Discussion Paper

Comparative statics for a 3 player differential game in resource economics – the case of exhaustible resources and varying allocations of initial stocks

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Abstract:

Differential games combine strategic interactions between agents and optimization concerning time. Decisions made in the past determine the present and even the future –in pay off as well as in the opportunities available – for oneself and for the rival players, eventually too. Unfortunately, due to high complexity it is hard to find a Nash-equilibrium within a differential game and it is even harder to get some results in comparative statics.

It is the purpose of the paper at hand to present findings concerning comparative statics in a differential game discussed by Wacker and Blank (1999). Comparative statics become available due to a routine solving for the open-loop Nash equilibrium for each parameter combination under consideration. A description of the routine – a 4 step simulation run which approximates the equilibrium numerically – was presented in an earlier Working Paper. In the earlier Paper Excel was applied as it is a wild spread tool. Here again Excel, its Solver and Macros constitute the main instruments; they are used to get repeated simulation runs for varying parameter constellations. The findings presented here concern varying allocations in initial stocks. Generalization to comparative statics in further parameters is in progress.♣

♣ JEL-classification: A22, C73, Q30

1 Introduction

More and more, spreadsheet tools like Excel are in use to enable middlebrows to rebuild ecological and economical models. The tools ease the understanding of interdependencies between time and variables related to each other. Thus, the solution of differential games may become an additional domain for tools like Excel. In the paper at hand, a model from resource economics is linchpin for demonstration.

The literature about software support in resource management is multiform nowadays. The early literature in the seventies mainly concentrated on tools used in system dynamics.[1] But with the diffusion of spreadsheet tools like Excel, more and more literature on modelling decisions and forecasts with the help of spreadsheets arose.

Literature explaining how to utilize spreadsheet tools to acquire a deeper understanding of decision analysis, biology and economics includes for instance Papadatos et al. (2002), Ragsdale (2001), Conrad (1999), Buongiorno and Gilles (2003) and Kirschke and Jechlitschka (2002).

One early publication on computation of optimal harvesting strategies is Kolberg (1993), who investigates renewable resource management, especially steady states, in a competitive industry. The author analyses backward and forward induction (Bellman Principle of Optimality[2]) and compares them to classical linear quadratic dynamic programming. The output is generated via a Quattro Pro program.

Conrad (1999) employs Excel within resource economics. Special emphasis is given to fishery- and forestry models, as these topics are the most common one in renewable resource

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[1] Examples for system dynamic tools are Dynamo, SIMPAS, DynSim, VenSim and PowerSim or Stella; concerning history and features of the software see Gilbert and Troitzsch (1999), chapter 3. These tools allow the definition of stocks and flows, to control feedback effects, and so on. They ease forecasting the development of variables linked through a system of differential equations.

[2] Bellmann equations handle a value function; they induce the same necessary conditions for optimality as the Maximum Principle of Pontryagin. For further explamations see: Intriligator (1971), chapter 13, or Fernández-Cara and Zuazua (0000), appendix 2.

economics. Further issues are depletable resources, like in the paper at hand, and pollution. Last but not least, option values in resource economics and sustainable development are analysed. In contrast to the present paper, in Conrad (1999) chapter 5, the market for the exhaustible resource is in perfect competition. Thus, the suppliers behave like price takers, unaware of any influence on the resource price. Therefore, there is not interaction between suppliers and the optimization problem is of a usual control theoretic type.

Ragsdale (2001), too, deals with Excel. In contrast to Conrad, Ragsdale concentrates on typical business and organisation problems, when applying Excel to Decision Analysis, Linear Programming, Forecasting and Queuing Problems. Optimization problems are mostly solved through the usual Solver. In special cases the Premium Solver (enhanced version of the usual Excel Solver) is in use.

Papadatos et al. (2002) give an example on how to apply Excel and Access to optimize net revenue through an appropriate product mix of cheese categories and other milk merchandise. Access is utilized as user interface. The optimization runs through the Premium Solver in Excel with an adjusted option setting for their nonlinear optimization problem.

Another example for an Excel application in decision analysis is Kirschke and Jechlitschka (2002), who present an introduction into worksheet design to model governmental interventions in agricultural markets. They integrate both, foreign markets and multi product situations, when solving for optimal tax and subvention rates or optimizing the budget allocation on different programs for structural/rural development. In contrast to Papadatos et al. (2002) they work with the usual Excel Solver and refer to the Premium Solver only for extended models including 12 product markets at once.

Buongiorno and Gilles (2003) deal with Forest management; they employ Excel and the usual Excel Solver for silvicultural decisions. Forest management is a field strongly influenced through long time decisions. A profound knowledge of the mayor effects current decisions have on future opportunities is indispensable. Furthermore, proper forecasts in margins should be available, and last but not least, the risks intrinsic in silviculture are enormous, thus risk management becomes essential. Buongiorno and Gilles (2003) explain how to deal with all these aspects, and in elucidating the design of the spreadsheets they take into account constraints through environmental policy, biodiversity requirements, and integer variables. Besides, they provide a comprehensive reference list for their work. But strategic thoughts on competitors' behaviour are missing. Insofar, their spreadsheet design is not adequate for an expansion to differential games.

Differential games combine game theoretic and dynamic aspects. Their advantage is due to the fact that most opportunities depend on past decisions – not only on ones own past decisions but on competitors' actions, too; thus it is not necessary to abstract reality beyond justifiable limits. The handicap of differential games is due the identification of equilibria and their features.

The paper presents some findings on comparative statics concerning the open-loop Nash equilibrium of the oligopolistic resource market described in Wacker & Blank (1999). The results are of numerical type and the reader can reproduce them without a profound knowledge in control theory or game theory.

The subsequent part summarizes the layout of the Excel file employed to find the open-loop Nash equilibrium[3] of the 3-player-game, investigated by Wacker and Blank (1999). An extended explanation of the worksheet design and the solution routine is given in an earlier Working Paper.

Then, chapter 3 presents some findings in comparative statics. The findings concern varying allocations in initial stocks. Finally opportunities for further research are discussed.

[3]. An open-loop Nash equilibrium is a strategy combination, where each player commits to his strategy at the beginning of the game, i.e. no one updates his decision during the course of the game. See Dockner et al. (2000), p. 59 for further discussion.

2 Model and Excel worksheet design

In the 3-player differential game under consideration each player owns a given initial stock S_0^i, $i = 1,2,3$, of an exhaustible resource. Extraction causes costs, proportional to the extracted quantity, i.e. marginal and average extraction costs are identical, time-invariant, and independent from residual stock size; they amount c^i.

Players face a time-invariant linear demand curve expressed through a willingness to pay for the total period output R_t, $p(R_t) = a - b \cdot R_t$, which is the sum of all players' output. The market is oligopolistic; players seek to maximize their individual total discounted profit taking into account the extraction paths of both competitors. Thus, the 'outcome' is an open-loop Nash equilibrium. But unfortunately, it is not possible to find a general analytical solution for this differential game, thus numerical solutions are in focus.

When looking for numerical solutions, it is advisable to check for their quality. In the model at hand this task is done through a look at the Euler-Equation.[4] The Euler-Equation states the identity of marginal discounted profits in all periods with non-diminishing extraction – in case each pay-off meets its optimum.[5]

With Excel and its Solver one can reconstruct the open-loop Nash equilibrium – i.e. the extraction paths – given at certain parameter constellation. Therefore, the qualitative findings presented in Wacker and Blank can be precised quantitatively.

Wacker and Blank choose the following parameter constellation for different initial stocks:

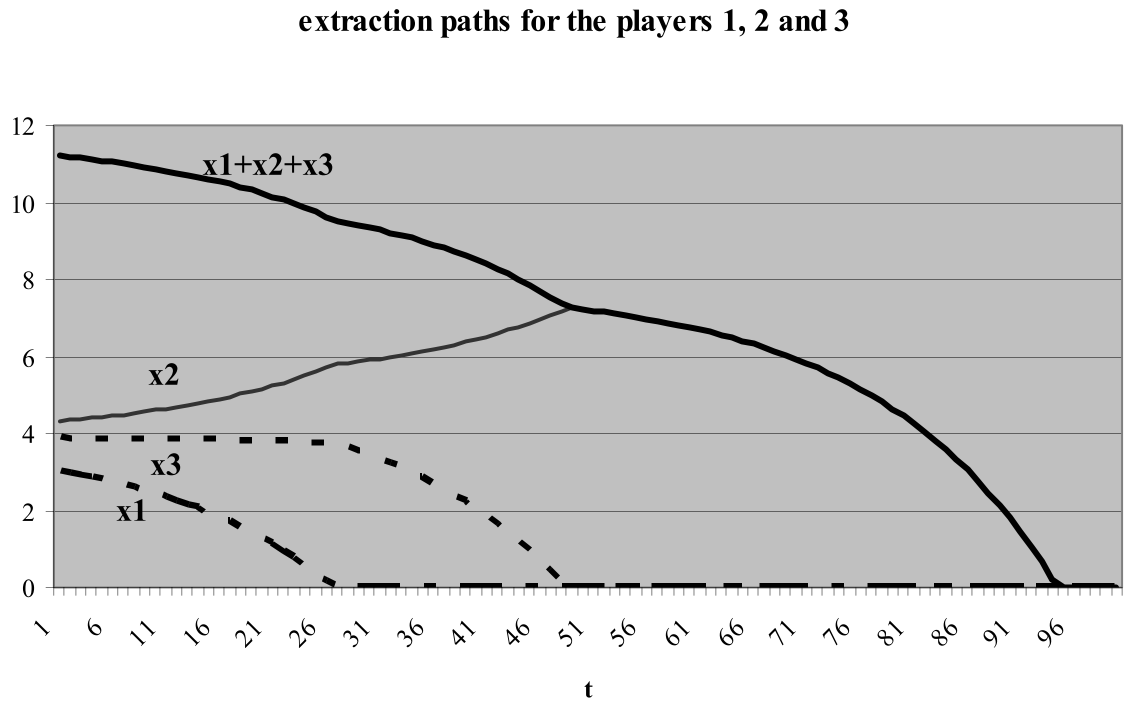
$a = 80;\ b = 5;\ c^i = 2\ and\ \delta^i = 6\%\ for\ i = 1,2,3;\ S_0^1 = 50;\ S_0^2 = 500;\ S_0^3 = 150$ [6]

[4] For a discussion of evaluation techniques in numerical methods, see Judd (1999), Chapter 2.10

[5] See for example: Sydsæter, Strøm, Berck (2000) Chapter 16.

[6] Notice that we changed index two and three.

Inserting these parameters into the worksheet and running the routine provides the extraction paths presented in the following picture:



Picture 1: total extraction and it's split between players

The following table displays the design of the worksheet used to reproduce the example with different initial stocks. The upper part, precisely A1:B9, contains all parameter values; to make formulas in the middle part more readable, cells are renamed:

B1 ≡ a; B2 ≡ b; B3 ≡ r; B4 ≡ c_1 ¸B5 ≡ c_2 B6 ≡ c_3; B7 ≡ S_01; B8 ≡ S_02 ; B9 ≡ S_03

A11:K14 from the middle part display the first four rows of the area used to reproduce the extraction paths. These rows content period index, price, extraction quantities x1, x2 and x3, non-discounted profits Π1, Π2, and Π3 and discounted profits, and these rows represent just the first four periods; it is omitted to fill up the table with all other rows of the original Excel worksheet, as they follow simply by copy and paste.

A	B	C	D	E	F	G	H	I	J	K
a	80									
b	5									
r	0,06									
c1	2									
c2	2									
c3	2									
S01	50									
S02	500									
S03	150									
t	p	x1	x2	x3	Π1	Π2	Π3	disc. Π1	disc. Π2	disc. Π3
1	a-b*(C11+D11+E11)	0,5	1,5	5	(B11-c_1)*C11	(B11-c_2)*D11	(B11-c_3)*E11	((1/(1+r))^A11)*F11	((1/(1+r))^A11)*G11	((1/(1+r))^A11)*H11
2	a-b*(C12+D12+E12)	0,5	1,5	5	(B12-c_1)*C12	(B12-c_2)*D12	(B12-c_3)*E12	((1/(1+r))^A12)*F12	((1/(1+r))^A12)*G12	((1/(1+r))^A12)*H12
3	a-b*(C13+D13+E13)	0,5	1,5	5	(B13-c_1)*C13	(B13-c_2)*D13	(B13-c_3)*E13	((1/(1+r))^A13)*F13	((1/(1+r))^A13)*G13	((1/(1+r))^A13)*H13
4	a-b*(C14+D14+E14)	0,5	1,5	5	(B14-c_1)*C14	(B14-c_2)*D14	(B14-c_3)*E14	((1/(1+r))^A14)*F14	((1/(1+r))^A14)*G14	((1/(1+r))^A14)*H14
…										
…										
98	a-b*(C108+D108+E108)	0,5	1,5	5	(B108-c_1)*C108	(B108-c_2)*D108	(B108-c_3)*E108	((1/(1+r))^A108)*F108	((1/(1+r))^A108)*G108	((1/(1+r))^A108)*H108
99	a-b*(C109+D109+E109)	0,5	1,5	5	(B109-c_1)*C109	(B109-c_2)*D109	(B109-c_3)*E109	((1/(1+r))^A109)*F109	((1/(1+r))^A109)*G109	((1/(1+r))^A109)*H109
100	a-b*(C110+D110+E110)	0,5	1,5	5	(B110-c_1)*C110	(B110-c_2)*D110	(B110-c_3)*E110	((1/(1+r))^A110)*F110	((1/(1+r))^A110)*G110	((1/(1+r))^A110)*H110
		SUMME (C11:C110)	SUMME (D11:D110)	SUMME (E11:E110)				SUMME(I11:I110)	SUMME(J11:J110)	SUMME(K11:K110)

Table 1: design of calculation worksheet

The lower part of table 1 contains just the last three periods and sum-formulas: in C112, D112 and E112 the sums of individual extraction quantities, in I112, J112, and K112 the sums of the discounted profits, i.e. the optimization targets.

Parts of the initial extraction paths with a uniform distribution of the initial stocks over a time horizon of 100 periods are displayed in table 1. Although, this is surely not a good initial guess, it allows reproducing the Nash equilibrium through 4 simulation runs.

Rotational optimization of discounted profits in 4 simulation runs provides approximations acceptable as depletion paths[7] and represented through picture 1. For more details[8], see the earlier Working Paper.

The functioning of the tatonement process relates greatly on the initial guess concerning extraction paths. In case of a bad choice, the algorithm might get stuck. Fortunately, jittering curves identify usually suboptimal extraction paths. In addition, a look at the Euler equations displays a problem for one or more players. In such a case it may be helpful to choose a related parameter constellation, run the algorithm, and apply the result as initial guess for the original problem.[9] For more details, see the earlier Paper.

3 Comparative statics on initial stock allocation

The automation of simulation runs by Macros eases the work on comparative statics. For example it allows analysing how the allocation of initial stocks[10] influences the distribution of discounted profits.

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[7] A check of the Euler-Equation provides information about convergence of the tatonement process. For additional details see the earlier Paper.

[8] To account for the non-linearity, the check mark for 'assume linear'8 is deactivated and 'quadratic' instead of 'linear' as 'estimates' option8 is chosen. Further, the presetting of 'max time'8 and 'iterations' are both increased by factor 5.

[9] For detailed description, see Judd (1999), p. 119 concerning 'hot starts'. Further a rearrangement of optimization sequence between players may be helpful; Judd (1999), p.71.

[10] To recall, in chapter 3 the allocation under consideration was $S_0^1 = 50; S_0^2 = 500; S_0^3 = 150$.

To guarantee some comparability, allocations with equal total initial stock volume are chosen, to be precise with a total initial volume of 600 units. Ten different allocations are analysed, see table 2 for details:

allocation_ID	S_01	S_02	S_03
1	200	200	200
2	180	190	230
3	160	180	260
4	140	170	290
5	120	160	320
6	100	150	350
7	80	140	380
8	60	130	410
9	40	120	440
10	20	110	470

Table 2: the 10 settings for allocation of initial stocks

The setting concerning the demand parameters, extraction costs and interest rate is as follows:

$a = 80; b = 5; c^i = 5 \text{ and } \delta^i = 6\% \text{ for } i = 1,2,3$

I.e., the only difference between players is in initial stock size. Beginning with an identical situation of all players, stepwise player 1 owns 20 units less in S_01 and player 2 owns 10 units less in S_02. Accordingly, with each step player 3 gets 30 units more than before.

Given the allocation of initial stocks, some type of initial 'competitiveness' of the market can be expressed by a Herfindahl index, using size of initial stocks instead of sales volume (a_i), which is the usual input in the formula.[11] [12]

[11] The Herfindahl index usually expresses the concentration in a delimited market during a delimited period. For a uniform distribution of market shares, the index corresponds to 1/N, with N as the number of firms. In case of a monopoly, the index degenerates to its limit value 1. Alternative measures are the concentration ratios CR1, CR3 or CR4 and the Lerner index.

[12] Concerning the formula, see for example Bleymüller et al. (1988), chapter 26.

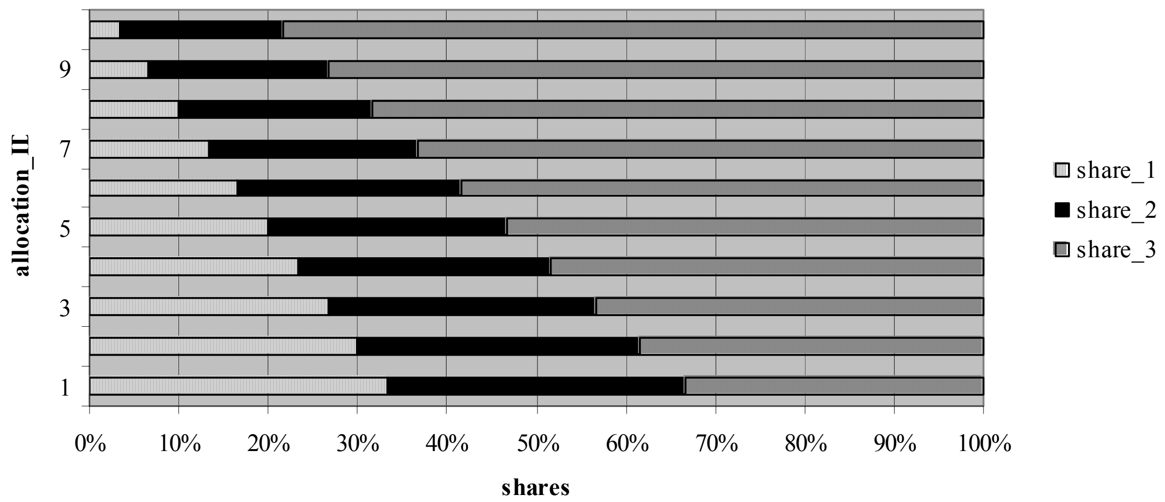
$$H := \frac{\sum_{i=1}^{N} a_i^2}{\left(\sum_{i=1}^{N} a_i\right)^2}$$

Where helpful, shares in initial stock size are employed as well.

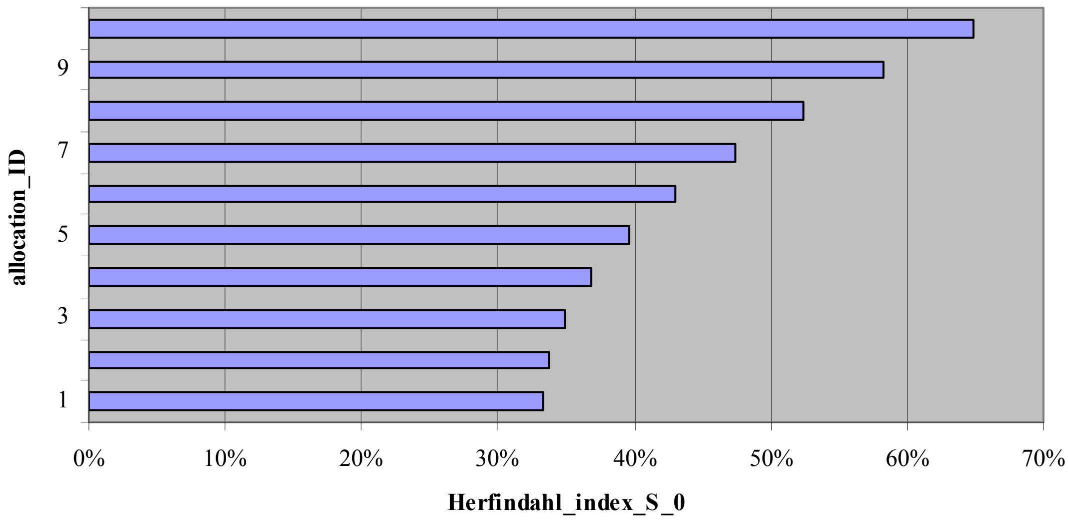
The following table 3 displays the Herfindahl index and the share in initial stock size related to the allocations above, and pictures 6 and 7 summarize the settings graphically:

allocation_ID	Herfindahl_Index_S_0	Share_1	Share_2	Share_3
1	33%	33%	33%	33%
2	34%	30%	32%	38%
3	35%	27%	30%	43%
4	37%	23%	28%	48%
5	40%	20%	27%	53%
6	43%	17%	25%	58%
7	47%	13%	23%	63%
8	52%	10%	22%	68%
9	58%	7%	20%	73%
10	65%	3%	18%	78%

Table 3: concentration and shares in initial stock volume



Picture 6: shares in initial stock volume



Picture 7: concentration

For each allocation a 4-step simulation run was initiated and for each player his discounted total profit as well as the period he leaves the market was documented. Further continuously, minimum and maximum discounted marginal profits are compared to check the quality of each particular simulation result.

The following table 4 summarizes the main findings:

allocation_ID	profit_1	profit_2	profit_3	Lambda_1_min	Lambda_2_min	Lambda_3_min	Lambda_1_max	Lambda_2_max	Lambda_3_max	t_1	t_2	t_3	S_01	S_02	S_03
1	1363,17	1363,17	1363,17	1,40	1,40	1,40	1,40	1,40	1,40	68	68	68	200	200	200
2	1311,92	1340,30	1443,22	1,60	1,50	1,30	1,60	1,50	1,30	63	65	70	180	190	230
3	1162,80	1219,86	1403,58	1,50	1,40	1,00	1,50	1,40	1,00	59	63	74	160	180	260
4	1109,05	1209,25	1495,63	1,90	1,50	0,90	1,90	1,50	0,90	54	60	76	140	170	290
5	1045,59	1204,06	1603,89	2,30	1,70	0,80	2,30	1,70	0,80	48	56	78	120	160	320
6	968,02	1205,19	1732,91	2,9	2,0	0,7	2,9	2,0	0,7	42	53	80	100	150	350
7	870,26	1214,39	1888,07	3,9	2,3	0,6	3,9	2,3	0,6	36	50	82	80	140	380
8	744,08	1234,11	2076,98	5,3	2,8	0,5	5,3	2,8	0,5	30	46	84	60	130	410
9	577,19	1268,59	2309,31	7,5	3,3	0,5	7,5	3,3	0,5	23	42	86	40	120	440
10	348,29	1326,20	2601,11	11,3	4,0	0,4	11,3	4,0	0,4	16	39	88	20	110	470

Table 4: individual discounted total profit, minimum and maximum of discounted marginal profit (Lambda), depletion period and initial stocks

The middle part of table 4 displays the quality of the simulation runs. The maximum and the minimum[13] of the discounted marginal profit indicate an acceptable approximation result.

Varying the setting from a uniform distribution of initial stocks to a more and more concentrated stock allocation one might expect the following outcomes:

- The higher the concentration the earlier player 1 and 2 leave the market
- The higher the concentration the longer player 3 stays in the market
- The higher the concentration the longer player 3 exists as a monopolist towards the end of the game
- The higher a player's share in initial total stock volume, the higher his discounted total profit
- Given a particular share in initial total stock volume, the higher the concentration the higher the discounted total profit

To give an intuition for some expectations from above one might argue from a static oligopoly situation:

Usually, suppliers with higher production capacity or higher sales volume experience lower average- and marginal production costs, thus average margin and units sold are both higher, resulting in higher profits.

Given a particular share in total industry capacity and an oligopoly with quantity setting enterprises, concentration and profits correlate positive. But this relationship might no longer exist in a competitive fringe situation. And it might even be erroneous in an oligopoly with price setters.

Concerning expectations on extraction paths, a static model is useless. Thus, the first statement above is based on 'gut instinct'. The second and third statement ground on the well known wisdom in resource management that a monopolist is a conserver of the exhaustible

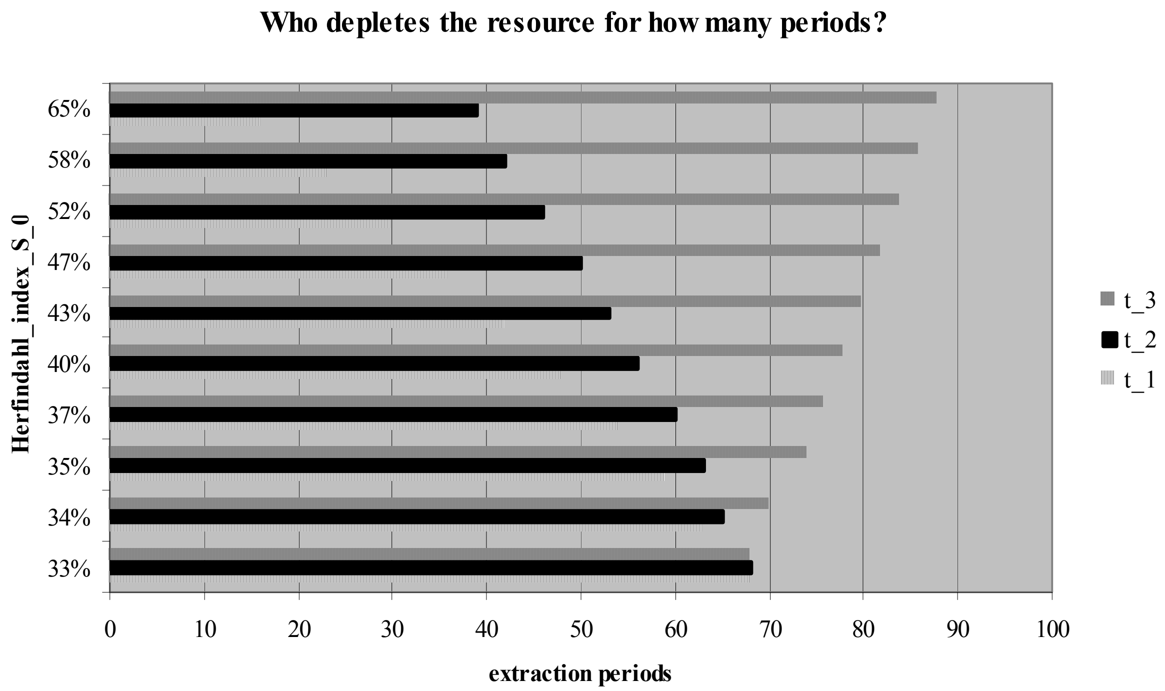
[13] Precisely, the maximum and the minimum for the discounted marginal profit of all periods with positive extraction quantity for the player under consideration at the end of a simulation run for a particular stock allocation. If maximum and minimum are equal even for the first decimal place, the simulation result was accepted as an approximation for the open-loop Nash equilibrium.

resource. As in a static model his per period sales volume is lower than the total per period sales volume in perfect competition – at least at the beginning of the periods of examination.

Coming back to table 4, a comparison of expectations and results is possible. First, extraction periods will be of interest. Next, discounted profits are under consideration – with a special glance at the influence of concentration on profits – and last but not least marginal discounted profits are going to be analysed.

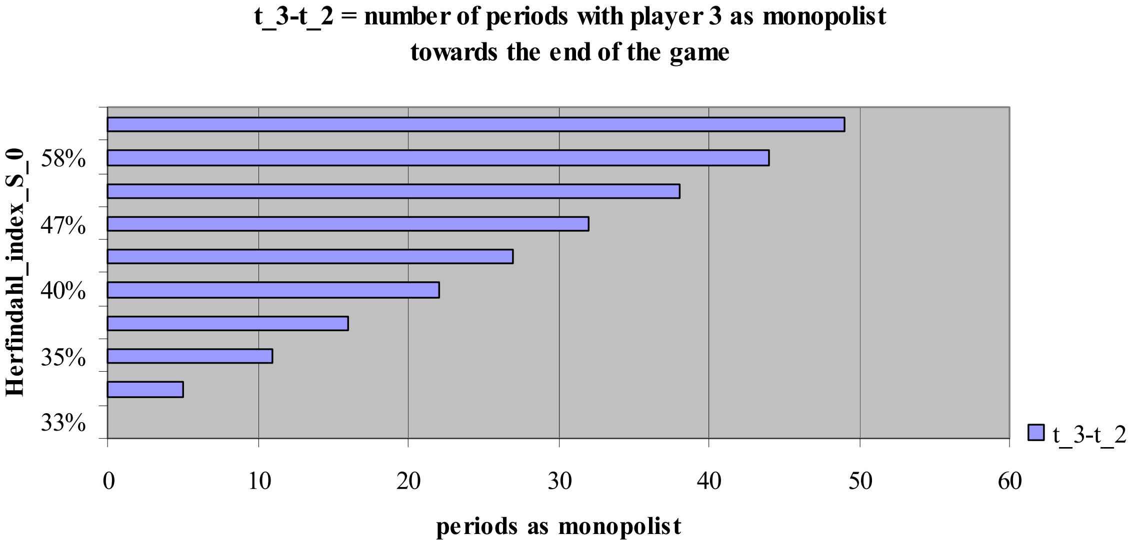
3.1 extraction periods

As visible in picture 2, players 'shrinking' with increased concentration, leave the market earlier than before. In contrast, player 3 stays longer in the market. Therefore, his phase as monopolist extends.



Picture 2: extraction periods related to the Herfindahl_index of initial stock allocation

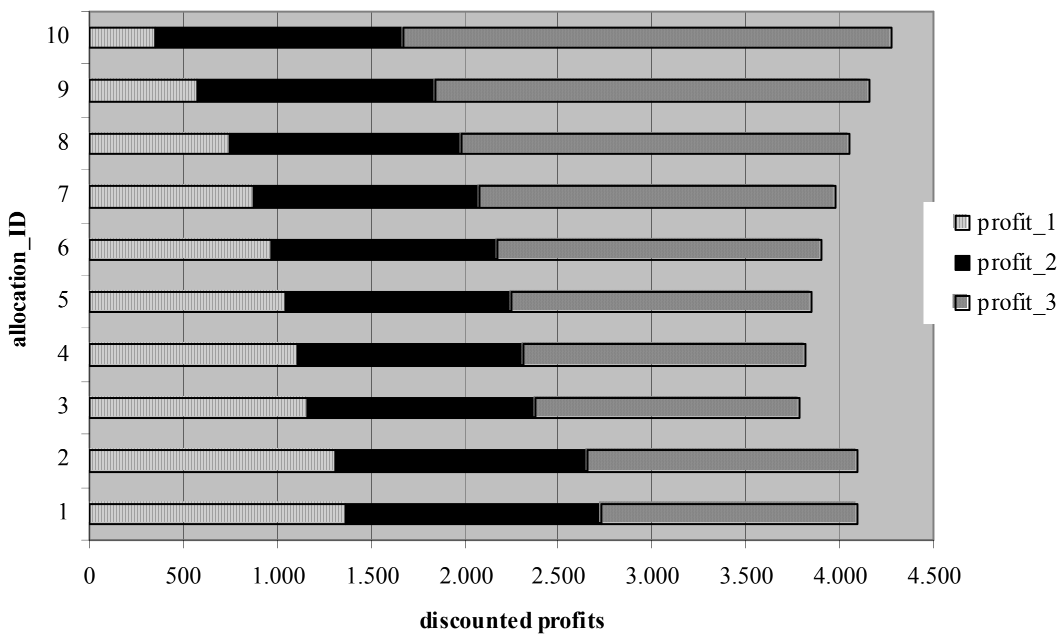
The monotonic relationship between monopolistic periods and the initial concentration in stock volume is presented in a condensed manner in picture 3:



Picture 3: monopolistic periods related to the Herfindahl_index of initial stocks

3.2 discounted profits

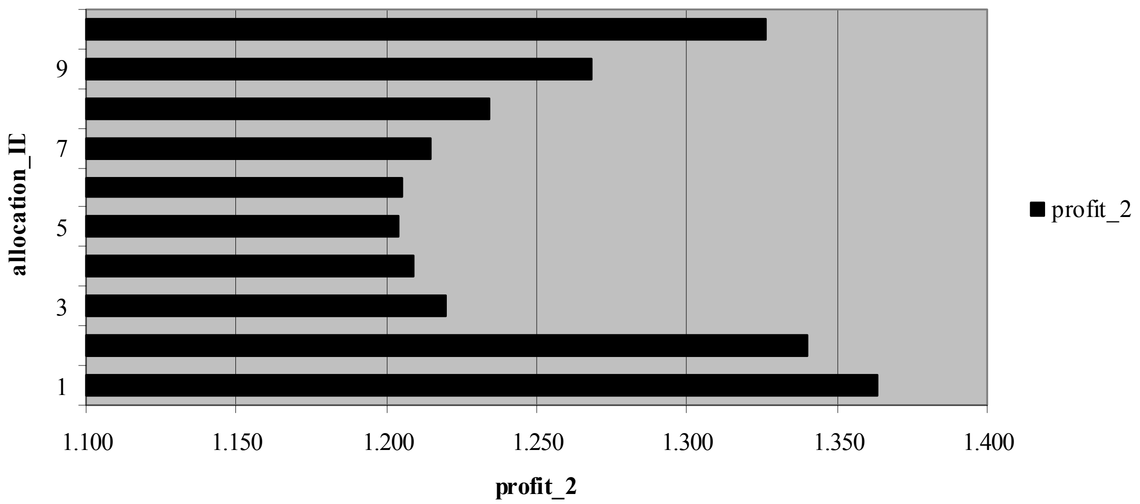
Further, discounted profits, i.e. net present values (NPVs) of stock ownership, are presented in picture 4:



Picture 4: discounted profits related to allocation_ID

First of all, the non-monotonic relationship between discounted total industry profit and concentration is ocular. Its causer is the way the allocation scenarios are constructed. For higher concentrated allocations, the expected monotonic relationship exists.

Second, the strictly negative relationship between player 1's profit and the concentration as well as the positive correlation between player 3's profit and the concentration is striking. Player 2 has the most interesting part in the game. Although his share in total initial stock volume decreases, his position relative to player 1 becomes stronger and stronger. On the other side, he looses strength compared to player 3. Therefore his discounted profit is u-formed in a plot against the concentration index; see picture 5 for a 'view' against the allocation_ID:



Picture 5: singled out profit of player 2 related to allocation_ID

Even so player 2 owns a smaller share of the total stock volume in later allocation_IDs, his discounted profit increases.

The design of the allocation setting allows to compare a particular share in total stock volume in different concentration situations. The available share/concentration combinations are depictured in the following table 5:

Share_1 (in %)	Share_2 (in %)	Herfindahl_index_S_0 (in %)
30	32	34
27	30	35
23	28	37
20	27	40
17	25	43
13	23	47
10	22	52
7	20	58

Table 5: combinations of allocations with coincident share data

A share of 30 % exists at a concentration of 34 % and of 35 %, a share of 27 % at a concentration of 35 % and of 40 % and so on.

Evidently the discounted profit depends not only on the share in total stock volume but on the market concentration, too. The following table 6 summarizes the profit / share / concentration combinations:

Share under consideration (in %)	Herfindahl_index_S_0 (in %)	Corresponding discounted profit
30	34	1311,92
30	35	1219,86
27	35	1162,80
27	40	1204,06
23	37	1109,05
23	47	1214,39
20	40	1045,59
20	58	1268,59

Table 6: combinations of allocations with coincident share data and the according discounted profit

Concerning table 6, one can claim: the higher the concentration the higher the profit - given a fixed share - except for the first case (30 % share). To what extent this statement can be generalized has to be analysed be further investigations. The line of argumentation might go according to: the higher the concentration, the more the big player (player 3) behaves like a monopolist, the larger the residual demand in early periods, and therefore the higher the profits of the other players.

3.3 marginal discounted profits

Another aspect for further investigations is the effect of concentration to marginal discounted profit – given a fixed share in the initial stock volume. As displayed in table 4, different concentration values generate different marginal discounted profits – given a fixed share in initial stock volume. The following table 7 summarizes the cases comparable:

Share under consideration (in %)	Herfindahl_index_S_0 (in %)	Corresponding marginal discounted profit
30	34	1,6
30	35	1,4
27	35	1,5
27	40	1,7
23	37	1,9
23	47	2,3
20	40	2,3
20	58	3,3

Table 7: combinations of allocations with coincident share data and the according marginal discounted profit

Again, the findings exclude the first case with the 30 % share. For the other three cases available, one can state: the higher the concentration the higher the marginal discounted profit - given a fixed share. As before, this indication is the one expected from capacity considerations in a static oligopoly model. Given a ‘fixed’ market size (here 600 units) and one big player (here player 3) with the tendency to behave like a monopolist, the residual demand remaining for the other players is comparatively large. Therefore their window of opportunity is large, too, resulting in higher marginal profits. As player 3 withholds a relevant fraction of his stock volume till the other players leave the market, the residual demand is even larger when one investigates early periods in an exhaustible resource game. Hence the positive effect of concentration on marginal discounted profits does not amaze.

But as before, further investigations are necessary to be able to formulate a general and comprehensive statement.

4 Discussion

On the one side comparative statics show some anticipated findings, and on the other side they provide some surprising outcomes. Thus, more work in this direction is recommended. One way is to reply the simulations for different cost levels and different discount rates. This allows for a multidimensional comparative static analysis. First attempts with different cost levels approve the findings from the parameter settings in chapter 4.

Given a sufficient number of parameter settings analysed, an econometrical examination becomes viable. It can discover the influence of stock size, concentration, costs and discount rate on the discounted profit. Additional interesting issues are the final period and the length of the monopoly phase.

Further, to enlarge the number of players is desirable, but this extension is time-consuming and thus calls for a revised computer program.

5 Conclusion

In order to demonstrate students of non-mathematical faculties differential games in resource economics - without any mathematical training - a simple procedure in Excel was presented. Provided that concavity requirements are fulfilled, extraction stops within acceptable periods and the initial guess about the extraction paths is not too bad, the procedure approximates the Nash equilibrium. Furthermore, having an easy algorithm to calculating the equilibrium, comparative statics become viable. First results on comparative statics confirm some but not all expectations grounded on considerations in static models. More precisely, the connection between total industry profit and concentration might be partly reverse – depending on the manner concentration develops.

References:

Bleymüller, J.; Gehlert, G.; Gülcher, H. (1988): *Statistik für Wirtschaftswissenschaftler*; Verlag Vahlen, München

Buongiorno, J., Gilles, J.K. (2003): *Decision Methods for Forest Resource Management*, Academic Press, Elsevier Science

Chiang, A. (1992): *Elements of Dynamic Optimization*, Waveland Press, Inc., Prospect Heights, Illinois

Conrad, J.M. (1999): *Resource Economics*, Cambridge University Press, New York

Dockner, Engelbert et al.(2000): *Differential games in Economics and Management Science*, Cambrige, Mass.

Dorfman, R. (1969): An Economic Interpretation of Optimal Control Theory, *American Economic Review*, 59 (5), 817 - 831

Fernández-Cara, E.; Zuazua, E. (0000) : *Control Theory: History, Mathematical Achievements and Perspectives*, Bol. Sic. Esp. Mat. Apl. n°0, 1 -63

Gilbert, N.; Troitzsch, G. (1999): *Simulation for the Social Scientist*; Open University Press, Buckingham, Philadelphia

Huck, P. (2005): *Solving a three player differential game in resource economics – the case of exhaustible resources*; Working Paper 04-2005

Intriligator, M. (1971): Mathematical Optimization and Economic Theory, Prentice-Hall, Inc., Englewood Cliffs, N.J.

Judd, K. L. (1999): *Numerical Methods in Economics*, MIT Press; Cambridge

Kamien, M.; Schwartz, N. (1981): *Dynamic Optimization, The calculus of Variations and Optimal Control in Economics and Management*; North Holland; New York, Oxford

Kirschke, D., Jelitschka, K. (2002): *Angewandte Mikroökonomie und Wirtschaftspolitik mit Excel*, Verlag Vahlen, München

Kirschke, D., Jelitschka, K. (2003): Interaktive Programmierungsansätze für die Gestaltung von Agrar- und Umweltprogrammen, *Agrarwirtschaft* 52 (4), 211 - 217

Kolberg, W.C. (1993): Quick and Easy Optimal Approach Paths for Nonlinear Natural Resource Models, *American Journal of Agricultural Economics* 75, 685 - 695

Krautkraemer, J. (1998): Nonrenewable Resource Scarcity, *Journal of Economic Literature* 36 (Dec.), 2065 – 2107

Krautkraemer, J.; Toman, M. (2003): *Fundamental Economics of Depletatle Energy Supply*; Resources for the Future; Discussion Paper 03-01 (http://www.rff.org))

Papadatos, A., Berger, A.M., Pratt, J.E., Barbano, D.M. (2002): A Nonlinear Programming Optimization Model to maximize Net Revenue in Cheese Manufacture, *Journal of Dairy Science* 85, 2768 - 2785

Pezzey, J.; Toman, M. (2992): *The Economics of Sustainability: A Review of Journal Articles*; Resources for the Future; Discussion Paper 02-03 (http://www.rff.org)

Ragsdale, C. (2001): *Spreadsheet Modeling and Decision Analysis*, South-Western College Publishing, Cincinnati

Sydsæter, K., Strøm, A., Berck, P. (2000): *Economists' Mathematical Manual*; Springer-Verlag, Berlin-Heidelberg

Wacker, H.; Blank, J. E. (1999): *Ressourceökonomie, Band II: Erschöpfbare natürliche Ressourcen,* Oldenbourg, München