Competing auctions with discrete valuations

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Summary: This paper considers a situation of two sellers of perfectly substitutable items competing in publicly announced reserve prices to induce potential bidders participation at their auction. After learning their own valuations and upon observing the reserve prices, potential bidders make a participation decision consisting of a unique auction to visit. The participation decision is modelled as a standard Bayesian-Nash game. Once participation decisions are realized, each bidder observes the aggregate number of bidders at the auction, and bidding games unfold. For extreme parameter values of the distribution of potential-bidders' valuation, it is shown that a unique, symmetric, pure-strategy equilibrium in reserve prices exists. For other values however, no equilibrium in pure strategy exists. The results share the non-existence feature of Hotelling location type models. The model also contributes to the recent literature on directed search.

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1 Introduction

Transactions in markets often involve auctions of various forms. Auctions are used, in particular, to establish a price when the item sold does not have a predetermined market value. A number of questions have been raised regarding what auction form to use and the respective properties of each. Of particular interest is which form generates the highest revenue to the seller, and how e±cient is the resulting allocation. Questions such as this have led to the development of an extensive literature on auctions. For a survey of the literature on auctions see Milgrom(1989) and McAfee and McMillan(1987a). More technical survey are presented in Wilson (1991) and Klemperer(1999). The most common ¯nding of this literature is that di®erent auction forms generate the same revenue when bidders' valuations are independent. Another ¯nding is that a seller can maximize revenue by setting a reserve price; i.e., a aminimum acceptable bid. Most of the literature on auction has been based on the assumptions that auctions are held in a monopolistic environment, and that the number of bidders at the auction is speci¯ed exogenously. A handful of exceptions which deals with a competitive environment are Burguet and Sakovics(2000), McAfee(1993), Peters(1994), and Peters and Severinov (2001). Regarding the former assumption, situations abound in which auctions are conducted in a more competitive environment. The degree of competition depends on the number of sellers, the availability of similar items sold in auctions or elsewhere, and how substitutable these items are with the item at a particular auction. For example, given the uniqueness of pieces of art, their auctioning will most likely be less competitive than the auctioning of widely available assets, such as treasury bills. As for the assumption of an exogenously speci¯ed number of bidders, there are several situations in which the number of bidders is likely to depend on certain attributes of the auction which are under the control of the seller. Consequently, the number of bidders could be endogenously speci¯ed. For example, when a relatively high reservation price is set, other things equal, fewer bidders would be expected to participate.

The purpose of this paper is threefold. First, to investigate how the presence of other auctions of similar items a®ects seller behavior and outcomes at a particular auction. Second, to examine how the allocation of resources is a®ected when potential bidders have more information than sellers, and bidders can choose in which auction to participate. Third, to obtain predictions on equilibrium reserve prices. For simplicity, the paper examines a small number of participants on each sides of the market. The auctioning environment structured in the paper can have several applications. For example, a limited number of communities competing with tax incentives to attract investment or the location of a production process from a limited number of multinational ¯rms, and the privatizing of public enterprises in economies in transition, where buyer interest is limited. However, competing auctions environment is now a common way by which prices are formed, and the internet as facilitated the development of such transaction mechanism.

This paper develops a simple static model of competition among two sellers holding ¯rst-price sealed-bid auctions with identical items. Each seller attempts to induce participation by two potential bidders. Sellers competition is modelled as a three-stage game. In the ¯rst stage, each seller publicly announces a reserve price under which no bid would be accepted. In the second stage, potential bidders consider how much they value the item, and decide whether to bid in an auction or not; if they decide to bid, they subsequently choose in which auction to bid. In the third stage, bidders observe the total number of bidders at the auction and make a bid. In this environment, con°icting interests arise between sellers and potential bidders. Sellers maximize expected revenue by noncooperatively choosing a reserve price to induce participation (i.e. increase competition at the auction). Higher participation, in turn, induces upward pressure on the expected winning bid. The objective of potential bidders is to minimize competition at the auction and, hence, the price they expect to pay. They achieve this through randomized participation.

The model introduces an additional in°uence on the choice of reserve price made by a seller, other than the standard trade-o® presented in the literature on monopoly auctions. The trade-o® involves the interaction of two con°icting in°uences on the choice of the optimal reservation price. On the one hand, by setting a higher reserve price, a seller raises the expected winning bid, thus raising expected revenue. On the other hand, a higher reserve price lowers the probability of bidder participation, hence lowering the probability of selling the item. This additional in°uence rises from the e®ect the choice of reserve price by the other seller has on the participation decision at the seller's auction. This e®ect is explicitly derived. Moreover, a feature of the model is the presence of discontinuity in sellers' expected payo® functions when gains from trade are strictly positive for all. This is occurs because equilibrium participation strategies change discontinuously with respect to the reserve prices.

The introduction of competition on the seller's side (the one controlling the auction rule) transfers expected rent from sellers to buyers, lowering the expected winning bids. However, sellers are left with positive expected rent. The model allows for explicit derivation of symmetric equilibrium reserve prices and to show that they are above the sellers' costs or outside option. This result di®ers from the standard Bertrand model in which the introduction of a second seller drives prices to marginal cost completely eroding the sellers rent. The results also suggest that the model has the capability of predicting a smoother relationship between concentration and industry pro¯ts, much like in a Cournot model. When the gains from trade are certain, that is when the lowest bidders' valuation is strictly higher than the sellers costs, the model exhibits payo® discontinuity similar to Hotelling type models. In a sense, this paper has an Hotelling type location interpretation. Sellers \locate" themselves on the reserve price space, and buyers choose with which seller to trade based on their reserve price \location". For extreme parameter values of the distribution of potential-bidders' valuation, it is shown that a unique, symmetric, pure-strategy equilibrium in reserve prices exists. For other values however, no equilibrium in pure strategy exists, and that existing Theorems on discontinuous games are non applicable.

The paper also shows how duopoly auctions can produce ine±cient outcomes in the sense that items may remain unsold even though bidders have a higher valuation than reserve prices. This di®ers from the e±cient outcomes produced by the monopoly auction and the standard Bertrand duopoly model in which sellers compete with prices and information is symmetric.

Few paper are related to this one. Burguet and Sakovics(1999), Peters and Severinov(1997) and McAfee(1993). They all also consider sellers competition in reserve prices with endogenous bidder participation through randomization. However, this paper introduces a model discrete distribution of bidders' valuations, while they consider models with continuous distribution for bidders valuation. Burguet and Sakovics(1999) also consider a duopoly auction situation with only two bidders. They show that in a two sellers two buyers environment, the symmetric equilibrium reserve prices are above the sellers' costs. Peters and Severinov(1997)'s main contribution is a limit equilibrium concept applied to markets with in¯nitely many buyers and sellers. Like McAfee(1993), they show that the symmetric equilibrium reserve prices are equal to sellers' costs. McAfee(1993) assumes ¯nitely many buyers and sellers, but resort to a perfectly competitive argument by ignoring deliberately how the change in mechanism a®ects buyers equilibrium selection. He shows that when sellers focus on direct revelation mechanisms, an equilibrium exists with all sellers o®ering a second-price auction with reserve prices driven down to sellers costs. Peters and Severinov(1997) account for how buyers respond to changes in mechanisms o®ered by sellers and get around the analytical di±culties of the two stage game at the limit by retaining the restrictions imposed by subgame perfection in the ¯nite version of the game. In this paper, focusing on small markets, I am able to explicitly derive potentila bidders' selection as a Bayesian-Nash equilibrium. This allows a detailed account of how buyers' respond to changes in the mechanisms (reserve price) that a deviating sellers would o®er. While the focus on mixed-strategy equilibrium selection is more compelling in large markets, the reason for focusing on small markets is in fact to show that with discrete valuations, the selection probability can be derived explicitely.As shown in Julien, Kennes and King(2000), the equilibrium payo®s converge very quickly to the limit values as the number of buyers and sellers increase, which means that the extent of coordination frictions carries through. Furthermore, it can be shown that the properties of the equilibrium selection extends to large market and one can derive the symmetric equilibrium reserve prices in such markets.

Finally, this paper is related to the recent and growing literature that has recently earned the name of directed search. This literature considers markets in which capacity constrained sellers commit to either a reserve or a posted price, and buyers select over sellers after observing the array of prices. Prices then direct buyers' selection. Like in this paper, but without informational asymmetries, this literature focuses on the symmetric-equilibrium mixed strategy to capture market frictions. This modelling strategy provides a microfoundation for the standard matching technology used in search models of the labor market. Therefore, this paper, along with Burguet and Sakovics(1999), Peters and Severinov(1997) and McAfee(1993), are models of directed search under informational asymmetry. For a dynamic directed search model with bidding, complete information and homogeneity (see Julien, Kennes and King (2000)). For static models of directed search with price posting as opposed to bidding, but with heterogeneity (see Burdett, Shi and Wright(2001), Coles and Eeckhout(1999) and Shimer(2001)), and Shi(2001) for dynamic models.[2]

The remainder of the paper is organized as follows. Section 2 introduces and solves a model of a monopoly auction. Section 3 extends the model of the monopoly auction to a duopolistic environment, solves for an equilibrium when one exists, and, compares the outcomes of the two environments. Finally, Section 4 contains a brief summary, the conclusions, and topics for further research. Most proofs are relegated to appendices.

2 A Monopoly Auction

As a benchmark, consider the following trading activity involving a single seller and two potential bidders, all risk neutral. The trading activity is modelled as a three-stage game. In the ¯rst stage, the seller holds a ¯rst-price sealed-bid auction to sell an indivisible item, in which he makes public a reserve price. In the second stage, potential bidders observe the reserve price and make a decision to bid or not at the auction. In the third stage, once at

[2] While competition in annouced reserve prices does not seem realistic, modelling competition in auctions with reserve prices remains an important theoretical exploration. For instance, it provides some basis to address the key issue on why are reserve prices not annouced publicly.

the auction, each bidder observes the total number of bidders and makes a bid. The item is sold to the bidder with the highest bid for a price equal to his bid. This formulation differs from the standard model of a monopoly auction in that participation decisions are explicitly modelled.

A solution of this game is obtained by backward induction, solving first for the bidding strategies, then for the participation decisions, and finally for the choice of reserve price. The following assumptions are made. The seller's value for the item is normalized to zero. Potential bidders are symmetric and obtain independently their valuation $\theta_i \in \{\underline{\theta}; \bar{\theta}\}$ from the following distribution function:

$$\Pr\{\theta = \underline{\theta}\} = \beta; \qquad \Pr\{\theta = \bar{\theta}\} = (1 - \beta) \tag{1}$$

where $\beta \in (0; 1)$; and $\underline{\theta} < \bar{\theta}$ are assumed common knowledge. The realized values of θ, however, are private information to potential bidders. This notation is similar to the one used in Fudenberg and Tirole(1991) and makes the results of this paper easily comparable with a textbook type exposition.

2.1 The Bidding Game

Let r be the publicly announced reserve price by the seller, and $n \in \{0; 1; 2\}$ be the total number of bidders at the auction. Individual rationality implies that a bidder never bids strictly over his valuation, i.e., $b \leq \theta$. A bidder strategy is a function $b : \{\underline{\theta}; \bar{\theta}\} \mapsto [0; \bar{\theta}]$. Each bidder chooses a bid to maximize:

$$\text{Expected Utility} \; = \theta \; [\text{Probability of winning}] - \text{Expected payment}$$

Using the bid function b, this expected utility can be expressed with the probability of winning conditional on the bid and the observed reserve price.

$$U(\Theta; r) \; = [\theta - b] \Pr\{\text{win}|n; b; r\} \tag{2}$$

where $\Theta = (\theta_1; \theta_2)$. Since valuations are discrete, bidding ties are broken by tossing a fair coin. The implication of using a discrete support of the distribution of types is that high valuation bidders play a mixed strategy in equilibrium (See Maskin and Riley (1985)). The equilibrium bids at the auction are described in the following lemma.

Lemma 1 In a first-price sealed-bids auction with discrete valuations, low valuation bidders bid their valuation net of their outside option and high valuation bidders randomize. Let

$\underline{b}(r)$ and $\bar{b}(r)$ be the bids submitted by a low and a high valuation bidder respectively. The equilibrium bids are

$$\underline{b}(r) = \begin{cases} r_i & \text{if } n^i = 1 \\ \underline{\theta} & \text{if } n^i = 2 \end{cases}$$

and

$$\bar{b}(r) = \begin{cases} r_i & \text{if } n^i = 1 \\ \bar{\theta} - \dfrac{(\bar{\theta} - \underline{\theta})\beta P^i_{\underline{\theta}}(r)}{\beta P^i_{\underline{\theta}}(r) + F(b)(1-\beta)P^i_{\bar{\theta}}(r)} & \text{if } n^i = 2 \end{cases}$$

where $F(b)$ is the equilibrium bid distribution with support $[\underline{\theta}; \bar{b}_H]$, with

$$\bar{b}_H(r) = \bar{\theta} - \frac{(\bar{\theta} - \underline{\theta})\beta P_{\underline{\theta}}(r)}{(1-\beta)P_{\bar{\theta}}(r) + \beta P_{\underline{\theta}}(r)}$$

being the supremum of the bid distribution.

Proof. If $r > \underline{\theta}$ 1and a bidder is not alone then he knows that the other bidder is type $\bar{\theta}$, perfect information induce bidders to bid their valuation. If $r \leq \underline{\theta}$; the demonstration of the Bayesian equilibrium bids when all types participate follows Maskin and Riley (1985) and adapted for explicit participation decisions. First focus on type-$\bar{\theta}$ behavior. Let $\rho(r) = E[P_\theta(r)] = [\beta P_{\underline{\theta}}(r) + (1-\beta)P_{\bar{\theta}}(r)]$ be the expected participation of a potential bidder held by all potential bidders. Given that there are only two potential bidders, the conditional probabilities of facing a competitor or not at the auction given the reserve price are

$$\Pr\{n|r\} = \begin{cases} \rho(r) & \text{if } n = 2 \text{ (facing a competitor)} \\ 1 - \rho(r) & \text{if } n = 1 \text{ (not facing a competitor)} \end{cases} \tag{3}$$

For a potential bidder, the probability of facing a competitor of type θ at the auction is

$$\Pr\{\theta \wedge n = 2|r\} = \begin{cases} \beta P_{\underline{\theta}}(r) & \text{if } \theta = \underline{\theta} \\ (1-\beta)P_{\bar{\theta}}(r) & \text{if } \theta = \bar{\theta}. \end{cases} \tag{4}$$

Once he decides to participate, a potential bidder observes the total number of bidders. Based on this information, priors about a competitor's type are revised according to Bayes' rule. The posterior probabilities over a competitor's type are

$$\begin{aligned} \Pr\{\theta|n = 2; r\} &= \frac{\Pr\{\theta \wedge n = 2|r\}}{\Pr\{n = 2|r\}} \\ &= \begin{cases} \dfrac{\beta P_{\underline{\theta}}(r)}{\rho(r)} & \text{if } \theta = \underline{\theta} \\ \dfrac{(1-\beta)P_{\bar{\theta}}(r)}{\rho(r)} & \text{if } \theta = \bar{\theta}. \end{cases} \end{aligned} \tag{5}$$

Since type $\bar{\theta}$ randomizes, let $F(\bar{b})$ be the cumulative distribution function on $[\bar{b}_L; \bar{b}_H]$ used in the mixed strategy. Clearly, $\bar{b}_L = \underline{\theta}$ for if $\bar{b}_L > \underline{\theta}$ a type-$\bar{\theta}$ bidder would be better off bidding just above $\bar{\theta}$ rather than $\bar{b}_L$ since this would not affect his probability of winning and would reduce his payment.

The conditional probability of winning for type $\bar{\theta}$ when he is facing a competitor at the auction is

$$
\begin{aligned}
\Pr\{win|n = 2; r; \bar{b}\} &= \Pr\{\theta = \underline{\theta}|n = 2; r\} + \Pr\{\theta = \bar{\theta}|n = 2; r; \bar{b}\}F(\bar{b}) \\
&= \left[\frac{\beta P_{\underline{\theta}}(r)}{\mu(r)} + F(\bar{b})\frac{(1-\beta)P_{\bar{\theta}}(r)}{\mu(r)}\right]
\end{aligned}
\tag{6}
$$

since a bid of $\bar{b} = \underline{\theta}$ is a measure-zero event. The expected utility of a type-$\bar{\theta}$ bidder is

$$
U((\bar{\theta}; \underline{\theta}); r) = (\bar{\theta} - \bar{b}) \left[\frac{\beta P_{\underline{\theta}}(r)}{\mu(r)} + F(\bar{b})\frac{(1-\beta)P_{\bar{\theta}}(r)}{\mu(r)}\right]: \tag{7}
$$

Any bid made as a part of a mixed strategy must yield the same expected payoff. The expected payoff at the infimum of the support of the mixed strategy distribution is

$$
U((\bar{\theta}; \underline{\theta}); r) = (\bar{\theta} - \underline{\theta}) \left[\frac{\beta P_{\underline{\theta}}(r)}{\mu(r)}\right]: \tag{8}
$$

where $F(\underline{\theta}) = 0$. Note that the support of a probability distribution is the smallest closed set that has probability 1, hence $\underline{\theta}$ can belong to the support of type-$\bar{\theta}$ bidder's equilibrium strategy even though playing $\underline{\theta}$ with positive probability would risk tying with a type-$\underline{\theta}$ bidder and yield a lower expected utility. The equilibrium bid $\bar{b}$ and its equilibrium distribution are solved for by equating 6 and 7. Last, because $F(b_H) = 1$; this implies that

$$
\bar{b}_H(r) = \bar{\theta} - \frac{(\bar{\theta} - \underline{\theta})\beta P_{\underline{\theta}}(r)}{(1-\beta)P_{\bar{\theta}}(r) + \beta P_{\underline{\theta}}(r)} < \bar{\theta} \tag{9}
$$

The equilibrium strategy for a type-$\underline{\theta}$ bidder to bid his valuation since he never wins again a type-$\bar{\theta}$ bidder. Furthermore, a bid below $\underline{\theta}$ is not an equilibrium since it yields a zero probability of winning. Therefore, a type-$\underline{\theta}$ bidder always obtains zero expected utility at the auction. ■

In this context, the equilibrium bids depend on the reserve price via the participation decision. Hence, the probability of winning depends on the equilibrium participation decisions.

2.2 Participation Decision

The decision whether to become a bidder or not, after a potential bidder observes his valuation and the reserve price, is a function $P_\theta(r) : [0,1] \mapsto \{0,1\}$ defined as

$$P_\theta(r) = \begin{cases} 1 & \text{if } \theta \geq r \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

This function is the probability that a type θ participates in the auction, given the observed reserve price.

Using equation 3 the ex ante expected utility before one decides to participate at the auction is

$$R(\Theta; r) = \sum_{n=1}^{2} \Pr\{n|r\} U(\Theta; r). \tag{11}$$

Using a standard individual rationality argument,

$$P_\theta(r) = \begin{cases} 1 & \text{if } R(\Theta; r) \geq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

Note that this formulation encompasses the case when $r > \underline{\theta}$. In such case, $P_{\underline{\theta}}(r) = 0$ and $P_{\bar{\theta}}(r) = 1$, which yields $U((\bar{\theta}, \theta); r) = 0$.

2.3 The Seller's Decision Problem

A seller who chooses to sell an indivisible item using a first-price sealed-bid auction in which he announces a reserve price r, solves the following decision problem

$$\max_{r \in \Re} \Pi(r) = \sum_{n=0}^{2} \pi_n(r) B(r; n) \tag{13}$$

where $\pi_n(r) = \binom{2}{n} \mu(r)^n (1 - \mu(r))^{2-n}$ is the probability that the seller be visited by n bidders and $B(r; n)$ is the expected winning bid conditional on having n bidders at the auction. Note that to simplify notation, Θ is omitted as an argument of $\pi_n(r), B(r; n)$ and $\Pi(r)$ for the monopoly and duopoly models. The probabilities that a seller is visited by 0, 1, and 2 bidders are, respectively

$$\begin{aligned} \pi_0(r) &= (1 - \mu(r))^2 \\ \pi_1(r) &= 2\mu(r)(1 - \mu(r)) \\ \pi_2(r) &= (\mu(r))^2. \end{aligned} \tag{14}$$

The following lemma describes the expected winning bid at the auction.

Lemma 2 The expected winning bid at the auction is

$$B(r;n) = \begin{cases} r & \text{if } n = 1 \\ \underline{\theta} + \left[\dfrac{(1-\beta)P_{\bar{\theta}}(r)}{\mu(r)}\right]^2 (\bar{\theta} - \underline{\theta}) & \text{if } n = 2 \end{cases} \tag{15}$$

Proof. See Appendix A. ∎

If only one bidder visits the auction, the seller is paid his reserve price. The expected winning bid when $n = 2$ is a weighted average of $\underline{\theta}$ and $\bar{\theta}$. For example, if $r > \underline{\theta}$ then $P_{\underline{\theta}}(r) = 0$ and $B(r;2) = \bar{\theta}$. It is easily shown that $B(r;2) \geq \underline{\theta}$ for all $r \in [0;\bar{\theta}]$.

The seller's maximization problem becomes

$$\max_{r \in \Re} \Pi(r) = \pi_1(r)r + \pi_2(r)B(r;2) \tag{16}$$

the solution of which is presented in the following proposition.

Proposition 1 Let $\rho(\underline{\theta};\bar{\theta}) = \frac{2(\bar{\theta}-\underline{\theta})}{(2\bar{\theta}-\underline{\theta})}$: The optimal reserve price is

$$r^* = \arg\max_r \Pi(r) = \begin{cases} \bar{\theta} & \text{if } \beta \in \left[0; \rho(\Theta)\right] \\ [0;\underline{\theta}] & \text{if } \beta \in \left[\rho(\Theta); 1\right] \end{cases} \tag{17}$$

Proof. The seller's expected payoff can be written as

$$\Pi(r) = \begin{cases} 2\beta(1-\beta)r + (1-\beta)^2\bar{\theta} & \text{if } r > \underline{\theta} \\ \underline{\theta} + (1-\beta)^2(\bar{\theta}-\underline{\theta}) & \text{if } r \leq \underline{\theta}: \end{cases} \tag{18}$$

The obvious optimizing values are $r^* = \bar{\theta}$ if $r > \underline{\theta}$, and $r^* \in [0;\underline{\theta}]$ otherwise. Let $\Delta(\beta;\underline{\theta};\bar{\theta}) = \Pi(r = \bar{\theta}) - \Pi(r \in [0;\underline{\theta}])$ and set $\Delta(\beta;\underline{\theta};\bar{\theta}) = 0$ to find $\rho(\underline{\theta};\bar{\theta}) = \frac{2(\bar{\theta}-\underline{\theta})}{(2\bar{\theta}-\underline{\theta})}$. ∎

Figure ?? illustrates the result.

[Insert Figure 1 here]

Proposition 1 simply states that, for relatively small probability that potential bidders are of low type, the seller maximizes expected revenue by choosing a unique reserve price which induces only type-$\bar{\theta}$ potential bidders to participate at the auction. When the probability that potential bidders are of low type is relatively large, the seller is indifferent between any reserve prices that induce both types to participate at the auction. In this case, uniqueness of the optimal reserve price no longer holds. This follows from the binary feature of the participation decision. When the seller chooses a reserve price below $\underline{\theta}$: he is certain he will face two bidders; he will realize a sale with certainty; and, he will always be paid a higher price than his reserve price. For a given distribution of valuations, the seller's optimal reserve price reflects the trade-off between a higher expected winning bid and an increased probability of no sale (no participation). A higher expected winning bid is induced through the information revealed from the observation of the total number of participants. However, it can be shown that the results of the bidding game do not depend on the observability of the number of participants. For example, if the number of participants is kept secret but the seller announces a reserve price of $r > \underline{\theta}$, a potential bidder can deduce from the observed reserve price that only type $\bar{\theta}$ participates, and thus bid $\bar{\theta}$. Therefore, the assumption made on the observability of the number of participants is innocuous. This is shown in McAfee and McMillan(1987b) for a more general environment.

In addition, Proposition 1 implies that there exists a specific value of the parameter β, i.e., $\beta^{\circ}(\underline{\theta}; \bar{\theta})$, such that the seller is indifferent between reserve prices that induce only type $\bar{\theta}$ to participate and reserve prices that induce both types to participate. Consider the following definition.

Definition 1 A **key parameter** is defined as a parameter value of β; such that a seller is indifferent between at least two reserve prices, given the potential bidders' equilibrium participation decisions and bidding strategies.

The value of the key parameter depends on the values of the support of the distribution of valuations. Therefore, the region of the parameter space of $\beta \in [0; 1]$ for which a particular reserve price is an equilibrium choice, depends on the values of $\underline{\theta}$ and $\bar{\theta}$. This yields the following comparative statics:

$$\frac{\partial \beta}{\partial \underline{\theta}} = -\frac{2\bar{\theta}}{(2\bar{\theta} - \underline{\theta})^2} = -\frac{\partial \beta}{\partial \bar{\theta}} < 0 \qquad (19)$$

For higher dispersion in potential bidders' valuations, the range of beliefs that induce all types of potential bidders to participate is smaller.

This notion of key parameters is particularly useful in order to rank a seller's payoff for all possible deviations when expected payoff is discontinuous. The following section introduces competition on the seller's side.

3 A Duopoly Auction Model

Consider two sellers who choose to sell identical items by simultaneously using a first-price sealed-bid auction in which they publicly announce a reserve price of $r_i \in [0;1]$, $i \in \{a;b\}$.[3] Each seller has a single item to sell. Potential bidders observe the reserve prices and decide to bid or not at an auction; if they decide to bid, they choose in which one of the two auctions to bid. At an auction, a bidder observes the total number of bidders and makes a bid. Solutions to the game are obtained by backward induction, solving first for the bidding game, then for the participation game, and, finally, for the sellers' game.

The formulation and the results of the bidding game Lemma 1 of the monopoly section is still valid with the only exception of a superscript to identify a particular auction. There using n^i for the total number of bidders in auction i; $P^i_\theta(r)$ the probability with which type-θ bidder selects auction i, $\lambda^i(r)$ the expected participation in auction i, and $U^i(\Theta; r)$ the expected payoff from choosing auction $i \in \{a;b\}$, we have the equilibrium bidding behavior in each auction.

3.1 The Participation Game

The principal implication of having two sellers is that potential bidders have an outside option in their desire to acquire one item. After potential bidders have observed their valuations and the vector of reserve prices $r = (r_a; r_b)$, they make a participation decision. Peters and Severinov(1997) analyse both cases when bidders make a participation decision before or after they learn their valuation. In this model, like in McAfee(1993), since bidders learn their valuation before participation, a participation decision involves whether to become a bidder or not, and where to bid. Formally, it is a function $P^i_\theta : [0;1] \mapsto [0;1]$ with $\sum_{i \in \{e;a;b\}} P^i_\theta(r) = 1$ and

$$P^e_\theta(r) = \begin{cases} 1 & \text{if } \theta < \min\{r_a; r_b\} \\ 0 & \text{otherwise.} \end{cases} \qquad (20)$$

[3] There is an issue of commitment in this model as well as all directed search models with finite and large markets. A seller selected by only one bidder has the incentive to increase the reserve price. This is one of the weaknesses of this modelling strategy.

The function $P_\theta^e(r)$ represents the decision to become a bidder and the superscript e (for entry) refers to the participation decision. If $P_\theta^e(r) = 1$, a type-θ potential bidder does not participate.

Participation strategies and their corresponding payoffs (to be defined later) constitute a Bayesian-Nash game: at the time of the participation decision, potential bidders have private information about their valuation.

Using equation 3 with the appropriate superscript, the ex ante expected payoff in auction i is

$$\begin{aligned} R^i(\Theta; r) &= \sum_{n^i=1}^{2} \Pr\{n^i | r\} U^i(\Theta; r; n^i) \\ &= \pi^i(r) U^i(\Theta; r; n^i = 2) + (1 - \pi^i(r)) U^i(\Theta; r; n^i = 1) \end{aligned} \tag{21}$$

where $U^i(\Theta; r; n^i = 1) = (\theta - r_i)$, the expected utility of type θ bidder when he faces no competitor in auction i; and $U^i(\Theta; r; n^i = 2) = U^i((\bar{\theta}; \theta); r) = (\bar{\theta} - \theta)\left[\frac{\beta P_\theta^i(r)}{\pi^i(r)}\right]$.

Once they choose to participate in the market, potential bidders maximize their ex ante expected payoff by choosing $P_\theta^i(r)$ such that

$$R^a(\Theta; r) = R^b(\Theta; r) \tag{22}$$

and

$$P_\theta^a(r) + P_\theta^b(r) = 1. \tag{23}$$

The following proposition summarizes the equilibrium participation strategies.

Proposition 2 The unique, symmetric-equilibrium of the Bayesian-Nash participation game is

$$\left\langle P^e_{\underline{\theta}}(r); P^a_{\underline{\theta}}(r); P^a_{\bar{\theta}}(r)\right\rangle = \begin{cases} \left\langle 1; 0; \frac{1}{2}\right\rangle & \text{if } r_a = r_b = \bar{\theta} \\ \left\langle 1; 0; \frac{\bar{\theta} - r_a - \beta(\bar{\theta} - r_b)}{(1-\beta)(2\bar{\theta} - r_a - r_b)}\right\rangle & \text{if } r_a > \underline{\theta}; r_b > \underline{\theta} \\ \left\langle 0; 1; \frac{(1-\beta)(\bar{\theta} - r_a) - \beta(\underline{\theta} - r_b)}{(1-\beta)(2\bar{\theta} - r_a - r_b)}\right\rangle & \text{if } r_a \le \underline{\theta} \le r_b < \hat{r}_b; r_a \ne r_b \\ \langle 0; 1; 1\rangle & \text{if } r_a \le \underline{\theta} < \hat{r}_b \le r_b; r_a \ne r_b \\ \left\langle 0; 0; \frac{\bar{\theta} - r_a - \beta(\bar{\theta} - \underline{\theta})}{(1-\beta)(2\bar{\theta} - r_a - r_b)}\right\rangle & \text{if } \hat{r}_a > r_a \ge \underline{\theta} \ge r_b; r_a \ne r_b \\ \langle 0; 0; 0\rangle & \text{if } r_a \ge \hat{r}_a > \underline{\theta} \ge r_b \\ \left\langle 0; \frac{1}{2} - \frac{(r_a - r_b)}{2\beta(2\underline{\theta} - r_a - r_b)}; \frac{1}{2}\right\rangle & \text{if } r_a < \underline{\theta}; r_b < \underline{\theta} \\ \left\langle 0; \frac{1}{2}; \frac{1}{2}\right\rangle & \text{if } r_a = r_b = \underline{\theta} \end{cases} \tag{24}$$

where

$$\hat{r}_i = \left[\bar{\theta} - \beta(\bar{\theta} - \underline{\theta})\right]; P^e_{\bar{\theta}}(r) = 0 \text{ for all } r \in [0; \bar{\theta}]$$

and

$$\left\langle P^b_{\underline{\theta}}(r); P^b_{\bar{\theta}}(r)\right\rangle = \left\langle (1 - P^a_{\underline{\theta}}(r)); (1 - P^a_{\bar{\theta}}(r))\right\rangle :$$

Proof. See Appendix A1. ■

This proposition is one of the main results of this paper. It shows precisely how the equilibrium participation decisions depend on the vector of reserve prices. For some vectors of reserve prices, a potential bidder maximizes his ex ante expected payoff by choosing explicitly which auction to visit. While for other vectors of reserve prices, he maximizes his ex ante expected payoff by choosing randomly which auction to visit. By making a random choice, a potential bidder minimizes the opportunity cost of visiting any auction. Furthermore, a random choice reflects the negative externality on the price he expects to pay, created by the presence of the other bidder at the auction. Suppose $r_a < r_b$, in a standard Bertrand model, all buyers would visit seller a. In this model, buyers visit seller a with a higher probability. A buyer understands that the other buyer randomizes. If he visit seller a, he will pay a lower price if he is alone than what he will pay if alone visiting seller b. However, he is more likely not to be alone at auction a than at auction b, in which case he will pay a higher price. This is the explicitely derived equilibrium mixed-strategy participation as in Julien, Kennes and King(2000) but with informational asymmetry.[4] Because of their focus on limit equilibrium and continuous valuations, Peters and Severinov(1997) are unable to do this.

An interpretation of the equilibrium participation strategies is to consider them as the potential-bidders demand function for the right to bid at an auction. From a seller's viewpoint, the equilibrium participation strategies are used to specify an expected demand to participate in his auction in the following way:

$$\text{Expected demand} = 2[\beta P^i_{\underline{\theta}}(r) + (1-\beta)P^i_{\bar{\theta}}(r)] = 2\rho^i(r). \tag{25}$$

3.2 Sellers' Game

A strategy for a seller is a public announcement of a reserve price $r_i \in \Re_+$ with $i = a, b$. The expected payoff of a seller i is

$$\Pi^i(r;n) = \sum_{n^i=1}^{2} \pi^i_{n^i}(r) B^i(r;n^i) \tag{26}$$

where $\pi^i_{n^i}(r)$ is the probability that a seller i is selected by n^i potential bidders, and $B^i(r;n^i)$ is the expected winning bid given the vector of reserve prices $r = (r_a; r_b)$, conditional on the number of bidders n^i at the auction. The expected winning bid is established in the following lemma.

Lemma 3 The expected winning bid at an auction for a competing seller is

$$B^i(r;n^i) = \begin{cases} r & \text{if } n^i = 1 \\ \underline{\theta} + \left[\dfrac{(1-\beta)P^i_{\bar{\theta}}(r)}{\rho^i(r)}\right]^2 (\bar{\theta} - \underline{\theta}) & \text{if } n^i = 2 \end{cases}$$

Proof. Same as Lemma 2. ■

This lemma is similar to Lemma 2, except that the participation decision is a participation strategy derived as an equilibrium of a Bayesian-Nash game among potential bidders.

The probability that a seller i is visited by n^i bidders is[5]

$$\pi^i_{n^i}(r) = \binom{2}{n^i} \rho(r)^{n^i}[1-\rho(r)]^{2-n^i} \tag{27}$$

---

[4] It is important to notice that there exist pure strategy equilibria in this game for which a bidder always visits auction a, and the other always visits auction b, and vice versa. The existence of these equilibria have been demonstrated in the directed search literature, namely, in Julien, Kennes and King (2000).

[5] In what follows, let $\rho^a(r) = \rho(r)$ and $\rho^b(r) = 1 - \rho(r)$.

The respective probabilities of being visited by 0, 1, and 2 bidders are

$$
\begin{aligned}
\pi_0^a(r) &= (1 - \mu(r))^2 = \pi_2^b(r) \\
\pi_1^a(r) &= 2\mu(r)(1 - \mu(r)) = \pi_1^b(r) \\
\pi_2^a(r) &= \mu(r)^2 = \pi_0^b(r)
\end{aligned} \tag{28}
$$

Since sellers choose their reserve price simultaneously, an equilibrium vector of reserve prices is found by a standard Nash argument,

$$
\Pi^i(r_i^*; r_{-i}^*) \geq \Pi^i(r_i; r_{-i}^*) \tag{29}
$$

for all $i = a; b$ and $r_i \neq r_i^* \in \mathbb{R}_+$. The equilibrium reserve price for a seller is not only the result of the standard trade-off, but also the result of the influence of the competing seller's reserve price on the optimal participation strategies of potential bidders.

Proposition 3 *Let the low-type valuation of a potential bidder be $\underline{\theta} = 0$. For every $\beta \in [0; 1]$, the vector of reserve prices $(r_a^*; r_b^*) = \left((1 - \beta)\bar{\theta}/2; (1 - \beta)\bar{\theta}/2\right)$ is the unique, symmetric equilibrium of the sellers' game.*

Proof. Given that sellers are symmetric, all proofs in the remainder of the paper are done only for one seller's payoff. (In order to ease notation, r is omitted as an argument of μ in the following proof.) ■

Since both reserve prices are above $\underline{\theta}$, $P_{\underline{\theta}}(r) = 0$ and $P_{\bar{\theta}}(r) = \frac{\bar{\theta} - r_a - \beta(\bar{\theta} - r_b)}{(1-\beta)(2\bar{\theta} - r_a - r_b)}$, which imply that $B(r; 2) = \bar{\theta}$. Seller a maximizes

$$
\begin{aligned}
\Pi(r_a; r_b) &= \pi_1(r) r_a + \pi_2(r) B(r; 2) \\
&= 2\mu(1 - \mu) r_a + \mu^2 \bar{\theta}
\end{aligned} \tag{30}
$$

by choosing r_a.

$$
\frac{\partial \Pi(r_a; r_b)}{\partial r_a} = 2\mu(1 - \mu) + 2(1 - 2\mu)\frac{\partial \mu}{\partial r_a} r_a + 2\mu \frac{\partial \mu}{\partial r_a}\bar{\theta} \tag{31}
$$

Using the fact that

$$
\frac{\partial \mu}{\partial r_a} = -\frac{(1 - \mu)}{(2\bar{\theta} - r_a - r_b)} \tag{32}
$$

yields

$$
\begin{aligned}
\frac{\partial \Pi(r_a; r_b)}{\partial r_a} &= 2\mu(1 - \mu)\left\{1 - \frac{(\mu\bar{\theta} + r_a(1 - 2\mu))}{\mu(2\bar{\theta} - r_a - r_b)}\right\} \\
&= \pi_1(r)\left\{1 - \frac{(\mu\bar{\theta} + r_a(1 - 2\mu))}{\mu(2\bar{\theta} - r_a - r_b)}\right\} :
\end{aligned} \tag{33}
$$

Setting it equal to zero

$$-r_a(1-\mu)+\mu(\bar{\theta}-r_b)=0. \tag{34}$$

Substituting the value of $P_{\bar{\theta}}(r)$ in μ and rearranging

$$\bar{\theta}(1-\beta)+\beta r_b-(2+\beta)r_a=0 \tag{35}$$

yields the reaction function

$$r_a(r_b) \quad = \quad \frac{\bar{\theta}(1-\beta)+\beta r_b}{(2+\beta)}. \tag{36}$$

A similar expression exists for seller b

$$r_b(r_a) \quad = \quad \frac{\bar{\theta}(1-\beta)+\beta r_a}{(2+\beta)}. \tag{37}$$

These reaction functions yield $r_i^* = \frac{(1-\beta)\bar{\theta}}{2}$ for $i = a, b$. Uniqueness is established from the slope of the reaction functions, which are less than one for all $\beta \in [0, 1]$. That means that these functions cross only once over the strategy space. Consider a deviation to $r_i = 0$ (a non-local deviation) by one seller when the other seller sets $r_{-i}^* = \frac{(1-\beta)\bar{\theta}}{2}$. Let $\Delta(\beta, \underline{\theta}, \bar{\theta}) = \Pi((1-\beta)\bar{\theta}/2, (1-\beta)\bar{\theta}/2) - \Pi(0, (1-\beta)\bar{\theta}/2)$. It can be shown that $\Delta(\beta, \underline{\theta}, \bar{\theta}) = 2\bar{\theta} + 3\beta + 8\beta^2 + 3\beta^3 > 0$ for all $\beta \in [0, 1]$. Notice that if $\beta = 1$ then $r_a = r_b = 0$ is an equilibrium. The second-order conditions are easily shown to be satisfied since

$$\frac{\partial^2 \Pi(r_a, r_b)}{\partial r_a^2} = -\frac{2(1-\mu)(\bar{\theta}-r_b+r_b)}{\mu(2\bar{\theta}-r_a-r_b)^2} < 0 \tag{38}$$

■

Figure 2 depicts this result.

[Insert Figure 2 here]

The equilibrium reserve prices can be represented by the use of the reaction functions defined in (36) and (37), and shown in Figure 3.

[Insert Figure 3 here]

These reaction functions exhibit strategic complementarity in a manner similar to the Bertrand-differentiated product environment. In the model of this paper, it is the heterogeneity of consumers that yields this form of reaction functions, not heterogeneity of the product. Strategic complementarity is an implication of the equilibrium participation strategies. When a seller reduces his reserve price, potential bidders place more weight into participating in this seller's auction. The other seller would then react by decreasing his price. The similarity with the Bertrand differentiated product is not surprising. Each seller is selling his auction through a reserve price. There is a probability of winning associated with each auction which depends on the equilibrium participation strategies, themselves dependent on the potential bidders types.

Consider the situation in which their is always gains from trade, i.e. $\underline{\mu} > 0$. Standard differential arguments cannot be used to solve the sellers' game in this case because of the discontinuous payoff functions that this assumption creates. A seller's expected payoff changes continuously as he increases his reserve price until it reaches a value $r_i = \underline{\mu}$, where it falls discontinuously. This is a direct implication of the discontinuity in the equilibrium participation strategies formulated in Proposition 2. It is also an implication of the assumption of discrete distribution of valuations.

In order to find an equilibrium vector of reserve prices, consider the following terminology.

Definition 2 A **local deviation** is a deviation by a seller from an initial reserve price, to a new reserve price that induces his expected payoff to change continuously.

Definition 3 A **non-local deviation** is a deviation by a seller from an initial reserve price, to a new reserve price that induces his expected payoff to change discontinuously.

If a vector of reserve prices is to be an equilibrium for the sellers' game, it must resist all local and non-local deviations by all sellers.

A discontinuity occurs in a seller's payoff function whenever he sets a reserve price equal to $\underline{\mu}$. An example of such a discontinuity is shown in Figure 9 Appendix C.

Since a symmetric equilibrium in reserve prices is sought for, the following reserve prices are the only candidates for such an equilibrium: $r_a = r_b \in [0; \underline{\mu})$; $r_a = r_b = \underline{\mu}$; and, $r_a = r_b \in (\underline{\mu}; \bar{\mu}]$. In a series of lemmas, the symmetric vectors of reserve prices that resist unilateral, local deviations will be established. In particular, the following lemma establishes that if there is an equilibrium in which $r_a = r_b < \underline{\mu}$ then it must be $r_a = r_b = \underline{\mu}/2$. This reserve price does not depend on the prior belief for the simple reason that when both reserve prices are below $\underline{\mu}$, all types participate and perfectly randomize. This can be seen by replacing these reserve prices in the equilibrium participation strategies in Proposition 2.

Since there are only two seller, each is visited with probability 1/2. If there were more than two sellers, such reserve prices would be expected to be decreasing in the number of sellers.

Lemma 4 Let $\theta > 0$, for all $\beta \in [0,1]$, the vector of reserve prices $(r_a, r_b) = (\theta/2, \theta/2)$ is the only symmetric vector that resists unilateral deviations over $[0, \theta)$.

Proof. The participation strategies impose restrictions on the payoff functions that result in non-differentiable points. From Proposition 2, when $r_i < \theta$ then $P_\theta(r) = \frac{1}{2} - \frac{(r_a - r_b)}{2\beta(2\theta - r_a - r_b)}$ and $P_{\bar\theta}(r) = 1/2$. The non-differentiable points are defined by $r_i^\ell(r_{-i})$ and $r_i^h(r_{-i})$, the solution of $P_\theta(r) = 1$ and $P_\theta(r) = 0$, respectively. For all $r_i < r_i^\ell(r_{-i})$, $P_\theta(r) = 1$ and for all $r_i > r_i^h(r_{-i})$, $P_\theta(r) = 0$. Deviations beyond the non-differentiable points induce low-type potential bidders to adopt pure strategies in the participation game. The expected payoffs for these deviations have the following property

$$\frac{\partial \Pi(r_i, r_{-i})}{\partial r_i} = \frac{(1 - \beta^2)}{2} \geq 0 \tag{39}$$

for all $r_i \in [0, r_i^\ell] \cup [r_i^h, \theta]$. As a result, a deviation is not profitable beyond a non-differentiable point. ■

Over the interval $[r_i^\ell(r_{-i}), r_i^h(r_{-i})]$, using the equilibrium participation strategies from Proposition 2 for the case when both reserve prices are below θ,

$$\frac{\partial \Pi(r_a, r_b)}{\partial r_a} = 2\mu(1 - \mu) + [2(r_a(1 - 2\mu) + 2\mu\theta]\frac{\partial \mu}{\partial r_a}. \tag{40}$$

A similar derivative exists for seller b given symmetry.

Setting this derivative equal to zero yields

$$2\theta - 4r_a = 0, \tag{41}$$

with a similar expression for b. Therefore, $(r_a, r_b) = (\theta/2, \theta/2)$. ■

Lemma 5 For all $\beta \in [0, 1 - 2\theta/\bar\theta]$, the vector of reserve prices $(r_a, r_b) = ((1 - \beta)/2, (1 - \beta)/2)$ is the only symmetric vector that resists unilateral deviations over $(\theta, \bar\theta]$.

Proof. The proof of this lemma is essentially the same as the proof of Proposition 3 which deals with local deviations. ■

Non-local deviations are handled by a comparison of sellers' expected payoff evaluated at all possible deviations. For that purpose, consider the following definition.

Definition 4 The following equalities define a set of **key parameters**:

(1) $\beta_1(\underline{\theta}, \bar{\theta})$ solves $\Pi^i((1-\beta)\bar{\theta}/2, (1-\beta)\bar{\theta}/2) = \Pi^i(\underline{\theta}, (1-\beta)\bar{\theta}/2)$.

(2) $\bar{\delta}$ satisfies $r^* = (1-\beta)\bar{\theta}/2 = \underline{\theta}$.

(3) β_2 solves $\lim_{r_i \to -\underline{\theta}} \Pi^i(r_i, \underline{\theta}/2) = \Pi^i(\underline{\theta}/2, \underline{\theta}/2)$.

(4) $\beta_3(\underline{\theta}, \bar{\theta})$ solves $\lim_{r_i \to -\underline{\theta}} \Pi^i(r_i, \underline{\theta}/2) = \Pi^i(BR_i(\underline{\theta}/2), \underline{\theta}/2)$.

(5) $\beta_4(\underline{\theta}, \bar{\theta})$ solves $\Pi^i(\underline{\theta}/2, \underline{\theta}/2) = \Pi^i(BR_i(\underline{\theta}/2), \underline{\theta}/2)$;

where $BR_i(\underline{\theta}/2) > \underline{\theta}$ is a seller's best response above $\underline{\theta}$, when the other seller sets $r_{-i} = \underline{\theta}$, and $BR_i(\underline{\theta}/2)$ is found from $BR_a(r_b) = \frac{(\bar{\theta}-r_b)(\bar{\theta}-\beta(\bar{\theta}-\underline{\theta}))}{2(\bar{\theta}-r_b)+\beta(\bar{\theta}-\underline{\theta})}$ (This best response is derived in Appendix B, equation (102).)

It follows from 1-5 in Definition 4 that three key parameters are functions of $\underline{\theta}$ and $\bar{\theta}$, as opposed to the monopoly case where only one key parameter was a function of $\underline{\theta}$ and $\bar{\theta}$. In order to determine for which subset of the parameter space of β there is an equilibrium in reserve prices or not, the key parameters of Definition 4 are used to partition the parameter space of β. Values of $\underline{\theta}$ and $\bar{\theta}$ induce a ranking over the key parameters. This ranking of key parameters, in turn, can be used to establish a ranking of the different payoffs from relevant deviations. For a given value of β, the expected payoffs from deviating from a certain vector of reserve prices can be found.

Lemma 6 The key parameters of Definition 4 exist and are unique.

Proof. See Appendix B. ■

The following results are obtained for the seller's game.

Proposition 4 The vector of reserve prices $(r_a, r_b) = (\underline{\theta}, \underline{\theta})$ is not a symmetric equilibrium of the sellers' game.

Proof. See Appendix B. ■

At the candidate vector of reserve prices, for all $\beta \in [0, 1]$, a seller can increase his expected payoff discontinuously by undercutting his rival. This is illustrated in Figure 9 in Appendix C.

Consider the two regions of the reserve price domain $[0, \underline{\theta})$ and $(\underline{\theta}, \bar{\theta}]$. Continuity of sellers' expected payoffs over these two regions, implies the following. Given Proposition 4, the vectors $(r_a, r_b) = (\underline{\theta}/2, \underline{\theta}/2)$ and $(r_a, r_b) = ((1-\beta)/2, (1-\beta)/2)$ are the only candidate vectors an equilibrium of the sellers' game.

Proposition 5 [Equilibrium reserve prices below θ:] For given values of θ and $\bar{\theta}$ in $\Re_+$, there exists a unique set of key parameters $\left\{\beta_2, \beta_4(\theta, \bar{\theta})\right\} \subset [0, 1]$, such that for all $\beta > \max\{\beta_2, \beta_4(\theta, \bar{\theta})\}$; the vector of reserve prices $(r_a^*, r_b^*) = (\theta/2, \theta/2)$ is a unique, symmetric equilibrium of the sellers' game.

Proof. See Appendix B. ■

Figures 4, 5, and 6 illustrate the equilibrium described in Proposition 5, for different values of θ. Figure 7 in Appendix C illustrates a seller's payoff function associated with the equilibrium of Proposition 5.

When the probability that potential bidders are low type is relatively high, the equilibrium reserve prices under competition are lower than θ. This is different from the monopoly case, where a reserve price below θ was not an equilibrium. In a monopolistic environment, there is no pressure, other than the distribution over potential bidders' valuations, that induces the seller to set a reserve price below θ (i.e., potential bidders' only interest is to acquire the seller's item). Because of the discontinuity in the payoff functions at θ, β needs to be sufficiently low relative to θ for a deviation to $r_i = \theta$ or $r_i > \theta$ to be worthwhile when the other seller sets $\theta/2$. Intuitively, the probability of low-type bidders has to be high enough, relative to θ and $\bar{\theta}$, for a seller to find it profitable to set r_i in order to attract marginal bidders.

Proposition 6 [Equilibrium reserve prices above θ:] For given values of θ and $\bar{\theta}$ in $\Re_+$, there exists a unique key parameter $\beta_1(\theta, \bar{\theta}) \in [0, 1 - 2\theta/\bar{\theta}]$, such that for all $\beta < \beta_1(\theta, \bar{\theta})$, the vector of reserve prices $(r_a^*, r_b^*) = \left((1-\beta)\bar{\theta}/2, (1-\beta)\bar{\theta}/2\right) \gg (\theta, \theta)$ is the unique, symmetric equilibrium of the sellers' game.

Proof. See Appendix B. ■

Figures 5 and 6 illustrate this equilibrium for different values of θ. Figure 8 shows the payoff function of a seller at such equilibrium.

The existence of symmetric equilibrium in reserve prices depend on the extent of heterogeneity as by the difference between θ and $\bar{\theta}$. For $\theta > \bar{\theta}/2$, that is for low level of heterogeneity of potential bidders, no equilibrium above θ exists, because $\beta_1(\theta, \bar{\theta}) \leq 0$. This is shown in Figure 4 below.

The following numerical examples may facilitate the interpretation of the main propositions and how the existence of symmetric equilibrium in reserve prices for different beliefs depends on the extent of heterogeneity.

Let $\bar{\theta} = 1$ and consider the following cases:

1. Suppose $\underline{\theta} = .75$, then $\beta_1(.75;1) = 0$ and $\beta_4(.75;1) = .07 < \beta_2 = .414$. Therefore $\max\{.07; .414\} = .414$. From Proposition 5, $r_a^* = r_b^* = .375$ is a unique, symmetric equilibrium of the sellers' game, for all $\beta \in [.414; 1]$. Figure 4 illustrates this case.

   [Insert Figure 4 here]

2. Suppose $\underline{\theta} = .4$, then $\beta_1(.4;1) > 0$ and $\beta_3(.4;1) = .21 < \beta_2 = .414$. Therefore $\max\{.21; .414\} = .414$. From Proposition 5, $r_a^* = r_b^* = .2$ is a unique symmetric equilibrium of the sellers' game, for all $\beta \in [.414; 1]$. From Proposition 6, for $\beta < \beta_1(.4;1)$, then $r_a^* = r_b^* = (1-\beta)/2$ is a unique symmetric equilibrium of the sellers' game. This is illustrated in Figure 5.

   [Insert Figure 5 here]

3. Suppose $\underline{\theta} = .05$, then $\beta_1(.05;1) > 0$ and $\beta_3(.05;1) = .55 > \beta_2 = .414$. Therefore $\max\{.55; .414\} = .55$. From Proposition 5, $r_a^* = r_b^* = .025$ is a unique symmetric equilibrium of the sellers' game, for all $\beta \in [.55; 1]$. From Proposition 6, for $\beta < \beta_1(.05;1)$, then $r_a^* = r_b^* = (1-\beta)/2$ is a unique symmetric equilibrium of the sellers' game. This is illustrated in Figure 6.

   [Insert Figure 6 here]

For other parameter values of the distribution of potential bidders valuation, the following result is obtained.

Proposition 7 [Non-existence of symmetric equilibrium in reserve prices.] For given values of $\underline{\theta}$ and $\bar{\theta}$ in $\Re_+$, there exists a unique set of key parameters, $\left\{\beta_1(\underline{\theta};\bar{\theta}); \beta_2; \beta_4(\underline{\theta};\bar{\theta})\right\} \subset [0;1]$; such that for all $\beta \in (\beta_1(\underline{\theta};\bar{\theta}); \max\{\beta_4(\underline{\theta};\bar{\theta}); \beta_2\})$, there is no vector of symmetric reserve prices constituting an equilibrium of the sellers' game.

Proof. See Appendix B. ■

The result of non-existence of a pure strategy equilibrium is common to discontinuous games. As mentioned earlier, this model shares this feature with Hotelling-location type models.

A standard implication of increasing competition arises. For extreme belief values, the introduction of a second seller transfers expected rent from the sellers to the bidders. Under a single auction a type $\underline{\theta}$, when he participates, bids his valuation and gets a zero expected payoff. However, under two auctions, he gets a positive expected surplus from submitting a bid in one auction. In equilibrium, there is a positive probability that he would face no competitors at an auction which announced a reserve price below $\underline{\theta}$. A type $\bar{\theta}$ always obtains a positive expected payo® under competition, compared to the monopoly situation where he obtains a positive expected payo® only when he faces a competitors, and the equilibrium reserve price is smaller or equal to µ.

In all auctions, the bidder with the highest valuation always gets the item, i.e., auctions are e±cient. In this environment however, when private information is held before participation decisions are made, the introduction of competition leads to potential ine±ciencies (in a Pareto sense). In equilibrium in this model, there is a positive probability that two potential bidders would end up at the same auction when both reserve prices are below the potential bidders' valuations. This means that in equilibrium, a seller may not sell his item even though all potential bidders' valuations are higher than the reserve price he announced. This kind of ine±ciency is common in models of trading activities in which participants on opposite sides of the market are matched by an exogenous matching technology. This model shows how a duopolistic environment under asymmetric information yields potential ine±ciencies, despite the feature that participants on both side of the market are matched endogenously. Of course, this is driven by the static nature of the model.

In the duopoly auction, the existence of mixed strategies can be investigated outside the region of pure strategies. However, the results from the literature on the existence of an equilibrium mixed strategy in discontinuous games do not apply in the environment of this paper. Another di±culty in applying the existence results for discontinuous games, is that the equilibrium participation strategies of Proposition 2 play the role of a highly discontinuous demand function. This form of discontinuity increases the di±culties in calculating the boundaries of the interval over which randomization can occur in the sellers' game.

To understand the di±culties involved in computing a mixed strategy equilibrium in this context, it is worthwhile to consider the usual Bertrand game under perfect information. In the typical Bertrand game, the demand function is also discontinuous. When one seller undercuts his rival, even by an in¯nitesimal amount, he serves the entire market. The discontinuity in the payo® functions such a demand induces, occurs only when sellers set the same price. Given that strategy spaces are a continuous, when sellers randomize, the probability of setting the same price is a measure-zero event. Therefore, the discontinuity in the demand does not pose problems in the calculation of the equilibrium mixed strategies. In the present model, the demand and the expected pro¯t of sellers become discontinuous when a seller sets a reserve price equal to µ, independently of the reserve price set by the other seller. The set of seller strategies that induce a discontinuity on the expected payo® can be written as $R^{¤}(i) = f(r_a; r_b)jr_a = µ; 0 \cdot r_b \cdot µg \bigcup f(r_a; r_b)jr_b = µ; 0 \cdot r_a \cdot µg$. The only existence result that could be applied in this context is the Theorem 5a formulated by Dasgupta and Maskin (1986). However, the payoffs associated with the sellers game in this context do not satisfy the condition (i) of their Theorem. Namely, just by considering $\lim_{r_i \to \underline{\theta}; r_{-i} \to \hat{r}_{-i}} \Pi^i(r_i; \hat{r}_{-i}) \neq \Pi^i(\underline{\theta}; \hat{r}_{-i})$, for all $\hat{r}_{-i} \in [0; \bar{\theta}]$: A typical form of a seller's expected payoff function is illustrated in Figure 9 in Appendix C. The question of whether or not there exist an equilibrium in mixed strategies despite this violation is open. An example of a discontinuous game that produces a set of discontinuity that is not of measure zero can be found in Dasgupta and Maskin(1986b).

Until now, the paper has focused on finding symmetric equilibrium reserve prices for given parameter values. However, from the definition of the key parameters, it is evident that there exist values of the parameter β for which an asymmetric equilibrium in reserve prices exists.

Corollary 1 For given values of $\underline{\theta}$ and $\bar{\theta}$ in $\Re_+$ and $BR_i(\underline{\theta}/2) = \frac{(\bar{\theta}-r_i)(\bar{\theta}-\beta(\bar{\theta}-\underline{\theta}))}{2(\bar{\theta}-r_i)+\beta(\bar{\theta}-\underline{\theta})} > \underline{\theta}$ (as defined in equation (102) in Appendix B), the following vector of reserve prices constitutes an equilibrium of the sellers' game.

$$r = (r_i; r_{-i}) = \begin{cases} ((1-\beta)\bar{\theta}/2; \underline{\theta}) & \text{if } \beta = \beta_1(\underline{\theta}; \bar{\theta}) \text{ and } \underline{\theta} \leq \bar{\theta}/2 \\ (\underline{\theta}/2; \underline{\theta}) & \text{if } \beta = \max\{\beta_4(\underline{\theta}; \bar{\theta}); \beta_2\} \\ (BR_i(\underline{\theta}/2); \underline{\theta}/2) & \text{if } \beta = \beta_4(\underline{\theta}; \bar{\theta}) = \beta_2 = \beta_3(\underline{\theta}; \bar{\theta}) \\ (BR_i(\underline{\theta}/2); \underline{\theta}) & \text{if } \beta = \beta_4(\underline{\theta}; \bar{\theta}) = \beta_2 = \beta_3(\underline{\theta}; \bar{\theta}) \end{cases}$$

Uniqueness of equilibrium reserve prices no longer holds when $\beta = \beta_4(\underline{\theta}; \bar{\theta}) = \beta_2 = \beta_3(\underline{\theta}; \bar{\theta})$. This model is therefore capable of generating equilibrium reserve price dispersion. Given ex post bidding, reserve price dispersion translates into further price dispersion

3.2.1 Alternative formulation of the results

An alternative way to present the implications of this model is to use the mean and variance of the underlying distribution of valuations. In this paper, a potential bidder's valuation is a Bernoulli random variable. Since $E[\theta] = \beta\underline{\theta} + (1-\beta)\bar{\theta}$, the variance of the distribution of potential bidders' valuations is

$$\begin{aligned} \sigma^2 &= \beta\underline{\theta}^2 + (1-\beta)\bar{\theta}^2 - (E[\theta])^2 \\ &= \beta(1-\beta)(\bar{\theta}-\underline{\theta})^2 \end{aligned} \tag{42}$$

For given values of $\underline{\theta}$ and $\bar{\theta}$,

$$\begin{aligned} \frac{\partial E[\theta]}{\partial \beta} &= -(\bar{\theta} - \underline{\theta}) < 0 \\ \frac{\partial \sigma^2}{\partial \beta} &= (\bar{\theta} - \underline{\theta})^2 (1 - 2\beta) \lesseqgtr 0 \quad \Leftrightarrow \quad \beta \gtreqless 1/2 \end{aligned} \tag{43}$$

and a for a given value of β;

$$\begin{aligned} \frac{\partial E[\theta]}{\partial \underline{\theta}} &= \beta > 0 \\ \frac{\partial \sigma^2}{\partial \underline{\theta}} &= -2\beta(1 - \beta)(\bar{\theta} - \underline{\theta}) < 0 \\ \frac{\partial E[\theta]}{\partial \bar{\theta}} &= (1 - \beta) > 0 \\ \frac{\partial \sigma^2}{\partial \bar{\theta}} &= 2\beta(1 - \beta)(\bar{\theta} - \underline{\theta}) > 0 \end{aligned} \tag{44}$$

For each β, $\underline{\theta}$; and $\bar{\theta}$; there is a unique σ^2. For each key parameter obtained earlier, there is an associated variance. Therefore, each possible values of $\underline{\theta}$ and $\bar{\theta}$ induce a ranking of the key values of the variance. The result that a pure-strategy Nash equilibrium in reserve prices exists for extreme values of β can be expressed in variance terms. The above derivatives show that the variance is small for extreme values of β:

Proposition 8 For given values of $\underline{\theta}$ and $\bar{\theta}$ and a sufficiently small variance of the distribution of valuations of potential bidders, there exists a symmetric Nash equilibrium in reserve prices, otherwise, no symmetric Nash equilibrium in reserve prices exist.

This proposition makes the results a little more intuitive. For known possible valuations, the smaller the believed dispersion of the underlying distribution of potential bidders' valuations, the smaller is the gain to a seller by deviating from a given equilibrium reserve price to attract marginal bidders.

3.3 Perfect information

It is possible to find the perfect information equilibrium reserve prices. The first possibility is when the two potential bidders are known to have the same valuation. From Proposition

5 and 6, as $\beta \mapsto 0$, then $r^* \mapsto \rho/2$ and as $\beta \mapsto 1$, then $r^* \mapsto \theta/2$. If types are known to be the same, given that they are assumed to make simultaneous participation decisions, there exists a unique symmetric equilibrium in reserve prices in which potential bidders perfectly randomize choosing each seller with probability $1/2$. There are also two pure strategy equilibria in which potential bidders choose different auctions. This case of bilateral homogeneity under perfect information has been explored in Julien, Kennes and King(2001). The second possibility is when the two potential bidders are know to have different valuations. If it is perfect information that one potential bidder is type θ and the other is type ρ, given that sellers are homogeneous, it is easily shown that potential bidders will perfectly randomize choosing each seller with probability $1/2$, and that $r^* = \theta/2$ is the unique, symmetric equilibrium reserve prices. It is interesting to note that for symmetric reserve prices such that all potential bidders choose a seller with positive probability, private information held by the potential bidders does not seem to matter for sellers equilibrium choice of reserve prices. This is explained by the fact that very low reserve prices reveal to potential bidders that all types will participate. Based on that information, each potential bidders adopt a perfectly random participation strategy.

4 Conclusion

This paper has formulated a model of auctions with discrete bidders valuations in a duopolistic environment. The model is able to produce symmetric equilibrium reserve prices when the lowest bidder's valuation coincide with the seller's opportunity cost. The model also makes predictions about symmetric equilibrium reserve prices for extreme beliefs about potential bidders' valuations and when the lowest bidder's valuation is strictly above the seller's opportunity cost. When sellers are very uncertain about potential bidders' valuations, the model has no symmetric equilibrium in reserve prices. This result is in line with Burguet and Sakovics(1999) in that with continuous bidders valuations they are also able to show that in a dupolistic environment the symmetric equilibrium reserve prices are above the seller's opportunity cost, but their framework does not allow for an explicit solution for the equilibrium reserve prices.

Despite the limitation that the model cannot produce equilibria for all parameter values of the distribution of valuations, the model, nevertheless, accounts for several stylized facts of competing auctioning processes; namely that, on average, bidders pay a lower price, and reserve prices are lower than in monopolistic auctions. Furthermore, the model shows that in a static environment, potential inefficiencies arise by introducing competition under the presence of informational asymmetries. In other words, items which are reserve priced below the valuation of potential buyers can, nonetheless, remain unsold.

The paper aims to contribute to a better understanding of competitive interactions under informational asymmetries, and to enrich the literature on price formation and directed search. Several extensions of the model on this paper are possible. The obvious extension is to consider ¯nite number of sellers and buyers, and asymptotic results. Other extensions would be to remove the assumption that the item is indivisible. Extending the model to consider divisible items and common-value auctions would bring it closer to applications such as foreign-exchange or treasury-bill auctions. (See Bartolini and Cottarelli (1994)). For a recent application of the competing auctions paradigm with open bids to trade over the internet (see Peters and Severinov (2001)). Further extensions would be to allow a choice of auction format by the sellers under the assumption of private and common value; to introduce intermediaries; and, to account for private information about sellers' opportunity costs.

Appendix A

Proof of lemma 2. Let $Eb(\theta; r)$ be the expected winning bid when the bidders are type $\Theta = (\theta_j; \theta_{-j})$ and the reserve price r is announced. Hence $Eb((\underline{\theta}; \bar{\theta}); r)$ and $Eb((\bar{\theta}; \bar{\theta}); r)$ are the expected winning bid when there is only one type-$\bar{\theta}$ bidder and the expected winning bid when there are two type-$\bar{\theta}$ bidders, respectively. $Eb((\underline{\theta}; \bar{\theta}); r) \geq \underline{\theta}$; and in case of equality we assume that type $\bar{\theta}$ wins. Denote by $B(r; 2)$ the expected winning bid by the seller when two bidders are present at the auction.

It is clear that $B(r; 1) = r$. Now it can be shown that $B(r; 2)$ is a convex combination of $\underline{\theta}$ and $\bar{\theta}$. From the randomization equations (6) and (7) with appropriate superscripts, $b(r)$ and $F(b)$ can be solved for, where b is the bid made by a type-$\bar{\theta}$ bidder.

$$b(r) = \bar{\theta} - \frac{\beta(\bar{\theta} - \underline{\theta}) P_{\underline{\theta}}(r)}{[\beta P_{\underline{\theta}}(r) + F(b(r))(1 - \beta) P_{\bar{\theta}}(r)]} \tag{45}$$

$$F(b(r)) = \frac{\beta(b(r) - \underline{\theta}) P_{\underline{\theta}}(r)}{(1 - \beta)(\bar{\theta} - b(r)) P_{\bar{\theta}}(r)}$$

From the same equation, since $F(b_H) = 1$

$$b_H(r) = \bar{\theta} - \frac{(\bar{\theta} - \underline{\theta})\beta P_{\underline{\theta}}(r)}{(1 - \beta) P_{\bar{\theta}}(r) + \beta P_{\underline{\theta}}(r)} \tag{46}$$

and

$$F'(b(r)) = \frac{(\bar{\theta} - \underline{\theta})\beta P_{\underline{\theta}}(r)}{(1 - \beta) P_{\bar{\theta}}(r)(\bar{\theta} - b(r))^2} \tag{47}$$

where $F'(b(r))$ is the partial derivative of F with respect to b. The expected bid from a type-$\bar{\theta}$ bidder is then

$$Eb((\underline{\theta}; \bar{\theta}); r) = \int_{\underline{\theta}}^{b_H(r)} b(r) F'(b(r)) db(r) \tag{48}$$

The expected bid can be transformed as follows. Let $\alpha = \beta(\bar{\theta} - \underline{\theta})$ and $\delta = (1 - \beta)$. Then

$$Eb((\underline{\theta}; \bar{\theta}); r) = \int_{\underline{\theta}}^{b_H(r)} \frac{b(r)\alpha P_{\underline{\theta}}(r)}{\delta P_{\bar{\theta}}(r)(\bar{\theta} - b(r))^2} db(r) \tag{49}$$

Let $a(r) = \frac{\alpha P_{\underline{\theta}}(r)}{\delta P_{\bar{\theta}}(r)}$ then

$$Eb((\underline{\theta}; \bar{\theta}); r) = a(r) \int_{\underline{\theta}}^{b_H(r)} \frac{b(r)}{(\bar{\theta} - b(r))^2} db(r) \tag{50}$$

Rewriting $\frac{b(r)}{(\bar{\theta} - b(r))^2} = \frac{b(r)}{(\bar{\theta} - b(r))} \frac{1}{(\bar{\theta} - b(r))} = (\frac{\bar{\theta}}{(\bar{\theta} - b(r))} - 1)(\frac{1}{(\bar{\theta} - b(r))})$ we have

$$Eb((\underline{\theta}; \bar{\theta}); r) = a(r) \int_{\underline{\theta}}^{b_H(r)} (\frac{\bar{\theta}}{(\bar{\theta} - b(r))} - 1)(\frac{1}{(\bar{\theta} - b(r))}) db(r) \tag{51}$$

$$= \Big[\frac{1}{(\bar{\theta} - b(r))} + \ln(\bar{\theta} - b(r))\Big]a(r)\Big|_{\underline{\theta}}^{b_H(r)}$$
$$= \Big[\frac{1}{\bar{\theta} - b_H(r)} + \ln(\bar{\theta} - b_H(r)) - \frac{1}{(\bar{\theta} - \underline{\theta})} - \ln(\bar{\theta} - \underline{\theta})\Big]a(r)$$

Using the above definition of $a(r)$, $b_H(r)$ and $\mu(r)$ to write

$$Eb((\underline{\theta};\bar{\theta});r) = \bar{\theta} + \frac{\beta(\bar{\theta} - \underline{\theta})P_{\underline{\theta}}(r)}{(1-\beta)P_{\bar{\theta}}(r)}\ln\Big(\frac{\beta P_{\underline{\theta}}(r)}{\mu(r)}\Big) \quad (52)$$

Where it can be shown using L'Hôpital's rule that

$$\lim_{\beta \to 0} Eb((\underline{\theta};\bar{\theta});r) = 1 \qquad \lim_{\beta \to \infty} Eb((\underline{\theta};\bar{\theta});r) = \underline{\theta} \quad (53)$$

furthermore $\frac{\partial Eb((\underline{\theta};\bar{\theta});r)}{\partial \underline{\theta}} > 0$ and $\frac{\partial Eb((\underline{\theta};\bar{\theta});r)}{\partial \beta} < 0$. The expected winning bid when two type-$\bar{\theta}$ bidders participate is

$$Eb((\bar{\theta};\bar{\theta});r) = \int_{\underline{\theta}}^{b_H} 2bF'(b)F(b)db \quad (54)$$

From the definition of $F(b)$ above

$$Eb((\bar{\theta};\bar{\theta});r) = 2\int_{\underline{\theta}}^{b_H} b\frac{\beta^2 P_{\underline{\theta}}(r)^2(\bar{\theta} - \underline{\theta})(b - \underline{\theta})}{(1-\beta)^2 P_{\bar{\theta}}(r)^2(\bar{\theta} - b)^3} \quad (55)$$

Let $\alpha = \frac{\beta^2 P_{\underline{\theta}}(r)^2(\bar{\theta} - \underline{\theta})}{(1-\beta)^2 P_{\bar{\theta}}(r)^2}$. In order to evaluate the integral, consider

$$\begin{aligned} Eb((\bar{\theta};\bar{\theta});r) &= 2\alpha\int_{\underline{\theta}}^{b_H} \frac{b^2 - b\underline{\theta}}{(\bar{\theta} - b)^3}db \quad (56)\\ &= 2\alpha\Big[\int \frac{b^2}{(\bar{\theta} - b)^3}db - \int \frac{b\underline{\theta}}{(\bar{\theta} - b)^3}db\Big] \\ &= 2\alpha\Big[\int \Big(\frac{b}{(\bar{\theta} - b)}\Big)^2 \frac{1}{(\bar{\theta} - b)}db - \underline{\theta}\int \Big(\frac{b}{(\bar{\theta} - b)}\Big)\frac{1}{(\bar{\theta} - b)^2}db\Big] \end{aligned}$$

using the same transformation as above for $(b/(\bar{\theta} - b))$;

$$\begin{aligned} Eb((\bar{\theta};\bar{\theta});r) &= 2\alpha\Big[\int \frac{\bar{\theta}(\bar{\theta} - \underline{\theta})}{(\bar{\theta} - b)^3}db - \int \frac{(2\bar{\theta} - \underline{\theta})}{(\bar{\theta} - b)^2}db + \int \frac{1}{(\bar{\theta} - b)}db\Big] \quad (57)\\ &= 2\alpha\Big[\frac{\bar{\theta}(\bar{\theta} - \underline{\theta})}{2(\bar{\theta} - b)^2} - \frac{(2\bar{\theta} - \underline{\theta})}{(\bar{\theta} - b)} - \ln(\bar{\theta} - b)\Big]_{\underline{\theta}}^{b_H} \end{aligned}$$

From the above definitions of b_H, $\mu(r)$ and $Eb((\underline{\theta};\bar{\theta});r)$ the expected winning bid from two type-$\bar{\theta}$ bidders is

$$Eb((\bar{\theta};\bar{\theta});r) = \bar{\theta} - \frac{2\beta P_{\underline{\theta}}(r)}{(1-\beta)P_{\bar{\theta}}(r)}[Eb((\underline{\theta};\bar{\theta});r) - \underline{\theta}] \quad (58)$$

Finally the expected winning bid when $n = 2$ is

$$B(r;2) = \frac{\beta^2 P_{\underline{\theta}}(r)^2}{\mu(r)^2}\underline{\theta} + \frac{2\beta(1-\beta)P_{\underline{\theta}}(r)P_{\bar{\theta}}(r)}{\mu(r)^2}Eb((\underline{\theta};\bar{\theta});r) + \frac{(1-\beta)^2 P_{\bar{\theta}}(r)^2}{\mu(r)^2}Eb((\bar{\theta};\bar{\theta});r) \tag{59}$$

$$B(r;2) = \frac{\beta^2 P_{\underline{\theta}}(r)^2}{\mu(r)^2}\underline{\theta} + \frac{2\beta(1-\beta)P_{\underline{\theta}}(r)P_{\bar{\theta}}(r)}{\mu(r)^2}\underline{\theta} + \frac{(1-\beta)^2 P_{\bar{\theta}}(r)^2}{\mu(r)^2}\bar{\theta}$$

$$B(r;2) = \left(1 - \frac{(1-\beta)^2 P_{\bar{\theta}}(r)^2}{\mu(r)^2}\right)\underline{\theta} + \frac{(1-\beta)^2 P_{\bar{\theta}}(r)^2}{\mu(r)^2}\bar{\theta} \qquad \blacksquare$$

Proof of Proposition 2:

The demonstration simply follows from equating the ex ante expected utility of a potential bidder among auctions, $R^a(\theta;r) = R^b(\theta;r)$, for all combinations of r relative to θ. Consider the following cases

1. $r_i > \underline{\theta}$ for all $i = a;b$. Then $P_{\underline{\theta}}(r) = 0$ is a dominant strategy for a low-type potential bidder. For a type $\bar{\theta}$, $P_{\bar{\theta}}(r)$ is found from

$$(1-\beta)(1-P_{\bar{\theta}}(r))(\bar{\theta}-r_a) + \beta(\bar{\theta}-r_a) = (1-\beta)P_{\bar{\theta}}(r)(\bar{\theta}-r_b) + \beta(\bar{\theta}-r_b) \tag{60}$$

therefore

$$P_{\bar{\theta}}(r) = \frac{\bar{\theta} - r_a - \beta(\bar{\theta}-r_b)}{(2\bar{\theta}-r_a-r_b)(1-\beta)} \tag{61}$$

2. $r_a = r_b = \bar{\theta}$. The equilibrium strategies are found using L'Hôpital's rule on $P_{\bar{\theta}}(r)$, yielding $P_{\bar{\theta}}(r) = 1/2$.

3. $r_a \leq \underline{\theta} \leq r_b$ with $r_a \neq r_b$. In this case $P_{\underline{\theta}}(r) = 1$ is a dominant strategy. Type $\bar{\theta}$'s strategy is found by

$$\begin{aligned}&[(1-\beta)(1-P_{\bar{\theta}}(r)) + \beta(1-P_{\underline{\theta}}(r))](\bar{\theta}-r_a) + \beta P_{\underline{\theta}}(r)(\bar{\theta}-\underline{\theta}) = \\ &[(1-\beta)P_{\bar{\theta}}(r) + \beta P_{\underline{\theta}}(r)](\bar{\theta}-r_b)\end{aligned} \tag{62}$$

hence

$$P_{\bar{\theta}}(r) = \frac{(1-\beta)(\bar{\theta}-r_a) - \beta(\underline{\theta}-r_b)}{(1-\beta)(2\bar{\theta}-r_a-r_b)} \tag{63}$$

4. $r_a \geq \underline{\theta} \geq r_b$ with $r_a \neq r_b$. For this case $P_{\underline{\theta}}(r) = 1$ is a dominant strategy. $P_{\bar{\theta}}(r)$ results from

$$\begin{aligned}&\{\beta(1-P_{\underline{\theta}}(r)) + (1-\beta)(1-P_{\bar{\theta}}(r))\}(\bar{\theta}-r_a) = \\ &\{(1-\beta)P_{\bar{\theta}}(r) + \beta P_{\underline{\theta}}(r)\}(\bar{\theta}-r_b) + \beta P_{\underline{\theta}}(r)(\bar{\theta}-\underline{\theta})\end{aligned} \tag{64}$$

therefore

$$P_{\bar\theta}(r) = \frac{\bar\theta - r_a - \beta(\bar\theta - \theta)}{(1-\beta)(2\bar\theta - r_a - r_b)} \tag{65}$$

5. $r_i < \theta$ for all $i = a, b$. In this case there are no dominant strategies. The equilibrium strategies are derived from the following equations

$$\{(1-\beta)(1-P_{\bar\theta}(r)) + \beta(1-P_\theta(r))\}(\theta - r_a) = \{(1-\beta)P_{\bar\theta}(r) + \beta P_\theta(r)\}(\theta - r_b)$$

$$\begin{aligned} &\{(1-\beta)(1-P_{\bar\theta}(r)) + \beta(1-P_\theta(r))\}(\bar\theta - r_a) + \beta(\bar\theta-\theta)P_\theta(r) \\ =\; &\{(1-\beta)P_{\bar\theta}(r) + \beta P_\theta(r)\}(\bar\theta - r_b) + \beta(\bar\theta-\theta)(1-P_\theta(r)) \end{aligned} \tag{66}$$

therefore

$$P_\theta(r) = \frac{1}{2} - \frac{(r_a - r_b)}{2\beta(2\theta - r_a - r_b)}; \qquad P_{\bar\theta}(r) = \frac{1}{2}. \tag{67}$$

6. Finally the case of $r_a = \theta = r_b$. The equilibrium strategies are found by taking $\lim_{r_a = r_b = r \to \theta} P_{\theta_j}(r)$ of the equilibrium strategies of case 4 using L'Hôpital's rule yields $P_\theta(r) = P_{\bar\theta}(r) = 1/2$.

Uniqueness of cases 1 ,2 ,3 and 5 is obvious. For case 4, the uniqueness of the completely mixed strategies is clear. It must be shown that no pure strategies are an equilibrium in this case. Consider the possible pure strategies,

$< P_\theta, P_{\bar\theta} > = \{< 0, 0 >, < 0, 1 >, < 1, 0 >, < 1, 1 >\}$. Without loss of generality let $r_a < r_b < \theta$.

(a) Suppose $< P_\theta, P_{\bar\theta} > = < 0, 0 >$; that is both types are participating in auction b, and let $R(\theta)_{(0,0)}$ be type θ's expected utility when both types play $< 0, 0 >$.

$$R(\theta)_{(0,0)} = 0; \qquad R(\bar\theta)_{(0,0)} = \beta(\bar\theta - \theta) \tag{68}$$

consider the deviation $\tilde P_{\bar\theta} = 1$

$$R(\bar\theta)_{(0,1)} = \beta(\bar\theta - r_b) > \beta(\bar\theta - \theta) \tag{69}$$

since $r_b < \theta$.

(b) Suppose $< P_\theta, P_{\bar\theta} > = < 0, 1 >$

$$R(\theta)_{(0,1)} = (1-\beta)(\theta - r_b); \qquad R(\bar\theta)_{(0,1)} = \beta(\bar\theta - r_a) \tag{70}$$

Consider a deviation $\bar P_{\bar\theta} \in [0,1]$

$$\begin{aligned}
&R(1)_{(0,\bar P_1)} - R(1)_{(0,1)} \\
&= [1 - \bar P_1 + \beta \bar P_1](\bar\theta - r_a) + (1-\beta)\bar P_1(\bar\theta - r_b) + \beta(\bar\theta - \theta) - [\beta(\bar\theta - r_a)] \\
&= (1-\beta)(1-\bar P_1)(\bar\theta - r_a) + (1-\beta)\bar P_1(\bar\theta - r_b) + \beta(\bar\theta - \theta) > 0
\end{aligned} \tag{71}$$

(c) Suppose $< P_\theta, P_{\bar\theta} > = < 1, 0 >$

$$R(\theta)_{(1,0)} = (1-\beta)(\theta - r_a); \qquad R(\bar\theta)_{(1,0)} = \beta(\bar\theta - r_b) \tag{72}$$

Consider the deviation $\bar P_\theta = 0$

$$R(\theta)_{(0,0)} = (\theta - r_a) > (1-\beta)(\theta - r_a) \tag{73}$$

(d) Suppose $< P_\theta, P_{\bar\theta} > = < 1, 1 >$

$$R(\theta)_{(1,1)} = 0; \qquad R(\bar\theta)_{(1,1)} = \beta(\bar\theta - \theta) \tag{74}$$

Consider the deviation $\bar P_{\bar\theta} = 0$

$$R(1)_{(1,0)} = \beta(\bar\theta - r_b) > \beta(\bar\theta - \theta) \tag{75}$$

Therefore no equilibrium in pure strategies exists. ■

Appendix B

Proof of Proposition 4: The proof consists of showing that it always pays a seller to deviate from the candidate (θ, θ) for all $\beta \in [0,1]$ and $\theta \in [0,1]$. In doing that, first $\Pi^i(\theta, \theta)$ needs to be evaluated. Given that discontinuity occurs at θ,

$$\Pi^i(r_i, r_{-i})|_{r_i = r_{-i} = r} = \frac{(2r + (\beta\theta(2-\theta) + (1-\beta)^2)}{4} \tag{76}$$

where $r < \theta$. Consider the following limit

$$\begin{aligned}
\lim_{r \to \theta^-} \Pi^i(r_i, r_{-i})|_r &= \frac{\theta + (1+\theta)(1-\beta)^2}{4} \\
&= \Pi^i(\theta, \theta)
\end{aligned} \tag{77}$$

In order to demonstrate that it always pays a seller to set a reserve price just below θ when his rival sets θ, the payoff $\Pi^i(r_i < \theta, \theta)$ evaluated at $r_i = \theta$ is needed

$$\Pi^i(r_i < \theta, \theta)|_{r_i = \theta} = \frac{(1-\beta^2)\theta}{2} + \theta + \frac{(1-\beta)^2(\bar\theta - \theta)}{4}. \tag{78}$$

Comparing the two payoffs, let

$$\begin{aligned}
\Delta(\beta;\underline{\theta};\bar{\theta}) &= \Pi^{-}(r_{-} < \underline{\theta};\underline{\theta})|_{r_{-}=\underline{\theta}} - \lim_{r \to -\underline{\theta}} \Pi^{-}(r_{-}; r_{--})|_{r} && (79)\\
&= \frac{(1-\beta^2)\underline{\theta}}{2} + \underline{\theta} + \frac{(1-\beta)^2(\bar{\theta}-\underline{\theta})}{4} - \left(\frac{\underline{\theta} + (\bar{\theta}+\underline{\theta})(1-\beta)^2}{4} \right)\\
&= \frac{\beta\underline{\theta}(1-\beta)}{2} + \frac{3\underline{\theta}}{4} > 0
\end{aligned}$$

This shows that when a seller sets $\underline{\theta}$, the other seller's payoff increases when he sets $r_{-} < \underline{\theta}$. There exists an arbitrary $\epsilon > 0$ such that for $r_{-} \in N_{\epsilon}^{-}(\underline{\theta})$, $\Delta(\beta;\underline{\theta};\bar{\theta})|_{r_{-}} > 0$ for all $\beta \in [0;1]$ and $\underline{\theta} \in [0;1]$. This result is illustrated in Figure 9 in Appendix C. ■

Proof of Lemma 6:

To show existence and uniqueness of the key parameters, consider:

1. For $\beta_1(\underline{\theta};\bar{\theta})$. Let $\Delta(\beta;\underline{\theta};\bar{\theta}) = \Pi(r^*_a; r^*_b) - \Pi(\underline{\theta}; r^*_b)$ where $r^*_{-} = (1-\beta)\bar{\theta}/2$. $\Delta(\beta;\underline{\theta};\bar{\theta})$ is easily shown to be monotonically decreasing in β. Rewrite $\Delta(\beta;\underline{\theta};\bar{\theta})$ while omitting the subscript of r^*;

$$\begin{aligned}
\Delta(\beta;\underline{\theta};\bar{\theta}) &= \frac{(1-\beta^2)r^*}{2} + \frac{(1-\beta)^2\bar{\theta}}{4} - \frac{\{(\bar{\theta}-\underline{\theta})^2[1-\beta^2\underline{\theta}]}{(2\bar{\theta}-r^*-\underline{\theta})^2} && (80)\\
&\quad + \frac{(\bar{\theta}-\underline{\theta})(\bar{\theta}-r^*)[2\underline{\theta}(1+\beta) - 2\beta(\bar{\theta}-\underline{\theta})] + \beta^2(1-r^*)^2(\bar{\theta}-\underline{\theta})}{(2\bar{\theta}-r^*-\underline{\theta})^2}
\end{aligned}$$

Using the envelope theorem

$$\begin{aligned}
\frac{\partial\Delta(\beta;\underline{\theta};\bar{\theta})}{\partial\beta} &= -\left(\frac{(1-\beta)^2(\bar{\theta}-\underline{\theta})^2 + (1-\beta^2)(1+\beta)^2}{2(2\bar{\theta}-\underline{\theta}-r^*)^2} \right) && (81)\\
&\quad - \left(\frac{(\bar{\theta}-\underline{\theta})((1-\beta^2)\beta + 4\underline{\theta}(1+\beta\underline{\theta}))}{2(2\bar{\theta}-\underline{\theta}-r^*)^2} \right) < 0
\end{aligned}$$

for all $\beta;\underline{\theta} \in [0;1]$. Observe that

$$\lim_{\beta \to 0} \Delta(\beta;\underline{\theta};\bar{\theta}) = \frac{1-4\underline{\theta}+4\underline{\theta}^2}{2(3\bar{\theta}-2\underline{\theta})^2} > 0 \qquad (82)$$

and

$$\lim_{\beta \to 1-2\underline{\theta}} \Delta(\beta;\underline{\theta};\bar{\theta}) = -\underline{\theta}(1-2\underline{\theta}) < 0 \qquad (83)$$

From the monotonically decreasing feature of $\Delta(\beta;\underline{\theta};\bar{\theta})$ and the limit values, it follows that there exists a $\beta_1(\underline{\theta};\bar{\theta})$ such that $\Delta(\beta;\underline{\theta};\bar{\theta}) = 0$ and for all $\beta \leq \beta_1(\underline{\theta};\bar{\theta})$

$\Leftrightarrow 4(\beta;\theta;\bar{\theta}) \geq 0$. This shows that for a relatively small β, a deviation to $r_i = \theta$ is not profitable when the other seller sets r^*. It is obvious from the above, that $\beta_1(\theta;\bar{\theta}) < (1 - 2\theta) = \hat{\beta}$.

2. For $\hat{\beta}$, clear by inspection of item 2 of Definition 4.

3. β_2 solves

$$\frac{(1-\beta^2)\theta}{2} + \frac{(1-\beta)^2}{4} = \frac{\theta(1+2\beta-\beta^2)+(1-\beta)^2}{4} \tag{84}$$

Therefore $\beta_2 = 0.414$.

4. $\beta_3(\theta;\bar{\theta})$ solves

$$\frac{(1-\beta^2)\theta}{2} + \frac{(1-\beta)^2\bar{\theta}}{4} = \frac{(\bar{\theta}-\beta(\bar{\theta}-\theta))^2}{(3\bar{\theta}-\theta)} \tag{85}$$

The solution is $\beta_3(\theta;\bar{\theta}) = \frac{(\bar{\theta}-3\theta)\pm 2\sqrt{\theta^3(3\bar{\theta}-\theta)}}{(\bar{\theta}-\theta+2\theta^2)}$. It is easily verified that for any $\theta \in [0;\bar{\theta}]$ and $\bar{\theta}$, $\beta_3(\theta;\bar{\theta})$ is unique.

5. $\beta_4(\theta;\bar{\theta})$ solves

$$\frac{\theta(1+2\beta-\beta^2)+(1-\beta)^2\bar{\theta}}{4} = \frac{(\bar{\theta}-\beta(\bar{\theta}-\theta))^2}{(3\bar{\theta}-\theta)} \tag{86}$$

The solution is $\beta_4(\theta;\bar{\theta}) = \frac{2(\bar{\theta}+\theta)\pm\sqrt{8\theta(3\bar{\theta}-\theta)}}{2(\bar{\theta}-3\theta)}$. It is easily verified that the solution is unique for all $\theta \in [0;\bar{\theta}]$ and $\bar{\theta}$. ■

Proof of Proposition 5:

The proof of this proposition consists of showing that no possible deviations upsets the candidate $(r_a^*;r_b^*) = (\theta/2;\theta/2)$ as an equilibrium of the sellers' game.

Lemma 4 establishes that local deviations over $[0;\theta)$ are not profitable. Deviations to $r_i = \theta$ and r_i above θ must be considered. In order to investigate these possible deviations, some properties of a seller's payoff function evaluated at each candidate equilibrium are needed. The following lemma shows that profit falls discontinuously as soon as a seller sets $r_i = \theta$ when the other sets $r_i = \theta/2$.

Lemma 7 For a seller, a deviation close to θ is more profitable than a deviation to $r_i = \theta$ when the other seller sets $r_{-i} = \theta/2$

$$\lim_{r_i \to \theta^-} \Pi^i(r_i;\theta/2) \geq \Pi^i(\theta;\theta/2); \forall \beta \in [0;1]. \tag{87}$$

Proof of Lemma 7: Consider the following limit

$$\begin{aligned}\lim_{r_i \to \underline{\theta}^-} P^i_{\underline{\theta}}(r) &= \frac{1}{2} - \lim_{r_i \to \underline{\theta}^-} \frac{(r_i - \underline{\theta}/2)}{2\beta(2\underline{\theta} - r_i - \underline{\theta}/2)} \\ &= \frac{1}{2} - \frac{1}{2\beta} < 0\end{aligned} \tag{88}$$

Therefore, the lower bound on the participation strategy is binding, and hence, at the limit $P^i_{\underline{\theta}}(r) = 0$. The expected payoff at the limit is

$$\lim_{r_i \to \underline{\theta}^-} \Pi^i(r_i; \underline{\theta}/2) = \frac{(1-\beta^2)\underline{\theta}}{2} + \frac{(1-\beta)^2\bar{\theta}}{4}. \tag{89}$$

The expected profit evaluated at $\underline{\theta}$ when the other seller sets $r_i = \underline{\theta}/2$ is

$$\Pi^i(\underline{\theta}; \underline{\theta}/2) = \frac{2(\bar{\theta} - \underline{\theta})(1-\beta)(1 - \underline{\theta}/2 + \beta(\bar{\theta} - \underline{\theta}))\underline{\theta} + (\bar{\theta} - \underline{\theta})^2(1-\beta)^2}{(2\bar{\theta} - \underline{\theta} - \underline{\theta}/2)^2}. \tag{90}$$

Rearranging

$$\Pi^i(\underline{\theta}; \underline{\theta}/2) = \frac{4(\bar{\theta} - \underline{\theta})(1-\beta)(2 - \underline{\theta} + 2\beta(\bar{\theta} - \underline{\theta})) + 4(\bar{\theta} - \underline{\theta})^2(1-\beta)^2}{(4\bar{\theta} - 3\underline{\theta})^2} \tag{91}$$

These payoffs have the following properties:

$$\begin{aligned}\frac{\partial \lim_{r_i \to \underline{\theta}^-} \Pi^i(r_i; \underline{\theta}/2)}{\partial \beta} &= -\beta\underline{\theta} - \frac{(1-\beta)\bar{\theta}}{2} < 0 \\ \frac{\partial \Pi^i(\underline{\theta}; \underline{\theta}/2)}{\partial \beta} &= \frac{-4(\bar{\theta} - \underline{\theta})(\underline{\theta}(1 + 2\beta(1 - \underline{\theta}) + 2(\bar{\theta} - \underline{\theta})(1-\beta))}{(4\bar{\theta} - 3\underline{\theta})^2} < 0.\end{aligned} \tag{92}$$

Let $\Delta(\beta; \underline{\theta}; \bar{\theta}) = \lim_{r_i \to \underline{\theta}^-} \Pi^i(r_i; \underline{\theta}/2) - \Pi^i(\underline{\theta}; \underline{\theta}/2)$. The function has the following limits

$$\begin{aligned}\lim_{\beta \to 0} \Delta(\beta; \underline{\theta}; \bar{\theta}) &= \frac{1 + 2\underline{\theta}}{4} - \frac{4(\bar{\theta} - \underline{\theta})(2 - \underline{\theta})\underline{\theta} + 4(\bar{\theta} - \underline{\theta})^2}{(4\bar{\theta} - 3\underline{\theta})^2} \\ &= \frac{2\underline{\theta}^3 + 7\underline{\theta}(\bar{\theta} - \underline{\theta}) + \underline{\theta}}{(4\bar{\theta} - 3\underline{\theta})^2} > 0\end{aligned} \tag{93}$$

and

$$\lim_{\beta \to 1} \Delta(\beta; \underline{\theta}; \bar{\theta}) = 0 \tag{94}$$

for all $\beta \in [0;1]$. This proves the result. ■

Consider a deviation above $\underline{\theta}$. The payoffs $\Pi^i(r_i; r_{-i})$ are continuous over $(\underline{\theta}; \bar{\theta}]$ because the equilibrium participation strategies are continuous over that range. A seller would choose $r_i > \underline{\theta}$ to maximize

$$\begin{aligned}\Pi^i(r_a; r_b) &= \pi^i_1(r) r_i + \pi^i_2(r) B^i(r; 2) \\ &= 2\mu^i(1 - \mu^i) r_i + (\mu^i)^2\end{aligned} \tag{95}$$

Lemma 8 Seller i's expected payoff $\Pi^i(r_i;\theta/2)$ is concave in $r_i \in (\theta,\bar\theta]$.

Proof of Lemma 8: Because of symmetry let $\Pi^a(.,.) = \Pi^b(.,.) = \Pi(.,.)$. Consider the payoff of seller a. The first order condition is

$$\frac{\partial\Pi(r_a,r_b)}{\partial r_a} = 2\rho(1-\rho) + 2r_a(1-2\rho)\frac{\partial\rho}{\partial r_a} + 2\rho\frac{\partial\rho}{\partial r_a} = 0 \tag{96}$$

Using the fact that

$$\frac{\partial\rho}{\partial r_a} = -\frac{(1-\rho)}{(2\bar\theta - r_a - r_b)} \tag{97}$$

yields

$$\frac{\partial\Pi(r_a,r_b)}{\partial r_a} = 2\rho(1-\rho)\left\{1 - \frac{(\rho + r_a(1-2\rho))}{\rho(2\bar\theta - r_a - r_b)}\right\} = 0. \tag{98}$$

The second-order condition is

$$\frac{\partial^2\Pi(r_a,r_b)}{\partial r_a^2} = \frac{2(1-\rho)}{(2\bar\theta - r_a - r_b)^2}\left\{2\rho(\bar\theta - r_b) - 2r_a(1-\rho) - (3-2r_a)(1-\rho)\right\} \tag{99}$$

Given that $r_b = \theta/2$, $\rho < 1/2$ and $r_a > r_b$,

$$\frac{\partial^2\Pi(r_a,r_b)}{\partial r_a^2} = -\frac{2(1-\rho)\{(1-\theta)(1-2\rho) - 2r_a(1-\rho) - (2-3\rho)\}}{(2\bar\theta - r_a - r_b)^2} < 0 \tag{100}$$

Similar expressions exist for seller b. ■

With some transformations, the first-order condition becomes

$$\begin{aligned} r_a(1-\rho) - \rho(\bar\theta - r_b) &= 0 \\ r_a &= \frac{\rho}{(1-\rho)}(\bar\theta - r_b). \end{aligned} \tag{101}$$

Substituting the value of $P^i_{\bar\theta}(r) = \frac{(\bar\theta - r_a - \beta(\bar\theta-\theta))}{(1-\beta)(2\bar\theta - r_a - r_b)}$ in ρ

$$BR_a(r_b) = \frac{(\bar\theta - r_b)(\bar\theta - \beta(\bar\theta - \theta))}{2(\bar\theta - r_b) + \beta(\bar\theta - \theta)} \tag{102}$$

which is seller a's best response above θ when seller b sets a reserve price below θ. A similar expression exists for seller b.

Since the candidate equilibrium $(r_a, r_b) = (\theta/2, \theta/2)$ is of particular interest, let $BR_i(\theta/2) > \theta$ be the best response above θ by seller i when $-i$ sets $\theta/2$.

Consider a deviation to $r_i = \theta$ by one seller when the other sets $r_{-i} = \theta/2$. The proof of Lemma 7 implies that as a seller increases his reserve price from an initial reserve price below θ, payoff falls discontinuously when it hits $r_i = \theta$. The payoff evaluated at a deviation to $r_i = \theta$ is $\lim_{r_i \to \theta^-} \Pi^i(r_i, \theta/2)$.

Lemma 6 shows the existence of a set of unique key parameters. The relevant properties of the payoffs involved in the definition of the key parameters are now established.

Lemma 9 The sellers' expected payoffs,

1. $\Pi^i(BR_i(\theta/2), \theta/2)$ is convex in β.
2. $\Pi^i(\theta/2, \theta/2)$ is concave in β.
3. $\lim_{r_i \to \theta^-} \Pi^i(r_i, \theta/2)$ is concave in β if $\theta/\bar{\theta} > 1/2$, linear in β if $\theta/\bar{\theta} = 1/2$ and convex in β if $\theta/\bar{\theta} < 1/2$.

Furthermore, all three payoffs are strictly decreasing in β for all $\beta \in [0, 1]$.

Proof of Lemma 9.

Using the participation strategies for $r_i > \theta > r_{-i}$ and simplifying the payoffs yields

$$\Pi^i(BR_i(\theta/2), \theta/2) = \frac{(\bar{\theta} - \beta(\bar{\theta} - \theta))^2}{(3\bar{\theta} - \theta)} \tag{103}$$

$$\frac{\partial \Pi^i(BR_i(\theta/2), \theta/2)}{\partial \beta} = \frac{-(\bar{\theta} - \theta)^2(1 - \beta(\bar{\theta} - \theta))^2 - 3\beta(\bar{\theta} - \theta))}{(3\bar{\theta} - \theta)} < 0$$

$$\frac{\partial^2 \Pi^i(BR_i(\theta/2), \theta/2)}{\partial \beta^2} = -\frac{(\bar{\theta} - \theta)(1 - 4\theta + 2\theta^2) + 2\beta(1 - \theta^4}{(3\bar{\theta} - \theta)} < 0$$

When both reserve prices are $\theta/2$

$$\Pi^i(\theta/2, \theta/2) = \frac{\theta(1 + 2\beta - \beta^2) + (1 - \beta)^2}{4} \tag{104}$$

$$\frac{\partial \Pi^i(\theta/2, \theta/2)}{\partial \beta} = -\frac{(\bar{\theta} - \theta)(1 - \beta)}{2} < 0$$

$$\frac{\partial^2 \Pi^i(\theta/2, \theta/2)}{\partial \beta^2} = \frac{(\bar{\theta} - \theta)}{2} > 0$$

$$\lim_{r_i \to \theta^-} \Pi^i(r_i, \theta/2) = \frac{(1 - \beta^2)\theta}{2} + \frac{(1 - \beta)^2\bar{\theta}}{4} \tag{105}$$

$$\frac{\partial \lim_{r_i \to \theta^-} \Pi^i(r_i, \theta/2)}{\partial \beta} = -\beta\theta - \frac{(1 - \beta)\bar{\theta}}{2} < 0$$

$$\frac{\partial^2 \lim_{r_i \to \theta^-} \Pi^i(r_i, \theta/2)}{\partial \beta^2} = -\frac{(\bar{\theta} - 2\theta)}{2} \lesseqgtr 0 \qquad \text{iff } \bar{\theta} \lesseqgtr 2\theta \qquad \blacksquare$$

Consider the limit values of the above payoffs,

$$\lim_{\beta \to 0} \Pi^i(BR_i(\theta/2);\theta/2) = \frac{1}{(3\bar{\theta} - \theta)} > \frac{\bar{\theta} + \theta}{4} \tag{106}$$

$$\lim_{\beta \to 0}[\lim_{r_i \to -\theta} \Pi^i(r_i;\theta/2)] = \frac{\bar{\theta} + 2\theta}{4}$$

$$\lim_{\beta \to 0} \Pi^i(\theta/2;\theta/2) = \frac{\bar{\theta} + \theta}{4}$$

and

$$\lim_{\beta \to 1} \Pi^i(BR_i(\theta/2);\theta/2) = \frac{\theta^2}{(3\bar{\theta} - \theta)} < \frac{\bar{\theta} + \theta}{4} \tag{107}$$

$$\lim_{\beta \to 1}[\lim_{r_i \to -\theta} \Pi^i(r_i;\theta/2)] = 0$$

$$\lim_{\beta \to 1} \Pi^i(\theta/2;\theta/2) = \frac{\theta}{2} > \frac{\theta^2}{(3\bar{\theta} - \theta)}$$

From these limit values and the properties of the payoff functions defined in Lemma 9 follows this corollary:

Corollary 2 For a given $\theta \in [0;\bar{\theta}]$, the following relations hold:

1. If $\beta \lesseqqgtr \beta_1(\theta;\bar{\theta})$ then $\Pi^i(\theta;(1-\beta)\bar{\theta}/2) \lesseqqgtr \Pi^i((1-\beta)\bar{\theta}/2;(1-\beta)\bar{\theta}/2)$.

2. If $\beta \lesseqqgtr \beta_1(\theta;\bar{\theta})$ then $\theta \lesseqqgtr r^* = (1-\beta)\bar{\theta}/2$.

3. If $\beta \lesseqqgtr \beta_2$ then $\Pi^i(\theta/2;\theta/2) \lesseqqgtr \lim_{r_i \to -\theta} \Pi^i(r_i;\theta/2)$.

4. If $\beta \lesseqqgtr \beta_3(\theta;\bar{\theta})$ then $\Pi^i(BR_i(\theta/2);\theta/2) \lesseqqgtr \lim_{r_i \to -\theta} \Pi^i(r_i;\theta/2)$.

5. If $\beta \lesseqqgtr \beta_4(\theta;\bar{\theta})$ then $\Pi^i(\theta/2;\theta/2) \lesseqqgtr \Pi^i(BR_i(\theta/2);\theta/2)$:

These relationships suggest that $\lim_{r_i \to -\theta} \Pi^i(r_i;\theta/2)$ and $\Pi^i(\theta/2;\theta/2)$, as well as $\Pi^i(BR_i(\theta/2);\theta/2)$ and $\Pi^i(\theta/2;\theta/2)$, will cross only once over the parameter space of $\beta \in [0;1]$. Similarly, it can be concluded that $\Pi^i(BR_i(\theta/2);\theta/2)$ and $\lim_{r_i \to -\theta} \Pi^i(r_i;\theta/2)$ will cross once, if and only if the following limits hold:

$$\lim_{\beta \to 0} \left\{ \Pi^i(BR_i(\theta/2);\theta/2) \right\} = \frac{(\bar{\theta} + 2\theta)}{4} > \frac{1}{(3\bar{\theta} - \theta)} = \lim_{\beta \to 0} \left(\lim_{r_i \to -\theta} \Pi^i(r_i;\theta/2) \right)$$

or equivalently when $\theta < 0.219$.

A ranking among some of the key parameters can be established. First, it follows that for all $\theta \leq 0.219$, $\beta_3(\theta, \bar{\theta}) = 0$, and for $\theta > 0.219$, $\beta_3(\theta, \bar{\theta}) > 0$. Consider the following lemma

Lemma 10 For a given value of θ, if $\beta_4(\theta, \bar{\theta}) \lesseqgtr \beta_2$ then $\beta_2 \lesseqgtr \beta_3(\theta, \bar{\theta})$.

Proof of Lemma 10: Consider the case $\beta_4(\theta, \bar{\theta}) < \beta_2 < \beta_3(\theta, \bar{\theta})$.

Suppose, by way of contradiction, that $\beta_4(\theta, \bar{\theta}) < \beta_2$ and $\beta_2 > \beta_3(\theta, \bar{\theta})$. Let $\beta = \beta_2$. On one hand, Corollary 2 implies that $\lim_{r_i \to \theta^-} \Pi^i(r_i, \theta/2) = \Pi^i(\theta/2, \theta/2)$

and $\Pi^i(BR_i(\theta/2), \theta/2) < \lim_{r_i \to \theta^-} \Pi^i(r_i, \theta/2)$ which yields $\Pi^i(BR_i(\theta/2), \theta/2) < \Pi^i(\theta/2, \theta/2)$. On the other hand $\beta = \beta_2 > \beta_3(\theta, \bar{\theta})$. Corollary 2 implies that $\Pi^i(BR_i(\theta/2), \theta/2) > \Pi^i(\theta/2, \theta/2)$, a contradiction.

Now let $\beta \in (\beta_2, \beta_3(\theta, \bar{\theta}))$ and $\beta < \beta_4(\theta, \bar{\theta})$. Corollary 2 implies that $\lim_{r_i \to \theta^-} \Pi^i(r_i, \theta/2) < \Pi^i(\theta/2, \theta/2)$ and $\Pi^i(BR_i(\theta/2), \theta/2) < \lim_{r_i \to \theta^-} \Pi^i(r_i, \theta/2)$ which yields $\Pi^i(BR_i(\theta/2), \theta/2) < \Pi^i(\theta/2, \theta/2)$. Since $\beta < \beta_4(\theta, \bar{\theta})$, $\Pi^i(BR_i(\theta/2), \theta/2) > \Pi^i(\theta/2, \theta/2)$, a contradiction.

Finally, let $\beta_2 < \beta = \beta_3(\theta, \bar{\theta})$ and $\beta < \beta_4(\theta, \bar{\theta})$. Corollary 2 implies that $\Pi^i(BR_i(\theta/2), \theta/2) = \lim_{r_i \to \theta^-} \Pi^i(r_i, \theta/2)$ and $\Pi^i(BR_i(\theta/2), \theta/2) > \Pi^i(\theta/2, \theta/2)$ which yields $\lim_{r_i \to \theta^-} \Pi^i(r_i, \theta/2) > \Pi^i(\theta/2, \theta/2)$. Since $\beta > \beta_2$, $\lim_{r_i \to \theta^-} \Pi^i(r_i, \theta/2) < \Pi^i(\theta/2, \theta/2)$, a contradiction.

The demonstration is similar for other inequalities. ■

Therefore, given values of θ and $\bar{\theta}$ induce a ranking of the three key parameters described in the above lemma. It is particularly convenient to compare $\beta_4(\theta, \bar{\theta})$ with β_2. In particular, consider $\max\{\beta_2, \beta_4(\theta, \bar{\theta})\}$.

Proposition 4 establishes that (θ, θ) is not a symmetric-equilibrium vector of reserve prices for all $\beta \in [0, 1]$. From Lemma 10, no symmetric reserve prices above θ could be an equilibrium for all $\beta > \beta_1(\theta, \bar{\theta})$. Hence existence and uniqueness of $r^* = \theta/2$ is established. This concludes the proof of the proposition. ■

Proof of Proposition 6:

(Given symmetry of the sellers' payoffs, the proof focuses on seller a's payoff. A similar result exists for seller b.) Because of the discontinuity of the payoff functions, a differential argument is not sufficient to show unprofitable deviations. Lemma 6 proves the existence of a unique parameter $\beta_1(\theta, \bar{\theta})$ for all θ. This shows that local deviations are not profitable for a seller.

The next possible deviation is $r_i \in [0, \theta]$. Consider a seller's expected payoff when $r_a < \theta < r_b$.

$$\Pi^a(r_a, r_b^*) = \pi_1(r) r_a + \pi_2(r) B(r, 2) \tag{108}$$

$$= \quad 2\rho(1-\rho)r_a + \beta\rho\underline{\theta} + \beta(1-\beta)P_{\bar{\theta}}(r)\underline{\theta} + (1-\beta)^2 P_{\bar{\theta}}(r)^2$$

$$\frac{\partial \Pi^a(r_a; r_b^*)}{\partial r_a} = 2\rho(1-\rho) - \{2r_a(1-2\rho) + 2\beta\underline{\theta} + 2(1-\beta)P_{\bar{\theta}}(r)\}\frac{\partial \rho}{\partial r_a} \tag{109}$$

Rearranging the derivative using the value of ρ when $r_a < \underline{\theta} < r_b$ and the fact that

$$\frac{\partial \rho}{\partial r_a} = -\frac{(1-\rho)}{(2\bar{\theta} - r_a - r_b)} \tag{110}$$

yields

$$\frac{\partial \Pi^a(r_a; r_b^*)}{\partial r_a} = \frac{2(1-\rho)}{(2\bar{\theta} - r_a - r_b)}\left\{\bar{\theta} + 2r_a\rho + \beta\bar{\theta} + \rho - \beta\right\} > 0 \tag{111}$$

since

$$\rho - \beta > 0. \tag{112}$$

The uniqueness of r^* follows from Lemma 10 and Proposition 4, where it is shown that the vectors $(r_a; r_b) = (\underline{\theta}; \underline{\theta})$ is not equilibrium vector of reserve prices for all $\beta \in [0; 1]$; hence for $\beta < \beta_1(\underline{\theta}; \bar{\theta})$. ■

Proof of Proposition 7:

The proof follows directly from the following lemma that completes the ranking of the key parameters.

Lemma 11 The interval $B = (\beta_1(\underline{\theta}; \bar{\theta}); \max\{\beta_2; \beta_4(\underline{\theta}; \bar{\theta})\})$ is not empty, for every $\underline{\theta} \in [0; \bar{\theta}]$.

Proof of Lemma 11:

Consider $\underline{\theta} \in [1/2; \bar{\theta}]$. Since $\beta_1(\underline{\theta}; \bar{\theta}) < \beta_1(\underline{\theta}; \bar{\theta}) = (1 - 2\underline{\theta}/\bar{\theta})$ and because for all $\beta > 1 - 2\underline{\theta}$, $r^* = \frac{(1-\beta)\bar{\theta}}{2} < \underline{\theta}$, it follows that $\beta_1(\underline{\theta}; \bar{\theta}) = 0$. From Lemma 6, the exact values of $\beta_4(\underline{\theta}; \bar{\theta})$ and β_2 are such that for all $\underline{\theta} > .14$, $\beta_4(\underline{\theta}; \bar{\theta}) < \beta_2$. Therefore, over $[1/2; 1]$, $\beta_1(\underline{\theta}; \bar{\theta}) = 0 < \beta_2 = \max\{\beta_4(\underline{\theta}; \bar{\theta}); \beta_2\}$.

Let $\underline{\theta} \in [.29; 1/2)$. It is still true that $\beta_4(\underline{\theta}; \bar{\theta}) < \beta_2$. Now, for all $\underline{\theta} \geq .29$, $\bar{\rho} \leq \beta_2$ it follows that $\beta_1(\underline{\theta}; \bar{\theta}) < \beta_2$.

If $\underline{\theta} \in (.14; .29)$ then $\beta_4(\underline{\theta}; \bar{\theta}) < \beta_2 < \bar{\rho}$. Evaluate $\Delta(\beta_2; \underline{\theta})$ for all $\underline{\theta} \in (.14; .29)$ to discover that $\Delta(\beta_2; \underline{\theta}) < 0$ which implies, given the limit values of $\Delta(\beta; \underline{\theta}; \bar{\theta})$ with respect to β, that $\beta_1(\underline{\theta}; \bar{\theta}) < \beta_2$.

Consider $\underline{\theta} \in (0; .14]$, it is found that $\beta_4(\underline{\theta}; \bar{\theta}) \geq \beta_2$. Evaluating $\Delta(\beta_4(\underline{\theta}; \bar{\theta}); \underline{\theta})$ to discover that $\Delta(\beta_4(\underline{\theta}; \bar{\theta}); \underline{\theta}) < 0$, which implies that $\beta_1(\underline{\theta}; \bar{\theta}) < \beta_4(\underline{\theta}; \bar{\theta})$. Finally let $\underline{\theta} = 0$. This implies that $\beta_4(\underline{\theta}; \bar{\theta}) = \beta_1(\underline{\theta}; \bar{\theta}) = 1$. ■

This lemma along with Proposition 4 proves the result. ■

Appendix C

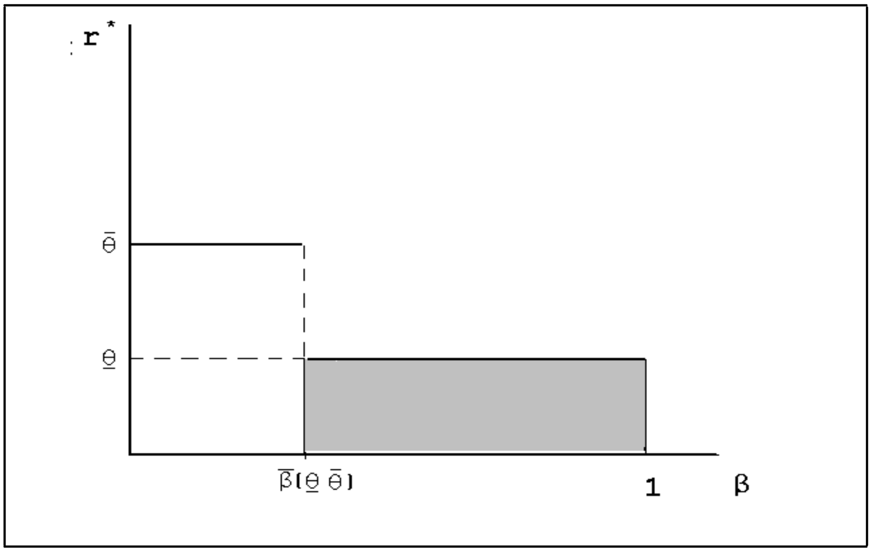

Figure 1: Reserve Price in a Monopoly Auction

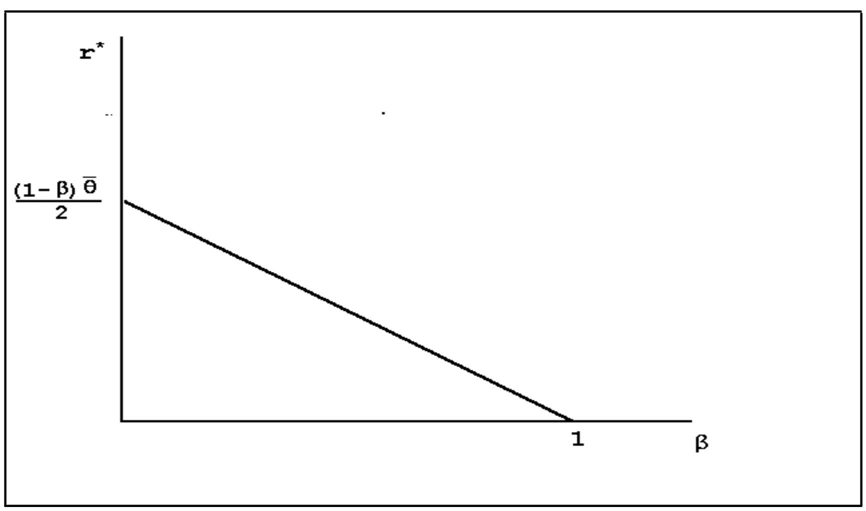

Figure 2: Reserve Price in a Duopoly Auctions with lowest bidder valuation equal to the seller's opostunity cost.

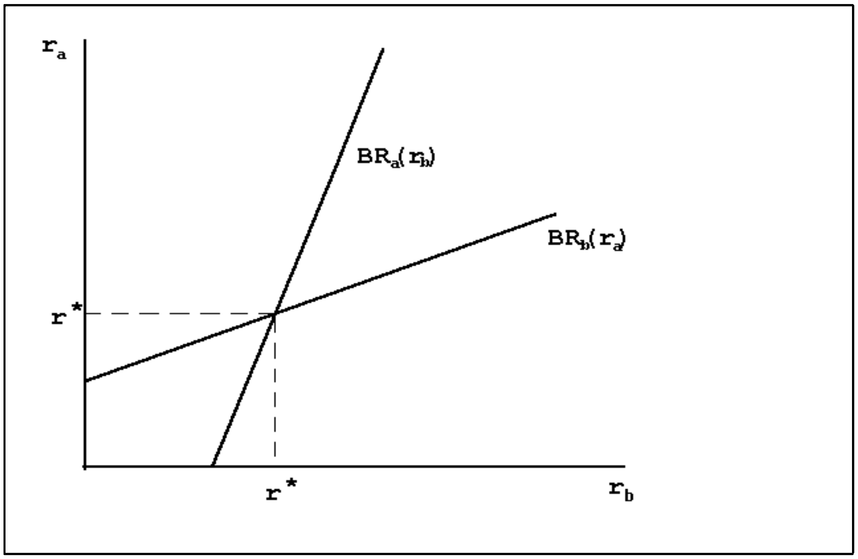

Figure 3: Reaction Functions for the Duopoly Auctions

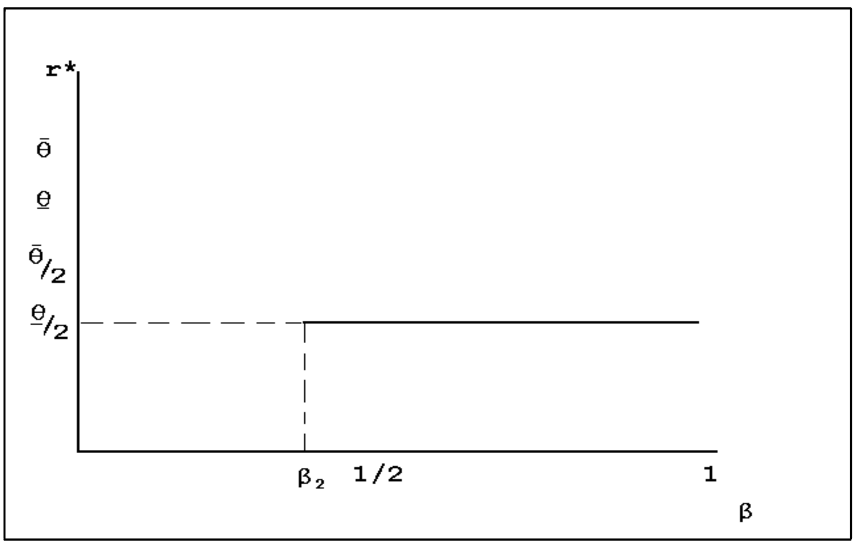

Figure 4: Symmetric equilibrium prices for large values of ¯

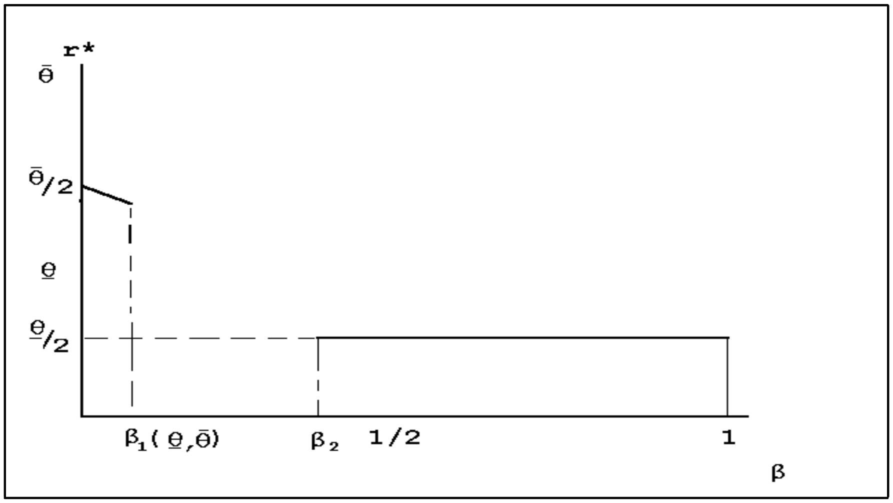

Figure 5: Equilibrium Reserve Prices for Low and High Values of ¯.

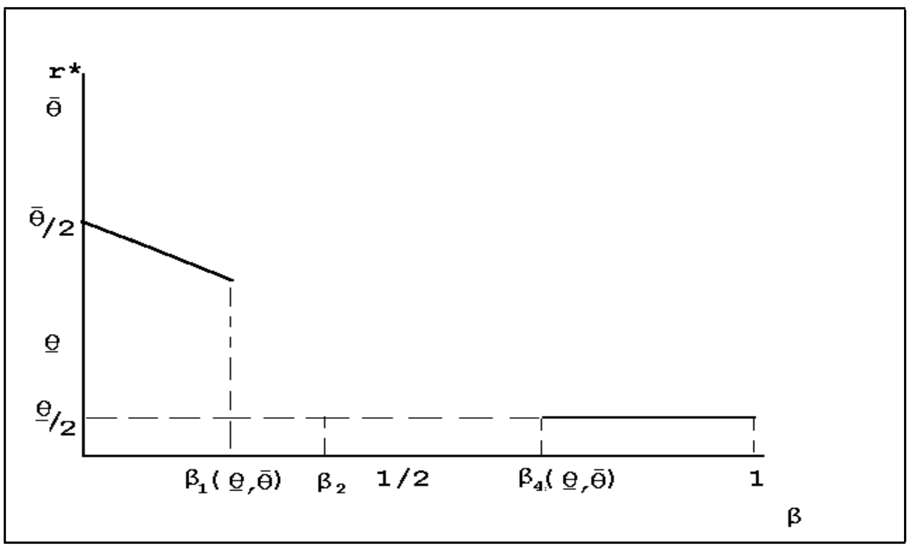

Figure 6: Equilibrium Reserve Prices for Low and High Values of ¯.

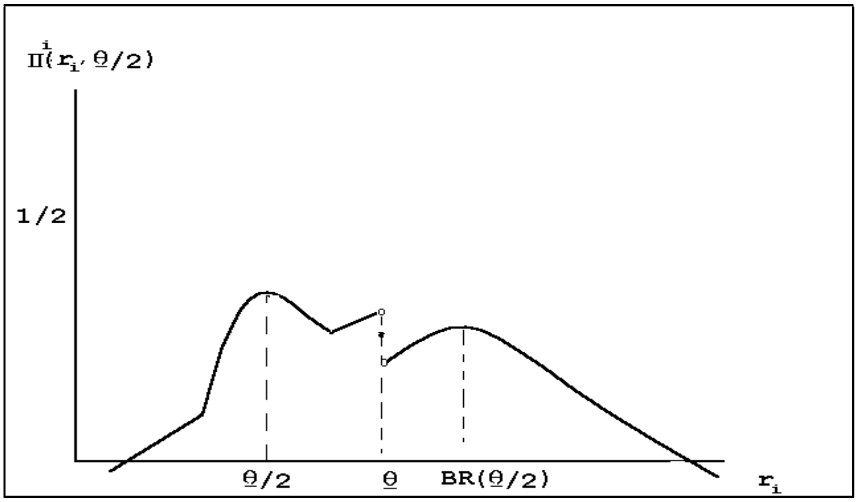

Figure 7: Typical seller's payoff when $r^* = \underline{\theta}/2$.

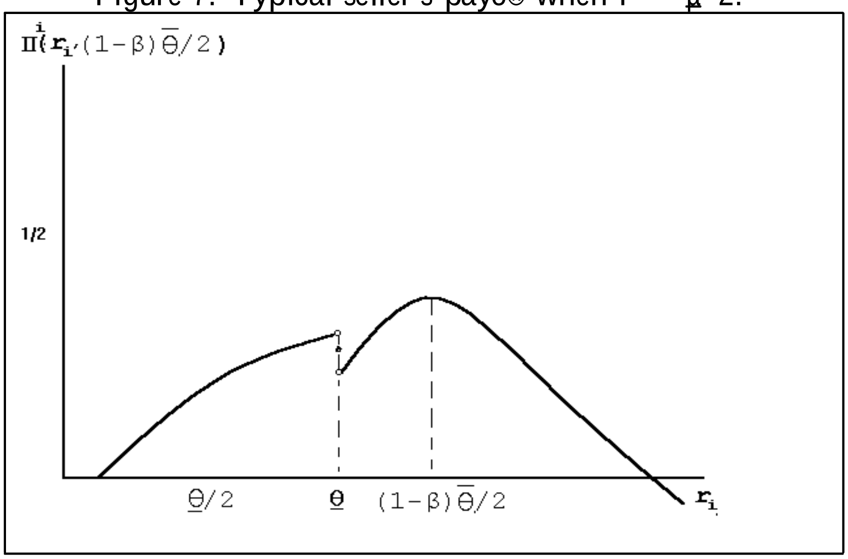

Figure 8: Typical seller's payoff when $r^* = (1-\beta)\bar{\theta}/2$.

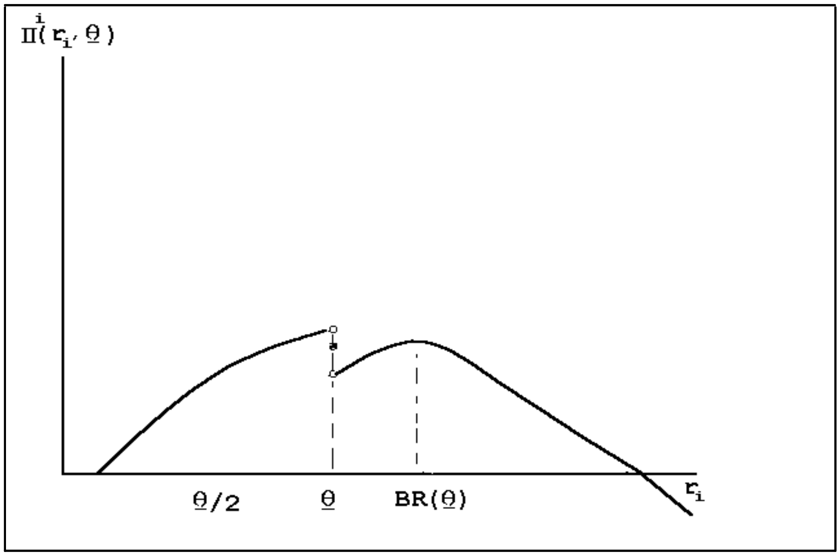

Figure 9: Typical seller's payo® when no equilirbium reserve prices exist.

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