Strongly Stable Networks*

by

Matthew O. Jackson† and Anne van den Nouweland‡

June 2001

draft: February 13, 2002

Abstract

We analyze the formation of networks among individuals. In particular, we examine the existence of networks that are stable against changes in links by any coalition of individuals. We show that to investigate the existence of such strongly stable networks one can restrict focus on a component-wise egalitarian allocation of value. We show that when such strongly stable networks exist they coincide with the set of efficient networks (those maximizing the total productive value). We show that the existence of strongly stable networks is equivalent to core existence in a derived cooperative game and use that result to characterize the class of value functions for which there exist strongly stable networks via a "top convexity" condition on the value function on networks. We also consider a variation on strong stability where players can make side payments, and examine situations where value functions may be non-anonymous – depending on player labels.

*An earlier and incomplete version of this paper was entitled "Efficient and Stable Networks and their Relationship to the Core," (August 1999). This supersedes that version. We began this work during a visit of Anne van den Nouweland to Caltech in April of 1999, and we thank Caltech for its hospitality. Support of the National Science Foundation under grant SBR-9729568 is also gratefully acknowledged.

†HSS 228-77, California Institute of Technology, Pasadena CA 91124, USA. Email: jacksonm@hss.caltech.edu

‡Department of Economics, University of Oregon, Eugene, OR 97403-1285, USA. Email: annev@oregon.uoregon.edu

1 Introduction

The importance of networks in a variety of social and economic settings is well-documented. Given that network relationships matter, it is important to understand which networks are likely to form and how this depends on the structure of the setting. This a growing area of research to which this paper contributes.[1]

In this paper we examine the formation of networks through a careful study of the existence and properties of strongly stable networks: those networks which are stable against changes in links by any coalition of individuals. Strong stability of networks is a demanding property, as it means that no set of players could benefit through any rearranging of the links that they are involved with (including those linking them to players outside the coalition). As such, we expect there to be contexts where such networks will not exist. However, we show that strongly stable networks do exist in many natural settings, including a number that pop up in the literature as examples of network situations. In situations where strongly stable networks exist they are quite compelling, in the sense that once formed such networks are essentially impossible to destabilize, as there is no possible reorganization that would be improving for all of the players whose consent is needed.

Strongly stable networks are those which are supported by strong Nash equilibria of an appropriate game of network formation. In network formation, individual or pairwise based solution concepts such as Nash equilibrium and pairwise stability (see Jackson and Wolinsky (1996)) often lead to many stable networks, so that they provide broad predictions. We study strongly stable networks as a natural way for making narrower predictions using coalitional considerations. One can think of a notion such as pairwise stability as a weak stability concept which is essentially a necessary condition for stability, while strong stability is essentially a sufficient condition for stability in almost any context.

Another reason for examining the existence of strongly stable networks, beyond their compelling stability properties, is that such networks exhibit additional properties. For instance, if a network is strongly stable and has more than one component it turns out that value must be allocated equally among members of each component, and in fact the per capita value must be equal across components. This is a strong equity property. More importantly, strongly stable networks have strong efficiency properties.

[1]For bibliographies on network study generally and network formation in particular, we refer the reader to Slikker and van den Nouweland (2001) and Dutta and Jackson (2001).

One obvious property is Pareto efficiency. But if the value of each component of a network is allocated equally among the members of that component of a network, then when strongly stable networks exist they exhibit even stronger efficiency properties. In this case, strongly stable networks maximize the overall value of the network. This statement actually takes a bit of proof, as although it is obvious if a network consists of just one component, it is more subtle when efficient networks consist of several components.

Part of the motivation for the study of the existence of networks that are efficient and satisfy some stability requirement comes out of the previous literature. From previous research, we know that there are a variety of contexts where the stability of networks can be at odds with efficiency. Jackson and Wolinsky (1996) show that for some settings the sets of pairwise stable networks and efficient networks do not intersect. Moreover, for some value functions they showed that this is true regardless of how value is allocated or transferred among players, provided the allocation respects component balance and anonymity (which are formally defined below). Jackson (2001) goes on to show that even a weaker form of efficiency is at odds with pairwise stability, and that in some very natural contexts even Pareto efficiency can be widely incompatible with pairwise stability.

The tension between stability and efficiency suggests several directions for further study. One is to examine whether the tension disappears if we are free to construct the allocation rule in careful and non-anonymous ways. This angle is pursued by Dutta and Mutuswami (1997) who show that careful construction of allocation rules that may be non-anonymous (on unstable networks) can restore the compatibility between efficiency and stability.[2] Another direction is to identify those settings for which there is no tension between stability and efficiency (or at least that there is an overlap between the two) when keeping with anonymity. That direction is pursued both in Jackson and Wolinsky (1996) and Jackson (2001), when the concept in question is pairwise stability. The current paper is in that same spirit, but moves beyond pairwise stability to strong stability. As we shall see, efficient networks and strongly stable networks will coincide when the latter exist. Of course, the existence of strongly stable networks is of interest beyond efficiency, given that such networks are robust to all kinds of deviations, as we

[2] Another interesting direction, not as closely related to what we examine here, is to study situations where the allocation rule and networks are formed simultaneously and endogenously. This is explored in Currarini and Morelli (2000), Mutuswami and Winter (2000) and Slikker and van den Nouweland (2001b). As shown by Currarini and Morelli (2000), at least for some bargaining protocols, efficiency can be regained in some settings.

have already discussed above.

The paper proceeds as follows. In the next section we provide definitions. In Section 3 we first show that the existence of strongly stable networks requires an egalitarian allocation. Next, we characterize the existence of strongly stable networks under the component-wise egalitarian allocation rule in terms of nonemptiness of the core of a closely related cooperative game. We use this in Section 4 to obtain a characterization of the value functions for which there exist strongly stable networks, showing that a "top convexity" condition is both necessary and sufficient. We provide applications of these results to a variety of settings. In Section 5 we move on to consider side payments, showing that the characterizations in the previous sections relating to the component-wise egalitarian allocation rule are in fact necessary for any allocation rule when strong stability allows for side payments. Finally, we close the paper with some results on non-anonymous value functions in Section 6 and some concluding remarks in Section 7.

2 Definitions

Networks

There is a set $N = \{1, \ldots, n\}$ of players who may be involved in network relationships.

Non-directed graphs are used to model the network relations between players.[3] In such a graph the nodes (vertices) correspond to the players and the links (edges) correspond to bilateral relationships between players. Let g^N be the set of all subsets of N of size 2, and similarly for any $S \subseteq N$ let g^S be the set of all subsets of S of size 2. $G = \{g \mid g \subseteq g^N\}$ is the set of all possible networks or graphs on N.

The link between players i and j is denoted by ij.

A network g induces a *partition* $\Pi(g)$ of the player set N, where two players i and j are in the same partition element if and only if there exists a path[4] in the graph connecting i and j (using the convention that there is a path from each player to

[3]For some analysis of network formation in directed networks see Bala and Goyal (2000) and Dutta and Jackson (2000). The general problem of strong stability in directed networks has not been studied.

[4]Formally, a path in g from i to j is a sequence of players $i_1, \ldots, i_K$ such that $i_k i_{k+1} \in g$ for each $k \in \{1, \ldots, K-1\}$, with $i_1 = i$ and $i_K = j$.

him or herself). For any $S \in \Pi(g)$, $g(S)$ denotes the subgraph of g on the set S, i.e. $g(S) = g \cap g^S$.

The *components* of a network g, denoted $C(g)$, are defined by $C(g) = \{g(S) | S \in \Pi(g), |S| \geq 2\}$. The restriction that $|S| \geq 2$ rules out empty networks as components.

The Value of a Network

The value of a network is given by a value function $v : G \rightarrow I\!R$. We normalize v so that $v(\emptyset) = 0$. The set of all such value functions is denoted V.

A value function is *anonymous* if for any permutation of the set of players π (a bijection from N to N), $v(g^\pi) = v(g)$, where $g^\pi = \{\pi(i)\pi(j) | ij \in g\}$.

Anonymity says that the value of a network is derived from the structure of the network and not the labels of the players who occupy certain positions. For many of the results we will restrict our attention to anonymous value functions, and we discuss extensions to non-anonymous value functions in a later section of the paper.

A value function is *component additive* if $v(g) = \sum_{h \in C(g)} v(h)$ for all $g \in G$.

Component additivity precludes that the value of a given component of a network depends on how other components are organized. This precludes externalities across components of a network. However, it still allows for externalities within components. That is, the value of a given component, and ultimately each player's payoff, can depend on the way that the network is structured. For example, the value of $\{12, 23\}$ can differ from $\{12, 23, 13\}$, and so, for instance, player 2's payoff may depend on whether 1 and 3 are linked.

Allocation Rules

An *allocation rule* is a function $Y : G \times V \rightarrow I\!R^n$ describes how the value of a network is distributed among the players. The payoff of player $i \in N$ in network g with a value function v under allocation rule Y is denoted $Y_i(g, v)$.

The allocation function may arise naturally, or might also represent additional transfers of value among players. We can be agnostic on whether the allocation rule arises naturally, is derived from some bargaining among players, or is forced by some government or other intervening party.

An allocation rule Y is *component balanced* if $\sum_{i \in S} Y_i(g, v) = v(g(S))$ for each component additive v, $g \in G$ and $S \in \Pi(g)$.

Component balance requires that the value of a given component of a network is allocated to the members of that component in cases where the value of the component is independent of how other components are organized. This would tend to arise naturally. It also is a condition that an intervening planner or government would like to respect if they wish to avoid secession by components of the network.

An allocation rule Y is *component decomposable* if $Y_i(g,v) = Y_i(g(S),v)$ for each component additive v, $g \in G$, $S \in \Pi(g)$, and $i \in S$.

Component decomposability requires that in situations where v is component additive, the way in which value is allocated within a component does not depend on the structure of other components. So, in situations where there are no externalities across components, the allocation within a component is independent of the rest of the network.

An allocation rule Y is *anonymous* if for any $v \in V$, $g \in G$, and permutation of the set of players π, $Y_{\pi(i)}(g^\pi, v^\pi) = Y_i(g,v)$, where the value function v^π is defined by $v^\pi(g) = v(g^{\pi^{-1}})$ for each $g \in G$.

Anonymity of an allocation rule requires that if all that has changed is the labels of the agents and the value generated by networks has changed in an exactly corresponding fashion, then the allocation only change according to the relabeling.

Given any $v \in V$, the *component-wise egalitarian allocation rule* Y^{ce} is defined by

$$Y_i^{ce}(g,v) = \frac{v(g(S_i))}{|S_i|},$$

where $S_i \in \Pi(g)$ is the unique partition element containing player i.

The component-wise egalitarian rule is one where the value of each component is split equally among the members of the component. This allocation rule is anonymous, component balanced, and component decomposable, and satisfies nice egalitarian properties in terms of equalizing payoffs (at least within the limit of component balance).

As we shall see, this allocation rule will actually emerge naturally if one wishes to have strongly stable networks, and will play a key role in the characterization of value functions that allow such networks.

Efficiency and Stability Notions

A network g is *efficient* with respect to v if $v(g) \geq v(g')$ for all $g' \in G$.

We denote the set of networks that are efficient with respect to value function v by $E(v)$.

Note that an efficient network always exists since there are only finitely many networks in G. This is a strong notion of efficiency as it requires the maximization of total value. It only corresponds to Pareto efficiency if the value is freely and fully transferable across all components of a network.[5]

[5]For discussion of this and some weaker notions of efficiency see Jackson (2001).

The following definition of coalitional deviation is used in defining the strong stability notion.

A network $g' \in G$ *is obtainable from* $g \in G$ *via deviations by* S if

(i) $ij \in g'$ and $ij \notin g$ implies $ij \subseteq S$, and

(ii) $ij \in g$ and $ij \notin g'$ implies $ij \cap S \neq \emptyset$.

The above definition identifies changes in a network that can be made by a coalition S, without the need of consent of any players outside of S. (i) requires that any new links that are added can only be between players in S. This reflects the fact that consent of both players is needed to add a link. (ii) requires that at least one player of any deleted link be in S. This reflects that fact that either player in a link can unilaterally sever the relationship.

A network g is *strongly stable* with respect to allocation rule Y and value function v if for any $S \subseteq N$, g' that is obtainable from g via deviations by S, and $i \in S$ such that $Y_i(g', v) > Y_i(g, v)$, there exists $j \in S$ such that $Y_j(g', v) < Y_j(g, v)$.

We denote the set of networks that are strongly stable with respect to Y and v by $SS(Y, v)$.

The definition of strong stability we use here is slightly stronger (i.e., harder to satisfy) than that originally introduced by Dutta and Mutuswami (1997). The definition of strong stability here allows for a deviation to be valid if some members are strictly better off and others are weakly better off, while the definition in Dutta and Mutuswami (1997) considers a deviation valid only if all members of a coalition are strictly better off. For many value functions these definitions coincide.

There are several reasons for working with this stronger definition of strong stability. First, it implies pairwise stability whereas the Dutta and Mutuswami (1997) version of strong stability does not quite imply pairwise stability.[6] Second, this stronger definition allows for a stronger implication in Theorem 2, where we conclude that under certain conditions on the value function all efficient networks are strongly stable. Third, the converse of this statement in Theorem 2 is only true with the stronger definition of

[6]Pairwise stability (from Jackson and Wolinsky (1996)) is defined as follows. A network $g \in G$ is *pairwise stable* with respect to allocation rule Y given a value function $v \in V$ if no player benefits from severing one of their links and no two players benefit from adding a link between them, with one benefiting strictly and the other at least weakly. This last part of the definition is what makes our version of strong stability compatible with pairwise stability but the Dutta and Mutuswami version incompatible.

strong stability. Finally, if all members of a coalition are weakly better off and some strictly better off, then any ability of members to make even tiny transfers will result in a deviation. As we compare the definition of strong stability with what happens when transfers are possible, this slightly stronger notion of stability is natural.

Such differences between weak and strong inequalities are common to definitions of Pareto efficiency, the core, strong Nash equilibrium, and coalitional stability properties; and the difference sometimes has consequences. In working with the stronger definition here, one ends up with a more attractive solution when it is non-empty, but in cases where it is empty one might also wish to examine the weaker solution.

We remark that the strongly stable networks correspond exactly to the strong Nash equilibria of the network formation game suggested by Myerson (1991). In that game players simultaneously announce the set of players with whom they wish to be linked and a link between two players forms if and only if both players have named each other.[7]

Cooperative Games and the Core

A *TU cooperative game* is a pair (N, w), where N is the set of players and $w : 2^N \to \mathbb{R}$ defines the productive value of each subset of N. In line with this interpretation $w(\emptyset) = 0$.

As we fix N throughout our analysis, we often refer to a characteristic function $w : 2^N \to \mathbb{R}$ as a cooperative game.

An allocation $x \in \mathbb{R}^N$ is in the *core* of w if $\sum_{i \in N} x_i = w(N)$ and $\sum_{i \in T} x_i \geq w(T)$ for all $T \subseteq N$.

3 The Existence of Strongly Stable Networks, Efficiency and the Core

Let us begin by showing that strong stability has some particular implications about the structure of the allocation rule that must be in place.

[7]The equivalence holds for the corresponding definition of strong Nash equilibrium which requires that there are no deviations by a coalition that make all members weakly better off and some strictly better off. There are some details to verify, as there are some strong Nash equilibria where one player names another but is not reciprocated. It is easy to check that the networks formed in such equilibria must be strongly stable networks.

Theorem 1 *Consider any anonymous and component additive value function $v \in V$. If Y is an anonymous, component decomposable, and component balanced allocation rule and $g \in G$ with $\Pi(g) \neq \{N\}$ is a network that is strongly stable with respect to Y and v, then $Y(g, v) = Y^{ce}(g, v)$ and $Y_i(g, v) = \frac{v(g)}{n}$ for each $i \in N$.*

The proof of this theorem and all other results are collected in the appendix.

Theorem 1 says that if we find strong stability of a network that has more than one component to it, then the allocation must be as it would be under the component-wise egalitarian rule and in fact must involve an equal split of the total value of the network. The idea is that otherwise, some player in some component (perhaps completely disconnected) is getting a payoff below some other player in some other component and could deviate together with the other members of the second component to provide an improving deviation.

Theorem 1 shows that a component-wise egalitarian allocation of value will necessarily play a prominent role in the analysis of strongly stable networks.

The condition in the theorem that $\Pi(g) \neq \{N\}$ is critical to the result. This is demonstrated in the following example.

Example 1 *A Strongly Stable Network with One Component.*

The are three individuals. Networks with two links have value 2.5, the complete network has value 3, and other networks have 0 value. Consider the allocation rule where the middle player in a two link network (e.g., player 2 in $\{12, 23\}$) gets a payoff of .1 and the other two players get a payoff of 1.2, in the complete network each player gets 1 and in networks with at most one link each player gets 0. In this example, any network with two links is strongly stable. This relies on all players being part of a single component in g.

As we will see below, the strong stability of the two link network under the non-egalitarian allocation rule in the previous example depends critically on the inability of players to make transfers to each other. Otherwise, they would deviate to form the complete network. We return to make this point formally in Section 5 below.

To get insight into the role of component decomposability in the theorem, consider the following example.

Example 2 *The Role of Component Decomposability.*

There are $n = 6$ players. Let v be defined by $v(\{12,23\}) = 10$, $v(\{12\}) = 4$, and v is anonymous and component additive, so permutations of the above networks have the same value, and $v(\{12,23,45\}) = 14$, $v(\{12,23,45,56\}) = 20$, $v(\{12,34,56\}) = 12$, and so on. For any other structure of a component (that has three or more links) we let v have a value of 0.

On efficient networks such as $\{12,23,45,56\}$, set $Y_1(\{12,23,45,56\}) = Y_3(\{12,23,45,56\}) = Y_4(\{12,23,45,56\}) = Y_6(\{12,23,45,56\}) = 3$, and $Y_2(\{12,23,45,56\}) = Y_5(\{12,23,45,56\}) = 4$. Also, set $Y_1(\{12,23,45\}) = Y_1(\{12,23\}) = Y_3(\{12,23\}) = Y_3(\{12,23,45\}) = 5$, $Y_2(\{12,23,45\}) = Y_2(\{12,23\}) = 0$, and, in accordance with anonymity and component balance, $Y_4(\{12,23,45\}) = Y_4(\{45\}) = Y_5(\{45\}) = Y_5(\{12,23,45\}) = 2$. Set Y elsewhere to respect anonymity and component balance.

Note that $\{12,23,45,56\}$ is strongly stable, and yet Y differs from the component-wise egalitarian rule. In particular, Y adjusts on $\{12,23\}$ depending on how 4, 5, and 6 are linked, if at all. We have done this in such a way to preclude blocking by a coalition involving some players from $\{1,2,3\}$ and players from $\{4,5,6\}$. However, the allocation rule Y violates component decomposability.

Given the implications of Theorem 1 we focus on the component-wise egalitarian rule in what follows. This is with some loss of generality, as Theorem 1 does not imply that Y must equal Y^{ce} on all networks, or in cases where all players are in a single component, as indicated above. We will return to consider more general allocation rules later.

Given a value function v, let the cooperative game (N, w^v) be defined by

$$w^v(S) = \max_{g \in g^S} v(g).$$

Thus, every value function $v \in V$ defines a cooperative game where the value of a coalition is the maximum value it can obtain by arranging its members in a network.

Note that if v is anonymous, then w^v is symmetric (so $w^v(S) = w^v(T)$ whenever $|S| = |T|$). Also, if v is component additive, then w^v is superadditive. That is, $w^v(S \cup T) \geq w^v(S) + w^v(T)$ whenever $S \cap T = \emptyset$.

Theorem 2 *Consider any anonymous and component additive value function $v \in V$. Some efficient $g \in G$ with respect to v is strongly stable with respect to $Y^{ce}(\cdot, v)$ if and only if the core of w^v is nonempty. Moreover, $SS(Y^{ce}, v) \neq \emptyset$ if and only if $E(v) = SS(Y^{ce}, v)$.*

Theorem 2 shows that our interest in guaranteeing that a society forms efficient networks is closely tied to the non-emptiness of the core of a related cooperative game. This allows us to make use of the substantial knowledge on core existence in cooperative game theory to analyze the efficiency of network formation.

On a superficial level Theorem 2 seems obvious, since both strong stability and the core notion allow for deviations by arbitrary subsets of players.[8] However, there are several levels on which Theorem 2 is not obvious (which can also be seen from the proof). Moreover, these less obvious points are those which result in some of the theorem's power and usefulness, as we will discuss in what follows.

In particular, some of the differences are as follows. Strong stability allows for a deviating coalition to maintain links with non-deviating players (and keeps the rest of the network intact), while the core notion requires complete separation by a deviating coalition. This gives better opportunities for a coalition to improve under the strong stability notion. Working in the other direction is that the core allows for transfers to be made among players in a deviating coalition regardless of how that coalition derives its value, while under component balance a deviating coalition under the strong stability notion cannot make transfers across components of a new network that is formed. With these two critical differences, there is no obvious reason to expect the relationship outlined in the theorem to hold in general. Moreover, the last part of the theorem shows that it is not simply that there exists a network that is strongly stable with respect to $Y^{ce}(\cdot, v)$, but that the efficient networks and strongly stable networks with respect to $Y^{ce}(\cdot, v)$ coincide.

Application to Communication Networks and Convex Games

A special type of value function are those derived from some anonymous production function that depends on the agents who can communicate. The production function is represented by a characteristic function $z(S)$ which indicates the productive value of any coalition S, provided they can communicate through the network. Each link in

[8]With the strong stability notion in Dutta and Mutuswami (1997), where a deviation is valid only if all members of a coalition are strictly better off, the equivalence does not hold. An example shows this. There are 5 players. We describe an anonymous and component additive value function v. A network encompassing 3 players has value 7 and a network encompassing 2 players has value 3. A network that consists of one 2-player component and one 3-player component has value 10. All other networks have value 0. In this setting, an efficient network consists of two components, one encompassing 2 players and the other 3. Under the component-wise egalitarian rule, such a network is strongly stable as defined by Dutta and Mutuswami (1997). However, it follows by standard game-theoretic arguments that the core of w^v is empty.

a network incurs a cost c.

To be specific: a given cooperative game z and cost per link c lead to the value function $v^{z,c} \in V$ defined by

$$v^{z,c}(g) = \sum_{S \in \Pi(g)} z(S) - \sum_{ij \in g} c.$$

A characteristic function z is *zero-normalized* if $z(\{i\}) = 0$ for each $i \in N$.

Given a symmetric cooperative game z, let $z_k = z(S)$ where $|S| = k$. So we write z as a function of coalition size given the anonymity inherent from symmetry. Let Z denote the class of zero normalized symmetric cooperative games.

A cooperative game $z \in Z$ is *convex* if

$$\forall k \geq 2 : z_k - z_{k-1} \geq z_{k-1} - z_{k-2}$$

Corollary 3 *Consider any convex cooperative game $z \in Z$ and any cost per link $c \geq 0$. Then $E(v^{z,c}) = SS(Y^{ce}, v^{z,c})$.*

Corollary 3 shows that Theorem 2 has powerful implications, as the class of communication games with convex production and costly links is a wide class.

The proof of Corollary 3 is achieved by showing that the cooperative game $w^{v^{z,c}}$ is convex and thus has a non-empty core. This is not immediate since although z is convex, one needs to show that the induced game is still convex when link costs are accounted for.

The scope of Corollary 3 does not extend arbitrarily to a class of games that is larger than the class of convex games. We demonstrate this in the following example.

Example 3 *A Non-Convex Game.*

Consider the cooperative 5-player game (N, z) defined by $z(S) = |S|$ if $|S| \geq 2$ and $z(S) = 0$ otherwise. This game is obtained from an additive game in which each player contributes 1 to every coalition by setting the worth of one-player coalitions equal to 0. Suppose that $0 < c < 1$. Then an efficient network g consists of two components, one with two players connected by a link and the other with three players connected by two links. A network that is strongly stable with respect to $Y^{ce}(\cdot, v^{z,c})$ partitions the player set into three components, two of which have two players connected by one link and one of which consists of an isolated player. Hence, no network that is efficient with respect to $v^{z,c}$ is strongly stable with respect to $Y^{ce}(\cdot, v^{z,c})$. In fact, it can be shown that for *any* anonymous and component balanced allocation rule Y it holds that $E(v^{z,c}) \cap SS(Y, v^{z,c}) = \emptyset$.

4 Primitive Conditions on Value Functions

While the non-emptiness of the core of the associated cooperative game w^v is an interesting and useful condition, as illustrated at the end of the last section, we are also interested in direct conditions on v which characterize the strong stability of efficient networks. Theorem 2 is still useful in this regard, as the characterization of v's that allow for strongly stable networks to exist (and then coincide with efficient networks) can be obtained through the conditions on w^v.

A value function v is *top convex* if some efficient network also maximizes the per-capita value among individuals.[9] Formally, let $p(v,S) = \max_{g \in g^S} \frac{v(g)}{|S|}$.

The value function v is *top convex* if $p(v,N) \geq p(v,S)$ for all S.

One implication of top convexity is that all components of an efficient network must lead to the same per-capita value. If some component led to a lower per capita value than the overall average, then some other component would have to lead to a higher per capita value. As we now see, top convexity plays a key role in the existence of strongly stable networks.

Theorem 4 *Consider any anonymous and component additive value function v. The core of w^v is nonempty if and only if v is top convex. Thus, $E(v) = SS(Y^{ce}, v)$ (or $SS(Y^{ce}, v) \neq \emptyset$) if and only if v is top convex.*

Theorem 4 shows that one needs strong conditions on v in order to get have nice properties in terms of the set of strongly stable networks. Nevertheless, the top convexity condition is satisfied by many v's, and we now point out several such value functions.

Example 4 *The Symmetric Connections Model*

[9] A related condition is called "domination by the grand coalition," as defined in the context of a cooperative game by Chatterjee, Dutta, Ray, and Sengupta (1993). That condition requires that the per capita value of the grand coalition be at least that of any sub-coalition. Shubik (1982, page 149) shows that for symmetric cooperative games this condition is a necessary and sufficient condition for nonemptiness of the core. The top convexity condition we identify here is defined for the network setting, but is equivalent to requiring that w^v be dominated by the grand coalition. In a bargaining context, Chatterjee, Dutta, Ray and Sengupta show that this condition is equivalent to existence of a sequence of limiting efficient stationary equilibria for each bargaining protocol in a wide class.

The symmetric connections model of Jackson and Wolinsky (1996) is one where links represent social relationships between individuals; for instance friendships.[10] These relationships offer benefits in terms of favors, information, etc., and also involve some costs. Moreover, individuals also benefit from indirect relationships. A "friend of a friend" also results in some benefits, although of a lesser value than a "friend," as do "friends of a friend of a friend" and so forth. The benefit deteriorates with the "distance" of the relationship. For instance, in the network $g = \{12, 23, 34\}$ individual 1 gets a benefit $\delta < 1$ from the direct connection with individual 2, an indirect benefit δ^2 from the indirect connection with individual 3, and an indirect benefit δ^3 from the indirect connection with individual 4. As $\delta < 1$, this leads to a lower benefit from an indirect connection than a direct one. Individuals only pay costs, however, for maintaining their direct relationships.

Formally, the payoff player i receives from network g is

$$u_i(g) = \sum_{j \neq i} \delta^{t(ij)} - \sum_{j: ij \in g} c,$$

where $t(ij)$ is the number of links in the shortest path between i and j (setting $t(ij) = \infty$ if there is no path between i and j). The value in the connections model of a network g is simply $v(g) = \sum_i u_i(g)$.

It is easily seen that v is top convex for all values of $\delta \in [0, 1)$ and $c \geq 0$, so that all networks that are strongly stable with respect to Y^{ce} and v are efficient with respect to v.[11]

We remark that $Y_i^{ce}(g, v) \neq u_i(g)$ for some networks g. Thus, our result is not in contradiction with the finding of Jackson and Wolinsky (1996) that sometimes none of the pairwise stable networks (under u_i) are efficient in the connections model. Here the reallocation of value under the component-wise egalitarian rule helps in guaranteeing stability of the efficient network.[12]

[10] For further study of variations on the connections model, see Johnson and Gilles (2000), Watts (2001), and Jackson (2001).

[11] The proof of Proposition 1 in Jackson and Wolinsky (1996) provides some hints to the interested reader in filling in omitted details. Most importantly, for intermediate cost ranges the per capita value of the (efficient) star network is growing in the number of players in the star.

[12] More generally (beyond the connections model) Jackson and Wolinsky (1996) study when using the component-wise egalitarian rule provides for the pairwise stability of some efficient network. The characterizing condition that they identify, critical link monotonicity, is necessarily a weaker condition than top convexity, as pairwise stability is correspondingly weaker than strong stability.

Example 5 *The Co-Author Model*

The co-author model (from Jackson and Wolinsky (1996)) is described as follows. Each individual is a researcher who spends time working on research projects. If two researchers are connected, then they are working on a project together. The amount of time researcher i spends on a given project is inversely related to the number of projects, n_i, that he is involved in. Formally, i's payoff is represented by

$$u_i(g) = \sum_{j:ij \in g} \left(\frac{1}{n_i} + \frac{1}{n_j} + \frac{1}{n_i n_j} \right)$$

for $n_i > 0$, and $u_i(g) = 0$ if $n_i = 0$.[13] The total value is $v(g) = \sum_i u_i(g)$.

Provided that n is even, it is easily seen that v is top-convex as the efficient network always involves pairs of players who are only linked to each other. Thus strongly stable networks exist in this situation, and correspond to the networks with evenly matched pairs. If n is odd, top convexity is violated (dropping some individual increases the per capita value obtainable), and no strongly stable networks exist.[14]

Example 6 *Bilateral Bargaining Model*

Corominas-Bosch (1999) considers a bargaining model where buyers and sellers bargain over prices for trade. A link is necessary between a buyer and seller for a transaction to occur, but if an individual has several links then there are several possibilities as to whom they might transact with. Thus, the network structure essentially determines bargaining power of various buyers and sellers.

More specifically, each seller has a single unit of an indivisible good to sell which has no value to the seller. Buyers have a valuation of 1 for a single unit of the good. If a buyer and seller exchange at a price p, then the buyer receives a payoff of $1 - p$ and

[13] It might also make sense to set $u_i(g) = 1$ when an individual has no links, as the person can still produce research. This is not in keeping with the normalization of $v(\emptyset) = 0$, but it is easy to simply subtract 1 from all payoffs and then view Y as the extra benefits above working alone.

[14] Our results tell us that efficient networks are the only candidates. If players are matched in pairs, there is always a player left out. A coalition of some matched player and the unmatched player can deviate, making the unmatched player better off with the matched player being indifferent. This provides an interesting example to discuss the precise definition of strong stability. If we instead require both players to be better off in a deviation, then there does exist a stable network here. However, with any sort of side payments (as we discuss below) there would not exist a stable network.

the seller a payoff of p. A link in the network represents the opportunity for a buyer and seller to bargain and potentially exchange a good.[15]

Regardless of any costs to links, it is clear that per-capita value is maximized with buyers and sellers paired up. So, if there is a matched number of buyers and sellers, then v is top convex and so strongly stable networks exist and coincide with the efficient ones. As with the co-author model, if there is not a matched number, then v is not top convex as a subcoalition excluding the extra unmatched players could increase per-capita value. So, in this case no strongly stable network exists.

5 Strong Stability with Side Payments

Once we allow for coalitional deviations, so presumably coalitions can coordinate their actions, in many contexts it is reasonable to assume that they will also be able to reallocate value. This leads to the formulation of an even stronger stability concept.

Say that g is SSS (strongly stable with side payments) relative to an allocation rule Y and value function v if $\sum_{i\in S} Y_i(g,v) \geq \sum_{i\in S} Y_i(g',v)$ for any $S \subseteq N$ and g' obtainable from g by S. We denote this set $SSS(Y,v)$.

Theorem 5 *Let $v \in V$ be component additive and anonymous. The following statements are equivalent:*

(i) there exists a component balanced allocation rule Y such that $SSS(Y,v) \neq \emptyset$,

(ii) there exists a component balanced allocation rule Y such that $SSS(Y,v) = E(v)$,

(iii) $E(v) = SS(Y^{ce},v)$,

(iv) $E(v) = SSS(Y^{ce},v)$.

Theorem 5 reinforces the implications of Theorem 1 that component-wise egalitarian allocation of value plays a key role in the existence of strongly stable networks, this time including the possibility of side payments. So beyond Y^{ce}'s natural appeal in terms of egalitarian properties, we find that it is a key allocation rule to understand

[15] In the Corominas-Bosch framework links can only form between buyers and sellers. One can fit this into the more general setting where links can form between any individuals, by having the value function and allocation rule ignore any links except those between buyers and sellers.

when it comes to finding existence of strongly stable networks and for strongly stable networks with side payments.

An example shows that the result is not true if one changes SSS to SS in part (i) or (ii) of Theorem 5.

Example 7 *Strong Stability with Side Payments*

There are 6 players. A circle encompassing all six players has value 6 and a star encompassing four players has value 5. All other networks have value 0. For the allocation rule Y that we describe momentarily the efficient networks (circles) are exactly the strongly stable networks. According to Y each player gets 1 if they are in a circle. If g is a four person star, then the player who is the center of the star gets 0 and the three outside players in the star each get $\frac{5}{3}$. Players get 0 according to Y otherwise. For this Y, it holds that $E(v) = SS(Y, v) \neq \emptyset$. Under the component-wise egalitarian rule, however, the circle is not strongly stable. Hence, $E(v) \cap SS(Y^{ce}, v) = \emptyset$ and the equivalence in Theorem 5 would not hold.

If a network is SSS then it is stable in a very strong sense and so Theorem 5, together with our other results, shows that any top convex value function v (and only such value functions!) will have networks that are stable in very strong ways.

6 Non-anonymous Value Functions

So far, we have limited our attention to anonymous value functions. Let us consider the extent to which similar results hold for non-anonymous value functions.

If we do not require the value function to be anonymous, then the component-wise egalitarian rule is not as appealing.

Example 8 *The Component-Wise Egalitarian Rule for a Non-Anonymous Value Function.*

Consider a situation with 3 players and denote $g^1 = \{13\}$, the network with only the link between players 1 and 3, and $g^2 = \{23\}$. The value function v is defined by $v(g^1) = v(g^2) = 1$ and $v(g) = 0$ for all other $g \subseteq g^N$. Then x defined by $x_1 = x_2 = 0$ and $x_3 = 1$ is in the core of w^v. However, $E(v) = \{g^1, g^2\}$ and $SS(Y^{ce}, v) = \emptyset$, so that no efficient network is strongly stable (or strongly stable with side payments) with respect to the component-wise egalitarian rule. The reason is that Y^{ce} gives too much to players 1 and 2 and not enough to player 3.

The following theorem provides an analog of the previous results if we do not require the value function to be anonymous.

Theorem 6 *Let $v \in V$ be a component additive value function. The following statements are equivalent:*

(i) the core of w^v is nonempty,

(ii) there exists a component balanced allocation rule Y such that $SSS(Y, v) \neq \emptyset$,

(iii) there exists a component balanced allocation rule Y such that $E(v) = SSS(Y, v)$.

Moreover, top convexity of v implies each of the above and also implies that $E(v) = SS(Y^{ce}, v)$.

In the setting of non-anonymous value functions, top convexity of v, nonemptiness of the core of w^v, and $E(v) = SS(Y^{ce}, v)$ are no longer equivalent. In the example with which we started the current section, the core of w^v is nonempty, while $E(v) \neq SS(Y^{ce}, v)$ and v is not top convex.

Example 9 *Non-Anonymity and Top Convexity*

For an example of a value function v such that $E(v) = SS(Y^{ce}, v)$ while the core of w^v is empty (and v is not top convex), consider 4 players and define $g^1 = \{12\}$, $g^2 = \{34\}$, $g^3 = \{13, 34\}$, and $g^4 = \{23, 34\}$. The non-anonymous value function v is defined by $v(g^1) = 4$, $v(g^2) = 8$, $v(g^1 \cup g^2) = 12$, $v(g^3) = v(g^4) = 11$, and $v(g) = 0$ for all other $g \subseteq g^N$. Then, network $g^1 \cup g^2 = \{12, 34\}$ is the unique efficient network and it is also the unique network that is strongly stable with respect to Y^{ce} and v. However, the core of w^v is empty because any core element x would have to simultaneously satisfy the requirements $x_1 + x_2 = 4$, $x_3 + x_4 = 8$, $x_1 + x_3 + x_4 \geq 11$, and $x_2 + x_3 + x_4 \geq 11$, which is clearly impossible.

7 Concluding Remarks

Our main results may be summarized as follows. First, Theorem 1 showed that the component-wise egalitarian rule plays a prominent role in the study of the existence strongly stable networks. This was reinforced in some of the other results which are, for anonymous value functions, all captured in the following theorem.

Theorem 7 *Let v be component additive and anonymous. The following statements are equivalent:*[16]

(i) $SS(Y^{ce}, v) \neq \emptyset$*,*

(ii) $SS(Y^{ce}, v) = E(v)$*,*

(iii) the core of w^v *is nonempty,*

(iv) v *is top convex,*

(v) $SSS(Y^{ce}, v) \neq \emptyset$*,*

(vi) $SSS(Y^{ce}, v) = E(v)$*,*

(vii) there exists a component balanced allocation rule Y *such that* $SSS(Y, v) \neq \emptyset$*,*

(viii) there exists a component balanced allocation rule Y *such that* $SSS(Y, v) = E(v)$*.*

Theorem 6 summarizes the results for non-anonymous value functions.

Throughout our analysis in this paper we have focused our attention on component additive value functions. These are natural in the context of some social relationships, exchange relationships, etc., but are not so natural when different components of the network might be in competition with each other (e.g., political or trade alliances). On one level, once we move beyond component additive value functions, Y^{ce} exhibits even stronger properties. That is because under our definitions, Y^{ce} can split value completely evenly among all players and thus result in exactly the set of efficient networks always being strongly stable. Thus strongly stable networks always exist and coincide with the efficient networks.

[16] Note that (v) was not included in our earlier statements, but is easily seen to be equivalent given that it is implied by (vi) and implies (i).

This conclusion, however, depends on how one defines component balance when v is not component additive. If one has further information about the value accruing to each component when v is not component additive, then one could require that Y allocate the value of each component to that component even when there exist externalities.[17] With externalities, how players are arranged when some group deviates matters in determining the value of the deviating coalition. This changes the nature of stable networks under a variety of different stability concepts, as is nicely demonstrated in a new paper by Currarini (2002). The general existence of strongly stable networks in such settings is a difficult and open problem.[18]

[17]The argument for doing this in the presence of externalities is not quite as clear cut as in the case where no externalities are present, unless one assumes that no transfers are made at all.

[18]The problem has some similarities to the existence of core stable partitions in coalition formation games when there are externalities. See Bloch (2001) for some discussion of that problem.

References

Bala, V. and S. Goyal (2000) "Self-Organization in Communication Networks," *Econometrica*, 68, 1131–1230.

Bloch, F. (2001) "Networks and Coalitions in Industrial Organization," Université d'Aix Marseille.

Chatterjee, K., B. Dutta, D. Ray, and S. Sengupta (1993) "A Noncooperative Theory of Coalitional Bargaining," *Review of Economic Studies*, 60, 463–477.

Corominas-Bosch, M. (1999) "On Two-Sided Network Markets," Ph.D. dissertation: Universitat Pompeu Fabra.

Currarini, S. (2002) "Stable Networks with Externalities," mimeo.

Currarini, S. and M. Morelli (2000) "Network Formation with Sequential Demands," *Review of Economic Design*, 5, 229–250.

Dutta, B. and M.O. Jackson (2000) "The Stability and Efficiency of Directed Communication Networks," *Review of Economic Design*, 5, 251–272.

Dutta, B. and M.O. Jackson (2001) Introductory chapter in *Models of the Formation of Networks and Groups*, edited by B. Dutta and M.O. Jackson, forthcoming from Springer-Verlag: Heidelberg.

Dutta, B. and S. Mutuswami (1997) "Stable Networks," *Journal of Economic Theory*, 76, 322–344.

Jackson, M.O. (2001) " The Stability and Efficiency of Economic and Social Networks," forthcoming in *Models of the Formation of Networks and Groups*, edited by B. Dutta and M.O. Jackson, Springer-Verlag: Heidelberg.

Jackson, M.O. and A. Wolinsky (1996) "A Strategic Model of Social and Economic Networks," *Journal of Economic Theory*, 71, 44–74.

Johnson, C. and R.P. Gilles (2000) "Spatial Social Networks," *Review of Economic Design*, 5, 273–300.

Mutuswami, S. and E. Winter (2000) "Subscription Mechanisms for Network Formation," mimeo: CORE and Hebrew University in Jerusalem.

Myerson, R (1977) "Graphs and Cooperation in Games," *Mathematics of Operations Research*, 2, 225-229.

Myerson, R. (1991) *Game Theory: Analysis of Conflict*, Harvard University Press: Cambridge, MA.

Shubik, M. (1982) *Game Theory in the Social Sciences: Concepts and Solutions*, The MIT Press: Cambridge, MA.

Slikker, M. and A. van den Nouweland (2001) *Social and Economic Networks in Cooperative Game Theory*, Kluwer Academic Publishers: Boston, MA.

Slikker, M. and A. van den Nouweland (2001b) "A One-Stage Model of Link Formation and Payoff Division," *Games and Economic Behavior*, 34, 153-175.

Watts, A. (2001) "A Dynamic Model of Network Formation," *Games and Economic Behavior*, 34, 331-341.

Appendix

Proof of Theorem 1: Consider an anonymous and component additive v and any anonymous, component decomposable, and component balanced allocation rule Y. Consider $g \in G$ that has more than one component and is strongly stable. It follows from component balance of Y that $\sum_{i \in N} Y_i(g, v) = v(g)$. Consider any S and $S' \in \Pi(g)$ such that $S \neq S'$. Without loss of generality, assume that $\max_{i \in S} Y_i(g, v) \geq \max_{i \in S'} Y_i(g, v)$. Find $j \in \text{argmax}_{i \in S} Y_i(g, v)$ and $k \in \text{argmin}_{i \in S'} Y_i(g, v)$. To prove that $Y_i(g, v) = \frac{v(g)}{n}$ for all i, we need only show that $Y_j(g, v) = Y_k(g, v)$. Suppose, to the contrary that $Y_j(g, v) > Y_k(g, v)$. Consider a deviation by $S \cup \{k\} \setminus \{j\}$ so that k severs all links under g, $S \setminus \{j\}$ severs all links with j, and $S \cup \{k\} \setminus \{j\}$ form a component h' that is a duplicate of $g(S)$ with k replacing j. By component decomposability and anonymity it follows that $Y_i(h', v) = Y_i(g, v)$ for all $i \in S \setminus \{j\}$ and $Y_k(h', v) = Y_j(g, v) > Y_k(g, v)$. This contradicts the strong stability of g via a deviation by $S \cup \{k\} \setminus \{j\}$. Thus our supposition was incorrect. Given that Y is component balanced and $Y_i(g, v) = \frac{v(g)}{n}$ for all i, it follows that $Y_i^{ce}(g, v) = \frac{v(g)}{n}$ for all i. ∎

Proof of Theorem 2: The following lemma is useful.

Lemma 8 *Consider an anonymous and component additive value function $v \in V$. If the core of w^v is nonempty, then x defined by $x_i = \frac{w^v(N)}{n}$ for each i is in the core of w^v.*

Proof of Lemma 8: Given the symmetry of w^v (implied by the anonymity of v), the core of w^v is symmetric. The core is also convex by standard arguments. The statement of the lemma follows from the convexity and symmetry of the core of w^v, as taking any x in the core and averaging all of its permutations leads to identical payoffs of $\frac{w^v(N)}{n}$.[19] ∎

To complete the proof of Theorem 2, we prove that for any anonymous and component additive value function v the following statements are equivalent

(1) $SS(Y^{ce}, v) \neq \emptyset$,

(2) $SS(Y^{ce}, v) = E(v)$,

(3) the core of w^v is nonempty.

[19] A similar proof in a different context appears in Shubik (1982, page 149).

It is clear that (2) implies (1). We start by showing that (1) implies (3).

Suppose to the contrary that g is strongly stable with respect to $Y^{ce}(\cdot, v)$, and that the core of w^v is empty. Since by supposition the core is empty, we know that $Y^{ce}(g, v)$ is not a core element. Because $g \in SS(Y^{ce}, v)$, it holds that $Y_i^{ce}(g, v) = \frac{v(g)}{n}$ for each i (this follows by Theorem 1 when there is more than one component, and directly otherwise). Thus, there exists a $T \subseteq N$ such that $w^v(T) > \sum_{i \in T} \frac{v(g)}{n}$, which implies that $\frac{w^v(T)}{|T|} > \frac{v(g)}{n}$. By the definition of w^v it then follows that there exists some $S \subseteq T$ and g' with $S \in \Pi(g')$ such that $\frac{v(g'(S))}{|S|} > \frac{v(g)}{n}$. This contradicts the strong stability of g. So, our supposition was incorrect and the conclusion is established.

Next, let us show that (3) implies (1).

We show the stronger statement that if the core of w^v is nonempty, then $E(v) \subseteq SS(Y^{ce}, v)$. Suppose that the core of w^v is nonempty and let g be efficient with respect to v. Define x by $x_i = \frac{w^v(N)}{n}$ for each i. Then $\sum_{S \in \Pi(g)} v(g(S)) = v(g) = \sum_{i \in N} x_i = \sum_{S \in \Pi(g)} \sum_{i \in S} x_i$. Also, Lemma 8 tells us that x is in the core of w^v, and so $\sum_{i \in S} x_i \geq w^v(S) \geq v(g(S))$ for each $S \in \Pi(g)$. Hence, all weak inequalities must hold with equality, so that $\sum_{i \in S} x_i = v(g(S))$ for each $S \in \Pi(g)$. Define a component balanced allocation rule Y by $Y_i(g', v) = x_i \frac{v(g'(S))}{\sum_{j \in S} x_j}$ for each $g' \in g^N$, $S \in \Pi(g')$, and $i \in S$. With this construction, it follows that $x_i \geq Y_i(g', v)$ for each $g' \in g^N$ and $i \in N$; and also that $Y_i(g, v) = x_i$ for any $g \in E(v)$ and $i \in N$. This implies that $Y_i(g, v) = x_i \geq Y_i(g', v)$ for each $g \in E(v)$, $S \subseteq N$, $g' \in g^N$ reachable from g by S, and $i \in S$; which proves that $g \in SS(Y, v)$. However, note that $Y(\cdot, v)$ coincides with $Y^{ce}(\cdot, v)$, because $Y_i(g', v) = x_i \frac{v(g'(S))}{\sum_{j \in S} x_j} = \frac{v(g'(S))}{|S|}$ for each $g' \in g^N$, $S \in \Pi(g')$, and $i \in S$. We therefore conclude that $g \in SS(Y^{ce}, v)$.

To complete the proof, let us show that (1) implies (2).

We have shown above that $E(v) \cap SS(Y^{ce}, v) \neq \emptyset$ implies (3) and that (3) implies that $E(v) \subseteq SS(Y^{ce}, v)$. Thus, we know that $E(v) \cap SS(Y^{ce}, v) \neq \emptyset$ implies $E(v) \subseteq SS(Y^{ce}, v)$. Next, we argue that (1) implies $\emptyset \neq SS(Y^{ce}, v) \subseteq E(v)$. Consider a strongly stable g. If it is not efficient, then there exists g' such that $v(g') > v(g)$. It follows that there exists some $S \in \Pi(g')$ such that $\frac{v(g'(S))}{|S|} > \frac{v(g)}{n}$. Since, as argued above $Y_i^{ce}(g, v) = \frac{v(g)}{n}$ for all i, this contradicts the strong stability of g and so we conclude that g must be efficient. Thus, (1) implies both $SS(Y^{ce}, v) \subseteq E(v)$ and $E(v) \subseteq SS(Y^{ce}, v)$, which is (2). ∎

Proof of Corollary 3: We show that $w^{v^{z,c}}$ is convex and then the result follows from Theorem 2 as the core of a convex game is non-empty. In what follows, we fix z and c

and so we write w to indicate $w^{v^{z,c}}$, and v to indicate $v^{z,c}$.

It follows directly from the definition of w and the symmetry and zero-normalization of z that w is symmetric and zero-normalized. Thus, we can also write w as a function w_k. For each $k \leq n$, let $v(k) = v(g)$ where $g = \{12, 23, \ldots, k-1k\}$. Thus $v(k)$ is the value of a coalition of size k connected in a network that is a line. The function $v(k)$ can also be viewed a zero-normalized symmetric cooperative game. Let $X(k) = \{X \subseteq \{1, \ldots, k\}^k \mid k = \sum_{k' \in X} k'\}$. We think of breaking k into a set of integers that sum to k, and $X(k)$ is the set of such decompositions. We can write

$$w_k = max_{X \in X(k)} \sum_{k' \in X} v(k'). \tag{1}$$

Since $v(k) = z_k - (k-1)c$ for $k \geq 1$, it follows from convexity of z that

$$v(k) - v(k-1) \geq v(k-1) - v(k-2) \tag{2}$$

for every $k \geq 3$. So, v is "almost" convex, except possibly that it may be that $v(2) = v(2) - v(1) < v(1) - v(0) = 0$. However, by standard arguments inequality (2) still implies that if $v(k') > 0$ then $v(k' + k'') \geq v(k') + v(k'')$ for any k''. This combined with equation (1) implies that

$$w_k = \max\{0, v(k)\}. \tag{3}$$

It then follows directly from (2) and (3) that w is convex. ∎

Proof of Theorem 4: Suppose that the core of w^v is nonempty. Then by Lemma 8, x defined by $x_i = \frac{w^v(N)}{n}$ for each i is in the core of w^v. Hence, for every $S \subseteq N$ we have $\sum_{i \in S} x_i = |S| \frac{w^v(N)}{n} \geq w^v(S) = |S| \, p(v, S)$. This results in $p(v, N) = \frac{w^v(N)}{n} \geq p(v, S)$, so that v is top convex.

Now suppose that v is top convex. It is a straightforward exercise to show that then x defined by $x_i = \frac{w^v(N)}{n}$ for each i is in the core of w^v. ∎

Proof of Theorem 5: It is clear that (iv) implies (ii) and (ii) implies (i). So we need only show that (i) implies (iii) implies (iv). To show that (i) implies (iii), first, note that for any component balanced Y, $SSS(Y, v) \subseteq E(v)$. So, consider Y and g such that $g \in SSS(Y, v) \subseteq E(v)$. This implies that the vector $Y(g, v)$ is in the core of w^v. From Theorem 2 , it then follows that (iii) holds.

Next, let us show that (iii) implies (iv). Let $g \in E(v) = SS(Y^{ce}, v)$. Since we know by Theorem 4 that v must be top-convex, it follows that $Y_i^{ce}(g, v) \geq Y_i^{ce}(g', v)$ for all i and g'. Thus, $\sum_{i \in S} Y_i^{ce}(g, v) \geq \sum_{i \in S} Y_i^{ce}(g', v)$ for any S and g', and so $g \in SSS(Y^{ce}, v)$.

So we have shown that $E(v) \subseteq SSS(Y^{ce}, v)$. Pairing this with $SSS(Y^{ce}, v) \subseteq E(v)$, it follows that $SSS(Y^{ce}, v) = E(v)$. ▮

Proof of Theorem 6: First, let us show the equivalence that (i) implies (iii) implies (ii) implies (i).

Let us show that (i) implies (iii). It is clear that for any component balanced Y, $SSS(Y, v) \subseteq E(v)$ simply from considering deviations by N. Thus, we need only show that (i) implies that there exists a Y such that $E(v) \subseteq SSS(Y, v)$. Let $g \in E(v)$ and let x in the core of w^v. Define a component balanced allocation rule Y by $Y_i(g', v) = x_i \frac{v(g'(S))}{\sum_{j \in S} x_j}$ for each $g' \in g^N$, $S \in \Pi(g')$, and $i \in S$. With this construction, it follows analogously to the part of the proof of Theorem 2 where it is proved that (3) implies (1), that for $S \subseteq N$ and $g' \in g^N$ reachable from g by S we have $\sum_{i \in S} Y_i(g, v) = \sum_{i \in S} x_i \geq \sum_{i \in S} Y_i(g', v)$. This proves that $g \in SSS(Y, v)$.

It is clear that (iii) implies (ii).

We complete the equivalence proof by showing that (ii) implies (i). Let Y be a component balanced allocation rule such that $SSS(Y, v) \neq \emptyset$. Since $SSS(Y, v) \subseteq E(v)$, we can find $g \in E(v) \cap SSS(Y, v)$. It follows directly that $Y(g, v)$ is in the core of w^v.

Next, let us show the remaining statements of the theorem. If v is top convex, then it is a straightforward exercise to show that then x defined by $x_i = \frac{w^v(N)}{n}$ for each i is in the core of w^v.

Finally, let us show that if v is top convex and component additive, then $E(v) = SS(Y^{ce}, v)$. Let $g \in E(v)$. Then $\frac{v(g)}{n} = p(v, N) = \max_{S \subseteq N} p(v, S)$ and, hence, $\frac{v(g(S))}{|S|} = \frac{v(g)}{n}$ for each $S \in \Pi(g)$. Then, for each $i \in N$ we have $Y_i^{ce}(g, v) = p(v, N)$, the maximum a player can get in any network. Hence, $g \in SS(Y^{ce}, v)$. Suppose $g \notin E(v)$. Then $Y_i^{ce}(g, v) \leq p(v, N)$ for all $i \in N$ with strict inequality for at least one $i \in N$. A $g' \in E(v)$ is reachable from g by N, and $Y_i^{ce}(g', v) = p(v, N)$ for each $i \in N$. This shows that $g \notin SS(Y^{ce}, v)$. ▮