THE LAW OF ONE PRICE IN DATA ENVELOPMENT ANALYSIS: RESTRICTING WEIGHT FLEXIBILITY ACROSS FIRMS

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ABSTRACT

The Law of One Price (LoOP) states that all firms face the same prices for their inputs and outputs in the competitive market equilibrium. This law has powerful implications for productive efficiency analysis, which have remained unexploited thus far. This paper shows how LoOP-based weight restrictions can be incorporated in Data Envelopment Analysis (DEA). Utilizing the relation between the industry level and the firm level cost efficiency measures, we propose to apply a set of input prices that is common for all firms and that maximizes cost efficiency of the industry. Our framework allows for firm-specific output weights and variable returns-to-scale, and preserves the linear programming structure of the standard DEA. We apply the proposed methodology for evaluating research efficiency of economics departments of Dutch Universities. This application shows that the methodology is computationally tractable for practical efficiency analysis, and that it helps in deepening the DEA analysis.

Keywords: Data Envelopment Analysis; Law of One Price; industry-level efficiency; weight restrictions; research efficiency.

JEL classification: C14, C61, D21, D24

1. INTRODUCTION

Data Envelopment Analysis (DEA; Charnes, Cooper and Rhodes, 1978) is a widely applied nonparametric mathematical programming approach for analyzing the productive efficiency of firms. The principle of DEA is to let the data speak for themselves rather than to enforce them to some rigid, arbitrarily specified functional form.[1] This principle is firmly rooted in the economic literature on activity analysis (e.g., Dantzig, 1949; Koopmans, 1951a,b) and nonparametric production analysis (e.g., Afriat, 1972; Varian, 1984); see, e.g., Banker and Maindiratta (1988), Färe, Grosskopf and Lovell (1994) and Färe and Grosskopf (1995) for discussion. Following this economic perspective, a number of authors (see e.g. Kuosmanen and Post, 2001) state the lack of prior price information for some inputs and/or outputs as the prime motivation for using DEA. Examples of inputs/outputs that are notoriously difficult to price include public goods and services, environmental bads, new goods introduced to the market within the time-frame of the analysis, and durable goods with life-time exceeding the length of "one period" in the analysis (most notably capital inputs). The multiplier weights generated by DEA can be interpreted as implicit/shadow prices that express the performance of the evaluated firm in the most favorable light. If the firms' choices of input-output bundles are guided by rational economic objectives, these shadow prices then reveal, in the spirit of the revealed preference theory of Samuelson (1948), the underlying economic prices (opportunity costs) which are unknown to the researcher.

Generally, DEA derives shadow prices exclusively from the quantity data. Such analysis can be strengthened by imposing additional price information in the form of additional constraints that define a feasible range for the relative prices. These constraints are useful for modeling prior knowledge or expectations about prices. At least, we may typically rule out the most bizarre extreme cases where a relative price of a commodity approaches to zero or infinity when we think it should be close to one. However, the technical questions related to incorporating such price restrictions have been mainly discussed in the DEA context, most commonly under the label 'weight restrictions' or 'assurance regions' (see, e.g., Allen et al., 1997; Pedraja-Chaparro et al., 1997, for surveys), without making a clear distinction between the alternative interpretation of the multiplier weights as economic prices or marginal substitution (transformation) rates between inputs (outputs) (one exception is Kuosmanen and Post, 2001). This almost exclusive focus on technical (rather than economic) issues may explain why the DEA literature has so far ignored the coordinating function of prices in guiding the allocation of resources within the economy.

The shadow prices obtained as the optimal solution of a DEA model tend to exhibit huge variations across firms. This is at odds with the common perception about price formation in competitive markets. The prevalence of price differences across firms supplying the same homogenous good entails mixed signals for both the producers and the consumers. Moreover, multiplicity of prices implies arbitrage possibilities, which signals inefficiency at the level of the economy as a whole. By definition, the economy reaches a state of competitive equilibrium when the prices of homogenous goods stabilize at some commonly known and accepted level at which no economic agent wishes to engage into further transactions. As a

[1] This makes DEA particularly attractive in comparison to parametric efficiency analysis (see, e.g., Kumbakhar and Lovell, 2000), which is ultimately constrained by the functional structure that is imposed on the production possibilities. This argument is all the more valid in the case of efficiency analysis: imposing minimal *a priori* structure minimizes the risk of rejecting efficiency in the case of truly efficient behavior (i.e., so-called *Type I errors*). Such concern is especially important when minimal a priori information about the technology is available, which is frequently the case in practical application settings.

result, firms that operate on the same markets trade (homogenous) input/output commodities against the same unique prices. This definitional fact is commonly referred to as the *Law of One Price* (LoOP).[2]

One might be tempted to dismiss LoOP as a theoretical property, which does not hold in the real life markets that are never perfectly competitive. Indeed, perfect competition would tend to eliminate any inefficiencies that DEA tries to identify and measure. Consequently, DEA is most powerful in application areas where competitive markets fail, for example, in the production of public goods. For this reason, we may doubt the validity of the LoOP as a *descriptive hypothesis* of the real markets. However, this is not a valid excuse for ignoring the microeconomic theory of competitive markets and its powerful concepts such as the LoOP. Stated conversely, the LoOP typically fails as a descriptive hypothesis in the real markets *because of* inefficiencies. To be able to identify inefficiencies, we first have to define what constitutes efficient performance. This is where DEA becomes useful. We may think of DEA as a "simulator" tool that enables us to estimate the underlying economic prices (based on the shadow prices) *as if* the sample firms under study would be operating under competitive markets. This is where the LoOP can be very useful as a *normative property* that any efficient market should satisfy. By introducing the LoOP, we require that performance should be efficient not just at the level of individual firms, but also at the level of the industry or sector as a whole. In this sense, the incorporation of the LoOP restrictions into DEA can bind it more closely to its intellectual roots in activity analysis.[3]

In fact, we find that many of the basic features and implicit assumptions behind DEA are geared towards the competitive markets, and are at odds with non-competitive settings. Firstly, non-competitive industries are characterized by a relatively small number of firms, which causes a fundamental data problem for a firm-level DEA analysis that requires a relatively large sample of firms. Secondly, in non-competitive markets the input-output prices are not exogenously defined for the firm, as implicitly assumed in DEA, but depend on the input-output quantities demanded and supplied by the firm (see Cherchye, Kuosmanen and Post, 2002, for discussion). Thirdly, in the non-competitive markets, firms tend to qualitatively differentiate their products from other available products, whereas DEA makes the implicit assumption that outputs are homogeneous and so only differ in terms of quantity. In light of the previous arguments, we believe that the LoOP is a reasonable normative efficiency requirement to consider in most of the traditional application areas of DEA. While we can certainly think of other application areas where the LoOP has no meaningful economic content, we think that the general idea of making the optimal shadow prices of a firm dependent on the optimal shadow prices of other firms by restricting weight flexibility across firms has potential applications even if we abstract from the microeconomic content of the LoOP.[4]

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[2] The history of the Law of One Price can be traced back at least to Jevons (1871), who referred to it as the "Law of Indifference".

[3] Compare with the notion of *efficiency prices* introduced by Koopmans (1951a,b).

[4] See Cherchye and Kuosmanen (2004) for an application of the DEA-inspired "benefit of the doubt"-weighting principle for constructing a meta-level index of sustainable development (that enables cross-country comparisons). These authors limit weight flexibility across countries so as to direct the country-specific weight-profiles towards a more universally accepted consensus. Interestingly, this first application of the approach did not interpret the multiplier weights as prices, and hence does not directly relate to the LoOP discussed in the current study.

This paper presents a first attempt to integrate the LoOP, taken as an additional normative requirement for efficiency, in the DEA assessment.[5] In contrast to the currently available price/weight restriction tools that impose bounds on the relative prices of inputs/outputs *within the evaluation of a single firm*, we propose to restrict price variations *across firms*. That is, we operationalize the LoOP conditions by imposing weight restrictions that limit the extent to which the shadow prices of one firm may deviate from the shadow prices of another firm. Given the important role of shadow prices in DEA, we believe that such an extension of the existing apparatus opens up promising new routes in a wide variety of applications. The LoOP offers an economically sound justification for restricting price flexibility and thus improving the discriminatory power of the DEA assessment. In fact, it is often much easier to put intuitive constraints on the variation of (shadow) prices across (directly comparable) firms than to bound the price variation across (often not directly comparable) input/output dimensions. In this respect, LoOP restrictions can enrich and complement the existing machinery for modeling weight restrictions in DEA.

The assimilation of the LoOP within DEA presents a number of theoretical and technical challenges:

1) Since it seems most natural to impose LoOP restrictions at the aggregate level of the industry, one critical question concerns the relation between the firm-specific efficiency indices and the industry efficiency. The recent work by Blackorby and Russell (1999) and Färe and Zelenyuk (2003) has shed light on this aggregation issue, which will be utilized in this paper.
2) The simultaneous implementation of LoOP constraints, maintained production assumptions, and other constraints requires a general and flexible enough framework. Following Kuosmanen and Post (2001), we adopt the general Free Disposable Hull (FDH) technology as the minimal reference technology, and enhance discriminatory power of the model by imposing constraints on relative prices and technical substitution/transformation rates.
3) The optimal multiplier weights/shadow prices generated by DEA need not be unique. While possible non-uniqueness does not affect the industry-level efficiency measures, it can influence the efficiency distribution at the firm level. To resolve this problem, we propose a framework for testing uniqueness of the LoOP prices.
4) To keep the assessment procedure tractable even in large data sets, it is desirable to preserve the Linear Programming (LP) structure of the conventional DEA programming problems. Inspired by Agrell and Tind (2001), we present a general LP formulation for FDH-based cost efficiency measures which permits additional economic constraints on relative prices, as well as additional production assumptions regarding convexity, technical substitution/transformation rates, and returns to scale.

Next, we apply the proposed methodology for evaluating research programs organized at Economics and Business Management faculties of Dutch universities. This study complements an earlier study by Cherchye and Vanden Abeele (2002), who evaluated the same research programs without imposing LoOP restrictions. We believe that the main potential of the proposed methodology lies in public sector applications, where unambiguous market prices are not available although it may reasonably be argued that the same notional rates should be used across all units for cost-effectiveness evaluations. As explained below, the latter indeed seems to apply to university research in the Netherlands. This application

[5] Parallel to this study, Cross and Färe (2003) investigate the relationship between DEA and the LoOP from the opposite perspective: these authors examine the empirical validity of the LoOP hypothesis to determine whether it is justified to use monetary expenditure and cost data as input indicators (or quantity index) in DEA.

should also give us a deeper insight into the computational tractability of the proposed methodology for practical efficiency analysis, and into the value added that it produces in terms of the qualitative conclusions of the efficiency analysis.

The remainder of the paper unfolds as follows. In section 2, we briefly review the conventional firm-specific DEA efficiency assessment, and introduce the problem of including LoOP restrictions into such an evaluation. In section 3, we tackle the problem of efficiency assessment at the industry level. Section 4 presents operational tools to deal with the possibility of non-uniqueness of the optimal shadow prices, and to investigate sensitivity of the efficiency estimates with respect to (minor) departures of the LoOP conditions that are imposed. Section 5 presents our empirical application. Section 6, finally, recaptures our main findings and concludes.

2. MEASURING ECONOMIC EFFICIENCY AT THE FIRM LEVEL

2.1 Preliminaries

Consider an industry consisting of N firms, indexed by $n \in \nu \equiv \{1,...,N\}$, which utilizes a common technology that transforms R inputs into S outputs, indexed by $r \in \rho \equiv \{1,...,R\}$ and $s \in \sigma \equiv \{1,...,S\}$ respectively. Using standard notation, the vector $x = (x_1 \cdots x_R)' \in \mathbb{R}_+^R$ represents an arbitrary input vector and $y = (y_1 \dots y_S)' \in \mathbb{R}_+^S$ an output vector; the input price vector (representing the marginal opportunity cost of each input) is denoted as $w = (w_1 \dots w_R) \in \mathbb{R}_+^R$. The empirical production data are represented by the input matrix $X = (X_1 \dots X_N)$ and the output matrix $Y = (Y_1 \dots Y_N)$, where vectors $X_n = (X_{n1} \cdots X_{nR})'$ and $Y_n = (Y_{n1} \dots Y_{nS})'$ represent the input and the output vectors of firm $n \in \nu$.

To enhance transparency of our discussion, we solely focus on *cost efficiency* analysis in presenting the methodology. (The reasoning is easily adapted to other notions of economic efficiency such as revenue efficiency.) Cost efficiency analysis typically starts from representing the *production technology* in terms of the input correspondence $L : \mathbb{R}_+^S \to 2^{\mathbb{R}_+^R}$,

(1) $$L(y) \equiv \left\{ x \middle| x \in \mathbb{R}_+^R \text{ can produce } y \in \mathbb{R}_+^S \right\}.$$

That is, the input set $L(y)$ contains all input vectors x that yield output y. Sets $L(y)$ are assumed to satisfy the following well-known regularity conditions: 1) closedness; 2) non-emptyness; and 3) no free lunch [$y \geq \vec{0}^S, y \neq \vec{0}^S \Rightarrow \vec{0}^R \notin L(y)$].

Using (1), we define cost efficiency of a firm $n \in \nu$ (associated with an input-output combination (X_n, Y_n)) as the ratio of minimum cost over actual cost for given output Y_n, input prices w and the input reference set $L(Y_n)$. Representing the *cost function* by

(2) $$C^L(y, w) \equiv \min_x \left\{ wx \middle| x \in L(y) \right\},$$

we can formally express cost efficiency of firm $n \in \nu$ as

(3) $$CE^L(X_n, Y_n, w) \equiv \frac{C^L(Y_n, w)}{wX_n}.$$

In practical efficiency analysis, the input price vector w is often not observed, or the observed input prices do not constitute reliable proxies for the true opportunity costs faced by the evaluated firm. In that case, cost efficiency analysis can proceed by resorting to an input price domain rather than a single-valued input price vector; that price domain should then capture the true (but unknown) input prices faced by the firm under study. Specifically, we consider a general price domain that is characterized as the polyhedral convex set

(4) $$W \equiv \left\{ w \in \mathbb{R}_+^R \middle| Aw \geq b \right\}.$$

This set represents the price domain in terms of l linear inequalities; A is an $l \times S$ matrix and b an l-dimensional column vector. (If A and b are void, then we have $W = \mathbb{R}_+^S$.) In the following, we assume the price domain W merely includes *relative* price restrictions, which cannot be violated by equiproportionate rescaling of the input prices. In such a setting, we obtain for input prices $w \in W$ an upper bound for $CE^L(x, y; w)$ as

(5) $$\overline{CE}^L(X_n, Y_n; W) = \max_{w \in W} CE^L(X_n, Y_n, w) = \max_{w \in W} \left(\min_{\tilde{x} \in L(Y_n)} \left\{ w\tilde{x} \middle| wx = 1 \right\} \right).$$

Hence, $\overline{CE}^L(X_n, Y_n; W)$ approximates $CE^L(X_n, Y_n; w)$ by maximizing it in terms of the prices $w \in W$ that are endogenously selected; the firm k is assessed in the most favorable light. See Kuosmanen and Post (2001) for a more in-depth discussion.[6]

Another challenge of the empirical efficiency analysis is that the input correspondence L is not known either, but must be empirically estimated. In DEA, an inner bound approximation of the set $L(Y_n)$ is constructed from the data by imposing some additional production assumptions. A minimal assumption in conventional DEA models is free disposability of inputs and outputs. Solely maintaining this production assumption yields the free disposable hull (FDH) reference technology (see Deprins et al., 1984, and Tulkens, 1993, for detailed discussions), represented as $L_{FDH}(Y_n) \equiv \{x : x \geq X_m; Y_m \geq Y_n; m \in \nu\}$. Using this reference technology, we obtain an estimate for the cost efficiency of firm $k \in \nu$ as the optimal solution to the following Linear Programming (LP) problem:

[6] Kuosmanen and Post (2001; 2003) actually also introduce a lower bound for $CE^L(x, y, w)$ based on $w \in W$. We will not explicitly discuss this measure here. Still, it is easy to see that the lower bound estimate will coincide with the upper bound under the LoOP condition (introduced below), in which case W reduces to a ray. Also, given that DEA typically works with an inner bound approximation of the input correspondence $L(y)$ generally makes the upper bound measure $\overline{CE}^L(x, y; W)$ more meaningful for empirical application than its lower bound counterpart (which can actually overestimate the true value of $CE^L(X_k, Y_k, w)$ when an inner bound approximation is used for $L(Y_k)$).

$$\overline{CE}^{L_{FDH}}(X_k, Y_k; W) = \max_{c, p_m, w, f_m} c$$

$$\begin{aligned}
& s.t. \\
& c \leq \sum_{s=1}^{S} Y_{ks} p_{ms} + f_m \quad \forall m \in \nu \\
(6) \qquad & \sum_{s=1}^{S} Y_{ms} p_{ms} - \sum_{r=1}^{R} X_{mr} w_r + f_m \leq 0 \quad \forall m \in \nu \\
& \sum_{r=1}^{R} X_{kr} w_r = 1 \\
& p_m \in \mathbb{R}_+^S \quad \forall m \in \nu \\
& w \in W
\end{aligned}$$

Model variables p_m and f_m can be interpreted as the vector of output shadow prices and fixed cost, respectively. This LP formulation exploits the fact that the free disposable hull reference technology can be expressed as the union of a set of convex monotone hull reference technologies, defined with respect to each separate firm in the sample (see also Lovell and Vanden Eeckaut, 1993; p. 448-449). Problem (6) essentially solves the usual DEA input efficiency problem separately for each possible reference firm $m \in \nu$; notice that output prices p_m and fixed costs f_m are allowed to differ across reference firms m. The first constraint effectively selects the reference firm which produces the target output with the lowest cost.

The LP relaxation of the FDH model is not well known. It is worth to stress that our LP formulation (6) provides a more immediate link between FDH and convex DEA model formulations than the earlier approach by Agrell and Tind (2001). To gain further intuition for the LP formulation of the non-convex FDH, the dual of problem (6) is derived in a step-by-step fashion in the Appendix.

Apart from free disposability, other production assumptions are frequently maintained in practical DEA assessment. For example, many models assume convexity of the production possibilities (see, e.g., the popular DEA model introduced by Banker, Charnes and Cooper, 1984) and non-increasing, non-decreasing or constant returns-to-scale (see, e.g., Seiford and Thrall, 1990). These additional assumptions are implemented in model (6) by adding the restrictions (*NIRS* stands for non-increasing returns-to-scale; *NDRS* stands for non-decreasing returns-to-scale; *CRS* stands for constant returns-to-scale)

$$(7) \qquad \begin{cases} Convexity: \ p_m = p_n, f_m = f_n \ \forall m, n = 1, \ldots, N \\ NIRS: \ f_m \geq 0 \ \forall m = 1, \ldots, N \\ NDRS: \ f_m \leq 0 \ \forall m = 1, \ldots, N \\ CRS: \ f_m = 0 \ \forall m = 1, \ldots, N \end{cases}$$

These constraints apply (with straightforward modifications) to all model formulations specified below. We will therefore focus on the general FDH technology, and consider alternative production assumptions and constraints only where those are not immediately obvious from (7).

As the first step towards aggregate analysis, note that instead of solving problem (6) separately for each firm, we may calculate the cost efficiency measures simultaneously for all N firms by solving a bigger LP problem:

$$
(8)\qquad
\begin{aligned}
&\max_{c, p_{mn}, w_n, f} \sum_{n=1}^{N} c_n \\
&s.t. \\
&c_n \le \sum_{s=1}^{S} Y_{ns} p_{mns} + f_{mn} \quad \forall m, n \in \nu \\
&\sum_{s=1}^{S} Y_{ms} p_{mns} - \sum_{r=1}^{R} X_{mr} w_{nr} + f_{mn} \le 0 \quad \forall m, n \in \nu \\
&\sum_{r=1}^{R} X_{nr} w_{nr} = 1 \quad \forall n \in \nu \\
&p_{mn} \in \mathbb{R}_{+}^{S} \quad \forall m, n \in \nu \\
&w_n \in W \quad \forall n \in \nu
\end{aligned}
$$

Notice that the model variables of (8) include N input price vectors w_n, N^2 output price vectors p_{mn}, and N^2 fixed cost variables f_{mn}: this guarantees that problem (8) encompasses all constraints of problem (6) simultaneously for all firms. Since the shadow prices of individual firms are independent of the shadow prices of other firms in the sample, we may maximize the sum of cost efficiency measures, and hence $\overline{CE}^{L_{DEA}}(X_n, Y_n; W)$ is obtained as the optimal c_n^* from (8).

It is important to observe that both problems (6) and (8) select input prices $w_n \in W$ that are specific to each firm n=1,…,N. Hence, there is no guarantee that the optimal (endogenously selected) input price vectors w_n would satisfy the LoOP. We next investigate how we can implement this LoOP condition in terms of additional weight restrictions.

As a preliminary step, we note that the cost efficiency measure is homogenous of degree zero in the input prices (see definition (3)), which means that multiplying the input prices by a positive scalar does not affect its value. Therefore, it does not suffice to compare the different values $w_n^* \in W$ over all firms n in order to check consistency with the LoOP; the same cost efficiency values are obtained when re-scaling the optimal input vector as αw_n^* for all $\alpha > 0$. In other words, given our focus on cost efficiency, it does not matter whether we impose the LoOP condition in terms of absolute input prices, i.e.

$$
(9)\qquad w_m = w_n \;\; \forall m, n \in \nu \;,
$$

or in terms of relative input prices, i.e.

$$
(10)\qquad \frac{w_{mr}}{w_{mq}} = \frac{w_{nr}}{w_{nq}} \;\; \forall m, n \in \nu; q, r \in \rho \;.
$$

It follows from our above argument that if the optimal input prices satisfy (10), then we can always normalize the input prices to satisfy (9) without influencing the firm-specific cost efficiency values. In fact, the zero-homogeneity property of cost efficiency is already utilized in the LP formulations (6) and (8) for normalizing the firm-specific input prices such that the total cost of production activity equals one for each firm. Imposing the LoOP in absolute terms as in (9) would typically conflict with these normalization constraints, so we have to formulate the LoOP in terms of relative prices as in (10). In the following, we thus focus on implementing condition (10) in the standard evaluation problem (8).

Unfortunately, direct implementation is difficult in practice due to the non-linearity of condition (10), which involves ratios of two unknown (decision) variables. To solve this technical problem, we need to expand our discussion to economic efficiency evaluation at the industry level.

3. MEASURING ECONOMIC EFFICIENCY AT THE INDUSTRY LEVEL

3.1 Theory

Aggregation of efficiency indices from the firm level to the industry level has attracted some deserved attention in the recent literature. Generally, guaranteeing consistency of efficiency indices defined and measured at different aggregation levels proves trickier than one might first expect; see Blackorby and Russell (1999) for some general results. Building on these results, Färe and Zelenyuk (2003) discuss aggregation conditions for revenue efficiency. In the present context, it is interesting to note that Färe and Zelenyuk have to assume the LoOP holds to make their aggregation work. In this section we re-interpret the Färe and Zelenyuk approach within the context of cost efficiency analysis.

Industry efficiency analysis starts from a specification of the *industry production technology*. Let the aggregated industry input-output combination be denoted by $(\mathfrak{X},\mathfrak{Y}) \equiv \left(\sum_{n\in v} X_n, \sum_{n\in v} Y_n\right)$. Following usual practice, we define the *industry input set* as the sum of the individual firm input sets[7]

$$(11) \qquad \mathfrak{L}(\mathfrak{Y}) \equiv \sum_{k\in N} L(Y_k)\,.$$

Assuming that the LoOP holds, all firms in the industry face the same input prices w. The *industry cost function* is then defined as

$$(12) \qquad \mathfrak{C}(\mathfrak{Y},w) \equiv \min_x \left\{ wx \,\middle|\, x \in \mathfrak{L}(\mathfrak{Y}) \right\},$$

and *industry cost efficiency* is defined directly analogously to (3) as

$$(13) \qquad ICE^{\mathfrak{L}}(\mathfrak{X},\mathfrak{Y},w) \equiv \mathfrak{C}(\mathfrak{Y},w) \Big/ w\sum_{n\in v} X_n\,.$$

[7] Note that this setting does not allow for reallocation of output targets. Färe and Zelenyuk (2003) allow for firm-specific technologies, but those are of little use in the present context.

Focusing on revenue efficiency, Färe and Zelenyuk (2003) show that the industry efficiency can be decomposed as the revenue-share weighted average of the individual firm efficiencies. This result is directly adapted to the present setting. Specifically, industry cost efficiency can be decomposed into the cost-share weighted average of the firm cost efficiencies

$$(14) \qquad ICE^{\mathfrak{L}}(\mathfrak{X},\mathfrak{Y},w) = \sum_{n\in v} \frac{C^L(Y_n,w)}{wX_n} \cdot Sh^n ,$$

where Sh^n is the cost share of firm $n \in v$ in the total costs of the industry, that is,

$$(15) \qquad Sh^n = \frac{wX_n}{w\sum_{n\in v} X_n} .$$

3.2 Industry efficiency under incomplete price information

The previous aggregation result was derived under the assumption that the price vector *w* is known. As discussed above, we consider DEA to be most useful in the situations where the prices are not (completely) known. We therefore next extend the DEA approach to finding the optimal shadow prices from the firm level to the industry level. In the absence of reliable information about prices *w*, we propose to select those (implicit or shadow) input prices *w* that maximize the cost efficiency of the industry $ICE^{\mathfrak{L}}(\mathfrak{X},\mathfrak{Y},w)$. Intuitively, individual firms may be more or less cost efficient, but in a competitive environment, the industry as a whole should operate close to its efficiency limits.

We see at least two approaches for empirical estimation of the price vector *w*, which directly exploit the result in (14) and (15). We shall refer to these alternative routes as respectively the *Top-Down* approach and the *Bottom-Up* approach.

3.2.1 Top-Down Approach

The Top-Down approach starts directly from the definition of $ICE^{\mathfrak{L}}(\mathfrak{X},\mathfrak{Y},w)$ in (14). This approach only applies if the technology exhibits constant returns-to-scale. Given that all firms are assumed to operate under the same technology, represented by the input correspondence *L*, the constant returns-to-scale property implies that

$$(16) \qquad \mathfrak{L}(\mathfrak{Y}) = L(\mathfrak{Y}) ,$$

which in turn entails

$$(17) \qquad ICE^{\mathfrak{L}}(\mathfrak{X},\mathfrak{Y};W) = CE^L(\mathfrak{X},\mathfrak{Y};W) .$$

We may thus take the DEA production set based on the observed input matrix *X* and output matrix *Y* as the reference set, and assess cost efficiency of the industry input-output combination $(\mathfrak{X},\mathfrak{Y})$ relative to that empirical set. Formally, the industry cost efficiency is estimated as

$$\overline{CE}^{L_{FDH}}(\mathfrak{X},\mathfrak{Y};W)=\max_{c,p_m,w} c$$

$$s.t.$$

$$c \leq \sum_{s=1}^{S} \mathfrak{Y}_s p_{ms} \quad \forall m \in \nu$$

$$\text{(18)} \quad \sum_{s=1}^{S} Y_{ms} p_{ms} - \sum_{r=1}^{R} X_{mr} w_r \leq 0 \quad \forall m \in \nu$$

$$\sum_{r=1}^{R} \mathfrak{X}_r w_r = 1$$

$$p_m \in \mathbb{R}_+^S \quad \forall m \in \nu$$

$$w \in W$$

Problem (18) is directly analogous to the firm-level problem (6). Note that we exclude the "fixed cost" variable f_m because we have to assume constant returns-to-scale to guarantee equality (16). Note that problem (18) uses the nonconvex FDH under constant returns-to-scale as an approximation of the technology. Like before, convexity can be imposed by inserting an additional constraint $p_m = p \quad \forall m \in \nu$.

The optimal solution to (18) gives us an estimate of the industry cost efficiency $ICE^{\mathfrak{L}}(\mathfrak{X},\mathfrak{Y},w)$, but it also provides us with the optimal input shadow prices w^* that maximize cost efficiency at level of the industry as a whole. We may subsequently use these industry shadow prices within the firm level efficiency analysis. Specifically, we can impose that the relative input prices at the firm level should equal the relative prices implied by w^*. In practice, this is simply implemented by specifying the price domain W as $W=\left\{w\in \mathbb{R}_+^R \middle| \alpha w = w^*; \alpha \in \mathbb{R}_+\right\}$ or, equivalently, inserting in problem (6) or (8) the following linear constraint:

$$\text{(19)} \quad \begin{array}{l} w_m = \alpha w^* \ \forall m \in \nu \\ \alpha \in \mathbb{R}_+ \end{array}.$$

Note, however, that in doing so we have to assume that the shadow prices w^* are unique. Unfortunately, there is no guarantee for such uniqueness: the optimal $ICE^{\mathfrak{L}}(\mathfrak{X},\mathfrak{Y},w)$ may equally well be obtained with alternative input prices, associated with a different distribution of the firm-level efficiency indices. We return to this problem in more detail in Section 4.1.

3.2.2 Bottom-Up Approach

The previous Top-Down approach only applies if the technology exhibits constant returns-to-scale. Under variable returns-to-scale, we can adopt a *Bottom-Up* approach, which starts from the firm-specific cost efficiencies. This alternative approach builds on the observation that (see (14) and (15))

$$(20) \qquad ICE^{\mathfrak{L}}(\mathfrak{X},\mathfrak{Y},w) = \sum_{n\in\nu} \frac{C^L(Y_n,w)}{wX_n} \cdot Sh^n = \frac{\sum_{n\in\nu} C^L(Y_n,w)}{w\mathfrak{X}} \ .$$

That is, the industry cost efficiency can be seen as the sum of the firm-specific minimum costs divided by the total cost of the industry. Therefore, we can estimate $ICE^{\mathfrak{L}}(\mathfrak{X},\mathfrak{Y},w)$ by the optimal solution to the LP problem that selects $w\in W$ that satisfies $w\mathfrak{X}=1$, and that maximizes the sum of minimum costs associated with the individual firms $n\in\nu$, that is,

$$(21) \qquad \begin{aligned} & ICE^{\mathfrak{L}}(\mathfrak{X},\mathfrak{Y},w) = \max_{\gamma,p_{mn},w,f_{mn}} \sum_{n=1}^{N} \gamma_n \\ & s.t. \\ & \gamma_n \le \sum_{s=1}^{S} Y_{ns} p_{mns} + f_{mn} \quad \forall m,n\in\nu \\ & \sum_{s=1}^{S} Y_{ms} p_{mns} - \sum_{r=1}^{R} X_{mr} w_r + f_{mn} \le 0 \quad \forall m,n\in\nu \\ & \sum_{r=1}^{R} \mathfrak{X}_r w_r = 1 \\ & p_{mn} \in \mathbb{R}_+^s \quad \forall m,n\in\nu \\ & w\in W \end{aligned}$$

Analogous to (7), the additional convexity and/or returns-to-scale postulates can be imposed by means of the following constraints:

$$(22) \qquad \begin{cases} Convexity: \ p_{ln} = p_{mn}, f_{ln} = f_{mn} \ \forall l,m,n = 1,...,N \\ NIRS: \ f_{mn} \ge 0 \ \forall m,n = 1,...,N \\ NDRS: \ f_{mn} \le 0 \ \forall m,n = 1,...,N \\ CRS: \ f_{mn} = 0 \ \forall m,n = 1,...,N \end{cases}$$

It is illustrative to compare this problem with problem (8) of Section 2. Problem (21) is consistent with the LoOP because it applies the same input prices for all firms. The earlier nonlinearity problem associated with the LoOP constraint (10) is avoided by normalizing the input prices such that the total cost at the industry level equals unity, in line with (20). Consequently, the optimal γ_n^* do not directly represent the firm level cost efficiency indices, but should be interpreted as the values of the cost function at the normalized prices. The firm-specific cost efficiencies, which are consistent with the industry level cost efficiency computed on the basis of (21), are obtained from the decomposition of Färe and Zelenyuk, viz.

$$(23) \qquad \overline{CE}^{L_{FDH}}(X_n,Y_n;W) = \frac{\gamma_n^*}{w^* X_n},$$

where the vector w^* represents the optimal input prices that are obtained as the solution of (21). Like in the Top-Down approach, however, these optimal shadow prices need not be unique. We turn to this non-uniqueness problem in the next section

4. Uniqueness and sensitivity analysis

4.1 Uniqueness

We concluded Sections 3.2 and 3.3 by noting that the optimal shadow prices obtained from either the top-down or the bottom-up approach need not be unique. This is not a problem if we are primarily interested in economic efficiency at the industry level, because non-uniqueness of shadow prices does not impact on the value of $ICE^{\mathfrak{L}}(\mathfrak{X},\mathfrak{Y},W)$. At the firm-level, however, non-uniqueness does matter. If the optimal input prices that maximize $ICE^{\mathfrak{L}}(\mathfrak{X},\mathfrak{Y},W)$ are not unique, then the cost efficiency levels of individual firms may vary considerably depending on which particular shadow prices we choose at the industry level. It is therefore important to test whether the industry-level optimal solution is indeed unique.

The simplest way to proceed is to search for alternative input price vectors that yield the same optimal solution to the industry level problems (18) and (21). In this context, we can specify uniqueness of the optimal input prices $w^* \in W$ as the null hypothesis. (Based on the fact that the number of firms is usually considerably larger than the number of input and output dimensions, we may indeed reasonably expect the null hypothesis to hold in most applications.) Consider then the following additional constraint for relative prices of inputs $q, r \in \rho$:

$$(24) \qquad w_r \geq \left(\frac{w_r^*}{w_q^*} + \varepsilon \right) w_q ,$$

where $\varepsilon > 0$ can be interpreted as an exogenously specified 'tolerance' parameter; we consider the difference between the prices $w, w^* \in W$ to be (economically) insignificant if the difference $w_r / w_q - w_r^* / w_q^*$ is less than or equal to ε. Condition (24) is linear in the unknown variables *w*, so that we may directly implement it in problems (18) or (21). Let the new price domain that includes (24) be denoted by W^+ and denote the corresponding optimal solution of the industry cost efficiency problem by $ICE(W^+)$. We can now distinguish two cases. First, if $ICE(W^+) = ICE(W)$, then we have identified an alternative set of input prices that yields the same industry-level cost efficiency as the original w^*, and the (uniqueness) null hypothesis is to be rejected. Alternatively, if $ICE(W^+) < ICE(W)$, then it is not possible to increase the price of input *r* and decrease price of input *q* without affecting the industry cost efficiency measure. The next step is to reverse the signs of the inequality and the tolerance parameter ε in (24) as

$$(25) \qquad w_r \leq \left(\frac{w_r^*}{w_q^*} - \varepsilon \right) w_q ,$$

and perform the same test again. This procedure should be repeated for every pair of inputs, until the null hypothesis is rejected or all pairs have been considered. If we fail to reject the null hypothesis in all cases, we may conclude uniqueness of the optimal industry-level shadow prices. Since the number of inputs is usually relatively small, the number of combinations to be tested should not present a major computational problem.

As explained above, non-uniqueness of the optimal LoOP prices may affect the firm-level (but not the industry-level) efficiency estimates. One possibility to extend the firm-level analysis in such a case consists of calculating the upper and lower bounds of cost efficiency for each firm $k \in \nu$, restricting to input prices that keep the industry cost efficiency to its minimum (i.e., solving $ICE^{\mathfrak{L}}(\mathfrak{X},\mathfrak{Y},W)$ in (18) or (21)). However, it turns out that calculating these bounds is extremely complicated: changing the input prices not only influences the actual cost, but also the minimum cost for producing the given output (i.e., the cost function value). Given these computational problems, we leave the development of an efficient algorithm for calculating the firm-specific upper and lower bounds in the case of a non-unique industry optimum as an interesting topic for follow-up research.

4.2 Sensitivity analysis

A related issue concerns the sensitivity of firm level efficiency indices to the LoOP condition. The motivation is that the individual firms may be allocatively inefficient, and hence the shadow prices may deviate from the true economic prices that the firms face. The key idea behind our empirical cost efficiency estimates is that these inefficiencies should cancel out at the level of the industry. If allocative efficiency is viewed as purely irrational behavior, there is no reason to expect allocative inefficiencies to be systematically biased in any direction. Therefore, the LoOP constraints should provide a meaningful approximation at the level of the industry. However, errors that are small at the level of the industry can be very significant at the level of the individual firms. Therefore, we might want to test whether minor departures from the LoOP condition make a difference at the firm level. To this end, we consider two approaches for "weakening" the strict LoOP condition adopted above.

The first approach is a two-stage procedure that applies in the Top-Down as well as the Bottom-Up cases. In the first stage we estimate the optimal shadow prices. In the second stage we calculate the firm-specific efficiency indices; we start from the relative prices obtained in the first stage, but allow for deviations of the firm-specific input prices from the common (first stage) shadow prices. In other words, we first solve either (18) or (21) to obtain shadow prices w*; subsequently, we calculate firm-specific efficiency indices by solving either (6) or (8) incorporating the price domain W specified as

$$(26) \qquad W = \left\{ w_n \in \mathbb{R}_+^R \left| \underline{\alpha}_{nqr} \frac{w_{nr}^*}{w_{nq}^*} \leq \frac{w_{nr}}{w_{nq}} \leq \overline{\alpha}_{nqr} \frac{w_{nr}^*}{w_{nq}^*} \ \forall q,r \in \rho \right. \right\},$$

with the parameters $(\underline{\alpha}_{nqr},\overline{\alpha}_{nqr})$ defining the minimum and maximum factors by which the relative price of inputs q and r of the firm n can deviate from the industry-level relative shadow prices (w*). Firm-specific and/or input-specific factors may be used, or the same factor (e.g. $\underline{\alpha}_{nqr} = 1/\overline{\alpha}_{nqr} = 0.95$) may be applied across all firms and inputs. Of course, the specification of the values $(\underline{\alpha}_{nqr},\overline{\alpha}_{nqr})$ is particular to the application setting under study.

From the sensitivity analysis perspective, it may be worthwhile to examine the robustness of efficiency indices with respect to different values for $(\underline{\alpha}_{nqr}, \overline{\alpha}_{nqr})$.

The second approach is to model the weak LoOP bounds directly in the cost efficiency problem. This approach only applies in the Bottom-Up models. Specifically, the weaker LoOP condition can be implemented by rewriting problem (21) in the form

$$
\begin{aligned}
&ICE^{\mathfrak{L}}(\mathfrak{X},\mathfrak{Y},w) = \max_{\gamma, p_{mn}, w_n, f_{mn}} \sum_{n=1}^{N} \gamma_n \\
&s.t. \\
&\gamma_n \leq \sum_{s=1}^{S} Y_{ns} p_{mns} + f_{mn} \quad \forall m,n \in \nu \\
(27) \quad &\sum_{s=1}^{S} Y_{ms} p_{mns} - \sum_{r=1}^{R} X_{mr} w_{nr} + f_{mn} \leq 0 \quad \forall m,n \in \nu \\
&\sum_{r=1}^{R} \mathfrak{X}_r w_{nr} = 1 \quad \forall n \in \nu \\
&\underline{\beta}_{mnr} w_{mr} \leq w_{nr} \leq \overline{\beta}_{mnr} w_{mr} \quad \forall m,n \in \nu; r \in \rho \\
&p_{mn} \in \mathbb{R}_{+}^{s} \quad \forall m,n \in \nu \\
&w_n \in W \quad \forall n \in \nu
\end{aligned}
$$

where the parameters $(\underline{\beta}_{mnr}, \overline{\beta}_{mnr})$ define the minimum and maximum factors by which the price of input *r* is allowed to deviate between firms *m* and *n*. The interpretation of these parameters is directly analogous to that of $(\underline{\alpha}_{nqr}, \overline{\alpha}_{nqr})$ above. Notice that in the special case where $\underline{\beta}_{mnr} = \overline{\beta}_{mnr} \; \forall m,n,r$ problem (27) reduces to (21). On the other hand, in the limiting case where $\underline{\beta}_{mnr} = 0, \overline{\beta}_{mnr} \to \infty \; \forall m,n,r$, problem (27) simplifies to problem (8).

The previous two approaches cover the entire continuum between the non-LoOP DEA/FDH specification (8) on the one hand and the strict LoOP specifications (18) and (21) on the other. Therefore, the sensitivity analysis of the LoOP condition can be based on the comparison of efficiency indices calculated with different values for the parameters $(\underline{\alpha}_{nqr}, \overline{\alpha}_{nqr})$ or $(\underline{\beta}_{mnr}, \overline{\beta}_{mnr})$ parameters. Allowing for some limited firm-specific deviations from the LoOP conditions might improve robustness of the firm-specific efficiency indices with respect to data errors and possible asymmetries in terms of allocative inefficiency. Note, however, that the firm-specific efficiency indices can be consistently aggregated to the industry level only when the LoOP is interpreted in the strict sense as in (18) and (21).

5. Application: research efficiency in Dutch universities

5.1 Data and setting

We apply the presented methodology for examining the productive efficiency of research in Economics and Business Management faculties of Dutch universities. Specifically, we evaluate the efficiency of 79 research programs organized at 8 universities. The same data set was studied by Cherchye and Vanden Abeele (2002), who motivate efficiency assessment within this setting by the argument that efficient research production is not guaranteed by the usual market correction mechanisms. These authors further claim that the cost efficiency model discussed above seems particularly appropriate within this application context.

Furthermore, it is reasonable to invoke LoOP as the price condition in this application. In the Netherlands, salaries of the university staff are centrally negotiated by unions and, hence, should generally be the same for all universities. However, differences in overheads, which are not directly accounted for in our input-output selection, and the fact that the exact salary values depend on other factors that are not explicitly included in the evaluation model (such as age/experience), make it nearly impossible to determine the exact "market prices" unambiguously from the outset. In our opinion, all this makes academic research assessment a suitable application area for illustrating the proposed LoOP-techniques. Moreover, it is interesting to compare the LoOP results with the 'non-LoOP'-results of Cherchye and Vanden Abeele.

Generally, a research program can be defined a "a group of researchers who join forces to investigate a particular theme, and in the process to educate researchers and to publish research results". Cherchye and Vanden Abeele argue that this definition institutes research programs as the natural production units for studying academic research efficiency. Building on that definition, they suggest the following input-output selection for characterizing the production of each program: [8]

Inputs: (**1**) junior research staff (=PhD candidates), (**2**) senior research staff (= other research personnel).

Outputs: (**1**) total number of doctoral dissertations, (**2**) total number of refereed articles in top international journals, and (**3**) total number of refereed articles in other international journals.

The input and output data are taken from the 'Quality Assessment Reports on Research 1996-2000', delivered by each Dutch university in the context of the quinquennial assessment by the VSNU (i.e., the Dutch association of universities). For each research program we have data for the years 1998, 1999 and 2000. Pooling the three cross-sections in the same sample, we have 237 (79 x 3) (two zero output cases excluded) observations in total. For further details about the data and the input-output selection, we refer to Cherchye and Vanden Abeele (2002).

We model the production technology as follows. Since larger research programs are generally expected to benefit of economies of scale, we take the non-convex FDH technology as the starting point. Since we focus on cost efficiency, convexity or non-convexity of input sets does not make any difference whatsoever (see Kuosmanen, 2003, Theorem 3.3). To enhance the discriminatory power of our efficiency measures, we impose further restrictions on the

[8] A careful reader will notice that our specification of output variables differs from that used by Cherchye and Vanden Abeele (CA: 2002). CA substitute output 3 by the sum of outputs 2 and 3 (i.e., total number of articles) in their efficiency assessment. This way, CA implicitly constrain the shadow price of output 2 (articles in top journals) to be greater than or equal to the price of output 3 (observe that $p_2 y_2 + p_3(y_2 + y_3)$ $= (p_2 + p_3)y_2 + p_3 y_3$, so in the treatment of CA, price p_2 can be interpreted as the extra premium for the top-journal articles, which is added to the normal article price p_3). In the present paper, the same constraint is modeled explicitly, since or LP formulation of the FDH technology allows for implementing additional constraints on the relative prices. The reference technology is exactly the same in both papers.

relative input prices and the output multipliers, permitted by our novel LP formulation of the nonconvex FDH technology (see section 2 and the Appendix). Specifically, for input prices we postulate that the first input (junior staff) cannot be more expensive than the second input (senior staff). Similarly, for output weights we postulate that the shadow price of output 2 (articles in top journals) must be greater than or equal to the weight of output 3 (articles in other journals) (i.e., articles in top journals are *at least* as good as articles in other journals).

5.2 Results for the basic FDH LoOP model

Table 1 provides a number of summary statistics for the LoOP and non-LoOP results. The industry efficiency for the LoOP model corresponds to the weighted average of the observation-specific efficiency values, which is computed on the basis of (14) and (15). As for the non-LoOP results, we use equal weights for aggregating the different research programs, given that there is no consistent weighting scheme for aggregating the firm-specific non-LoOP efficiency indices to the industry efficiency. A first observation from the table is that, when comparing the LoOP and non-LoOP efficiency values, we observe that efficiency values decrease when imposing additional LoOP structure on the endogenously defined prices. Recall that the LoOP model differs from the non-LoOP model only in that it imposes an additional constrain on input prices. Still, the industry-level efficiency does not decrease very drastically. We further find that also the coefficient of variation is fairly similar in both cases. This leads to a first (tentative) conclusion that imposing the LoOP conditions in this particular setting does not substantially alter the distribution of efficiencies over the different production units; in other words, the computed efficiency values are fairly robust with respect to the specific restrictions on the distribution of shadow prices across observations.

By contrast, the computed input shadow prices are heavily influenced by the LoOP restrictions. Specifically, we obtain a very unbalanced picture in the non-LoOP case: the relative shadow prices of junior staff (see ‘rel. price PhD’) and senior staff (see ‘rel price other’) respectively equal (on average) approximately 0% and 100%; non-surprisingly, these extreme values are associated with huge coefficients of variation over the different evaluated research programs. By contrast, we get a much more balanced 40% (junior staff) versus 60% (senior staff) shadow price structure in the LoOP case (where we impose equality of shadow across all research programs under evaluation). In our opinion, these LoOP prices have a much more “reasonable” interpretation. More generally, we believe that imposing LoOP conditions may often generate more realistic estimates for the shadow prices. This is an important point to make, as the estimation and interpretation of these shadow prices often constitutes a core issue in practical DEA applications; in such instances the presented LoOP methodology may provide substantial value-added.

Table 1: Summary statistics for cost efficiency with and without LoOP

	Industry efficiency	**Coefficient of variation**	**Relative price Junior staff**	**Relative price Senior staff**
LoOP	0.567	0.568	0.402	0.598
Non-LoOP	0.623	0.510	0.002 (42.789)	0.998 (144.203)

Note: between brackets are coefficients of variation associated with the reported average values

Next, we compare the qualitative conclusions with respect to efficiency differences across universities (Table 2), considering both the LoOP and the non-LoOP efficiency indices. For each university, Table 2 reports the number of observations (see ‘# observations’) and the ‘university efficiency’: for the LoOP results we use the Färe-Zelenyuk aggregation described in Section 3.1, for non-LoOP results we report the unweighted averages. The column ‘p-value’ reports the (two-sided) probability value for the hypothesis that the aggregate efficiency of that university equals the aggregate efficiency of all other universities. Assuming the deviations of the program efficiencies from the overall university efficiency are normally distributed, we use the t-tests of sample means for examining systematic efficiency differences across universities. The use of t-tests allows for exploiting the weighting structure of the industry efficiency in the LoOP case (see (14) and (15)); for the LoOP results we use weighted t-tests. Given that consistent aggregation of the non-LoOP results is impossible, we use the standard unweighted t-tests of sample means in that case.[9]

The numerical values in Tables 2 differ slightly when comparing the LoOP and non-LoOP results. As can be expected, university efficiency tends to be higher in the latter case than in the former case; only universities of Tilburg and Nijmegen form exceptions. Still, although some variation is observed, we generally obtain the same qualitative results in the two exercises. As for differences over universities, we specifically find that Tilburg University is performing well; Wageningen University is no longer identified as generally outperforming the (average of the) remaining universities when LoOP constraints are taken up in the efficiency evaluation. On the other hand, the University of Amsterdam and Maastricht University are performing relatively poorly in terms of both the LoOP and the non-LoOP results.

Table 2: Efficiency differences across universities

		LoOP		**non-LoOP**	
	# obs.	**University Efficiency**	**p-value**	**Average Efficiency**	**p-value**
Tilburg Univ.	27	**0.806**	**0.000**	**0.788**	**0.005**
Wageningen Univ.	21	0.616	0.286	**0.752**	**0.057**
Free Univ. Amsterdam	36	0.596	0.656	0.643	0.746
Univ. Groningen	18	0.560	0.872	0.584	0.548
Erasmus Univ. Rotterdam	59	0.552	0.703	0.652	0.473
Univ. Maastricht	27	**0.478**	**0.100**	**0.515**	**0.050**
Univ. Nijmegen	6	0.465	0.339	0.449	0.162
Univ. Amsterdam	41	**0.365**	**0.000**	**0.525**	**0.022**

We next applied the same procedure to examine the differences across fields of specialization (Table 3). We find good performances in the fields Econometrics and Theoretical and Applied Microeconomics. On the other hand, research programs in the field of Applied Labor Economics appear to perform relatively badly in comparison to the other programs. When imposing LoOP conditions, also the research in the Economic of Public Policy does not seem to “pay off” to the same extent as the research in other domains. (We refer to Cherchye and Vanden Abeele for further discussion and various interpretations of these findings.)

[9] Cherchye and Vanden Abeele (2002) use Wilcoxon rank signed tests for identifying such differences. Their qualitative conclusions are generally the same as those obtained from the t-tests. These authors also investigate efficiency changes over time; comparing their non-LoOP results to our LoOP results leads to the same qualitative conclusions as the comparisons based on Tables 2 and 3. Finally, Cherchye and Vanden Abeele consider combinations of the two categorizations that are considered in Tables 2 and 3; e.g., they examine the relative efficiency of research programs (i) within a specific specialization unit that (ii) are organized at one particular university. To keep our discussion focussed, we abstract from such (admittedly interesting) study in this paper, where the application illustration mainly serves to illustrate the proposed methodology.

Table 3: Efficiency differences across field of specialization

		LoOP		non-LoOP	
	# obs.	**Field Efficiency**	**p-value**	**Average Efficiency**	**p-value**
Theoretical and Applied Microeconomics	21	**0.832**	**0.000**	**0.746**	**0.069**
Econometrics	15	**0.810**	**0.014**	**0.817**	**0.015**
Spatial and Environmental Economics	18	0.689	0.130	0.741	0.109
Macroeconomics, Money and International Issues	18	0.659	0.161	0.639	0.863
Development, Growth and Transition	15	0.590	0.752	0.604	0.771
Applied Mathematics	18	0.536	0.695	0.644	0.814
Marketing and Business Economics	66	0.534	0.273	0.601	0.433
Accounting and Finance	36	0.500	0.203	0.570	0.245
Applied Labor Economics	13	**0.388**	**0.054**	**0.434**	**0.023**
Economics of Public Policy	9	**0.324**	**0.042**	0.500	0.219

The close similarity of the LoOP and non-LoOP results additionally supports our earlier (tentative) conclusion: in this specific application, the distribution of the computed efficiency values is fairly robust with respect to the specific restrictions on the distribution of shadow prices across production units. (From our above discussion: the sole exceptions concern the different qualitative conclusions regarding the research at Wageningen University and the research programs operating in the area of Economics of Public Policy.) Generally, it seems interesting to investigate such robustness. The next section provides a more detailed discussion of how the proposed methodology can be used for the purposes of sensitivity analysis.

5.3 Uniqueness and sensitivity analysis

We next complement the analysis by investigating uniqueness of the efficiency of the optimal shadow prices and the sensitivity of the computed efficiency values with respect to the LoOP condition and the hypothesized production properties. Firstly, we tested for uniqueness of the industry-level optimal solution using the algorithm described in Section 4.1, using a tolerance value of 0.0001 for ε in (24) and (25). Since our application includes only two inputs, we only need to solve two LP problems. In both these LP problems the optimal solution (i.e., the newly computed industry efficiency value) lies below the original solution (that does not meet the restrictions (24) and (25)). Hence, we cannot reject the null hypothesis of unique input prices: forcing the relative shadow prices to deviate minimally from the original shadow prices entails a lower industry efficiency value.

We next study the sensitivity of the efficiency results with respect to the strong nature of the LoOP condition *stricto sensu*. As discussed before, there may indeed be small variations in input prices across university departments due to differences in overheads; in addition, salary deviations may follow from age, experience and competence differences between staff members. While we do not expect the differences to be very large, these considerations do call for some analysis of the sensitivity of the obtained efficiency values with respect to relaxations of the LoOP condition.

Table 4 presents a number of summarizing statistics for alternative formulations of the LoOP condition; we specifically consider the original LoOP constraint and some weaker alternatives that respectively allow (for each individually evaluated observation) for 5%, 10% and 20% deviations from the original LoOP relative prices (compare with the specification of $(\underline{\alpha}_{nqr}, \overline{\alpha}_{nqr})$ in Section 4.2 above).[10] Generally, we find that the FDH results are fairly robust with respect to the specific LoOP formulation, both in terms of the efficiency values that are obtained and in terms of the shadow price values: the average values hardly change; also, the coefficients of variation for the relative prices remain of fairly similar magnitude even when we allow for deviations of 20% from the unique LoOP prices.

The table contains some further information on the computational burden of the alternative efficiency assessments. While we generally find that the computational burden increases when tolerating more weight flexibility, the programming problems do remain easily tractable; the maximum execution time does not exceed 40 seconds. Generally, we may safely conclude that the (seemingly complex) linear programming problems associated with alternative LoOP specifications should not pose insurmountable computational problems for the PC configurations that have nowadays become standard. This argument seems all the more valid when considering that our sample consists of 237 research programs; this size exceeds that of most samples that are more commonly subject to DEA evaluation.

Table 4: LoOP efficiency results; summary statistics for alternative price conditions

	Industry efficiency	**Coefficient of variation**	**Relative price Junior staff**	**Relative price Senior staff**	**Execution time (sec)**	**# of iterations**
0% dev.	0.567	0.568	0.402 (0.000)	0.598 (0.000)	14.711	15700
5% dev.	0.559	0.566	0.401 (0.031)	0.599 (0.021)	18.977	22100
10% dev.	0.562	0.564	0.401 (0.057)	0.599 (0.068)	17.034	23200
20% dev.	0.568	0. 563	0.400 (0.108)	0.600 (0.072)	37.956	24600

Note: (i) between brackets are coefficients of variation associated with the reported average values. (ii) execution time ("exec. time"), expressed in seconds, and the number of iterations ("# iterations") refer to a PC with a Pentium 4 processor (2 GHz CPU, 256 Mb RAM).

Our above discussion restricts to the basic case of the FDH technology with variable returns-to-scale As a final exercise, we examine the impact of additional assumptions about the production technology on the computed industry and firm level cost efficiency measures. The additional assumptions of convexity and/or returns-to-scale were implemented by imposing constraints (7) in the Top-Down model (17) or constraints (21) in the Bottom-Up model (20). Table 5 presents summary statistics for a selection of alternative specifications of the production technology: FDH with and without the assumption of constant returns-to-scale (respectively referred to as FDH and FDH-crs) and the convexified counterpart of FDH with and without the same returns-to-scale assumption (respectively referred to as conv and conv-crs). For each alternative, we restrict attention to the strict LoOP version (i.e., we impose equality of shadow prices across all research programs).

The first column in Table 5 reports the industry efficiencies. The second column reports the coefficients of variation in the firm-level efficiency values. Generally, we find that the model specification has a strong impact on the efficiency measures, both at the industry level and at the firm level. Also the estimation of the relative prices varies substantially according to the

[10] Like for the non-LoOP results in Table 1, we use the equally weighted average for computing the industry efficiency under a relaxed formulation of the LoOP condition. Indeed, also in these cases the aggregation *à la* expressions (14) and (15) does not apply; see also our discussion in Section 4.2.

model specification; especially the shadow prices under conv-crs differ considerably from those under the other configurations. Generally, these findings seem to call for a careful inspection of the production assumptions (like convexity and constant returns-to-scale) prior to the actual efficiency analysis. In this respect, we believe that the non-parametric perspective advocates imposing production properties only when they are convincingly verified. Such practice minimizes the risk of specification error, which in our opinion constitutes the basic conceptual advantage of the nonparametric approach *vis-à-vis* its parametric counterpart.

Like before, Table 5 includes figures on the computational burden of the different efficiency assessments. Generally, we observe that the computational burden decreased substantially when using a convex reference production set. Imposing constant returns-to-scale increased computational burden in the FDH specification, while it decreased the burden in the convex DEA specification. But again, the execution time remains rather small in each variant. Based on these findings, we may safely argue that the computational effort should not be considered as a pivotal decision criterion when choosing amongst alternative model specifications.

Table 5: LoOP efficiency results; summary statistics for alternative technology assumptions

	Industry efficiency	Coefficient of variation	Relative price Junior staff	Relative price Senior staff	Execution time (sec)	# of iterations
FDH	0.567	0.568	0.402	0.598	14.711	15700
FDH-crs	0.419	0.638	0.267	0.733	21.180	20300
conv	0.358	0.707	0.311	0.698	5.878	1500
conv-crs	0.300	0.658	0.090	0.910	4.085	1200

Note: execution time (“exec. time”), expressed in seconds, and the number of iterations (“# iterations”) refer to a PC with a Pentium 4 processor (2 GHz CPU, 256 Mb RAM).

6. Summary and concluding discussion

In this paper we have argued that prevalence of The Law of One Price (LoOP), taken as a normative property, could be a powerful efficiency criterion in the context of productive efficiency analysis in general and DEA in particular. LoOP emphasizes the coordinating function of prices in the allocation of resources, and hence naturally relates the efficiency of individual firms to the efficiency at the aggregate level of the industry or sector as a whole. Specifically, the LoOP condition modeled in this paper guarantees consistency of the firm level and the industry level efficiency scores. The LoOP constraints restrict shadow price flexibility *across* firms, whereas all exiting tools restrict multiplier weights *within* the evaluation of a single firm. We believe that this new approach of restricting weight flexibility, when properly applied, can considerably enhance the discriminatory power of DEA; its particular strength can often be motivated by meaningful and intuitive economic rationale.

To demonstrate the potential of the LoOP condition in the context of DEA, we considered the standard case of cost efficiency analysis under incomplete price information. Utilizing the results of Färe and Zelenyuk (2003) on the aggregation of efficiency indices, our approach was to apply input prices that maximize the industry level cost efficiency within the firm level cost efficiency analysis. Specifically, we considered two dinstinct approaches to integrate the firm level and the industry level perspectives within the same DEA framework: the (less sophisticated) Top-Down approach, which requires constant returns-to-scale, and the (more sophisticated) Bottom-Up approach, which also applies under variable returns to scale. (The two approaches are equivalent under constant returns-to-scale.) In addition, we have introduced operational tools for investigating sensitivity of the firm-specific efficiency results with respect to the LoOP condition *stricto sensu*.

Two points of attention deserve special mentioning. Firstly, it is well known that the optimal 'shadow' prices produced by DEA need not be unique. Therefore, it is generally interesting to investigate the uniqueness of the optimal LoOP-consistent shadow prices; non-unique shadow prices may affect the firm-level efficiency estimates. In the current paper, we have proposed an easily implemented procedure to test such uniqueness. We suggest the examination of the robustness of firm-specific efficiencies in case of non-unique LoOP-prices as an interesting avenue for follow-up research; e.g., it may be possible to determine upper and lower bounds for the firm-specific efficiencies in such a case. Secondly, the more general bottom-up approach for implementing the LoOP in the efficiency assessment requires solving the efficiency scores simultaneously for all firms in the sample, which increases the computational burden as compared to the standard DEA models. In our experience, this extra comptational burden is of marginal importance in the usual applications; it becomes significant only in large data sets when combined with other computationally intensive techniques such as bootstrapping (e.g., Simar and Wilson, 1998).

To illustrate the practical aspects of our approach, we applied the methodology for evaluating the research efficiency of economic research programs organized at Dutch universities; we compared our LoOP results with the earlier non-LoOP results (based on the same data) obtained by Cherchye and Vanden Abeele (2002). We demonstrated that the presented techniques can be useful for investigating robustness with respect to LoOP conditions of (i) the efficiency rankings and (ii) the calculated shadow prices. In addition, we have illustrated our test for uniqueness of the LoOP-consistent shadow prices. Finally, we have studied sensitivity of the efficiency results with respect to (i) different degrees of price flexibility across production units and (ii) alternative models of the production technology.

Our results show that, for this particular application, the qualitative conclusions regarding the efficiency rankings are not very sensitive with respect to the LoOP conditions. By contrast, the calculated shadow prices are heavily affected by these conditions; generally, we believe that utilizing the LoOP may often generate more realistic estimates of the true opportunity costs faced within the production processes under evaluation. (Interestingly, we could not reject uniqueness of the optimal shadow prices in this application.) Further, we find that the efficiency results are fairly robust with respect to alternative specifications of (limited) LoOP weight flexibility. By contrast, they are heavily sensitive to the imposed technological properties; in particular constant returns-to-scale and convexity assumptions seem to have a substantial impact on the obtained efficiency results (both in terms of average efficiencies and estimated shadow prices). In our opinion, this pleads for not imposing such assumptions if they are not convincingly verified.

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APPENDIX: an LP formulation for the free disposable hull (FDH) reference technology

Since the LP relaxation of the cost efficiency measures based on the non-convex FDH technology is a new contribution of this paper, we find it useful to include an appendix that offers further insights by deriving the dual problem of (6) in a step-by-step fashion. For clarity, let us first re-write problem (6) in the form:

$$
\text{(i)} \qquad
\begin{aligned}
& \max_{\gamma,p,w,f} \gamma \\
& s.t. \\
& \gamma \le \sum_{s=1}^{S} Y_{ks} p_{ms} + f_m \quad \forall m \in v \\
& \sum_{r=1}^{R} X_{kr} w_r \le 1 \\
& \sum_{s=1}^{S} Y_{ms} p_{ms} - \sum_{r=1}^{R} X_{mr} w_r + f_m \le 0 \quad \forall m \in v \\
& p_m \in \mathbb{R}_+^S \quad \forall m \in v \\
& w \in \mathbb{R}_+^R \\
& \gamma, f_m \text{ free} \quad \forall m \in v
\end{aligned}
$$

The fact that we model γ as a free variable has no effect on the optimal solution of (i), but it will make the interpretation of the dual problem more transparent below.

First, recall that every (primal) LP problem has an equivalent dual problem. The primal/dual pair is generally represented as

$$
\text{(ii)} \qquad
\begin{array}{lcl}
\textbf{PRIMAL} & & \textbf{DUAL} \\
\max_{\mathbf{x}} \mathbf{c}'\mathbf{x} & & \min_{\mathbf{y}} \mathbf{b}'\mathbf{y} \\
\mathbf{Ax} + \mathbf{s} = \mathbf{b} & & \mathbf{A}^T\mathbf{y} - \mathbf{t} = \mathbf{c} \\
x_j \begin{cases} \ge 0 \\ free \\ \le 0 \end{cases} & \Leftrightarrow & t_j \begin{cases} \ge 0 \\ \equiv 0 \\ \le 0 \end{cases} \\
s_j \begin{cases} \ge 0 \\ \equiv 0 \\ \le 0 \end{cases} & & y_j \begin{cases} \ge 0 \\ free \\ \le 0 \end{cases}
\end{array}
$$

where **x** and **y** are the unknown model variables (not inputs and outputs!), vectors **b**, **c**, and matrix **A** include constant model parameters, and vectors **s** and **t** are slack variables. When re-expressing problem (i) in the matrix form (ii), we get the following: (1) the vector of unknowns **x** becomes $\mathbf{x} = (\gamma \quad \mathbf{p} \quad \mathbf{w} \quad \mathbf{f})'$ (a column vector with dimensions $(1+NS+R+N)\times 1$), which includes sub-vectors $\mathbf{p} = ((p_{11} \quad p_{12} \quad \cdots \quad p_{1S})$ $(p_{21} \quad p_{22} \quad \cdots \quad p_{2S}) \cdots (p_{N1} \quad p_{N2} \quad \cdots \quad p_{NS}))'$ (with dimensions $NS\times 1$),

$\mathbf{w} = \begin{pmatrix} w_1 & w_2 & \cdots & w_S \end{pmatrix}'$ (with dimensions $R \times 1$), and $\mathbf{f} = \begin{pmatrix} f_1 & f_2 & \cdots & f_N \end{pmatrix}'$ (with dimensions $N \times 1$); (2) vector **c** is $\mathbf{c} = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}'$ (with dimensions $(1+NS+R+2N)\times 1$); (3) matrix **A** becomes

$$\mathbf{A} = \left(\begin{array}{c|cccc|c|cccc}
1 & -\mathbf{Y}_k' & \mathbf{0}' & \cdots & \mathbf{0}' & \mathbf{0}' & -1 & 0 & \cdots & 0 \\
1 & \mathbf{0}' & -\mathbf{Y}_k' & \cdots & \mathbf{0}' & \mathbf{0}' & 0 & -1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \mathbf{0}' & \mathbf{0}' & \cdots & -\mathbf{Y}_k' & \mathbf{0}' & 0 & 0 & \cdots & -1 \\
\hline
0 & \mathbf{0}' & \mathbf{0}' & \cdots & \mathbf{0}' & \mathbf{X}_k' & 0 & 0 & \cdots & 0 \\
\hline
0 & \mathbf{Y}_1' & \mathbf{0}' & \cdots & \mathbf{0}' & -\mathbf{X}_1' & 1 & 0 & \cdots & 0 \\
0 & \mathbf{0}' & \mathbf{Y}_2' & \cdots & \mathbf{0}' & -\mathbf{X}_2' & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \mathbf{0}' & \mathbf{0}' & \cdots & \mathbf{Y}_N' & -\mathbf{X}_N' & 0 & 0 & \cdots & 1
\end{array}\right)$$

(with dimensions $(N+1+N)\times(1+NS+R+N)$) (note that all bold elements should be interpreted as row vectors with dimensions $1\times S$ for columns 2-5 (outputs) and $1\times R$ for column 6 (input)); (4) the slack vector **s** has dimensions $(N+1+N)\times 1$, with all elements non-negative; and (5) the vector **b** consists of N zeros, one 1, and again N zeros, i.e., $\mathbf{b} = \begin{pmatrix} 0 & 0 & \cdots & 0 \mid 1 \mid 0 & 0 & \cdots & 0 \end{pmatrix}'$ (with dimensions $(N+1+N)\times 1$).

Next, if we write the dual of (i) in matrix form then the vectors **c** and **b** remain unchanged. In addition, the transpose $\mathbf{A}^T$ becomes

$$\mathbf{A}^T = \left(\begin{array}{cccc|c|cccc}
1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\
\hline
-\mathbf{Y}_k & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{Y}_1 & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{0} & -\mathbf{Y}_k & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{Y}_2 & \cdots & \mathbf{0} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & -\mathbf{Y}_k & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{Y}_N \\
\hline
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{X}_k & -\mathbf{X}_k & -\mathbf{X}_k & \cdots & -\mathbf{X}_N \\
\hline
-1 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 \\
0 & -1 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & -1 & 0 & 0 & 0 & \cdots & 1
\end{array}\right)$$

(with dimensions $(1+NS+R+N)\times(N+1+N)$) (the bold elements should now be interpreted as column vectors with dimensions $S\times 1$ for rows 2-5 (outputs) and $R\times 1$ for row 6 (input)). The dual model variables of vector **y** can be interpreted as

$\mathbf{y} = \begin{pmatrix} \kappa_1 & \kappa_2 & \cdots & \kappa_N \mid \theta \mid \lambda_1 & \lambda_2 & \cdots & \lambda_N \end{pmatrix}'$

(with dimensions $(N+1+N)\times 1$). Since the slacks **s** in the primal problem are non-negative, the same holds for the corresponding **y** variables in the dual problem. The slack vector **t** has the dimensions $(1+NS+R+N)\times 1$ with the following signs $\begin{pmatrix}0 & + & + & 0\end{pmatrix}'$. We can consequently re-write the dual LP in sum notation as

$$\text{(iv)}\quad \begin{aligned} &\min_{\gamma,p,w,f} \theta \\ &s.t. \\ &\sum_{n=1}^{N} \kappa_n = 1 \\ &Y_{ns}\lambda_n \geq Y_{ks}\kappa_n \quad \forall n \in \nu, s \in \delta \\ &\theta X_{kr} \geq \sum_{n=1}^{N} X_{nr}\lambda_n \quad \forall r \in \rho \\ &\lambda_n = \kappa_n \geq 0 \ \forall n \in \nu \end{aligned} .$$

or, equivalently,

$$\text{(v)}\quad \begin{aligned} &\min_{\gamma,p,w,f} \theta \\ &s.t. \\ &Y_{ns}\lambda_n \geq Y_{ks}\lambda_n \quad \forall n \in \nu, s \in \delta \\ &\theta X_{kr} \geq \sum_{n=1}^{N} X_{nr}\lambda_n \quad \forall r \in \rho \\ &\sum_{n=1}^{N} \lambda_n = 1 \\ &\lambda_n \geq 0 \ \forall n \in \nu \end{aligned} .$$

This last formulation has an interpretation that comes close to the more conventional interpretation of the FDH model (that is most commonly expressed in dual form). Specifically, the first constraint guarantees that any firm *n* with positive weight λ_n must dominate the evaluated firm *k* in the outputs. On the input side, we allow for convex combinations (see the second constraint). This effectively provides an LP formulation of the Bogetoft (1996) convex input set model. Recall that convexity is a harmless property for measuring cost efficiency (see Kuosmanen, 2003, Theorem 3.3). Note also that the sum of intensity variables λ_n must equal unity, which has the standard interpretation of variable returns-to-scale.

To conclude, we discuss the alternative returns-to-scale assumptions. We cannot directly include such assumptions in (v); they are to be incorporated in the more general problem (iv). Let us first omit the "fixed cost" vector **f** in problem (i). In terms of the primal–dual conversion, this would exclude the constraint $\lambda_n = \kappa_n \ \forall n \in \nu$ in problem (ii). While the weights κ_n are constrained to sum up to unity, the weights λ_n allow for scaling the reference firms upwards or downwards as the sum of λ_n weights is unconstrained; this implements constant returns-to-scale (*CRS*), while imposing no convexity on the output set. Non-increasing returns-to-scale (*NIRS*) and non-increasing returns-to-scale (*NDRS*) are analogously modeled, viz. by imposing sign constraints on **f** in the primal problem (i) or including the inequalities $\lambda_n \leq \kappa_n \;\; \forall n \in \nu$ or $\lambda_n \geq \kappa_n \;\; \forall n \in \nu$ in the dual problem (iv).