in a parametric or in a nonparametric way.

Within a parametric framework, in the recent financial literature continuous-time stochastic processes are used to model asset prices. A benchmark model is described in terms of the following stochasitic differential equation

$$dS_t = \mu(S_t, t)\ S_t\ dt + \sigma(S_t, t)\ S_t\ dW_t\ , \qquad t \geqslant 0\ , \tag{6.11}$$

where S_t is the asset price at time t, μ is the instantaneous drift function, σ is the local volatility function and W_t is a Wiener process. Models described by (6.11) can be approximated by a binomial tree [see Nelson and Ramaswamy (1990)]. In view of this, our binomial tree can be used to approximate stationary models of the form (6.11) where the asymptotic stationary density has been predetermined. In this sense, our model is one of many possible discrete processes that have the desired stationary density.

Among recent popular choices distributions, that seem to better capture many features of market rate of returns, we find members of the generalized hyperbolic family (for instance Barndorff-Nielsen (1998), Eberlein and Keller (1995) and Eberlein (2001)). We can envision constructing a random map having approximately a stationary generalized hyperbolic density. We can always discretize the desired density over a finite interval and produce a piecewise constant density f for which Theorem 6.9 applies. The algorithm described in this section would produce a random map with a stationary density that can be arbitrarily close (in the L^1 distance) to a generalized hyperbolic density or to any continuous density for that matter. The resulting tree is, in a stationary sense, an approximation to a generalized hyperbolic diffusion process, or to any stationary process of the form (6.11) depending on the chosen density.

Within a nonparametric setting, we could find our starting density from actual option prices. Our model would then be a stationary implied binomial tree whose stationary density is the one implied in a set of option prices. The problem of extracting risk-neutral probability densities from option prices has been studied in the last decade [see for instance Jackwerth and Rubinstein (1996)]. A recent discussion of this type of constructions can be found in Jackwerth and Rubinstein (2004). In these studies, such a risk-neutral density is then used to construct a binomial tree to recover a market-consistent stochastic process for the underlying [see Cakici and Foster (2002) and Skiadopoulos (2001)]. Since our binomial tree can easily be constructed from this implied risk-neutral density, our model provides a way of constructing an implied binomial tree having the risk-neutral probability density as its stationary distribution. We call it an implied binomial tree in the sense that it is consistent with a set of option prices through the inferred stationary risk-neutral density. It remains yet to explore if our construction can lead to a binomial tree consistent with the implied volatility surface.

In other words, it seems to be desirable to have a binomial tree that not only has the risk-neutral density as its stationary distribution, but that is also consistent with the volatility smile. This would be the topic of further research.

6.3 Example

We have shown that our binomial tree can be constructed from any given discrete density. Here, we give an illustration of such a construction. We had briefly discussed that this starting density could be either a parametric density, chosen because of certain desirable features, or it could be a risk-neutral density extracted from actual option prices. The second choice deserves a more detailed study that would include the ability of our construction to produce trees consistent with the volatility smile and not only with a terminal stationary risk-neutral density. This will be the object of further research.

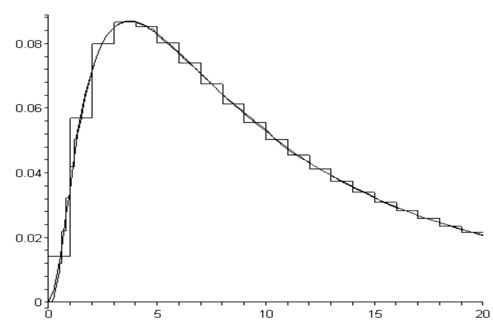
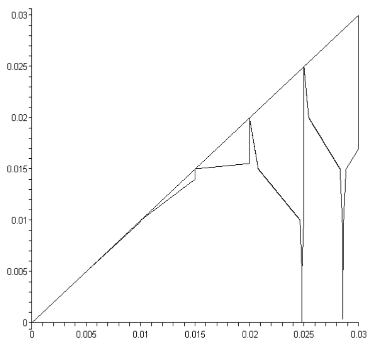


Figure 3

Lognormal density ($\mu = \ln 10$ and $\sigma = 1$) and its discretizations. 20, 100 and 200 intervals.

Instead, in our example we assume that a parametric density is given. We have chosen a lognormal density since it is a benchmark model for financial prices. We start with a lognormal density (parameters for illustrative purposes are: $\mu = \ln 10$ and $\sigma = 1$) which is then discretized over a finite interval $[0, 20]$. We then apply our algorithm that produces a set of τ_u , τ_d and p_u functions. We show empirically how such a tree follows a discrete stationary density close to the original desired lognormal.

In Figure 3 we have the original lognormal and its discretized version



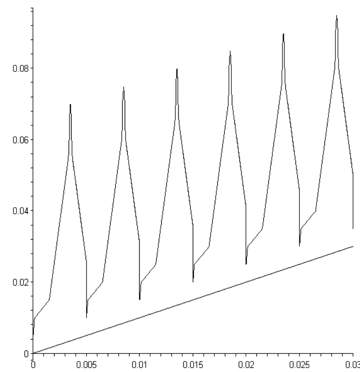


Figure 4

Down and up functions over the interval $[0, 0.03]$.

with twenty, a hundred and two hundred intervals. Notice that already a hundred intervals is very close to the continuous density. For two hundred intrevals the discretization coincides with the continuous plot in the picture.

Our algorithm is then applied producing the functions depicted in Figure 4. These graphs show the up- and down- maps over a small interval and not over the whole domain. Recall that these functions are piece-wise with as many pieces as intervals in the discretization, this means that in order to have a meaningful picture we need to zoom in into a small interval. In Figure 3, this interval is chosen to be $[0, 0.03]$.

Some trajectories of this implied binomial tree are shown in Figure 5. We can see how this trajectories resemble those of a stationary lognormal diffusion of the form (6.11).

An empirical distribution after $100,000$ iterations is shown Figure 6. We can see how the empirical distribution is very close to the original lognormal.

These graphs can be obtained with our program posted at:

http://www.mathstat.concordia.ca/pg/Economics200s.zip

This program also illustrates the effect of the parameters s, c's and d's in the resulting trajectories.

In the following section we discuss further some relevant features of our model that make it potentially useful in financial applications.

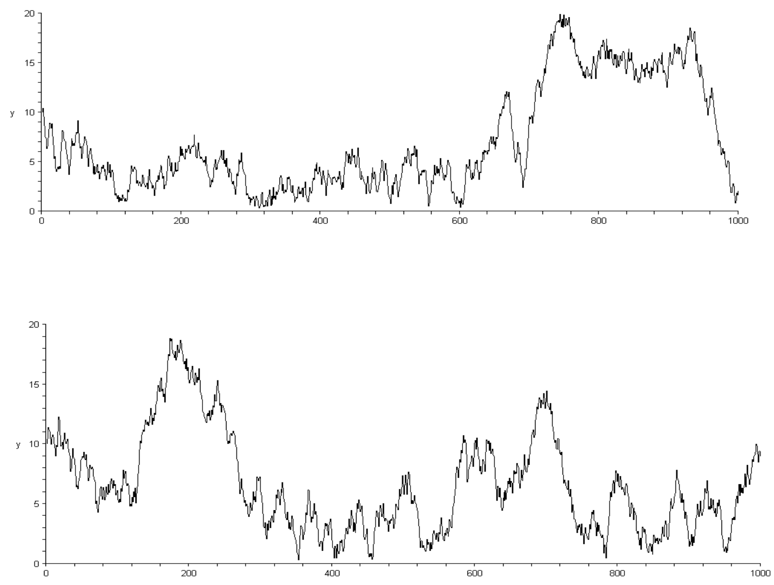


Figure 5

Trajectories of the binomial tree in Example 6.3.

7 Arbitrage Opportunities and Implied Binomial Trees

One important feature to be considered when looking at potential models for asset prices is its suitability to be used in option pricing. In this section we explore the conditions needed for our model s_1 to be arbitrage free, which implies the existence of an equivalent martigale measure, so that derivatives on this asset can be priced. We also discuss informally, and as a motivation for further study, some features of our model that make it an implied binomial tree in the sense of Rubinstein (1994), Derman and Kani (1994) and Dupire (1994).

7.1 Arbitrage Opportunities

Let (B, s) be the finite market defined on the filtered probability space $(\Omega, \mathcal{F}_t, \overline{P}, \mathbb{F}, \mathbb{T})$ as defined in Section 5.

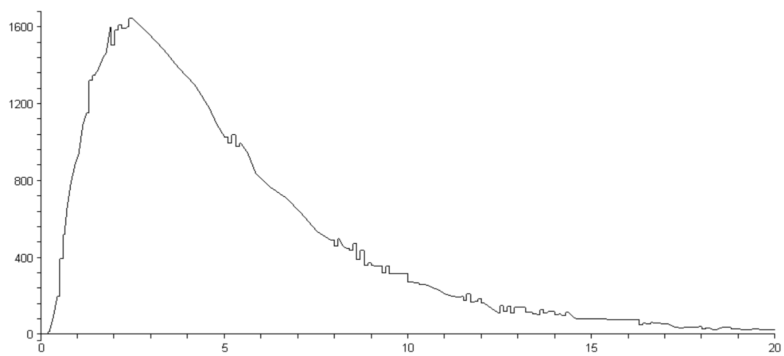


Figure 6

Empirical distribution of the binomial tree in Example 6.3 after 100,000 iterations.

We define a trading strategy $H = (H_0, H_1 \dots H_L)$ as a vector of stochastic processes $H_l = \{H_l(n); n = 0, \dots t\}$, $l = 0, 1, \dots, L$. $H_l(n)$, $l = 1, \dots L$, is the number of units of security with price s_l that the investors owns from time $n-1$ to time n, whereas $H_0(n)B_{n-1}$ is the amount of money invested in the bank account at time $n-1$. Negative values indicate short positions. We also define a value process associated with a trading strategy H by $V = \{V_n; n = 0, \dots, t\}$:

$$V_0 = H_0(1)B_0 + \sum_{l=1}^{L} H_l(1)s_l(0) \tag{7.1}$$

$$V_n = H_0(n)B_n + \sum_{l=1}^{L} H_l(n)s_l(n) \quad n \geq 1. \tag{7.2}$$

A trading strategy H is said to be self-financing if

$$V_n = H_0(n+1)B_n + \sum_{l=1}^{L} H_l(n+1)s_l(n), \quad n = 1, \dots, t-1,$$

i.e., the time n values of the portfolio just before and just after any time n transactions are equal. Intuitively, if no money is added to or withdraw from the portfolio between times $n = 0$ and time t, then any change in the portfolio's value must be due to a gain or loss in the investments.

Definition 7.1 *An arbitrage opportunity in the case of a multiperiod securities market is some trading self-financing strategy H such that its associated value process V satisfies:*

1. $V_0 = 0$, *a.s.*
2. $V_s \geq 0$, *for all* $s \in \mathbb{T}$,*a.s.*
3. $E[V_t] > 0$, *a.s.*

Definition 7.2 *A risk-neutral (or equivalent martingale) probability measure is a probability measure Q on $(\Omega, \mathcal{F}_t, \mathbb{F})$ such that the discounted price process*

$$s_l^*(n) \equiv \frac{s_l(n)}{B_n} \quad n = 0, \ldots, t \quad l = 0, \ldots, L$$

is a $\mathbb{F}$-martingale with respect to Q for every $l = 0, \ldots, L$.

In other words, a risk-neutral probability measure Q satisfies

$$E_Q[s_l^*(n+t)|\mathcal{F}_n] = s_l^*(n), \quad n \geq 0, t \geq 1. \tag{7.3}$$

One of the principal results in finance is the first fundamental asset pricing theorem (see Elliot and Kopp (1999) or Pliska (1994)):

Theorem 7.3 *In a finite market model (B, s_l), there are no arbitrage opportunities if and only if there exists an equivalent martingale measure Q for s_l.*

In our case, there are no arbitrage opportunities if and only if the process s_1 satisfy

$$E_Q[\frac{s_1(n+1)}{B_n}|\mathcal{F}_n] = \frac{s_1(n)}{B_n}, \quad n \geq 0. \tag{7.4}$$

If we suppose that the interest rate, r, is constant over time, then by using (5.1) and (7.4) we obtain

$$q(x)\left[\frac{u(x)-1-r}{1+r}\right] + (1-q(x))\left[\frac{d(x)-1-r}{1+r}\right] = 0 \tag{7.5}$$

for all x.

Following Cox, Ross and Rubinstein (1979) we can easily see that the one-step equivalent martingale measure at k is given by

$$q_k(s_1(k)) = \frac{1 + r - d(s_1(k))}{u(s_1(k)) - d(s_1(k))} , \qquad k = 0, 1, 2, \dots t - 1 .$$

Observe that the probabilities q_k depend on $s_1(k)$ because the functions u and d depend on the price as well.

Since q's are probabilities, it is easy to see that:

$$u(x) > 1 + r > d(x) \quad \text{for all } x. \tag{7.6}$$

Thus, as long as $u(x)$ and $d(x)$ satisfy (7.6) there is a unique equivalent martingale measure under which $s_1^*(n)$ is a martingale and s_1 is an arbitrage free price.

If the interest rate changes with the time, the we require the functions u and d to satisfy a more general condition:

$$u(x) > 1 + r_n > d(x) \quad \text{for all } x \text{ and } n = 0, \dots, t. \tag{7.7}$$

As far as the existence of equivalent martingale measure is concerned, our random map behaves in very much the same way as the classical binomial model. Option prices can then be computed for our random map binomial model.

7.2 Implied Binomial Trees

Rubinstein (1994), Derman and Kani (1994) and Dupire (1994) introduced implied binomial trees in order to study option prices as quoted in the market. Given a discretized density function for the asset returns they give a backward construction of a binomial model that is price dependent. Their construction extracts a risk-neutral density from actual option prices. Such a discretized density can be obtained in an involved way from actual market quotes for a set of options with similar strike prices and maturities [Jackwerth and Rubinstein (1996), Bahra (1997), Cakici and Foster(2002), Jackwerth and Rubinstein (2004)]. Some of the features that these implied binomial trees posses are: risk-neutral steps and price dependent local volatilities [we refer to Dumas, Fleming and Whaley (1998) for an empirical study of this latter feature]. We will briefly discuss how our binomial model behaves in very much the same way as an implied binomial tree.

An implied binomial model constructed as in Rubinstein (1994) is a finite market model (B, s_1) on a probability space $(\Omega, \mathcal{F}_t, \overline{P}, \mathbb{F}, \mathbb{T})$. It is constructed in a backward fashion such that the asset price $s_1(k)$ remains risk-neutral with respect to the up- and down-probabilities and the local volatilities are fitted to reproduce the volatility smile. Our model has these two features by construction.

From the expression for the one-step equivalent martingale probability, we can see that at each step

$$1+r = [q_k(s_1(k))]\ u(s_1(k)) + [1-q_k(s_1(k))]\ d(s_1(k))\ , \qquad k = 0, 1, \ldots, t-1\ ,$$

and our implied binomial tree is risk-neutral at each branch in the sense of Rubinstein (1994) when the up- and down-probabilities are set to be q_k and $1 - q_k$.

As for the local volatility at the k^{th} step, this is defined as

$$\begin{aligned}\sigma^2(k) \quad &= \quad [q_k(s_1(k))]\ [\ln u(s_1(k)) - (\mu(k))^2] \\ &\quad + [1 - q_k(s_1(k))]\ [\ln d(s_1(k)) - (\mu(k))^2]\ , \qquad k = 0, 1, \ldots, t-1\ ,\end{aligned}$$

where

$$\mu(k) = [q_k(s_1(k))]\ \ln u(s_1(k)) + [1 - q_k(s_1(k))]\ \ln d(s_1(k))\ .$$

We can see that this local volatility depends on the current price $s_1(k)$ and our implied binomial tree has state-dependent volatilities in the sense of Rubinstein (1994).

In summary, our construction produces a binomial tree from an input distribution. If this distribution is a somehow inferred risk-neutral distribution, then the resulting tree could be thought of as an implied binomial tree consistent with the market smile in a stationary way. This consistency can be seen as inherited from the original risk-neutral distribution and the stationarity of the process. In other words, our binomial tree follows the desired distribution after a large number of steps. Therefore, if the time window in question is long enough, we can consider the tree to be in the stationary state and in consequence risk-neutral for sufficiently long time windows. It remains to explore the question of whether the tree can be constructed in such a way that it is consistent with the volatility smile at each branch. This is, we would like that all (or most) local volatilities are consistent with a given volatility smile. This is subject of future research

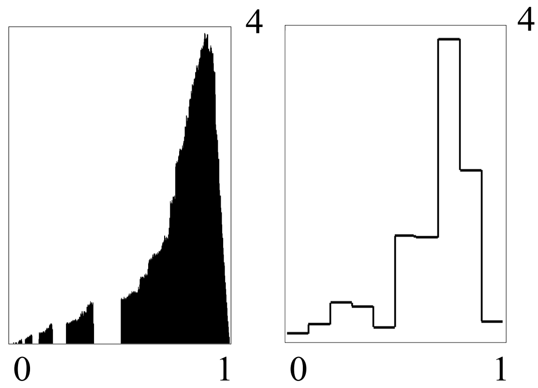


Figure 7

On the left: The histogram of invariant density of T of Example 5.2.
On the right: The density of Markov random map T_M.

8 Approximation of the Invariant Density

In this section we look at the problem of finding the stationary density of a given random map. This could be useful to find estimates of stationary densities in the classical binomial model and its generalizations, in particular those in Nelson and Ramaswamy (1990). For this purpose, $u(x), d(x), p_u(x)$ and $p_d(x)$ are assumed to be given, however, the invariant density f^* is unknown. We present a method for approximating f^*. This will endow us with a way of estimating probabilities of the form $\mu\{x\ ;\ T(x) \in (a, b)\}$ for any interval. Note that, if the transformations τ_u and τ_d are Markov, and the probabilities p_u and p_d are piecewise constant, we can find the exact unique invariant density f^* using the methods of Boyarsky and Góra (1997). When the transformations are not Markov, the invariant density can be approximated by using invariant vectors of matrix operators Góra and Boyarsky (2003). Example 5.2 is one such transformation. Now, we are going to approximate the invariant density of T in Example 5.2. First, we find two Markov transformations τ_{u_m} and τ_{d_m} which approximate τ_u and τ_d respectively. Let

$$\tau_{u_m}(x) = \begin{cases} 2x & \text{for } 0 \le x < 0.5 \\ 2x - 0.3 & \text{for } 0.5 \le x < 0.6 \\ x + 0.1 & \text{for } 0.6 \le x < 0.7 \\ x & \text{for } 0.7 \le x \le 1 \end{cases}, \tag{8.1}$$

and

$$\tau_{d_m}(x) = \begin{cases} x & \text{for } 0 \le x < 0.1 \\ x - 0.1 & \text{for } 0.1 \le x < 0.3 \\ x - 0.2 & \text{for } 0.3 \le x < 0.5 \\ 2x - 0.9 & \text{for } 0.7 \le x < 0.9 \\ 2x - 0.1 & \text{for } 0.9 \le x \le 1 \end{cases} . \tag{8.2}$$

Observe that τ_{u_m} and τ_{d_m} are Markov transformations on the common partition $[i/10, (i+1)/10)_{i=0}^{9}]$. The Perron-Frobenius operator of a Markov transformation can be represented by a matrix [Góra and Boyarsky (2003)]. Also, the Perron-Frobenius operator of the random map T_M, $T_M = \{\tau_{u_m}, \tau_{d_m}; p_u, p_d\}$, is represented by the following matrix

$$M = \Pi_u M_u + \Pi_d M_d, \tag{8.3}$$

where M_u, M_d are the matrices of $P_{\tau_{u_m}}$ and $P_{\tau_{d_m}}$ respectively, and Π_u, Π_d are the diagonal matrices of $p_u(x)$ and $p_d(x)$ respectively. We have

$$M_u = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$M_d = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 \end{pmatrix}$$

and

$$M = \begin{pmatrix} 0.6 & 0.2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.4 & 0 & 0.2 & 0.2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.4 & 0 & 0 & 0.2 & 0.275 & 0 & 0 & 0 & 0 \\ 0 & 0.4 & 0 & 0 & 0 & 0 & 0.275 & 0 & 0 & 0 \\ 0 & 0 & 0.4 & 0. & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.4 & 0 & 0 & 0 & 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 0 & 0.3625 & 0.725 & 0.4 & 0.3 & 0 \\ 0 & 0 & 0 & 0 & 0.4 & 0.3625 & 0 & 0 & 0.7 & 0.3 \\ 0 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0 & 0.7 \end{pmatrix},$$

where

$p_u = (0.8, 0.8, 0.8, 0.8, 0.8, 0.725, 0.725, 0.4, 0.4, 0.4);$, $p_d = 1 - p_u$. The invariant density of T_M is

$$f = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9, f_{10}), \quad f_i = f_{|I_i}, \quad i = 1, 2, \dots, 9, \tag{8.4}$$

normalized by

$$f_1 + f_2 + f_3 + f_4 + f_5 + f_6 + f_7 + f_8 + f_9 + f_{10} = 10, \tag{8.5}$$

and satisfying equation $fM = f$. Then,

$$f_1 = 0.11591, \quad f_2 = 0.23183, \quad f_3 = 0.48548, \quad f_4 = 0.44184, \quad f_5 = 0.19419,$$

$$f_6 = 1.28694, \quad f_7 = 1.26949, \quad f_8 = 3.64250, \quad f_9 = 2.07290, \quad f_{10} = 0.25892 \ .$$

The T_M-invariant density is shown on the right hand side of Figure 7.

Comparing left and right parts of Figure 7, we see that the invariant density of T_M approximates the invariant density of T in Example 5.2. Notice that, had we used Markov transformations on a finer partition than that in the above construction, we would have obtained a better approximation for the invariant density of T in Example 5.2 [Góra and Boyarsky (2003)].

9 Perturbed Random Maps and Arbitrage Opportunities

Finally, in this last section, we discuss a perturbed random map that can be suggested as a discrete-time binomial model that accepts arbitrage opportunities. It turns out that if we perturbed our random tree as it branches out, under certain conditions, it still has an stationary density. Since this new feature is not compatible with option pricing we do not advance at this point any possible application of this perturbed model. We simply present it here as an interesting extension of our position dependent binomial model.

We define our multiperiod model as in Section 5, the prices will then be driven by the random map T_G which is a perturbation of the random map T. We will see that (7.6) is not always satisfied as a result of the perturbations.

Let G be a family of functions g that are piecewise C^2 on the partition $\mathcal{P}$, $g(x) : [0,1] \to [0,1]$. We further assume that G is endowed with a regular probability measure η. Usually G will be a family of functions with parameter in a bounded region of $\mathbb{R}^d$ having normalized Lebesgue measure. At each iteration, to the map τ_k we add a function g_k from G chosen at random. This function will account for other sources of randomness that might exist in the absence of a perfect flow of information in the market. Thus, at each iteration, with probability $p_k(x)$, the next map is

$$\tau_{k,g}(x) = \tau_k(x) + g(x) \pmod 1,$$

where each g is chosen from G according to the probability η. The perturbed random map is denoted by T_g if the perturbing maps $\{g_k\}$ are fixed and by T_G if $\{g\}$'s are chosen at random from G. The iteration of the random map T_G is performed as follows:

$$T_G^n(x) = \tau_{k_n,g} \circ \tau_{k_{n-1},g} \circ \cdots \circ \tau_{k_1,g}(x)$$

with probability

$$p_{k_n}(\tau_{k_{n-1},g} \circ \cdots \circ \tau_{k_1,g}(x)) \cdot p_{k_{n-1}}(\tau_{k_{n-2},g} \circ \cdots \circ \tau_{k_1,g}(x)) \dots p_{k_1}(x),$$

where the perturbations are chosen in dependently at each step. T_G can be viewed as a Markov process with the transition function

$$\mathbb{P}(x, A) = \sum_{k=1}^{K} p_k(x) \int_G \chi_A(\tau_{k,g}(x)) d\eta(g),$$

where A is a measurable set and χ_A denotes the characteristic function of the set A.

We say that a measure μ is T_G-invariant if it is invariant for the above Markov process.

Thus, the price of stock l at time n will be given by

$$s_l(n) = T_G(s_l(n-1)),$$

where $n = 1, \dots, t$.

In this model, the g's create the arbitrage opportunities since condition (7.6) does not always hold. There might be steps where the value g will violate this condition creating an atbitrage opportunity. Hence, the prices will be really driven by the random map T_G, associated with the random map $T = \{\tau_u, \tau_p; p_u(x), p_d(x)\}$. The g's can be considered as small perturbations which are unknown to the investor. Moreover, since in our model we assume $\tau_{u,g}(x) > x$ and $\tau_{d,g}(x) < x$ for all x, and we want the same properties of the perturbed maps, the perturbation T_G is slightly modified. Assume τ_u is increasing. For τ_u we define $\tau_{u,g}(x)$ as follows. Let $x \in (a_i, a_{i+1}]$

$$\tau_{u,g}(x) = \begin{cases} \tau_u(x) + g(x), & x < \tau_u(x) + g(x) \leq 1 \\ \tau_u(x) + g(x) - (\tau_u(a_{i+1}) + g(a_{i+1})), & \tau_u(x) + g(x) > 1 \\ \tau_u(x) + g(x) - (\tau_u(a_i) + g(a_i)), & \tau_u(x) + g(x) < x \end{cases}. \tag{9.1}$$

The definitions for decreasing τ_u and for τ_d are similar. All perturbation results from Bahsoun, Góra and Boyarsky (2003) hold for this model as well.

The following results were proved in Bahsoun, Góra and Boyarsky (2003):

Theorem 9.1 *Let $T_G = \{\tau_{u,g}, \tau_{d,g}; p_u(x), p_d(x)\}$, where $\tau_{u,g}$ is defined as in equation (9.1), then P_{T_G} admits and invariant denisty.*

Theorem 9.2 *Let g, G, η, T_G and T be as above. Let us consider a family of sets $G : \{G_\varepsilon\}_{\varepsilon>0}$ such that $\sup_{g\in G_\varepsilon} \sup_x |g(x)| \leq \varepsilon$. Let f_ε be an invariant density of $P_{T_{G_\varepsilon}}$. Then, the family $\{f\}_{\varepsilon\geq 0}$ is precompact in L^1 and any weak limit point f^* of invariant densities f_ε as $\varepsilon \to 0$ is an invariant density of T.*

10 Conclusions

In this note, we develop techniques in theory of dinamical systems which can be applied in finance. We discuss the concept of position dependent random maps and some of their properties. We believe that these objects have properties that make them of interest in mathematical finance. We argue how these random maps can be implemented as generalized binomial models in ways that had not been explored before in the financial literature. The motivation behind our presentation lies on the fact that the proposed random maps accept a stationary density. In this respect, our model parallels the proposal of ergodic stationary diffusions in finance. We provide a very simple example that illustrates the model.

A second important feature of our generalized binomial models is that it can be constructed from a given stationary density. We explore in this note the inverse problem of finding a generalized binomial model having a predetermined invariant density. We present an algorithm for such a construction and a program that carries it out. As it turns out, our generalized binomial model has piecewise stationary densities. This suggests that we can approximate any desired continuous density with a piecewise constant function. This would endow us with a way of constructing a random map with a piece-wise constant density that can be as close as needed to any continuous density. This could be applied to popular choices for densities of price returns yielding a discrete stationary model of returns. As an illustration, we produce a program that constructs a binomial tree from a lognormal distribution. This program is available at:
http://www.mathstat.concordia.ca/pg/Economics200s.zip

Our construction could also find applications as an implied binomial tree. Implied binomial trees are constructed from historical data and then used to price less liquid options. These trees are grown to be consistent with the market volatility smile at every step. If the initial distribution is a risk-neutral density inferred from option prices then our construction yields an implied binomial tree. This tree is consistent with the market in a stationary way, i.e. after a long time, our tree follows the risk-neutral density previously inferred from option prices. It remains yet to be investigated if our construction can produce implied binomial trees consistent with a volatiliy smile at every step. Our model could be studied further in these type of applications.

We also discuss the problem of approximating the stationary density of a given random map. This could be used to approximate stationary densities for some simple discrete models.

Finally, we present a modified binomial model that accepts arbitrage opportunities. We present this as an interesting application of a perturbed random map. No comments on applications of such a model are made, nonetheless, the fact that it still has a stationary density makes it an interesting object.

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