System-based Economics

(second edition)

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Abstract

An exposition of economics using linear algebra and Lagrange multipliers to describe production and consumption. The structure of this work is based on the framework presented in *Systems*.

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Economic Network Structure

Section 1: Composition

Let an economic network be defined as a set of agents and the exchanges of goods they execute. Let A_t be the set of agents encompassed by an economic network at time t, such that

$$A_t = [A_1 ... A_m] \quad (2.1)$$

Within A_t, let B_t be the set of agents that can act as producers at time t, such that

$$B_t = [B_1 ... B_i] \quad (2.2)$$

where $i \leq m$.

Within A_t, let C_t be the set of agents that can act as consumers at time t, such that

$$C_t = [C_1 ... C_r] \quad (2.3)$$

where $r \leq m$. From these definitions, it is possible for a given agent to belong to subsets B_t and C_t at the same point in time.

Let N_t be the set of types of goods that can be produced by the agents in B_t, such that

$$N_t = [N_1 ... N_I] \quad (2.4)$$

Let n_t be the set of types of goods that can be consumed by the agents in C_t, such that

$$n_t = [n_1 ... n_R] \quad (2.5)$$

Let b_t be the set of goods produced by the agents in B_t, such that

$$b_t = [B_t][N_{B,t}] \quad (2.6)$$

where B_t is a 1 x i matrix and $N_{B,t}$ is an i x I matrix. From these definitions, the number of units of goods produced is given by

$$\sum_{g=1}^{i} \sum_{h=1}^{I} N_{gh} \quad (2.7)$$

and the number of units of goods of the E[th] type produced is given by

$$\sum_{g=1}^{i} N_{gE} \quad (2.8)$$

Let c_t be the set of goods consumed by the agents in C_t, such that

$$c_t = [C_t][n_{C,t}] \quad (2.9)$$

where C_t is a 1 x r matrix and $n_{C,t}$ is a r x R matrix. From these definitions, the number of units of goods consumed is given by

$$\sum_{j=1}^{r} \sum_{k=1}^{R} n_{jk} \quad (2.10)$$

and the number of units of goods of the E[th] type consumed is given by

$$\sum_{j=1}^{r} n_{jE} \quad (2.11)$$

Section 2: Architecture

Given the structure of economic networks, there are four possible production configurations:

a) protocol-based production determination,

b) consumption-based production determination,

c) consumption-regulated protocol-based production determination, and
d) protocol regulated returns-based production determination.

Similarly, there are four possible consumption configurations:
a) protocol-based consumption determination,
b) production-based consumption determination,
c) production-regulated protocol-based consumption determination, and
d) protocol regulated returns-based consumption determination.

Protocol-based determination configurations can be characterized as vertical streams, while consumption- and productions-based determinations can be considered horizontal streams. Since these streams are not inherently exclusive, any set of production and consumption configurations can exist in an economic network.

Section 3: Change

Let an agent reaction be defined as an agent's response to a given set of information. Let an environmental response be defined as a set of changes in the set of information available to agents. Given these definitions, there are four types of responses:
a) agent responses to other agent responses (1),
b) agent responses to environmental responses (2),
c) environmental responses to agent responses (3), and
d) environmental responses to other environmental responses (4).

Responses of either type can be initiated by
a) past agent responses (AP),
b) expected future agent responses (AF),
c) past environmental responses (EP), and
d) expected future environmental responses (EF).

Accordingly, there are eight possible causality types in economic networks:
a) 1-AP,
b) 1-AF,
c) 2-EP,
d) 2-EF,
e) 3-AP,
f) 3-AF,
g) 4-EP, and
h) 4-EF.

Since no combinations are inherently exclusive, any set of causality types can exist in a given process.

These changes in economic networks can be recorded in a two dimensional chart. Let there be two row types, one for environmental components and one for agents or agent types. Let columns represent time frames.

The time frame each column represents can be selected to correspond to the initiation and termination of process stages. Where parallel tracks converge and diverge at different points in time, columns can be constructed according to a common denominator.

Cells in the first column of this chart show the initial state of an economic network, and all subsequent cell show the changes. Agents can enter and exit an economic network during various time frames, so the set of agents present in a given time frame is not necessarily identical to the set of agents in any other time frame.

First-Order Metrics

Section 1: Ratios

Given the structure of an economic network, a set of ratios can be constructed:

i_t / m_t (r - 1.1)

r_t / m_t (r - 1.2)

i_t / r_t (r - 1.3)

I_t / R_t (r - 1.4)

$$\sum_{g=1}^{i}\sum_{h=1}^{I} N_{gh} / i \quad \text{(r - 1.5)}$$

$$\sum_{g=1}^{i} N_{gE} / i \quad \text{(r - 1.6)}$$

$$\sum_{j=1}^{r}\sum_{k=1}^{R} n_{jk} / r \quad \text{(r - 1.7)}$$

$$\sum_{j=1}^{r} n_{jE} / r \quad \text{(r - 1.8)}$$

$$\sum_{g=1}^{i} N_{gE} / \sum_{g=1}^{i}\sum_{h=1}^{I} N_{gh} \quad \text{(r - 1.9)}$$

$$\sum_{j=1}^{r} n_{jE} / \sum_{j=1}^{r}\sum_{k=1}^{R} n_{jk} \quad \text{(r - 1.10)}$$

$$\sum_{g=1}^{i} N_{gE} / \sum_{j=1}^{r} n_{jE} \quad \text{(r - 1.11)}$$

$$\sum_{g=1}^{i}\sum_{h=1}^{I} N_{gh} / \sum_{j=1}^{r}\sum_{k=1}^{R} n_{jk} \quad \text{(r - 1.12)}$$

Section 2: Properties

Let z be a given matrix of any given agent metrics. Let Z be a given matrix of any given exchange metrics.

A: Category 1

Let category 1 level 1 property 1 (C1-L1-P1) be $A_t z$.
Let category 1 level 1 property 2 (C1-L1-P2) be $B_t z$.
Let category 1 level 1 property 3 (C1-L1-P3) be $C_t z$.

Let category 1 level 2 property 1 (C1-L2-P1) be $N_t Z$.
Let category 1 level 2 property 2 (C1-L2-P2) be $n_t Z$.

Let category 1 level 3 property 1 (C1-L3-P1) be $b_t Z$.
Let category 1 level 3 property 2 (C1-L3-P2) be $c_t Z$.

B: Category 2

Let category 2 level 1 property 1 (C2-L1-P1) be $i_t z_1 / m_t z_2$.

Let category 2 level 1 property 2 (C2-L1-P2) be $r_t z_3 / m_t z_2$.
Let category 2 level 1 property 3 (C2-L1-P3) be $i_t z_1 / r_t z_3$.
Let category 2 level 1 property 4 (C2-L1-P4) be $I_t Z_1 / R_t Z_2$.

Let category 2 level 2 property 1 (C2-L2-P1) be $[\sum_{g=1}^{i}\sum_{h=1}^{I} N_{gh}][Z_1]/i$.

Let category 2 level 2 property 2 (C2-L2-P2) be $[\sum_{g=1}^{i} N_{gE}][Z_{1E}]/i$.

Let category 2 level 2 property 3 (C2-L2-P3) be $[\sum_{j=1}^{r}\sum_{k=1}^{R} n_{jk}][Z_2]/r$.

Let category 2 level 2 property 4 (C2-L2-P4) be $[\sum_{j=1}^{r} n_{jE}][Z_{2E}]/r$.

Let category 2 level 3 property 1 (C2-L3-P1) be $([\sum_{g=1}^{i} N_{gE}][Z_{1E}])/([\sum_{g=1}^{i}\sum_{h=1}^{I} N_{gh}][Z_1])$.

Let category 2 level 3 property 2 (C2-L3-P2) be $([\sum_{j=1}^{r} n_{jE}][Z_{2E}])/([\sum_{j=1}^{r}\sum_{k=1}^{R} n_{jk}][Z_2])$.

Let category 2 level 3 property 3 (C2-L3-P3) be $([\sum_{g=1}^{i} N_{gE}][Z_{1E}])/([\sum_{j=1}^{r} n_{jE}][Z_{2E}])$.

Let category 2 level 3 property 4 (C2-L3-P4) be $([\sum_{g=1}^{i}\sum_{h=1}^{I} N_{gh}][Z_1])/([\sum_{j=1}^{r}\sum_{k=1}^{R} n_{jk}][Z_2])$.

C: Category 3

Let category 3 property 1 (C3-P1) be the relationship between exchange types. C3-P1 can be determined by identifying rows in b_t and c_t that correspond to the same agent.

Let category 3 property 2 (C3-P2) be the relative value of exchanges. C3-P2 can be determined by comparing elements in a given Z.

Section 3: Accounting

A: Money Supply

Let the domestic currency of a given economic network be good N_D in N_t, and good n_D in n_t. Where interest exists, the interest rate charged on loans influences the value of N_D, and the interest rate paid on investments influences the value of n_D.

Where financial institutions can loan deposits, the multiplier effect on the supply of currency is given by

$$([\sum_{g=1}^{i} N_{gD}][Z_{1D}])/([\sum_{j=1}^{r} n_{jD}][Z_{2D}]) \qquad (3.1)$$

where Z_{1D} is the value of loans and Z_{2D} is the value of deposits, measured in units of the domestic currency. (1.1) is a form of C2-L3-P3.

Where financial institutions must retain a given percentage of the deposits they hold,

$$0 \le ([\sum_{g=1}^{i} N_{gD}][Z_{1D}])/([\sum_{j=1}^{r} n_{jD}][Z_{2D}]) \le 1/rr \qquad (3.2)$$

where rr is the reserve ratio.

Where the value of goods measured in units of the domestic currency changes, the price level is given by

$$\sum_{g=1}^{i} N_{gD} / \sum_{g=1}^{i}\sum_{h=1}^{I} N_{gh} \qquad (3.3)$$

(1.3) is a form of (r - 2.9).

Accordingly, the inflation rate is given by

$$[(\sum_{g=1}^{i} N_{gD,t} / \sum_{g=1}^{i}\sum_{h=1}^{I} N_{gh,t}) - (\sum_{g=1}^{i} N_{gD,t-1} / \sum_{g=1}^{i}\sum_{h=1}^{I} N_{gh,t-1})] / (\sum_{g=1}^{i} N_{gD,t-1} / \sum_{g=1}^{i}\sum_{h=1}^{I} N_{gh,t-1}) \qquad (3.4)$$

B: Balance of Payments

Let E be the set of goods exported by the agents in an economy, and let e be the set of goods imported. Using these definitions, the current account balance is given by

$$[\sum_{g=1}^{i} N_{gE}][Z_{1E}] - [\sum_{j=1}^{r} n_{je}][Z_{2e}] \qquad (3.5)$$

where Z_{1E} is an index of the prices of the goods in E and Z_{2e} is an index of the prices of the goods in e, measured in units of the domestic currency.

Let F be the set of domestically owned foreign assets and let f be the set of foreign owned domestic assets. Using these definitions, the capital account balance is given by

$$[\sum_{g=1}^{i} N_{gF}][Z_{1F}] - [\sum_{j=1}^{r} n_{jf}][Z_{2f}] \qquad (3.6)$$

where Z_{1F} is an index of the market values of the assets in F and Z_{2f} is an index of the market values of the assets in f, measured in units of the domestic currency.

C: Income

Let p be price and q be quantity. Where R is revenue,

$$R = (p_1)(q_1)$$
$$= [b_t][Z_{1t}] \qquad (3.7)$$

where Z_{1t} is a I x 1 matrix.

Similarly, where C is cost,

$$C = (p_2)(q_2)$$
$$= [c_t][Z_{2t}] \qquad (3.8)$$

where Z_{2t} is a R x 1 matrix.

For a given economic network,

$$\pi_t = R - C$$
$$= [b_t][Z_{1t}] - [c_t][Z_{2t}] \qquad (3.9)$$

Accordingly,

$$\pi_T = \sum_{t=1}^{T} \pi_t \qquad (3.10)$$

for $t = 1...T$ periods.

Where loans exist, the values of any loans are stated in Z_{1t}, and the costs of loans are stated in Z_{2t}. The revenues generated by investments are stated in Z_{1t}, and the costs of investments are stated in Z_{2t}.

Once interest rates, price levels, and inflation are determined, their effects on the value of loans and investments can be calculated using conventional equations. The results can be incorporated into Z_{1t} and Z_{2t} as required.

Second-Order Metrics

Section 1: Type 1 Statistics

Let v be velocity.

$$V_{xt} = \frac{(\sum_{u=1}^{I} b_{1u}) + (\sum_{w=1}^{R} c_{1w})}{t} \quad (1.1)$$

$$V_{yt} = \frac{b_t Z_1 + c_t Z_2}{t} \quad (1.2)$$

where Z_1 is a I x 1 matrix and Z_2 is a R x 1 matrix.

Let M be mass.

$$M_{xt} = (\sum_{u=1}^{I} b_{1u}) + (\sum_{w=1}^{R} c_{1w}) \quad (1.3)$$

$$M_{yt} = b_t Z_1 + c_t Z_2 \quad (1.4)$$

Section 2: Type 2 Statistics

Let a be acceleration.

$$a_{xt} = \frac{V_{xt} - V_{xt-1}}{t} \quad (2.1)$$

$$a_{yt} = \frac{V_{yt} - V_{yt-1}}{t} \quad (2.2)$$

Let g be growth.

$$g_{xt} = \frac{M_{xt} - M_{xt-1}}{t} \quad (2.3)$$

$$g_{yt} = \frac{M_{yt} - M_{yt-1}}{t} \quad (2.4)$$

Section 3: Type 3 Statistics

Let P be momentum.

$P_{xt} = (M_{xt})(V_{xt})$

$$= \frac{[(\sum_{u=1}^{I} b_{1u}) + (\sum_{w=1}^{R} c_{1w})]^2}{t} \quad (3.1)$$

$P_{yt} = (M_{yt})(V_{yt})$

$$= \frac{(b_t Z_1 + c_t Z_2)^2}{t} \quad (3.2)$$

Let F be force.

$F_{xt} = (g_{xt})(a_{xt})$

$$= \frac{1}{t^3}[(\sum_{u=1}^{I} b_{1ut} + \sum_{w=1}^{R} c_{1wt}) - (\sum_{u=1}^{I} b_{1ut-1} + \sum_{w=1}^{R} c_{1wt-1})]^2 \quad (3.3)$$

$F_{yt} = (g_{yt})(a_{yt})$

$$= \frac{1}{t^3}[(b_t Z_{1t} + c_t Z_{2t}) - (b_{t-1} Z_{1t-1} + c_{t-1} Z_{2t-1})]^2 \quad (3.4)$$

<u>Section 4: Type 4 Statistics</u>

A: Lower Thresholds for Type 1 Statistics

For a given A_t, assume a minimum set of reciprocators, $C_{t(min)}$, and a minimum set of goods they require, $n_{t(min)}$, where

$n_{t(min)} = [n_{1(min)} \ldots n_{\alpha(min)}]$ (4.1)

From these definitions,

$c_{t(min)} = [C_{t(min)}][n_{t(min)}]$ (4.2)

At minimum velocity b_t is identical to $c_{t(min)}$, so

$b_{t(min)} = c_{t(min)}$ (4.3)

and

$I = R$ (4.4)

If $i > r$, then $\sum_{g=1}^{i} N_{gE} < \sum_{j=1}^{r} n_{jE}$, for all E goods.

If $i = r$, then $\sum_{g=1}^{i} N_{gE} = \sum_{j=1}^{r} n_{jE}$, for all E goods.

If $i < r$, then $\sum_{g=1}^{i} N_{gE} > \sum_{j=1}^{r} n_{jE}$, for all E goods.

$$V_{xt(\min)} = \frac{(\sum_{u=1}^{\alpha} b_{1u}) + (\sum_{w=1}^{\alpha} c_{1w})}{t} = \frac{2\sum_{w=1}^{\alpha} c_{1w}}{t} \quad (4.5)$$

$$V_{yt(\min)} = \frac{b_{t(\min)} Z_1 + c_{t(\min)} Z_2}{t} = \frac{2c_{t(\min)} Z_2}{t} \quad (4.6)$$

$$M_{xt(\min)} = (\sum_{u=1}^{\alpha} b_{1u}) + (\sum_{w=1}^{\alpha} c_{1w}) = 2\sum_{w=1}^{\alpha} c_{1w} \quad (4.7)$$

$$M_{yt(\min)} = b_{t(\min)} Z_1 + c_{t(\min)} Z_2 = 2c_{t(\min)} Z_2 \quad (4.8)$$

B: Upper Thresholds for Type 1 Statistics

For a given A_t, assume a maximum set of initiators, $B_{t(max)}$, and a maximum set of goods they produce, $N_{t(max)}$, where

$N_{t(max)} = [N_{1(max)} \ldots N_{\Omega(max)}]$ (4.9)

From these definitions,

$b_{t(max)} = [B_{t(max)}][N_{t(max)}]$ (4.10)

At maximum velocity c_t is identical to $b_{t(max)}$, so

$$b_{t(max)} = c_{t(max)} \tag{4.11}$$

and

$$I = R \tag{4.12}$$

If $i > r$, then $\sum_{g=1}^{i} N_{gE} < \sum_{j=1}^{r} n_{jE}$, for all E goods.

If $i = r$, then $\sum_{g=1}^{i} N_{gE} = \sum_{j=1}^{r} n_{jE}$, for all E goods.

If $i < r$, then $\sum_{g=1}^{i} N_{gE} > \sum_{j=1}^{r} n_{jE}$, for all E goods.

$$V_{xt(\max)} = \frac{(\sum_{u=1}^{\Omega} b_{1u}) + (\sum_{w=1}^{\Omega} c_{1w})}{t} = \frac{2\sum_{u=1}^{\Omega} b_{1u}}{t} \tag{4.13}$$

$$V_{yt(\max)} = \frac{b_{t(\max)}Z_1 + c_{t(\max)}Z_2}{t} = \frac{2b_{t(\max)}Z_1}{t} \tag{4.14}$$

$$M_{xt(\max)} = (\sum_{u=1}^{\Omega} b_{1u}) + (\sum_{w=1}^{\Omega} c_{1w}) = 2\sum_{u=1}^{\Omega} b_{1u} \tag{4.15}$$

$$M_{yt(\max)} = b_{t(\max)}Z_1 + c_{t(\max)}Z_2 = 2b_{t(\max)}Z_1 \tag{4.16}$$

Section 5: Type 5 Statistics

A: Lower Thresholds for Type 2 Statistics

$$a_{xt(\min)} = \frac{V_{xt(\min)} - V_{xt-1}}{t} \tag{5.1}$$

$$a_{yt(\min)} = \frac{V_{yt(\min)} - V_{yt-1}}{t} \tag{5.2}$$

$$g_{xt(\min)} = \frac{M_{xt(\min)} - M_{xt-1}}{t} \tag{5.3}$$

$$g_{yt(\min)} = \frac{M_{yt(\min)} - M_{yt-1}}{t} \tag{5.4}$$

B: Upper Thresholds for Type 2 Statistics

$$a_{xt(\max)} = \frac{V_{xt(\max)} - V_{xt-1}}{t} \tag{5.5}$$

$$a_{yt(\max)} = \frac{V_{yt(\max)} - V_{yt-1}}{t} \tag{5.6}$$

$$g_{xt(\max)} = \frac{M_{xt(\max)} - M_{xt-1}}{t} \tag{5.7}$$

$$g_{yt(\max)} = \frac{M_{yt(\max)} - M_{yt-1}}{t} \qquad (5.8)$$

Section 6: Type 6 Statistics

A: Lower Thresholds for Type 3 Statistics

$$P_{xt(min)} = (M_{xt(min)})(V_{xt(min)})$$

$$= \frac{[2\sum_{w=1}^{\alpha} c_{1w}]^2}{t} \qquad (6.1)$$

$$P_{yt(min)} = (M_{yt(min)})(V_{yt(min)})$$

$$= \frac{(2c_{t(\min)}Z_2)^2}{t} \qquad (6.2)$$

$$F_{xt(min)} = (g_{xt(min)})(a_{xt(min)})$$

$$= \frac{1}{t^3}[(2\sum_{w=1}^{\alpha} c_{1wt}) - (\sum_{u=1}^{I} b_{1ut-1} + \sum_{w=1}^{R} c_{1wt-1})]^2 \qquad (6.3)$$

$$F_{yt(min)} = (g_{yt(min)})(a_{yt(min)})$$

$$= \frac{1}{t^3}[(2c_{t(\min)}Z_{2t}) - (b_{t-1}Z_{1t-1} + c_{t-1}Z_{2t-1})]^2 \qquad (6.4)$$

B: Upper Thresholds for Type 3 Statistics

$$P_{xt(max)} = (M_{xt(max)})(V_{xt(max)})$$

$$= \frac{[2\sum_{u=1}^{\Omega} b_{1u}]^2}{t} \qquad (6.5)$$

$$P_{yt(max)} = (M_{yt(max)})(V_{yt(max)})$$

$$= \frac{(2b_{t(\max)}Z_1)^2}{t} \qquad (6.6)$$

$$F_{xt(max)} = (g_{xt(max)})(a_{xt(max)})$$

$$= \frac{1}{t^3}[(2\sum_{u=1}^{\Omega} b_{1ut}) - (\sum_{u=1}^{I} b_{1ut-1} + \sum_{w=1}^{R} c_{1wt-1})]^2 \qquad (6.7)$$

$$F_{yt(max)} = (g_{yt(max)})(a_{yt(max)})$$

$$= \frac{1}{t^3}[(2b_{t(\max)}Z_{1t}) - (b_{t-1}Z_{1t-1} + c_{t-1}Z_{2t-1})]^2 \qquad (6.8)$$

First-Order Dynamics

Section 1: Elasticity

Let elasticity be defined as the relationship between the price of a good and quantity produced or consumed, or

$$\frac{\partial Q_t}{\partial P_t} = x_t \qquad (1.1)$$

Let consumer determined elasticity (CDE) be defined as an elasticity where $x_t < -1$. Let neutrally determined elasticity (NDE) be defined as an elasticity where $x_t = -1$. Let producer determined elasticity (PDE) be defined as an elasticity where $x_t > -1$.

Section 2: Returns to Scale

Let returns to scale (RS) be defined as the relationship between the quantity of a good produced or consumed and its price, or

$$\frac{\partial P_t}{\partial Q_t} = \bar{x}_t \qquad (1.2)$$

Let increasing returns to scale (IRS) be defined as an RS where $\bar{x}_t < 0$. Let constant returns to scale (CRS) be defined as an RS where $\bar{x}_t = 0$. Let decreasing returns to scale (DRS) be defined as an RS where $\bar{x}_t > 0$.

Section 3: Externality

Let externality be defined as the relationship between the returns to different economic activities, or

$$\frac{\partial \pi_{1,t}}{\partial \pi_{2,t}} = \overset{\square}{x}_t \qquad (1.3)$$

Let increasing externality (IE) be defined as an externality where $\overset{\square}{x}_t > 0$. Let neutral externality (NE) be defined as an externality where $\overset{\square}{x}_t = 0$. Let decreasing externality (DE) be defined as an externality where $\overset{\square}{x}_t < 0$.

<u>Second-Order Dynamics</u>

<u>Section 1: Production Constraints</u>

Let the elasticity constraint be expressed as

$$b_t = x_{bt}Z_{1t} \quad (1.1)$$

Let the RS constraint be expressed as

$$Z_{1t} = \bar{x}_{bt}\, b_t \quad (1.2)$$

Let the externality constraint be expressed as

$$\pi_{1t} = \boxed{x}_{bt}\, \pi_{2t} \quad (1.3)$$

<u>Section 2: Consumption Constraints</u>

Let the elasticity constraint be expressed as

$$c_t = x_{ct}Z_{2t} \quad (2.1)$$

Let the RS constraint be expressed as

$$Z_{2t} = \bar{x}_{ct}\, c_t \quad (2.2)$$

Let the externality constraint be expressed as

$$\pi_{1t} = \boxed{x}_{ct}\, \pi_{3t} \quad (2.3)$$

<u>Section 3: Economic Network Constraints</u>

Given the production and consumption constraints, an economic network's income equation can be expressed as

$$\pi_t = [b_tZ_{1t} - \lambda_1(b_t - x_{bt}Z_{1t}) - \lambda_2(Z_{1t} - \bar{x}_{bt}\, b_t) - \lambda_3(\pi_{1t} - \boxed{x}_{bt}\, \pi_{2t})] - [c_tZ_{2t} - \lambda_4(c_t - x_{ct}Z_{2t}) - \lambda_5(Z_{2t} - \bar{x}_{ct}\, c_t) - \lambda_6(\pi_{1t} - \boxed{x}_{ct}\, \pi_{3t})]$$

$$= [b_tZ_{1t} - \lambda_1(b_t - x_{bt}Z_{1t}) - \lambda_2(Z_{1t} - \bar{x}_{bt}\, b_t) - \lambda_3(b_tZ_{1t} - c_tZ_{2t} - \boxed{x}_{bt}\,(b_{2t}Z_{21t} - c_{2t}Z_{22t}))] - [c_tZ_{2t} - \lambda_4(c_t - x_{ct}Z_{2t}) - \lambda_5(Z_{2t} - \bar{x}_{ct}\, c_t) - \lambda_6(b_tZ_{1t} - c_tZ_{2t} - \boxed{x}_{ct}\,(b_{3t}Z_{31t} - c_{3t}Z_{32t}))] \quad (3.1)$$

Accordingly,

$$\frac{\partial \pi}{\partial b_t} = Z_{1t} - \lambda_1 + \lambda_2\bar{x}_t - \lambda_3 Z_{1t} + \lambda_6 Z_{1t} \quad (3.2),$$

$$\frac{\partial \pi}{\partial Z_{1t}} = b_t - \lambda_1 x_t + \lambda_2 - \lambda_3 b_t + \lambda_6 b_t \quad (3.3),$$

$$\frac{\partial \pi}{\partial c_t} = \lambda_3 Z_{2t} - Z_{2t} + \lambda_4 - \lambda_5\bar{x}_t - \lambda_6 Z_{2t} \quad \text{(3.4), and}$$

$$\frac{\partial \pi}{\partial Z_{2t}} = \lambda_3 c_t - c_t - \lambda_4 x_t + \lambda_5 - \lambda_6 c_t \quad (3.5).$$

A: Production-Specific Constraints

$$\frac{\partial\pi}{\partial x_{bt}} = \lambda_1 Z_{1t} \tag{3.6}$$

$$\frac{\partial\pi}{\partial \bar{x}_{bt}} = \lambda_2 b_t \tag{3.7}$$

$$\frac{\partial\pi}{\partial \tilde{x}_{bt}} = \lambda_3 (b_{2t} Z_{21t} - c_{2t} Z_{22t}) \tag{3.8}$$

$$\frac{\partial\pi}{\partial b_{2t}} = \lambda_3 \tilde{x}_{bt} Z_{21t} \tag{3.9}$$

$$\frac{\partial\pi}{\partial Z_{21t}} = \lambda_3 \tilde{x}_{bt} b_{2t} \tag{3.10}$$

$$\frac{\partial\pi}{\partial c_{2t}} = -\lambda_3 \tilde{x}_{bt} Z_{22t} \tag{3.11}$$

$$\frac{\partial\pi}{\partial Z_{22t}} = -\lambda_3 \tilde{x}_{bt} c_{2t} \tag{3.12}$$

$$\frac{\partial\pi}{\partial \lambda_1} = -b_t + x_{bt} Z_{1t} \tag{3.13}$$

$$\frac{\partial\pi}{\partial \lambda_2} = -Z_{1t} + \bar{x}_{bt} b_t \tag{3.14}$$

$$\frac{\partial\pi}{\partial \lambda_3} = -b_t Z_{1t} + c_t Z_{2t} + \tilde{x}_{bt} (b_{2t} Z_{21t} - c_{2t} Z_{22t}) \tag{3.15}$$

B: Consumption-Specific Constraints

$$\frac{\partial \pi}{\partial x_{ct}} = -\lambda_4 Z_{2t} \tag{3.16}$$

$$\frac{\partial \pi}{\partial \bar{x}_{ct}} = -\lambda_5 c_t \tag{3.17}$$

$$\frac{\partial \pi}{\partial \overset{\square}{x}_{ct}} = -\lambda_6 (b_{3t} Z_{31t} - c_{3t} Z_{32t}) \tag{3.18}$$

$$\frac{\partial \pi}{\partial b_{3t}} = -\lambda_6 \overset{\square}{x}_{ct} Z_{31t} \tag{3.19}$$

$$\frac{\partial \pi}{\partial Z_{31t}} = -\lambda_6 \overset{\square}{x}_{ct} b_{3t} \tag{3.20}$$

$$\frac{\partial \pi}{\partial c_{3t}} = \lambda_6 \overset{\square}{x}_{ct} Z_{32t} \tag{3.21}$$

$$\frac{\partial \pi}{\partial Z_{32t}} = \lambda_6 \overset{\square}{x}_{ct} c_{3t} \tag{3.22}$$

$$\frac{\partial \pi}{\partial \lambda_4} = c_t - x_{ct} Z_{2t} \tag{3.23}$$

$$\frac{\partial \pi}{\partial \lambda_5} = Z_{2t} - \bar{x}_{ct} c_t \tag{3.24}$$

$$\frac{\partial \pi}{\partial \lambda_6} = b_t Z_{1t} - c_t Z_{2t} - \overset{\square}{x}_{ct} (b_{3t} Z_{31t} - c_{3t} Z_{32t}) \tag{3.25}$$