The set of equilibria of first-price auctions

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Abstract

In this note I specify the class of functions that are equilibria of symmetric first-price auctions.

1 Introduction

Suppose we somehow obtained a bidding function $b(\cdot)$. It could, for example, be originated from some laboratory auction or be a linear interpolation of some auction data. Is this bidding function theoretically possible? If so, does it come from a model adequate to the situation at hand? In this paper I study this problem in a sealed bid first-price auction set up. To be more concrete we are in the independent private values model and suppose we have 3 bidders with signals in the interval $[0,1]$ and that our estimated bidding function is $b(x) = \frac{x}{2}$. If the distribution of signals is uniform with three bidders the equilibrium is $b^*(x) = \frac{2x}{3}$. With two bidders the equilibrium is exactly $b^*(x) = \frac{x}{2}$. Do we have a dummy bidder? Colusion? Here I focus on the distribution of signals. The uniform distribution that we supposed in this example is usually used more for convenience than for theoretical reasons. In this example if we change the distribution to $F(x) = \sqrt{x}$ we have that the equilibrium bidding function is exactly $b(x)$. I show in this paper that for any number of bidders and pratically any strictly increasing function $b(\cdot)$ it is possible to find a strictly increasing continuous distribution function such that the equilibrium bidding function is exactly $b(\cdot)$. This result is similar in spirit to the Sonneschein-Mantel-Debreu theorem on excess demand. I also analyze a second aspect of this problem. If we insist that the distribution of signals is given and vary the bidder valuation from $V_i = x_i$ to $V_i = u(x_i)$. The conditions to find an appropriate $u(\cdot)$ are however harder to be met.

2 The model

We consider first-price sealed bid auctions. There are n bidders with independent private values and each bidders' signal are in the interval $[0, \bar{v}]$. We consider distribution of bidders signals in the set

$$\mathcal{F} = \{F : [0, \bar{v}] \to [0, 1] ; F \text{ is continuous strictly increasing and onto}\}.$$

Thus $\mathcal{F}$ is the set of strictly increasing distributions with support $[0, \bar{v}]$. Let $F \in \mathcal{F}$ be the distribution of bidder $i \leq n$ signal x_i. If bidder i has signal x_i he valuates the object as $V_i = x_i$. Define $b_F(0) = 0$ and if $x \in (0, \bar{v}]$,

$$b_F(x) = x - \frac{\int_0^x F^{n-1}(v)\, dv}{F^{n-1}(x)}.$$

We first prove the

Lemma 1 *b_F is continuous and strictly increasing.*

Proof: The continuity of b_F for $x' > 0$ is immediate. At $x' = 0$ it follows from $b_F(x) < x$. Let us now prove that it is strictly increasing. Suppose that $0 \leq x < y \leq \bar{v}$. If $x = 0$ it is immediate from $\int_0^y F^{n-1}(v)\, dv < F^{n-1}(y)\, y$ that $b_F(y) > 0 = b_F(x)$. If $x > 0$ then:

$$b_F(y) - b_F(x) = y - \frac{\int_0^y F^{n-1}(u)\, du}{F^{n-1}(y)} - x + \frac{\int_0^x F^{n-1}(u)\, du}{F^{n-1}(x)} =$$
$$y - x - \frac{\int_x^y F^{n-1}(u)\, du}{F^{n-1}(y)} + \left(\frac{1}{F^{n-1}(x)} - \frac{1}{F^{n-1}(y)}\right) \int_0^x F^{n-1}(u)\, du > 0.$$

Note that $\int_x^y F^{n-1}(u)\, du \leq F^{n-1}(y)(y - x)$. QED

We now show that b_F is an equilibrium bidding function.

Proposition 1 *Let $F \in \mathcal{F}$. If there are n bidders with independent signals distributed accordingly to F then $b_F(\cdot)$ is a symmetric equilibrium of the first-price auction.*

Proof: Define $b = b_F$ and $x = x_i$. Suppose bidder $j \neq i$ with signal x_j bids $b(x_j)$. We have to prove that for any $y \in [0, \bar{v}]$,

$$(x - b(x)) \Pr\left(b(x) \geq \max_{j \neq i} b(x_j)\right) \geq (x - b(y)) \Pr\left(b(y) \geq \max_{j \neq i} b(x_j)\right).$$

Since b is strictly increasing and the signals are independent,

$$\Pr\left(b(x) \geq \max_{j \neq i} b(x_j)\right) = \prod_{j \neq i} \Pr(b(x) \geq b(x_j)) =$$
$$\prod_{j \neq i} \Pr(x \geq x_j) = F^{n-1}(x).$$

Therefore

$$(x - b(x)) \Pr\left(b(x) \geq \max_{j \neq i} b(x_j)\right) = \int_0^x F^{n-1}(v)\, dv.$$

Thus we have to prove that for every $y \in [0, \bar{v}]$,

$$\int_0^x F^{n-1}(v)\, dv \geq (x - y) F^{n-1}(y) + \int_0^y F^{n-1}(v)\, dv.$$

This is equivalent to

$$\int_y^x F^{n-1}(u)\, du \geq (x - y) F^{n-1}(y).$$

Considering separately the cases $x > y$ and $y \geq x$ we see that this inequality is true. QED

Define

$$\mathcal{B} = \{b_F(\cdot); F \in \mathcal{F}\}.$$

We may now prove our main theorem.

Theorem 1 *Suppose* $b : [0, \bar{v}] \to \mathbb{R}$. *Then* $b \in \mathcal{B}$ *if and only if:*

1. $b(\cdot)$ *is strictly increasing;*
2. $b(0) = 0$;
3. $b(x) < x$ *if* $x > 0$;
4. $\lim_{x \to 0^+} (x - b(x))\, e^{\int_x^{\bar{v}} \frac{dy}{y - b(y)}} = \infty$.

Proof: Suppose $b \in \mathcal{B}$. It is clear that it satisfies (1), (2) and (3). To see that it also satisfy (4) let $F \in \mathcal{F}$ be such that $b = b_F$. Then

$$\frac{1}{y - b(y)} = \frac{F^{n-1}(y)}{\int_0^y F^{n-1}(v)\,dv} = \left(\log\left(\int_0^y F^{n-1}(v)\,dv\right)\right)'.$$

Therefore

$$\int_x^{\bar{v}} \frac{dy}{y - b(y)} = \int_x^{\bar{v}} \left(\log\left(\int_0^y F^{n-1}(v)\,dv\right)\right)' dy = \log\left(\frac{\int_0^{\bar{v}} F^{n-1}(v)\,dv}{\int_0^x F^{n-1}(v)\,dv}\right)$$

and

$$(x - b(x))\, e^{\int_x^{\bar{v}} \frac{dy}{y - b(y)}} = \frac{\int_0^x F^{n-1}(v)\,dv}{F^{n-1}(x)} \frac{\int_0^{\bar{v}} F^{n-1}(v)\,dv}{\int_0^x F^{n-1}(v)\,dv} = \frac{\int_0^{\bar{v}} F^{n-1}(v)\,dv}{F^{n-1}(x)}.$$

Thus b satisfy (4). Now suppose $b(\cdot)$ satisfy (1), (2), (3) and (4). I show that $b = b_G$ where $G(0) = 0$ and if $x \in (0, \bar{v}]$,

$$G(x) = e^{-\frac{1}{n-1}\int_x^{\bar{v}} \frac{1}{y - b(y)} dy} \left(\frac{\bar{v} - b(\bar{v})}{x - b(x)}\right)^{\frac{1}{n-1}}.$$

First note that (4) imply that G is continuous at 0. And obviously $G(\bar{v}) = 1$. It is also clear that G is continuous if $x > 0$. We now show that G is strictly increasing. It is equivalent to prove that

$$\phi(x) := \log G^{n-1}(x) = -\int_x^{\bar{v}} \frac{1}{y - b(y)} dy - \log(x - b(x))$$

is strictly increasing. If $h > 0$ then

$$\frac{\phi(x+h) - \phi(x)}{h} = \frac{1}{h}\int_x^{x+h} \frac{1}{y - b(y)} dy + \frac{1}{h}\log\left(\frac{x - b(x)}{x + h - b(x+h)}\right).$$

If $\frac{x - b(x)}{x + h - b(x+h)} \geq 1$ then

$$\frac{\phi(x+h) - \phi(x)}{h} \geq \frac{1}{h}\int_x^{x+h} \frac{1}{y - b(y)} dy.$$

If $\frac{x + h - b(x+h)}{x - b(x)} > 1$, then

$$\log\left(\frac{x + h - b(x+h)}{x - b(x)}\right) = \log\left(1 + \frac{h - b(x+h) + b(x)}{x - b(x)}\right) < \frac{h - b(x+h) + b(x)}{x - b(x)}$$

and therefore

$$\frac{\phi(x+h)-\phi(x)}{h} > \frac{1}{h}\int_x^{x+h} \frac{1}{y-b(y)}dy - \frac{1}{h}\frac{h-b(x+h)+b(x)}{x-b(x)} \geq$$
$$\frac{1}{h}\int_x^{x+h} \frac{1}{y-b(y)}dy - \frac{1}{x-b(x)}.$$

Thus

$$\liminf_{h\to 0^+} \frac{\phi(x+h)-\phi(x)}{h} \geq 0$$

which implies that ϕ is increasing. To show that it is strictly increasing suppose not. Then ϕ is constant in an interval (c,d) and therefore it is differentiable and hence $b(\cdot)$ is differentiable in (c,d) as well. Thus from

$$\frac{d}{dx}\phi(x) = \frac{1}{x-b(x)} - \frac{1-b'(x)}{x-b(x)} = \frac{b'(x)}{x-b(x)}, x \in (c,d)$$

we have that $b'(x) = 0$ if $x \in (c,d)$ and this contradicts that $b(\cdot)$ is strictly increasing. It remains only to check that

$$b_G(x) = x - \frac{\int_0^x G^{n-1}(v)\,dv}{G^{n-1}(x)}$$

is equal to $b(x)$. Now note that

$$\int_0^x G^{n-1}(u)\,du = \int_0^x e^{-\int_u^{\bar{v}} \frac{1}{y-b(y)}dy}\left(\frac{\bar{v}-b(\bar{v})}{u-b(u)}\right)du =$$
$$(\bar{v}-b(\bar{v}))\int_0^x \frac{d}{du}\left(e^{-\int_u^{\bar{v}} \frac{1}{y-b(y)}dy}\right)du = (\bar{v}-b(\bar{v}))\left(e^{-\int_u^{\bar{v}} \frac{1}{y-b(y)}dy}\right)\Big|_0^x =$$
$$(\bar{v}-b(\bar{v}))\, e^{-\int_x^{\bar{v}} \frac{1}{y-b(y)}dy}.$$

Since

$$G^{n-1}(x) = e^{-\int_x^{\bar{v}} \frac{1}{y-b(y)}dy}\left(\frac{\bar{v}-b(\bar{v})}{x-b(x)}\right) = \frac{\int_0^x G^{n-1}(u)\,du}{x-b(x)}$$

we conclude that

$$b_G(x) = x - \frac{\int_0^x G^{n-1}(u)\,du}{G^{n-1}(x)} = x - (x-b(x)) = b(x).$$

QED

Example 1 *Let us consider* $b(x) = x/2, x \in [0,1]$. *Then if there are* n *bidders,*

$$G(x) = e^{-\frac{1}{n-1}\int_x^1 \frac{2}{y}dy}\frac{1}{x^{\frac{1}{n-1}}} = x^{\frac{1}{n-1}}.$$

If $n = 3$ *then* $G(x) = \sqrt{x}$.

3 The model with a more general valuation

Suppose now that the set of signals of bidder i is an abstract probability space $(X, \mathcal{T}, P)$ and if bidder i has a signal $x_i \in X$ his valuation is $V_i = u(x_i)$ where $u : X \to [0, \bar{v}]$. If the distribution

$$F_u(l) = \Pr(u(x) \leq l), l \in [0, \bar{v}]$$

belongs to $\mathcal{F}$ then we can easily see that

$$b_u(x) = b_{F_u}(u(x)) \tag{1}$$

is a symmetric equilibrium bidding function.

Is it possible to fix a distribution F and vary the valuation $u(\cdot)$ to obtain a pre-specified bidding function $b(\cdot)$? Suppose the set of signals is $X = [0, \bar{v}]$ with distribution $F(x)$ with a continuous density $f(x) > 0$. Then we have the

Theorem 2 *Suppose $b : [0, \bar{v}] \to \mathbb{R}$ is continuously differentiable, strictly increasing such that $b(0) = 0$ and*

$$u(x) = b(x) + \frac{b'(x) F^{n-1}(x)}{(F^{n-1})'(x)} = \frac{(b(x) F^{n-1}(x))'}{(F^{n-1})'(x)} \text{ is increasing.}$$

Then $b(\cdot)$ is the symmetric equilibrium of the first-price auction if bidders have valuation $V_i(x) = u(x)$ and the distribution of signals is F.

Proof: First note that $F_u(u(x)) = F(x)$ and therefore using (1) that

$$b_u(x) = u(x) - \frac{\int_0^{u(x)} F_u^{n-1}(l)\, dl}{F_u^{n-1}(u(x))} =$$
$$u(x) - \frac{\int_0^x F^{n-1}(l)\, u'(l)\, dl}{F^{n-1}(x)} = \frac{\int_0^x u(l)\,(F^{n-1})'(l)\, dl}{F^{n-1}(x)} =$$
$$\frac{\int_0^x (b(l) F^{n-1}(l))'\, dl}{F^{n-1}(x)} = b(x).$$