Risk Aversion, Beliefs, and Prediction Market Equilibrium

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Abstract

Manski [2004] analyzes the relationship between the distribution of traders' beliefs and the equilibrium price in a prediction market with risk neutral traders. He finds that there can be a substantial difference between the mean belief that an event will occur, and the price of an asset that pays one dollar if the event occurs and otherwise pays nothing. This result is puzzling, since these markets frequently produce excellent predictions. This paper resolves the apparent puzzle by demonstrating that both risk aversion and the distribution of traders' beliefs significantly affect the equilibrium price. For coefficients of relative risk aversion near those estimated in empirical studies and for plausible belief distributions, the equilibrium price is very near the traders' mean belief.

JEL Classification: D84, G10

1 Introduction

Prediction markets have been used frequently to estimate the probability of future events. In 1988 the Iowa Political Stock Market predicted the outcome of the U.S. presidential election more accurately than all major polls. (See Forsythe, Nelson, Neumann, and Wright [1992].) The methodology has been extended to numerous political and economic events, and provides accurate predictions in many contexts. Plott, Wit, and Yang [2003] demonstrate experimentally that prediction markets can aggregate widely dispersed information effectively, even when there are many states. Several informal explanations for the success of these markets are presented in the prediction market literature. These explanations include the importance of financial incentives, self-selection toward better informed traders, the information aggregation capabilities of markets, and the feedback loop that a market creates, in which the market price is used by traders to update beliefs, and these updated beliefs lead to a price that more accurately reflects the asset value.

Manski [2004] argues that these informal explanations don't provide an adequate basis for interpretation of prices in prediction markets. He demonstrates the potential interpretation problem by comparing the equilibrium price to the mean belief in a prediction market populated with traders who are risk-neutral and have heterogeneous beliefs, and finds conditions under which the mean belief and the equilibrium price differ substantially.

Manski's approach, which starts from traders' expected utility functions and beliefs and derives the equilibrium price, is an important step toward understanding the relationship between agents' characteristics and the equilibrium price in a prediction market. The purpose of this paper is to extend analysis of prediction markets to include risk aversion, and also to examine the role of the distribution of traders' beliefs in the determination of the equilibrium price. This analysis demonstrates several useful relationships between risk attitudes, beliefs, and equilibrium prices for these markets. In the risk neutral case with a symmetric distribution of beliefs, the equilibrium price lies between the mean of the belief distribution and the price one half. When traders have CRRA expected utility functions, this bias diminishes as the coefficient of relative risk aversion increases, and the bias disappears when the coefficient of relative risk aversion is one. At that point, the equilibrium price is equal to the mean belief for any distribution of beliefs. If the coefficient of relative risk aversion exceeds one, the direction of the bias reverses: when the mean belief is greater than one half, the price is above the mean belief, and when the mean belief is below one half, the price is below the mean belief. For coefficients of relative risk aversion between 0.5 and 1.5, which are typical estimates (see for example Hansen and Singleton [1982], Cox and Oaxaca [1996], and Holt and Laury [2002]), and for several unimodal distributions of traders' beliefs, numerical analysis demonstrates that prediction market forecasts are close to the mean of the distribution of traders' beliefs.

Section 2 describes the prediction market assets, and derives agents' asset demands. In Section 3, market asset demand is derived, and the relationship between the traders' coefficient of relative risk aversion and the equilibrium price is determined for symmetric belief distributions. In Section 4, an example of Manski's model is constructed, and this is compared to properties of the equilibrium price for prediction markets with unimodal belief distributions. Section 5 reviews the conclusions of the paper, and describes possible empirical tests of the main results of the paper.

2 Prediction market assets and demands

Prediction markets are designed with two or more assets, each of which corresponds to a discrete event. In practice, markets are created with mutually exclusive and unambiguous events, such as "the Democratic party candidate will win the 2004 U.S. presidential election," or "the Democratic party candidate will not win the 2004 election." In mathematical terms, we can call these events 'state A', denoted s_A, and 'state B', denoted s_B. Assets A and B that correspond to these states are created by the market designer. Asset A has dividend \$1 in state s_A and dividend \$0 in state s_B; asset B has dividend \$0 in s_A and dividend \$1 in s_B. Agents are able to purchase bundles of assets A and B from the market for \$1 or sell bundles of assets A and B to the market for \$1, and they also are able to trade assets with other agents in the market.

Simple arbitrage opportunities in these markets are rare. If the market prices of A and B are p_A and p_B, and if $p_A + p_B > \$1$, then an agent can purchase a bundle of assets A and B from the market for \$1 and sell the assets individually at a profit. If $p_A + p_B < \$1$ then an agent is able to purchase assets separately from other agents and sell the bundle to the market at a profit. Consequently in the analysis that follows, we assume the no arbitrage condition $p_A + p_B = \$1$.

For agent i with initial wealth level w_i her final wealth level at prices p_A and p_B, if she holds m_i units of A and n_i units of B, is $w_i + m_i(1 - p_A) - n_i p_B$ in state s_A and

$w_i - m_i\, p_A + n_i\, (1 - p_B)$ in state s_B. Under the no arbitrage condition, these wealth levels are $w_i + (m_i - n_i)\,(1 - p_A)$ in s_A and $w_i - (m_i - n_i)\, p_A$ in s_B.

The representation of final wealth levels can be simplified by observing that an allocation of $m_i > 0$ units of A and $n_i > 0$ units of B, with $m_i > n_i$, is equivalent to the allocation with $m_i - n_i$ units of A, no unit of B, and n_i units of a riskless asset. Similarly, an allocation of $m_i > 0$ units of A and $n_i > 0$ units of B, with $m_i < n_i$, is equivalent to the allocation with $n_i - m_i$ units of B, no unit of A, and m_i units of a riskless asset. Consequently, we restrict attention to asset holdings $m_i \geq 0$ and $n_i \geq 0$ with $m_i\, n_i = 0$.

Suppose that agent i wants to reach the asset position (m_i, n_i) with $m_i > 0$ units of asset A and no unit of asset B. One way to reach this position is to purchase m_i units of A from other agents. If agent i wants to hold $m_i = 0$ units of A and no unit of B, one way to do this is with a purchase of n_i bundles of assets A and B from the market and the sale of n_i units of asset A. This observation can be used to simplify representation of the decision problem. If agent i has a positive demand for n_i units of asset B, this is equivalent to a supply of n_i units of asset A. Therefore the decision problem for agent i can be represented in terms of a single decision variable $q_i = m_i - n_i$, and final wealth levels are $w_i + q_i\,(1 - p_A)$ in s_A and $w_i - q_i\, p_A$ in s_B.

Assume that, for some $\theta \in (-\infty, \infty)$, agent i has the constant relative risk averse (CRRA) expected utility function

$$u_i(w_i) = \begin{cases} \frac{w_i^{1-\theta}}{1-\theta}, & \text{if } \theta \neq 1, \\ \ln w_i, & \text{if } \theta = 1. \end{cases}$$

If agent i has a subjective belief π_i that s_A will occur, and belief $1 - \pi_i$ that s_B will occur, then the agent's decision problem is to choose q_i to maximize

$$E[u_i(w_i, q_i)] = \begin{cases} \frac{1}{1-\theta}\left((w_i + q_i\,(1 - p_A))^{1-\theta}\,\pi_i \;+\; (w_i - q_i\, p_A)^{1-\theta}\,(1 - \pi_i)\right), & \text{if } \theta \neq 1; \\ \pi_i \ln(w_i + q_i\,(1 - p_A)) + (1 - \pi_i) \ln(w_i - q_i\, p_A), & \text{if } \theta = 1. \end{cases}$$

For $\theta > 0$, the expected utility maximization problem has an interior maximum that solves the first-order condition

$$\frac{\partial E[u_i(w_i, q_i)]}{\partial q_i} = (1 - p_A)\,(w_i + q_i\,(1 - p_A))^{-\theta}\pi_i \;-\; p_A\,(w_i - q_i\, p_A)^{-\theta}\,(1 - \pi_i) = 0.$$

The solution to this equation is

$$q_i^*\,(p_A, \pi_i, \theta, w_i) \;=\; \frac{\left((1 - p_A)^{\frac{1}{\theta}}\,\pi_i^{\frac{1}{\theta}} - p_A^{\frac{1}{\theta}}\,(1 - \pi_i)^{\frac{1}{\theta}}\right) w_i}{(1 - p_A)\, p_A^{\frac{1}{\theta}}\,(1 - \pi_i)^{\frac{1}{\theta}} + p_A\,(1 - p_A)^{\frac{1}{\theta}}\,\pi_i^{\frac{1}{\theta}}}. \tag{1}$$

Equation (1) represents the demand of agent i with belief $\pi_i \in (0,1)$, coefficient of relative risk aversion $\theta > 0$, and initial wealth $w_i > 0$. For $(p_A, \pi_i, \theta) \in (0,1) \times (0,1) \times (0,\infty)$ the denominator in equation (1) is positive. When $\pi_i > p_A$ the numerator in equation (1) is positive, and when $\pi_i < p_A$ the numerator is negative. This result is intuitive: if agent i has belief π_i that s_A will occur exceeds the price p_A of asset A, then agent i will hold asset A, whereas if $\pi_i < p_A$, agent i will prefer to sell asset A.

For $\theta \leq 0$, the expected utility maximizing choice is at the boundary of the choice set. The demand correspondence for the risk neutral case ($\theta = 0$) is easily determined by examining equation (1) as $\theta \to 0$. When $\pi_i > p_A$, $p_A^{\frac{1}{\theta}} (1-\pi_i)^{\frac{1}{\theta}} \Big/ \left((1-p_A)^{\frac{1}{\theta}} \pi_i^{\frac{1}{\theta}} \right) \to 0$ as $\theta \to 0$. Therefore $q_i^*(p_A, \pi_i, \theta, w_i) \to \frac{w_i}{p_A}$ as $\theta \to 0$ for $\pi_i > p_A$. When $\pi_i < p_A$, $(1-p_A)^{\frac{1}{\theta}} \pi_i^{\frac{1}{\theta}} \Big/ \left(p_A^{\frac{1}{\theta}} (1-\pi_i)^{\frac{1}{\theta}} \right) \to 0$ as $\theta \to 0$, so $q_i^*(p_A, \pi_i, \theta, w_i) \to -\frac{w_i}{1-p_A}$ as $\theta \to 0$. A risk neutral agent with $\pi_i = p_A$ is indifferent between holding asset A, asset B, and a riskless asset. Since risk seeking agents ($\theta < 0$) also choose to invest all of their initial wealth in asset A if $\pi_i > p_A$ and in asset B if $\pi_i < p_A$, the demand correspondence for asset A is

$$q_i^*(p_A, \pi_i, \theta, w_i) = \begin{cases} \frac{w_i}{p_A}, & \text{if } \pi_i > p_A; \\ \left[-\frac{w_i}{1-p_A}, \frac{w_i}{p_A} \right], & \text{if } \pi_i = p_A; \\ -\frac{w_i}{1-p_A}, & \text{if } \pi_i < p_A \end{cases} \tag{2}$$

for risk neutral and risk seeking agents ($\theta \leq 0$).

3 Risk attitudes and market equilibrium

Individual asset demands determine market asset demand, and hence market equilibrium, once agents' coefficients of relative risk aversion and the joint distribution of beliefs and initial wealth levels are specified. Under the assumptions – maintained throughout this paper – that all traders have the same CRRA expected utility function with coefficient θ and that the distributions of beliefs and wealth levels are independent, market demand is

$$Q^*(p_A, \theta) = \int_0^\infty \int_0^1 q^* \left(p_A, \pi, \theta, w\right) f(\pi)\, g(w)\, d\pi\, dw. \tag{3}$$

This section uses this market demand to describe the relationship between an asset's equilibrium price and the mean of traders' beliefs that the event will occur.

Mean beliefs and equilibrium with logarithmic utility ($\theta = 1$)

The logarithmic expected utility function, which is the limit of a CRRA expected utility function as $\theta \to 1$, is an interesting special case. For this specification of expected utility, Theorem 1 shows that the equilibrium price is equal to the belief distribution mean, regardless of the specification of the distribution of traders' beliefs.

Theorem 1 If the distributions of agents' beliefs and wealth levels have densities $f(\pi)$ and $g(w)$, and these distributions are independent, and if all agents have the expected utility function $u(w) = \ln w$, then $Q^*(p_A, 1)$ is linear and $Q^*(\mu, 1) = 0$, so $p_A^* = E[\pi]$ is the unique equilibrium price.

Proof With $\theta = 1$, agent i has the demand function $q_i^*(p_A, \pi_i, 1, w) = \frac{(\pi_i - p_A)\, w}{p_A(1-p_A)}$ for asset A. Substitute individual demand into the market demand from equation (3) to get

$$\begin{aligned} Q^*(p_A, 1) &= \frac{1}{p_A\,(1-p_A)} \int_0^\infty \left[\int_0^1 (\pi - p_A)\, f(\pi)\, d\pi \right] w\, g(w)\, dw \\ &= \frac{E[w]}{p_A\,(1-p_A)} (E[\pi] - p_A). \end{aligned}$$

The unique equilibrium price for this market is therefore $p_A^* = E[\pi]$, i.e., the equilibrium price p_A^* is equal to the mean μ of the distribution of traders' beliefs. ■

Mean beliefs and equilibrium with general CRRA preferences

For values of θ other than $\theta = 1$, the relationship between the belief distribution mean and the equilibrium price is not as sharp as it is with logarithmic utility, yet it is still possible to provide a useful characterization of this relationship when the belief distribution is symmetric about its mean. The remainder of this section examines the relationship between the equilibrium price p_A^* and the mean belief μ for $\theta \in (-\infty, 0] \cup (^1/_2, 1) \cup (1, \infty)$. Theorem 2 below shows that for a symmetric distribution of beliefs and risk neutral utility, which corresponds to CRRA expected utility with $\theta = 0$, and for risk-seeking utility ($\theta < 0$), the equilibrium price p_A^* lies between the mean of the distribution of beliefs and the price $p_A = {}^1/_2$. Theorem 2 also shows that this result holds for $\theta \in (^1/_2, 1)$. Theorem 1 above has shown that for $\theta = 1$, the equilibrium price is exactly equal to the mean of the distribution of beliefs. For $\theta > 1$ Theorem 2 shows that the equilibrium price is above the mean μ of the belief distribution if $\mu > {}^1/_2$, and the equilibrium price is below the mean μ of the belief distribution if $\mu < {}^1/_2$. Figure 1 illustrates these results for several unimodal belief distributions that are approximately uniform. Along the curve labelled $\mu = 0.7$ in the figure, the belief distribution is $Beta(1.01, 1.01)$ on the interval $[0.4, 1]$, and the curve shows the equilibrium price for $\theta \in [0, 2]$. The other four curves show the equilibrium price as a function of θ for four other belief distributions. These distributions are $Beta(1.01, 1.01)$ on the intervals $[0.2, 1]$ ($\mu = 0.6$), $[0, 1]$ ($\mu = 0.5$), $[0, 0.8]$ ($\mu = 0.4$), and $[0, 0.6]$ ($\mu = 0.3$). The figure shows that for $\theta \in [0, 1)$, p_A^* is between μ and $p = {}^1\!/_2$; for $\theta = 1$ the equilibrium price is equal to μ; and for $\theta \in (1, 2]$ the equilibrium price is further from ${}^1\!/_2$ than μ is from ${}^1\!/_2$.

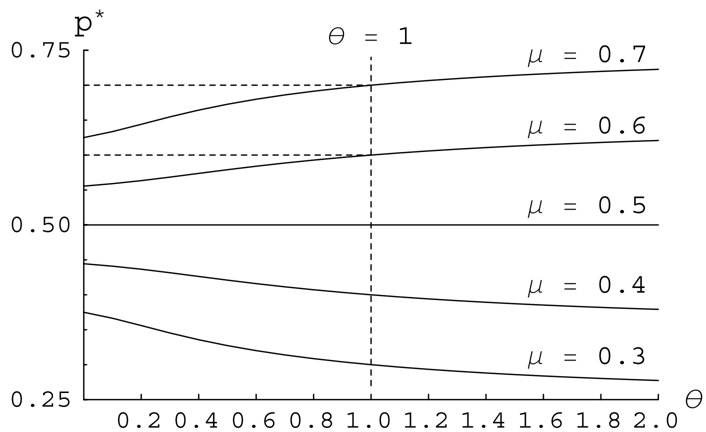


Figure 1: Equilibrium price as a function of θ for several values of μ

For a demand distribution that is symmetric about its mean belief μ, these results are demonstrated by determining the sign of the sum of the demands for an agent with belief $\pi = \mu + \delta$ and for an agent with belief $\pi = \mu - \delta$, with the demands evaluated at $p_A = \mu$. The net demand for asset A by a risk-averse agent with belief π is given by equation (1). From equation (1) it follows that these traders have asset demands

$$\begin{aligned} q^*(\mu, \mu+\delta, \theta, w) &= \frac{\left((1-\mu)^{\frac{1}{\theta}}(\mu+\delta)^{\frac{1}{\theta}} - \mu^{\frac{1}{\theta}}(1-\mu-\delta)^{\frac{1}{\theta}}\right) w}{(1-\mu)\,\mu^{\frac{1}{\theta}}(1-\mu-\delta)^{\frac{1}{\theta}} + \mu\,(1-\mu)^{\frac{1}{\theta}}(\mu+\delta)^{\frac{1}{\theta}}} \\ &= \frac{\left((1+\frac{\delta}{\mu})^{\frac{1}{\theta}} - (1-\frac{\delta}{1-\mu})^{\frac{1}{\theta}}\right) w}{(1-\mu)\,(1-\frac{\delta}{1-\mu})^{\frac{1}{\theta}} + \mu\,(1+\frac{\delta}{\mu})^{\frac{1}{\theta}}} \end{aligned} \tag{4}$$

and

$$\begin{aligned} q^*(\mu, \mu-\delta, \theta, w) &= \frac{\left((1-\mu)^{\frac{1}{\theta}}(\mu-\delta)^{\frac{1}{\theta}} - \mu^{\frac{1}{\theta}}(1-\mu+\delta)^{\frac{1}{\theta}}\right) w}{(1-\mu)\,\mu^{\frac{1}{\theta}}(1-\mu+\delta)^{\frac{1}{\theta}} + \mu\,(1-\mu)^{\frac{1}{\theta}}(\mu-\delta)^{\frac{1}{\theta}}} \\ &= \frac{\left((1-\frac{\delta}{\mu})^{\frac{1}{\theta}} - (1+\frac{\delta}{1-\mu})^{\frac{1}{\theta}}\right) w}{(1-\mu)\,(1+\frac{\delta}{1-\mu})^{\frac{1}{\theta}} + \mu\,(1-\frac{\delta}{\mu})^{\frac{1}{\theta}}}. \end{aligned} \tag{5}$$

The idea behind the proof of Theorem 2 is straightforward. If the belief distribution is symmetric about μ and $q^*(\mu, \mu+\delta, \theta, w) + q^*(\mu, \mu-\delta, \theta, w) > 0$ holds for each $\delta \in (0, \min\{\mu, 1-\mu\})$, then market demand $Q^*(p_A, \theta)$ is positive at $p_A = \mu$, so $p_A^* > \mu$. On the other hand, if $q^*(\mu, \mu+\delta, \theta, w) + q^*(\mu, \mu-\delta, \theta, w) < 0$ for all $\delta \in (0, \min\{\mu, 1-\mu\})$, then $p_A^* < \mu$. Therefore the equilibrium price relative to the mean μ of a symmetric belief distribution is determined by the sign of $q^*(\mu, \mu+\delta, \theta, w) + q^*(\mu, \mu-\delta, \theta, w)$.

Comparison between $q^*(\mu, \mu+\delta, \theta, w)$ and $-q^*(\mu, \mu-\delta, \theta, w)$ is simplified by separately comparing the numerators and the denominators of these two expressions. To simplify notation, write $q^*(\mu, \mu+\delta, \theta, w) = \frac{n_L(\mu,\theta,\delta)\, w}{d_L(\mu,\theta,\delta)}$ and $q^*(\mu, \mu-\delta, \theta, w) = \frac{n_R(\mu,\theta,\delta)\, w}{d_R(\mu,\theta,\delta)}$. Lemma 1 compares $n_L(\mu, \theta, \delta)$ to $-n_R(\mu, \theta, \delta)$; Lemma 2 compares $d_L(\mu, \theta, \delta)^{-1}$ to $d_R(\mu,\theta,\delta)^{-1}$; Lemma 3 combines these two results to compare $q^*(\mu, \mu+\delta, \theta, w)$ to $-q^*(\mu, \mu-\delta, \theta, w)$. Lemma 4, which shows that the asset demand function is a decreasing function of p_A, is used to prove uniqueness of equilibrium.

Lemma 1 If $n_L(\mu, \theta, \delta)\, w$ is the numerator in equation (4) and $n_R(\mu, \theta, \delta)\, w$ is the numerator in equation (5), then $n_L(\mu, \theta, \delta) > -n_R(\mu, \theta, \delta)$ when $\mu > 1/2$ and $\theta > 1$ or when $\mu < 1/2$ and $\theta < 1$. When $\mu < 1/2$ and $\theta > 1$ or when $\mu > 1/2$ and $\theta < 1$, the inequality is reversed, i.e., $n_L(\mu, \theta, \delta) < -n_R(\mu, \theta, \delta)$.

Proof The proof is provided in the appendix.

Lemma 2 If $d_L(\mu, \theta, \delta)$ is the denominator in equation (4) and $d_R(\mu, \theta, \delta)$ is the denominator in equation (5), then $d_L(\mu, \theta, \delta)^{-1} > d_R(\mu, \theta, \delta)^{-1}$ when $\mu > 1/2$ and $\theta > 1$ or when $\mu < 1/2$ and $\theta \in (1/2, 1)$. When $\mu < 1/2$ and $\theta > 1$ or when $\mu > 1/2$ and $\theta \in (1/2, 1)$, the inequality is reversed, i.e., $d_L(\mu, \theta, \delta)^{-1} < d_R(\mu, \theta, \delta)^{-1}$.

Proof The proof is provided in the appendix.

Lemma 3 For $\mu > 1/2$ and $\theta > 1$, $q^*(\mu, \mu+\delta, \theta, w) + q^*(\mu, \mu-\delta, \theta, w) > 0$. The same inequality also holds for $\mu < 1/2$ and $\theta \in (1/2, 1)$. For $\mu > 1/2$ and $\theta \in (1/2, 1)$ or for $\mu < 1/2$ and $\theta > 1$, $q^*(\mu, \mu+\delta, \theta, w) + q^*(\mu, \mu-\delta, \theta, w) < 0$.

Proof The sign of $q^*(\mu, \mu+\delta, \theta, w) + q^*(\mu, \mu-\delta, \theta, w)$ can be determined from the comparisons between $n_L(\mu, \theta, \delta)$ and $-n_R(\mu, \theta, \delta)$ in Lemma 1 and between $d_L(\mu, \theta, \delta)^{-1}$ and $d_R(\mu, \theta, \delta)^{-1}$ in Lemma 2. When $\mu > 1/2$ and $\theta > 1$ or when $\mu < 1/2$ and $\theta < 1$ Lemma 1 shows that $n_L(\mu, \theta, \delta) > -n_R(\mu, \theta, \delta)$. When $\mu > 1/2$ and $\theta > 1$ or when $\mu < 1/2$ and $\theta \in (1/2, 1)$ Lemma 2 shows that $d_L(\mu, \theta, \delta)^{-1} > d_R(\mu, \theta, \delta)^{-1}$. This implies that

$n_L(\mu,\,\theta,\,\delta)/d_L(\mu,\,\theta,\,\delta) > -n_R(\mu,\,\theta,\,\delta)/d_R(\mu,\,\theta,\,\delta)$ for $\mu > {}^1\!/_2$ and $\theta > 1$ or for $\mu < {}^1\!/_2$ and $\theta \in ({}^1\!/_2, 1)$.

A similar argument shows that the opposite inequality holds for $\mu > {}^1\!/_2$ and $\theta \in ({}^1\!/_2, 1)$ or for $\mu < {}^1\!/_2$ and $\theta > 1$. ■

Lemma 4 The asset demand function $q^*(p_A, \pi, \theta, w)$ in equations (1) and (2) is a decreasing function of p_A for all $(\pi, \theta, w) \in (0,1) \times (-\infty, \infty) \times (0, \infty)$.

Proof The proof is provided in the appendix.

Theorem 2 Suppose that all agents have the same CRRA expected utility function, that the distributions of beliefs and wealth levels are independent, and that the distribution of agents' beliefs is symmetric about its mean μ. Then for $\theta \in (-\infty, 0] \cup ({}^1\!/_2, 1)$ the equilibrium price p_A^* lies between μ and ${}^1\!/_2$ (i.e., $p_A^* \in (\min\{\mu, {}^1\!/_2\}, \max\{\mu, {}^1\!/_2\})$). For $\theta > 1$ the equilibrium price lies in $(0, \mu)$ if $\mu < {}^1\!/_2$ and it lies in $(\mu, 1)$ if $\mu > {}^1\!/_2$.

Proof Market demand for asset A evaluated at $p_A^* = \mu$ is

$$\begin{aligned} Q^*(\mu,\theta) &= \int_0^\infty \int_0^1 q^*(\mu, \pi, \theta, w)\, f(\pi)\, g(w)\, d\pi\, dw \\ &= \int_0^\infty \int_0^{\min\{\mu, 1-\mu\}} \left(q^*(\mu, \mu+\delta, \theta, w) + q^*(\mu, \mu-\delta, \theta, w)\right) f(\mu+\delta)\, g(w)\, d\delta\, dw \\ &= \int_0^\infty \left[\int_0^{\min\{\mu,1-\mu\}} \left(\frac{n_L(\mu,\theta,\delta)}{d_L(\mu,\theta,\delta)} + \frac{n_R(\mu,\theta,\delta)}{d_R(\mu,\theta,\delta)}\right) f(\mu+\delta)\, d\delta\right] w\, g(w)\, dw \\ &= E[w] \int_0^{\min\{\mu,1-\mu\}} \left(\frac{n_L(\mu,\theta,\delta)}{d_L(\mu,\theta,\delta)} + \frac{n_R(\mu,\theta,\delta)}{d_R(\mu,\theta,\delta)}\right) f(\mu+\delta)\, d\delta. \end{aligned}$$

The first equality above holds by definition. The second holds because $f(\pi)$ is symmetric around μ. The third holds from equations (4) and (5), and the definitions of $n_L(\cdot)$, $n_R(\cdot)$, $d_L(\cdot)$, and $d_R(\cdot)$. The fourth equality follows from integration with respect to w. Finally, $Q^*(\mu,\theta)$ has the same sign as the integrand $n_L(\mu,\,\theta,\,\delta)/d_L(\mu,\,\theta,\,\delta)+n_R(\mu,\,\theta,\,\delta)/d_R(\mu,\,\theta,\,\delta)$, which is determined in Lemma 3. For $\mu > {}^1\!/_2$ and $\theta > 1$ and for $\mu < {}^1\!/_2$ and $\theta \in ({}^1\!/_2, 1)$ the integrand is positive. Therefore, by Lemma 4 there is an equilibrium price $p_A^* > \mu$ and p_A^* is unique. When $\mu > {}^1\!/_2$ and $\theta \in ({}^1\!/_2, 1)$ or when $\mu < {}^1\!/_2$ and $\theta > 1$, the integrand is negative so there is a unique equilibrium price $p_A^* < \mu$.

For $\theta \in (-\infty, 0]$, by equation (2) $q^*(\mu, \mu+\delta, \theta, w) = \frac{w}{\mu}$ and $q^*(\mu, \mu-\delta, \theta, w) = -\frac{w}{1-\mu}$, so $q^*(\mu, \mu+\delta, \theta, w) + q^*(\mu, \mu-\delta, \theta, w) = \frac{(1-2\mu)\,w}{\mu\,(1-\mu)}$. When $\mu > {}^1\!/_2$, this is negative so $p_A^* < \mu$ and when $\mu < {}^1\!/_2$, this is positive so $p_A^* > \mu$. ■

Theorems 1 and 2 describe the relationship between the market equilibrium price and the mean belief for $\theta \in (-\infty, 0]$ and for $\theta > {}^1/_2$. Lemma 2 doesn't hold for $\theta \in (0, {}^1/_2)$, so the technique used in the proof of Theorem 2 can't be extended to this interval. Numerical analysis however suggests that the inequality $q^* (\mu, \mu + \delta, \theta, w) + q^* (\mu, \mu - \delta, \theta, w) < 0$ holds for all values of $\theta \in (0, {}^1/_2)$ when $\mu > {}^1/_2$, and the opposite inequality holds when $\mu < {}^1/_2$.

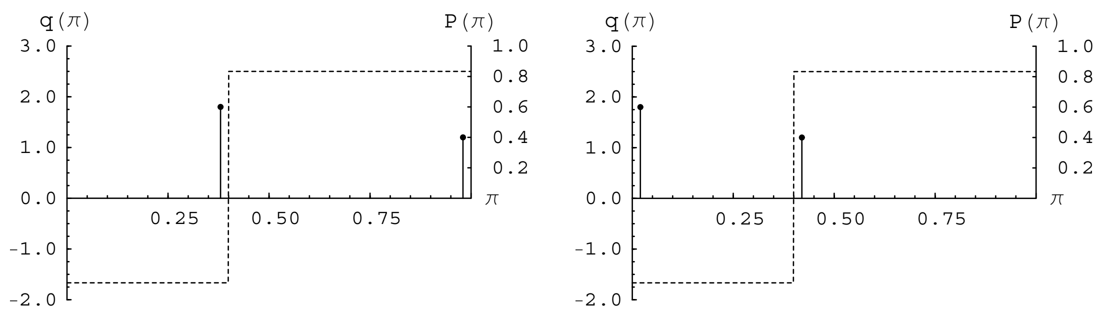
Theorem 2 has an interesting application to the favorite-longshot bias, which has been established in numerous empirical studies of gambling. Typically, longshots win less frequently than the probability implicit in their odds (or prices) suggest, and favorites win more frequently than their odds suggest. One common explanation for this is risk-seeking bettors. See, for example, Ali [1977]. This explanation is somewhat puzzling since risk aversion is prevalent. Only a very strong self-selection among bettors would lead to a population of bettors that is risk-seeking on average. Theorem 2 indicates that the favorite-longshot bias is consistent with risk-aversion, provided that the coefficient of risk aversion is less than one and the belief distribution is symmetric. The last example in the next section demonstrates further that there are asymmetric distributions that also lead to the favorite-longshot bias. Another explanation of the favorite-longshot bias, proposed by Thaler and Ziemba [1988], is that traders have biased probability judgments. Yet Theorem 2 and the examples in the next section show that the favorite-longshot bias is consistent with a standard expected utility model under natural assumptions about risk attitudes and beliefs.

4 Beliefs and market equilibrium

The first example in this section demonstrates the role of beliefs in Manski's model. Manski considers belief distributions that are highly dispersed, and are concentrated in the limit on two points (e.g., a fraction p of the traders believe the event will occur with certainty, and a fraction $1 - p$ believe it will occur with probability p). The first example demonstrates the effect that these limiting two-point distributions have on the relationship between belief distribution mean and equilibrium price. It is more plausible that belief distributions are unimodal, so this section also includes examples of unimodal *Beta* distributions. In these latter examples, equilibrium prices are close to mean beliefs for all values of the coefficient of relative risk aversion.

Example of Manski's model

Demand of a risk neutral agent is given by equation (2). If the market price of asset A is p_A, then an agent with belief $\pi_i > p_A$ will invest her entire initial wealth in asset A and have demand w_i/p_A, and an agent with belief $\pi_i < p_A$ will invest her entire initial wealth in asset B and have demand $-w_i/(1-p_A)$ for asset A. The dashed line in figure 2 (a) shows the demand for agents with income $w_i = 1$, asset price $p_A = 0.4$, and beliefs $\pi_i \in (0,1)$. The scale on the left axis represents the level of the agents' net demands for asset A.



(a) High mean belief (b) Low mean belief

Figure 2: Asset demand and belief distributions in Manski's model

Figure 2 (a) also depicts a belief distribution for agents, in which 3/5 of the traders expect that s_A will be realized with probability that is approximately (but just below) 0.4 and 2/5 of the traders expect that s_A will be realized with probability that is approximately 1.0. (The scale for the probability density is shown on the right axis in figure 2 (a).) With these beliefs, $p_A = 0.4$ is the equilibrium price for asset A, since 3/5 of the agents demand $-5/3$ units, and 2/5 of the agents demand 5/2 units. Finally, the mean of this belief distribution is approximately $\mu = 0.64$. Figure 2 (b) depicts another belief distribution for traders. In this figure, $p_A = 0.4$ is still the equilibrium price, and the mean of this belief distribution is approximately $\mu = 0.16$. This example illustrates Manski's main result: for risk neutral agents if the observed market price is p_A, then the mean of the belief distribution lies in the interval $(p_A^2, 2\,p_A - p_A^2)$.

Numerical analysis of equilibrium with unimodal belief distributions

Unimodal belief distributions are more plausible than the bimodal belief distributions, or their limiting two point distributions, that Manski considers. Returning to the example of the 2004 U.S. presidential election, it is unlikely that a fraction $1-p$ of the voters believe that the democratic candidate will win with certainty, and a fraction p believe that the democratic candidate will win with probability $1-p$. This section concludes with numerical analysis of equilibrium prices for several distributions, including $Beta(1.01, 1.01)$ belief distributions, $Beta(2,2)$ belief distributions, and skewed $Beta$ distributions. The $Beta(1.01, 1.01)$ distributions are approximately uniform but are unimodal. The densities of $Beta(2,2)$ distributions are parabolic. In all of these examples, equilibrium prices are very close to the mean of the belief distribution under natural assumptions about the level of traders' risk aversion.

Figure 1 (in Section 3) shows the relationship between μ and p_A^* as a function of θ for five different values of μ, when the distributions of beliefs are $Beta(1.01, 1.01)$ (approximately uniform). Figure 3 shows examples of the relationship between the mean of the belief distribution and the equilibrium price for five different $Beta(2,2)$ distributions. The five different $Beta(1.01, 1.01)$ distributions used in figure 1 have their supports on the intervals $[0.4, 1.0]$ (labelled $\mu = 0.7$), $[0.2, 1]$ (labelled $\mu = 0.6$), $[0, 1]$, $[0, 0.8]$, and $[0, 0.6]$. In order to facilitate comparison with the $Beta(1.01, 1.01)$ belief distributions, the five $Beta(2,2)$ distributions used in the example in figure 3 have the same supports as the five $Beta(1.01, 1.01)$ distributions in figure 1. Comparison of these two figures shows that the equilibrium price with a $Beta(2,2)$ belief distribution is closer to the mean of the belief distribution than in the case of $Beta(1.01, 1.01)$ belief distributions. For the coefficients of relative risk aversion $\theta \in [0.68, 0.97]$ that Hansen and Singleton [1982] estimate, the equilibrium price is within \$0.008 of μ for each of the $Beta(2,2)$ distributions shown in figure 3.

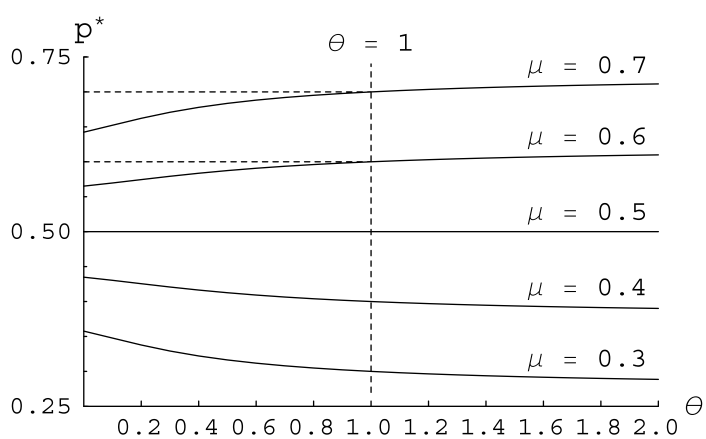


Figure 3: Equilibrium price as a function of θ with $Beta(2,2)$ belief distributions

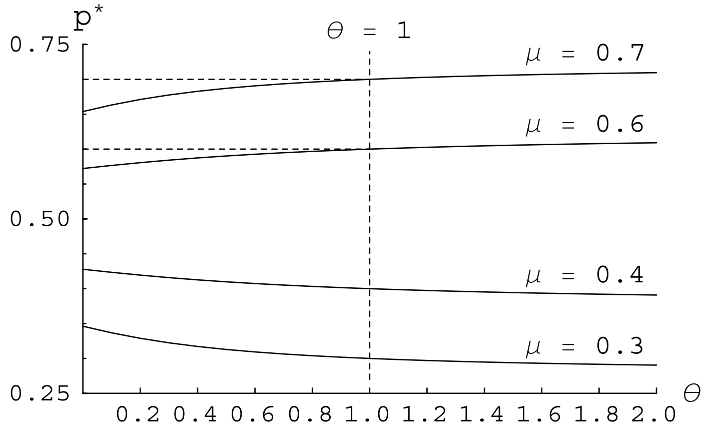


Figure 4: Equilibrium prices with asymmetric (e.g, $Beta(7, 3)$) belief distributions

With the exception of Theorem 1, which applies to any belief distribution, the analysis in Theorem 2 and the numerical examples above treat the case of symmetric belief distributions. The final example in this paper, in figure 4, shows the equilibrium price as a function of θ for four different asymmetric belief distributions. In this figure, the distributions are $Beta(7, 3)$ ($\mu = 0.7$), $Beta(3, 2)$ ($\mu = 0.6$), $Beta(2, 3)$ ($\mu = 0.4$), and $Beta(3, 7)$ ($\mu = 0.3$). These examples demonstrate that the main results of this paper hold at least for some distributions that are quite skewed, so that the main result isn't restricted only to symmetric distributions. In fact, we can see from this figure that not only are equilibrium prices related to belief distribution means in the same way as with the $Beta(1.01, 1.01)$ and $Beta(2, 2)$ distributions, but the equilibrium prices are closer to mean belief than they are for the $Beta(1.01, 1.01)$ and $Beta(2, 2)$ distributions.

5 Conclusions

Prediction markets are remarkably accurate information aggregation mechanisms. The equilibrium model in Manski [2004] is an important first step toward understanding the relationship between equilibrium prices and economic primitives – such as agents' preferences and the distribution of their beliefs – in prediction markets. This paper extends his analysis to include both risk aversion and unimodal belief distributions. Since empirical estimates of coefficients of relative risk aversion are frequently near one, which is the coefficient of relative risk aversion for logarithmic utility, this case is important. With logarithmic utility, the equilibrium price equals the mean of the belief distribution, regardless of the distribution of traders' beliefs. To the extent that the mean belief is itself a good predictor of an event, this is an encouraging result.

Under more restrictive conditions on the distribution of beliefs (i.e., the distribution of beliefs is symmetric), this paper has also demonstrated that the bias in the equilibrium price relative to the mean belief diminishes as the coefficient of relative risk aversion approaches one. In addition, even when the coefficient of relative risk aversion differs from one, the magnitude of the difference between the equilibrium price and the mean belief is small when the belief distribution is approximately uniform, and smaller yet when the traders' beliefs have a $Beta(2,2)$ distribution. Based on the numerical analysis for several $Beta(1.01, 1.01)$ distributions, for several $Beta(2,2)$ distributions, and for several skewed $Beta$ distributions, it is reasonable to expect that this difference between equilibrium price and belief distribution mean decreases as the variance of the belief distribution decreases.

Important problems remain to be solved before we understand how prediction markets aggregate information. By indicating relationships that exist among traders' beliefs, their risk aversion, and the resulting market price, the analysis in this paper should stimulate empirical questions about prediction markets. A belief elicitation procedure similar to the one employed by Nyarko and Schotter [2002] in their normal form game experiment, combined with elicitation of coefficients of risk aversion, as in Holt and Laury [2002], could be used to examine the consistency of the predictions of this model. A simpler consistency test could be developed using a controlled laboratory experiment similar to Plott et al. [2003], in which beliefs are induced through the differentiated information revealed to each subject. In such an experiment, a consistency check for the model could be performed by eliciting risk coefficients and then comparing observed prices to the predicted prices from the prediction market equilibrium model. In addition, since the difference between the mean belief and the equilibrium price is a function of the coefficient of relative risk aversion, the model predicts that the accuracy of the market price is related to the coefficient of relative risk aversion. Consequently, prediction market experiments that also estimate coefficients of risk aversion will simultaneously produce estimates of the probability of an event (its market price) and an estimate of the difference between the price and the traders' mean belief that the event will occur. Further examination of these questions may be useful for understanding the role of agents' characteristics – such as risk aversion, belief distributions, and changes to traders' beliefs during the course of trading – in the formation of asset market prices.

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Appendix

Lemma 1 If $n_{L}(\mu,\,\theta,\,\delta)\,w$ is the numerator in equation (4) and $n_{R}(\mu,\,\theta,\,\delta)\,w$ is the numerator in equation (5), then $n_{L}(\mu,\,\theta,\,\delta) > -n_{R}(\mu,\,\theta,\,\delta)$ when $\mu > {}^1\!/_2$ and $\theta > 1$ or when $\mu < {}^1\!/_2$ and $\theta < 1$. When $\mu < {}^1\!/_2$ and $\theta > 1$ or when $\mu > {}^1\!/_2$ and $\theta < 1$, the inequality is reversed, so that $n_{L}(\mu,\,\theta,\,\delta) < -n_{R}(\mu,\,\theta,\,\delta)$.

Proof From the assumption that $\mu > {}^1\!/_2$ it follows that $\frac{\delta}{\mu} < \frac{\delta}{1-\mu}$. For $\theta > 1$ the function $g(x) = x^{\frac{1}{\theta}}$ is concave so

$$\left(1+\tfrac{\delta}{\mu}\right)^{\frac{1}{\theta}} + \left(1-\tfrac{\delta}{\mu}\right)^{\frac{1}{\theta}} > \left(1+\tfrac{\delta}{1-\mu}\right)^{\frac{1}{\theta}} + \left(1-\tfrac{\delta}{1-\mu}\right)^{\frac{1}{\theta}}. \tag{A.1}$$

This inequality can be rewritten as

$$\left(1+\tfrac{\delta}{\mu}\right)^{\frac{1}{\theta}} - \left(1-\tfrac{\delta}{1-\mu}\right)^{\frac{1}{\theta}} > -\left(\left(1-\tfrac{\delta}{\mu}\right)^{\frac{1}{\theta}} - \left(1+\tfrac{\delta}{1-\mu}\right)^{\frac{1}{\theta}}\right)$$

which is equivalent to the inequality $n_{L}(\mu,\,\theta,\,\delta) > -n_{R}(\mu,\,\theta,\,\delta)$.

The three other cases are easily treated by examining inequality (A.1). For $\theta > 1$ with $\mu < {}^1\!/_2$, inequality (A.1) is reversed, since in that case the order of $1-\frac{\delta}{1-\mu}$ and $1-\frac{\delta}{\mu}$ is reversed, as is the order of $1+\frac{\delta}{1-\mu}$ and $1+\frac{\delta}{\mu}$. For $\mu > {}^1\!/_2$ and $\theta < 1$, inequality (A.1) is again reversed, this time because $g(x) = x^{\frac{1}{\theta}}$ is convex rather than concave. Finally, for $\mu < {}^1\!/_2$ and $\theta < 1$, inequality (A.1) holds so that $n_{L}(\mu,\,\theta,\,\delta) > -n_{R}(\mu,\,\theta,\,\delta)$. ■

Lemma 2 If $d_{L}(\mu,\,\theta,\,\delta)$ is the denominator in equation (4) and $d_{R}(\mu,\,\theta,\,\delta)$ is the denominator in equation (5), then $d_{L}(\mu,\,\theta,\,\delta)^{-1} > d_{R}(\mu,\,\theta,\,\delta)^{-1}$ when $\mu > {}^1\!/_2$ and $\theta > 1$ or when $\mu < {}^1\!/_2$ and $\theta \in \left({}^1\!/_2, 1\right)$. When $\mu < {}^1\!/_2$ and $\theta > 1$ or when $\mu > {}^1\!/_2$ and $\theta \in \left({}^1\!/_2, 1\right)$, the inequality is reversed, so that $d_{L}(\mu,\,\theta,\,\delta)^{-1} < d_{R}(\mu,\,\theta,\,\delta)^{-1}$.

Proof From equations (4) and (5), the inequality $d_{L}(\mu,\,\theta,\,\delta)^{-1} > d_{R}(\mu,\,\theta,\,\delta)^{-1}$ is equivalent to

$$(1-\mu)\left(\left(1+\tfrac{\delta}{1-\mu}\right)^{\frac{1}{\theta}} - \left(1-\tfrac{\delta}{1-\mu}\right)^{\frac{1}{\theta}}\right) > \mu\left(\left(1+\tfrac{\delta}{\mu}\right)^{\frac{1}{\theta}} - \left(1-\tfrac{\delta}{\mu}\right)^{\frac{1}{\theta}}\right). \tag{A.2}$$

Note that the left side of equation (A.2) is $2\,\delta$ times the slope of the segment from a to e in figure 5, and the right side of equation (A.2) is $2\,\delta$ times the slope of the segment from b to d in the same figure, so the argument proceeds by a comparison of these slopes.

Let m_{ij} be the slope from i to j in figure 5, for $i, j \in \{a, b, c, d, e\}$ and $i \neq j$. The slope from a to e is $m_{ab} = {}^1\!/_2\,(m_{ac} + m_{ce})$. Similarly, $m_{bd} = {}^1\!/_2\,(m_{bc} + m_{cd})$.

For $\theta > 1$, the derivative of $g(x) = x^{\frac{1}{\theta}}$ is decreasing, so $m_{ac} > m_{bc}$. For the same reason, $m_{cd} > m_{ce}$. Since $g''(x) = \frac{1}{\theta}\left(\frac{1}{\theta}-1\right)x^{\frac{1}{\theta}-2}$ is negative for $\theta > 1$, $g(x)$ is decreasing at a decreasing rate so $m_{ac} - m_{bc} > m_{cd} - m_{ce}$, or $m_{ac} + m_{ce} > m_{bc} + m_{cd}$. Therefore the inequality (A.2) holds for $\mu > {}^1\!/_2$ and $\theta > 1$.

The direction of the inequality is reversed if $\mu < {}^1\!/_2$, since in that case the order of $1-\frac{\delta}{1-\mu}$ and $1-\frac{\delta}{\mu}$ is reversed, as is the order of $1+\frac{\delta}{1-\mu}$ and $1+\frac{\delta}{\mu}$.

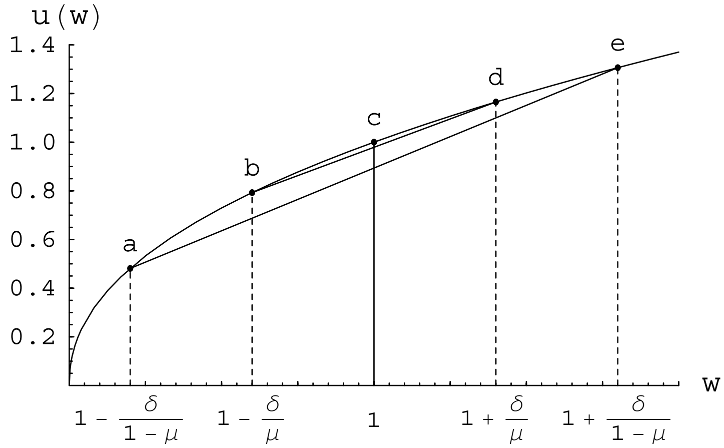

Figure 5: Representation of inequality (A.2) with $\theta > 1$

If $\mu > {}^1\!/_2$ and $\theta \in \left({}^1\!/_2, 1\right)$, then the direction of inequality (A.2) is reversed if $m_{ae} < m_{bd}$ in a diagram analogous to the one in figure 5 with $\theta \in \left({}^1\!/_2, 1\right)$. In this case, the function $g(x) = x^{\frac{1}{\theta}}$ is convex rather than concave, so $m_{bc} > m_{ac}$. Similarly, $m_{ce} > m_{cd}$.

The second derivative, $g''(x) = \frac{1}{\theta}\left(\frac{1}{\theta} - 1\right) x^{\frac{1}{\theta}-2}$, is a decreasing function of x, so $m_{bc} - m_{ac} > m_{ce} - m_{cd}$. This inequality can be rearranged to get $m_{bc} + m_{cd} > m_{ac} + m_{ce}$ or $m_{bd} > m_{ae}$. Therefore, inequality (A.2) is reversed for $\mu > {}^1\!/_2$ and $\theta \in \left({}^1\!/_2, 1\right)$.

Finally, inequality (A.2) holds in the case $\mu < {}^1\!/_2$ and $\theta \in \left({}^1\!/_2, 1\right)$, since the direction of the inequality is reversed when $\mu < {}^1\!/_2$ and it is reversed again when $\theta \in \left({}^1\!/_2, 1\right)$. ■

Lemma 4 The asset demand function $q^*(p_A, \pi_i, \theta, w)$ in equation (1) is a decreasing function of p_A for all $(\pi, \theta, w) \in (0,1) \times (0,\infty) \times (0,\infty)$.

Proof For $\theta \leq 0$ the proof is immediate from inspection of equation (2). For $\theta > 0$, the slope of the demand function with respect to p_A is

$$
\begin{aligned}
\frac{\partial q^*(p_A, \pi_i, \theta, w)}{\partial p_A} &= \frac{-(1-p)^{\frac{2}{\theta}}\pi^{\frac{2}{\theta}} + \frac{-1+2\theta p - 2\theta p^2}{\theta p(1-p)} p^{\frac{1}{\theta}}(1-p)^{\frac{1}{\theta}}\pi^{\frac{1}{\theta}}(1-\pi)^{\frac{1}{\theta}} - p^{\frac{2}{\theta}}(1-\pi)^{\frac{2}{\theta}}}{\left((1-p)p^{\frac{1}{\theta}}(1-\pi)^{\frac{1}{\theta}} + p(1-p)^{\frac{1}{\theta}}\pi^{\frac{1}{\theta}}\right)^2} \\
&= \frac{-(1-p)^{\frac{2}{\theta}}\pi^{\frac{2}{\theta}} + \left(2 - \frac{1}{\theta p(1-p)}\right) p^{\frac{1}{\theta}}(1-p)^{\frac{1}{\theta}}\pi^{\frac{1}{\theta}}(1-\pi)^{\frac{1}{\theta}} - p^{\frac{2}{\theta}}(1-\pi)^{\frac{2}{\theta}}}{\left((1-p)p^{\frac{1}{\theta}}(1-\pi)^{\frac{1}{\theta}} + p(1-p)^{\frac{1}{\theta}}\pi^{\frac{1}{\theta}}\right)^2} \\
&= \frac{-\left((1-p)^{\frac{1}{\theta}}\pi^{\frac{1}{\theta}} - p^{\frac{1}{\theta}}(1-\pi)^{\frac{1}{\theta}}\right)^2 - \frac{1}{\theta p(1-p)} p^{\frac{1}{\theta}}(1-p)^{\frac{1}{\theta}}\pi^{\frac{1}{\theta}}(1-\pi)^{\frac{1}{\theta}}}{\left((1-p)p^{\frac{1}{\theta}}(1-\pi)^{\frac{1}{\theta}} + p(1-p)^{\frac{1}{\theta}}\pi^{\frac{1}{\theta}}\right)^2}.
\end{aligned}
$$

This slope is negative for all $p \in (0,1)$ and $\theta > 0$. ■