Rationalizing Boundedly Rational Choice

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Abstract

A Rational Shortlist Method (RSM) translates into economic language some sequential choice heuristics studied in psychology. We show that the use of this boundedly rational procedure can be detected from observed choice data through tests that are very similar to those used to detect 'rational' choice (such as Samuelson's WARP). Yet, RSMs are compatible with some highly 'irrational' patterns of choice observed in experiments, such as pairwise cycles. We also provide partial results on a generalization of RSMs.

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1 Introduction

Cyclical choice is persistently observed in experimental evidence. It typically occurs in simple decision problems (involving only binary comparisons and few alternatives) and in significant proportions, sometimes nearing or even exceeding 50%.[1] This is obviously incompatible with the classical model of rational choice, in which choice is constructed as the maximizer of a single preference relation (which we call a *rationale*), or of a utility function. If a decision maker exhibits cycles of choice over some set of alternatives, for any candidate 'best' alternative there is always another one in the set which is judged better still: it is not possible to express his preferences by a utility function, since it is not possible to find a maximizer for it. In this paper we propose and study a family of boundedly rational choice procedures that can account for these observed anomalies.

A notable aspect of these procedures is that they are testable based on a 'revealed preference' type of analysis that, despite the highly non-standard choices to be explained, does not depart too much from the standard one[2]. In other words, we ask the following question: when are observed choices compatible with the use of our boundedly rational choice procedure? The answer is: if and only if the choice data satisfy two testable conditions. Of these conditions one is a standard Expansion axiom, and the other is a modification of Samuelson's Weak Axiom of Revealed Preference (WARP)[3]. We believe this is the first application of revealed preference analysis to infer boundedly rational procedures. The simplicity of our tests stands in contrast with the indirect estimation algorithms normally used (notably in the marketing literature) to infer boundedly rational procedures.[4]

In line with some prominent psychology and marketing studies (see below) in our model

[1] See e.g. Roelofsma and Read [18], Tversky [22] and Loomes *et al.* [13]. Roelofsma and Read [18] find that the majority (52%) of choices exhibited binary cycles in a universal choice set of four alternatives. In the experiment carried out in Loomes *et al.* [13] between 14% and 29% of choices made by all subjects were cyclical, and a staggering 64% of subjects exhibited at least one binary cycle in a universal choice set of just three alternatives. More recent results in this same line are in Blavatskyy [4], who finds that 55% of his experimental subjects violate transitivity of choice. Humans seem to fare better than non human animals: for instance, in an experiment of choice behavior on gray jays, Waite [25] finds that *all* the birds preferred choices a to b and b to c, but none preferred a over c, where all alternatives (n, l) consisted in going and getting n raisins at the end of a $l$cm long tube, with $a = (1 \text{ raisin}, 28 \text{ cm})$, $b = (2 \text{ raisins}, 42 \text{ cm})$ and $c = (3 \text{ raisins}, 56 \text{ cm})$. Thus none of the birds exhibited transitive choice; moreover, 25% of them exhibited consistently intransitive choice.

[2] See Varian [24] for a recent survey on standard revealed preference theory.

[3] Recall that the Weak Axiom of Revealed Preferences, in its general form, states that if an alternative a is chosen from some menu of alternatives when some other alternative b is present (i.e. a is directly revealed preferred to b), then it can never be the case that alternative b is selected from any other menu including both a and b.

[4] For recent examples see e.g. Kohli and Jedidi [12] and Yee et al. [26].

we assume that the decision maker uses sequentially two rationales to discriminate among the available alternatives. These rationales are applied in a fixed order, independently of the choice set, to remove inferior alternatives. This procedure 'sequentially rationalizes' a choice function if, for any feasible set, the process identifies the unique alternative specified by the choice function. In this case we say that a choice function is a *Rational Shortlist Methods* (RSM). Intuitively, the first rationale identifies a shortlist of candidate alternatives that the second rationale then selects among. The special case in which the first rationale always yields a unique maximal element corresponds to the standard model of rationality.

Typically, RSMs will lack standard menu-independence properties, so that it may be possible for an alternative to be revealed preferred to another alternative in some choice set, but for that preference to be reversed in a different choice set (thus violating WARP). Because of this feature, RSMs can exhibit cyclical patterns of choice; however, they still rule out other types of irrational choice. In this sense, an RSM is a non-vacuous notion and this gives it empirical content: it can be tested by observable choice data.

For a simple example of how an RSM works, suppose that an arbitrator has to pick one out of the available allocations a, b or c. Suppose that c Pareto dominates a, while no other Pareto comparisons are possible. Assume further that the arbitrator deems a fairer than b and b fairer than c. The arbitrator decides first on the basis of the Pareto criterion, invoking the fairness criterion only when Pareto is not decisive. Then the arbitrator's choice function γ would be such that $\gamma(\{a, b, c\}) = b$, since first a is eliminated by c using the Pareto criterion, and second c is eliminated by b using the fairness criterion. On the other hand, $\gamma(\{a, b\}) = a$, given that the Pareto criterion has no bite, and the arbitrator would select on the basis of fairness. Similarly, $\gamma(\{b, c\}) = b$, whereas $\gamma(\{a, c\}) = c$ by Pareto. This seems an entirely reasonable way for the arbitrator to come to a decision. In fact, this procedure has been proposed in a social choice setting by Tadenuma [21]. Yet it produces a violation of WARP and pairwise cyclical pattern of choice.

One can think of a wide array of other practical situations where RSMs may apply. A cautious investor comparing alternative portfolios first eliminates those that are too risky and then ranks the surviving ones on the basis of expected returns. A recruiting selector first excludes candidates without some desired skills and then selects based on merit from the remaining ones. The notion of RSM is relevant also in other fields in the social sciences. For instance, psychologists have often insisted on sequential 'non-compensatory'[5] heuristics, as opposed to one single rationale, to explain choices (though axiomatic characterizations of such boundedly rational procedures are lacking). Notable in this respect are Tversky's [23] Elimination by Aspects procedure and Gigerenzer and

[5] That is, in which the several 'criteria' used for choice cannot be traded off against each other.

Todd's (e.g.[10]) idea of 'fast and frugal heuristics'. Similarly, this type of model is widely used and documented in the management/marketing literature. Yee et al. [26] provide recent and compelling evidence of the use by consumers of "two stage consideration and choice" decision making procedures, and also refer to firms taking account of this fact in product development.

In summary, RSMs are simple boundedly rational procedures that are introspectively plausible and can explain empirically relevant 'anomalies' of choice patterns. Above all, whether or not the choice pattern of a decision maker can be explained by a RSM is a testable hypothesis. Last but not least, RSMs provide rigorous formal underpinnings to the heuristics approach central to much psychology and marketing literature.

In addition to providing a characterization of RSMs, we also consider a natural extension whereby the decision maker applies sequentially more than two rationales, much in the same way as they are used in the elimination procedure described before for RSMs. We call choice functions recoverable in this way *sequentially rationalizable*. Although a full characterization of sequentially rationalizable choice functions remains a non-trivial open problem, we are able to present some partial results. Interestingly, even when the number of rationales allowed is unboundedly large, not all choice functions are sequentially rationalizable.

The rest of the paper is organized as follows. In the next section we define and characterize RSMs. In section 3 we extend RSMs to sequential rationalizability. Some examples are provided in section 4. We conclude in section 5, while in the Appendices we report the proof of the main result and some technical examples.

2 Rational Shortlist Methods

2.1 Basic definitions

Let X be a set of alternatives, with $|X| > 2$. Given $S \subseteq X$ and an asymmetric binary relation $P \subseteq X \times X$, denote the set of P-maximal elements of S by

$$\max (S; P) = \{x \in S | \nexists y \in S \text{ for which } (y, x) \in P\}$$

Let $\mathcal{P}(X)$ denote the set of all nonempty subsets of X. A *choice function* on X selects one alternative from each possible element of $\mathcal{P}(X)$: so it is a function $\gamma : \mathcal{P}(X) \rightarrow X$ with $\gamma(S) \in S$ for all $S \in \mathcal{P}(X)$. We abuse notation by often suppressing set delimiters, e.g. writing $\gamma(xy)$ in place of $\gamma(\{x, y\})$.

The main characterization in this section goes through (as can be easily checked by an inspection of the proof) whether the choice sets S are finite or not. However for simplicity of notation, we confine ourselves to the case where X is finite.

Since Samuelson's [19] paper, economists have looked to express choice as the outcome of maximizing behavior. Formally, a choice function γ is *rationalizable* if there exists an acyclic binary relation P such that

$$\gamma(S) = \max(S; P) \text{ for all } S \in \mathcal{P}(X)$$

The main new concept we introduce is the following:

Definition 1 *A choice function γ is a **Rational Shortlist Method (RSM)** whenever there exists an ordered pair (P_1, P_2) of asymmetric relations, with $P_i \subseteq X \times X$ for $i = 1, 2$, such that:*

$$\{\gamma(S)\} = \max(\max(S; P_1); P_2) \text{ for all } S$$

In that case we say that (P_1, P_2) sequentially rationalize γ. We call each P_i a **rationale**.

So the choice from each S can be represented as if the decision maker went through two sequential rounds of elimination of alternatives. In the first round he retains only the elements which are maximal according to rationale P_1. In the second round, he retains only the element which is maximal according to rationale P_2: that is his choice. Note that, crucially, the rationales and the sequence are invariant with respect to the choice set.

2.2 An Example

To glean some intuition on what RSMs can and cannot do, let us consider an example where two types of 'pathologies of choice' are displayed. We show in the next section that the decomposition of pathologies illustrated in the example is very general; of these, only one can be accommodated by an RSM.

Suppose that the decision maker can conceivably choose between three alternative routes to go to work, A, B and C. Because of periodic road closures, we can observe his choices also between subsets of the grand set $\{A, B, C\}$. Up to a relabelling of the alternatives, it is not difficult to check that there are only three possible configurations of choice behavior. Fix the route which is taken when all are available, say route A. Then consider the situation when at any one time only two routes are available. Those that follow exhaust all choices possible[6]:

[6] Let XtY denote 'route X is taken when route Y is also available'. Then it is easy to see that, once we fix the route selected when all are available, there are eight possible combinations of routes chosen in each of the three possible pairwise comparisons between A and B, A and C and B and C, namely: (1) AtB, AtC and BtC; (2) AtB, AtC and CtB; (3) AtB, BtC and CtA; (4) BtA, AtC and CtB; (5) BtA, BtC, AtC; (6) BtA, BtC, CtA; (7) CtA, CtB, AtB; and (8) CtA, CtB, BtA. Of these possibilities, (1)

Case 1 (Dominance of the best route): Route A (the choice form the grand set) is also taken whenever only one other route is available, regardless of the choice when A is not available.

Case 2 (Pairwise cycle of choice): Route A is taken when B is the only other available route; route B is taken when C is the only other available route; route C is taken when A is the only other available route.

Case 3 (Always chosen): Some route different than A is always taken when only one other route is available, regardless of the choice when A is available.

These cases are depicted in figure 1, where arrows point away from the selected route to the unselected one in pairwise choice.

Case 1 can be rationalized in the standard way, with only one preference relation such that A is preferred to both B and C .

Case 2 is pathological from the point of view of standard economic rationality. Nonetheless it can be sequentially rationalized by two rationales, let us call them 'traffic' and 'length', as follows. The decision maker prefers less traffic to more, and prefers shorter routes. Route A and C have the same level of traffic, but route C is shorter. Route A has less traffic than route B, and route B has less traffic than route C. The decision maker looks first at traffic to eliminate routes, and then at length. It is immediate to see that the criteria applied in the given sequence generate the choice behavior of Case 2.[7]

In case 3, a different pathology of choice is observed. There is one route, say B to fix ideas, that is revealed preferred in pairwise choices to all other routes, yet it is not chosen when all routes are available. This pattern of choice is not an RSM. To see this, suppose to the contrary that this were an RSM, again with rationales 'traffic' and 'length' applied in that order. If so, the fact that B is chosen in pairwise comparison over A means that if B and A are comparable by traffic, then B has less traffic than A. Otherwise, B must be shorter than A. Similarly, since B is chosen in pairwise comparison over C, either B has less traffic than C, or is shorter (or both). But then, when all three routes are available, B can never be eliminated by either the traffic or the length criterion. This contradicts

and (2) correspond to case 1 in the text; (3) and (4) are the same subject to relabelling by switching B and C, and correspond to case 2 in the main text; finally both (5) and (7), and (6) and (8) are the same subject to swapping B for C, and correspond to Case 3 in the main text.

[7] An alternative possibility for rationalization is identical to the one just described, except that routes A and B have the same level of traffic but A is shorter than B. Again, it is easy to check that these criteria also generate the stated choices. This serves to illustrate the fact that even in simple situations such as this one, an external observer may have more than one way to explain choices as an RSM. This stands in contrast to standard rationalization, where the choice of a preference relation explaining choices is uniquely determined.

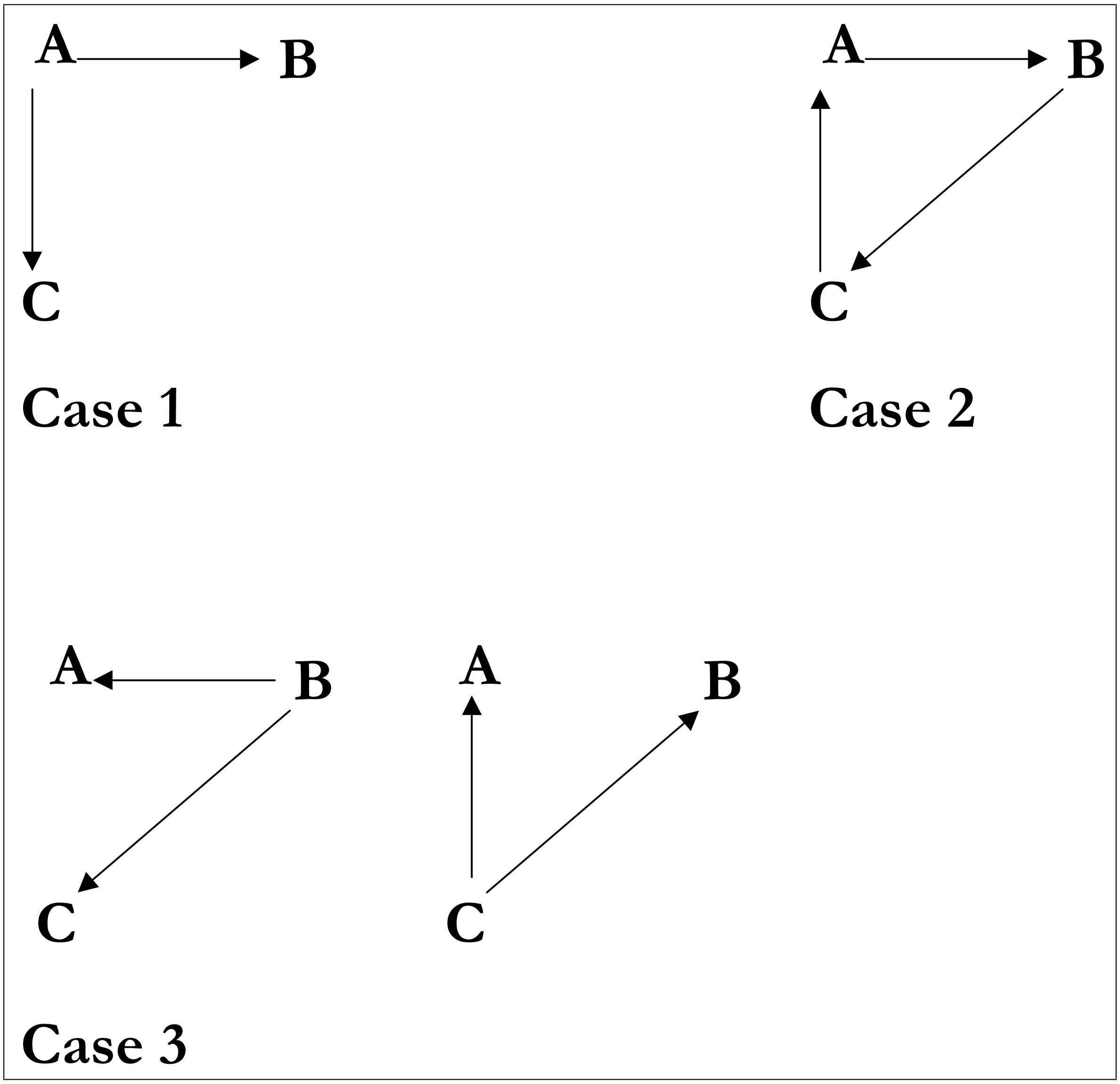

Figure 1: The patterns of choice in the route example

the initial hypothesis that the choice was an RSM. We shall see later that this reasoning can be generalized to more complex cases, and in fact it would stand even if the number of possible criteria were not limited to two. It is this type of pathological behavior that gives our theory empirical content.

2.3 Characterization of Rational Shortlist Methods

In general, suppose that we observed the choices of a decision maker. How could we test whether his behavior is consistent with the sequential maximization of two rationales? Surprisingly, it turns out that RSM's can be simply characterized through familiar observable properties of choice.

Recall first the standard Weak Axiom of Revealed Preference pioneered by Samuelson [19] for consumer theory:

WARP: If an alternative x is chosen when y is available, then y is not chosen when x is available. Formally, for all $S, T \in \mathcal{P}(X) : [x = \gamma(S), y \in S, x \in T] \Rightarrow [y \neq \gamma(T)]$.

It is well-known that (in the present setting) WARP is a necessary and sufficient condition for choice to be rationalized by an ordering (i.e. a transitive binary relation).[8] WARP essentially asserts the absence of a certain type of 'menu effects' in choice: if an alternative is revealed preferred to another within a certain 'menu' of alternatives, changing the menu cannot reverse this judgement. The property we introduce allows menu effects, but requires some consistency in the way they operate. It is in the following spirit: if you are observed to choose steak over fish when they are the only items on the menu, and also when a *large* selection of pizzas is on the menu, then you do not choose fish over steak when a *small* selection of pizzas is on the menu. A pairwise preference for x over y does not exclude in principle that in larger menus some reason can be found to reject x and choose y instead. However, if a large menu does not contain any such reason, no smaller menu contains such a reason either. Although this property may look introspectively plausible, here we are not interested in issues of plausibility: we simply propose this property as an observable test for the RSM model.

WARP*: If an alternative x is chosen *both* when only y is also available and when y and other alternatives $\{z_1, ..., z_K\}$, are available, then y is not chosen when x and a subset of $\{z_1, ..., z_K\}$ are available. Formally, for all $R, S \in \mathcal{P}(X) : [\{x, y\} \subset R \subset S, x = \gamma(xy) = \gamma(S)] \Rightarrow [y \neq \gamma(R)]$.

The second property is called Expansion, and it directly rules out pathologies of the type considered in Case 3 of the route example above:

[8] See e.g. Moulin [16] and Suzumura [20].

Expansion: An alternative chosen from each of two sets is also chosen from their union. Formally, for all $S, T \in \mathcal{P}(X) : [x = \gamma(S) = \gamma(T)] \Rightarrow [x = \gamma(S \cup T)]$.

Our main result is that Expansion and WARP* are necessary and sufficient for a choice function be an RSM. While we relegate the formal proof to the Appendix, we provide a verbal intuition of the sufficiency part of the proof, which will also give the reader a clue on how to actually *construct* the rationales from choice data, when it is possible to do so.

Informal construction of rationales for choice functions that satisfy Expansion and WARP*: Recall that $(y, x) \in P_i$ means that alternative y eliminates alternative x at the i-th round of elimination, i.e. when rationale P_i is applied. For any alternative z which 'beats' the choice x from some set S in pairwise comparisons, the pair (z, x) must be in the second rationale: otherwise, since the pair (z, x) must be in *some* rationale[9], it would be impossible rationalize the choice of x from S, as x would be eliminated in the first round. Moreover, it must be the case that such a z is eliminated in the first round (otherwise it would eliminate x in the second round). Therefore there must exist some 'neutralizing' alternative y in S which beats z in pairwise comparisons and for which the pair (y, z) is in the first rationale. However, we must be careful to select the neutralizing alternative y: it cannot be the case that in some other set T that contains y the choice is z (for z is eliminated in the first round by y). Can we find a neutralizing alternative y with these desired characteristics? Suppose not: then, for every alternative y in S that beats z, there is a set $T(y, z)$ that contains y where z is chosen. Expansion implies that y is chosen from the union (over y) of all such sets $T(y, z)$. But z also beats x in pairwise choice, so that by WARP* z should be chosen also from S, which is 'intermediate' between the union of all the $T(y, z)$ and $\{x, y\}$. This is a contradiction. So there must exist at least *some* neutralizing alternative y such that (y, z) can be safely assigned to the first rationale. We can repeat this procedure for any other set where alternatives such as x, y and z exist. This completes the set of 'forced' assignments of rationales, and we are free to explain all remaining pairwise choices with either of the two rationales.

Theorem 2 *Let X be any (not necessarily finite) set. A choice function γ on X is a Rational Shortlist Method if and only if it satisfies Expansion and WARP*.*

Proof: See Appendix A[10].

[9] If (z, x) belonged to no rationale, the choice of z in pairwise comparisons with x could never be rationalized.

[10] We are grateful to Ariel Rubinstein for suggesting the simple construction in the Sufficiency part of the proof, which replaces a previous more complicated argument.

As discussed above, the strength of this characterization lies in the fact that it connects what would be traditionally considered highly 'irrational' choice patterns to well-known and easy to check rationality properties. The only relaxation from standard tests is to allow a limited form of *menu-dependence* in the WARP* axiom.

In Appendix B we establish by means of examples that the set of axioms in Theorem 2 is tight.

3 Beyond two rationales

3.1 Sequential Rationalizability

The concept of an RSM suggests an immediate generalization. Instead of using only two rationales, the decision maker might use a larger number of them. For example, in the routes scenario of the previous section, one can conceive that the decision maker uses not only traffic and length, but also scenery as a criterion for choice. This leads us to the following definition.

Definition 3 *A choice function γ is **sequentially rationalizable** whenever there exists an ordered list $P_1, ..., P_K$ of asymmetric relations, with $P_i \subseteq X \times X$ for $i = 1...K$, such that, defining recursively*

$$M_0(S) = S$$
$$M_i(S) = \max(M_{i-1}(S); P_i), \ i = 1, ..., K$$

we have

$$\{\gamma(S)\} = M_K(S) \text{ for all } S$$

In that case we say that $(P_1, ..., P_K)$ sequentially rationalize γ. We call each P_i a **rationale**.

So the choice from each S can be constructed through sequential rounds of elimination of alternatives. At each round only the elements which are maximal according to a round-specific rationale survive. Like for RSMs (which can now be viewed as special sequentially rationalizable choice functions where only two rationales are used) the rationales and the sequence are invariant with respect to the choice set.

Are there choices which are not sequentially rationalizable? At first sight, it may seem that if we are free to use as many rationales as we like, *any* choice can be rationalized by a sufficiently large number of rationales. On the contrary, the answer may be negative even for very simple choice functions (on a domain X with as few as thee alternatives). Examples are provided in section 4.1 below.[11]

[11] See also example 8 in Appendix B.

3.2 Violations of Economic Rationality Are of Just Two Types

To delve deeper into the notion of sequential rationalizability, let us recall another well-known property of choice.

Independence of Irrelevant Alternatives[12]: If an alternative is chosen from a set, it remains chosen when some rejected alternatives are discarded from the set. Formally, for all $S, T \in \mathcal{P}(X) : [\gamma(T) \in S, S \subset T] \Rightarrow [\gamma(S) = \gamma(T)]$.

Recall that, at least for the finite case, Independence of Irrelevant Alternatives is equivalent to WARP and therefore is a necessary and sufficient condition for rationalizability with a single ordering.[13]

What types of boundedly rational behavior does sequential rationalizability allow? To answer this question consider the following two very basic rationality requirements. The first one requires that if an alternative 'beats' all others in a set in binary choices, then this same alternative is chosen from the set - this is obviously a weakening of Expansion. The second property requires that there are no pairwise cycles of choice - this is a weakening of Independence of Irrelevant Alternatives and WARP. Formally:

Always Chosen*:* If an alternative is chosen in pairwise choices over all other alternatives in a set, then it is chosen from the set. Formally, for all $S \in \mathcal{P}(X)$: $[x = \gamma(xy) \text{ for all } y \in S] \Rightarrow [x = \gamma(S)]$.

No Binary Cycles: There are no pairwise cycles of choice. Formally, for all $x_1, ..., x_n \in X$: $[\gamma(x_i x_{i+1}) = x_i, i = 1, ..., n] \Rightarrow [x_1 = \gamma(x_1 x_{n+1})]$.

The reason for highlighting these two properties is that the class of choice functions that do not satisfy WARP (i.e. are not rationalizable by a single standard economic preference relation) can be classified very simply: they are partitioned into just three subclasses: the choice functions that violate exactly one of No Binary Cycles or Always Chosen, and those that violate both. This is established in the next Proposition, which is of independent interest.

Proposition 4 *A choice function that violates WARP also violates Always Chosen or No Binary Cycles.*

Proof. It is easier to conduct the proof in terms of IIA instead of the equivalent property WARP. Let γ be a choice function on X. We argue by induction on the cardinality of

[12] For single-valued choice functions this conflates several properties of correspondences such as Chernoff's property ($S \subset T \Rightarrow \gamma(T) \cap S \subseteq \gamma(S)$) and Arrow's condition ($S \subset T$, $\gamma(T) \cap S \neq \emptyset \Rightarrow \gamma(S) = \gamma(T) \cap S$).

[13] See e.g. Moulin [16] and Suzumura [20].

X. Let $X = \{x, y, z\}$. Suppose that $x = \gamma(X)$ and $y = \gamma(xy)$, so that Independence of Irrelevant Alternatives is violated. Then there are two possibilities: if $y = \gamma(yz)$, then Always Chosen is violated; if instead $z = \gamma(yz)$, then either Always Chosen is violated (if $z = \gamma(xz)$), or No Binary Cycles is violated (if $x = \gamma(xz)$, so that $x = \gamma(xz)$, $z = \gamma(yz)$, $y = \gamma(yx)$).

Assume now that the statement holds for all sets X with $|X| \leq K$. Take X' such that $|X'| = K + 1$. Suppose that $x = \gamma(X')$ but there exists $\{x, y\} \subseteq S \subset X'$ such that $y = \gamma(S)$. If the restriction of γ to S violates Independence of Irrelevant Alternatives, then we are done by the inductive hypothesis. Suppose then that the restriction of γ to S satisfies Independence of Irrelevant Alternatives. Consider the set $V = X' \backslash S$. Obviously $V \neq \varnothing$, and let $z = \gamma(V)$.

If the restriction of γ to V violates Independence of Irrelevant Alternatives, then we are done by the inductive hypothesis. Suppose it satisfies Independence of Irrelevant Alternatives. Then $z = \gamma(vz)$ for all $v \in V \backslash z$.

Suppose $z = \gamma(yz)$. If $z = \gamma(sz)$ for all $s \in S$, then Always Chosen is violated. If there exists some $t \in S$ such that $t = \gamma(tz)$, then this generates the cycle $t = \gamma(tz)$, $z = \gamma(yz)$, $y = \gamma(ty)$, where the last relation follows from Independence of Irrelevant Alternatives on S.

Suppose alternatively $y = \gamma(yz)$. By a reasoning similar to the one above we can show that Always Chosen or No Binary Cycles is violated. ■

3.3 Sequential Rationalizability Excludes One Type of Irrational behavior

Next we show that sequential rationalizability restricts violations of the two basic rationality properties introduced in this section:

Lemma 5 *If a choice function is sequentially rationalizable it satisfies Always Chosen.*

Proof. Let γ on X be sequentially rationalizable by the rationales $P_1, P_2...P_K$. For any two alternatives $a, b \in X$, let $i(a, b)$ be the smallest i such that P_i relates a and b, that is

$$i(a, b) = \min\{i \in \{1, ..., K\} \mid (a, b) \in P_i \text{ or } (b, a) \in P_i\}$$

Given $S \subseteq X$ and $x \in S$, let $x = \gamma(xy)$ for all $y \in S \backslash x$. For each $y \in S \backslash x$ we must have $(x, y) \in P_{i(x,y)}$, so that the successive application of the rationales eliminates all $y \in S \backslash x$, and no rationale can eliminate x. Therefore $x = \gamma(S)$, as desired. ■

Our partial characterization result shows the equivalence of WARP and No Binary Cycles on the domain of sequentially rationalizable choice functions; it follows from Proposition 4 and Lemma 5 by observing that WARP is violated if there is a binary cycle:

Theorem 6 *A sequentially rationalizable choice function violates WARP if and only if it exhibits binary cycles.*

Thus, the results in this section generalize the message of the basic 'routes' example of the previous section. We have established that in general, and not only in that example, *all* violations of 'rationality' can be traced back to two elementary pathologies of choice, corresponding to case 2 and 3 of the routes example: violations of 'Always chosen' and 'No Binary Cycles'. Like RSMs, even the more general notion of sequential rationalizability is intimately connected with pairwise cycles of choice, and cannot possibly explain the other pathology.

A full characterization of sequential rationalizability remains a nontrivial open question.

4 More Examples

One may wonder whether other choice procedures generate choices which can be also explained 'as if' generated by an RSM. In this section we start with a 'negative' result, exhibiting some notable choice procedures which have been proposed in the literature and that are not RSM - this highlights how sequential rationalizability is not a vacuous notion of rationality. Next, we turn to a 'positive' results and present an application of RSM in the domain of time preferences, where, besides cycles, they can account for a context-specific anomaly of choice such as preference reversal.

4.1 Notable Choice Functions That Are Not Sequentially Rationalizable

Interestingly, violations of Always Chosen[14] can be generated by several well-known procedures that have attracted economists' attention. Such procedures are therefore non rationalizable not only in the classical sense, via a single binary relation, but also in the weaker sense of sequential rationalizability. In all examples below the set of alternatives is $X = \{x, y, z\}$.

The first procedure is (a refinement of) the '*choose the median*' procedure defined as follows. There is a 'fundamental' order B on X (e.g. given by ideology from left to right) such that $(x, y), (y, z) \in B$. The decision-maker chooses the median according to B, breaking ties by picking the highest element in the set of median elements. We have $z = \gamma(xz), = \gamma(yz)$ and yet $y = \gamma(xyz)$, violating Always Chosen.

[14] Of the simple type studied in in example 8 of Appendix B.

The same choice pattern is consistent with the '*never choose the uniquely largest*' procedure. There is again a fundamental order B on alternatives and the chosen alternative cannot be the unique maximizer of B. However, to interpret the choice pattern in this way the fundamental ordering must be exactly the reverse of the one used for the choose the median procedure, namely $(z,y),(y,x) \in B$. Baigent and Gaertner [3] and Gaertner and Xu ([7], [8]) have axiomatized this type of procedure.

A third procedure generating the choice pattern is the one described in the dinner example by Luce and Raiffa [14] (see also Kalai *et al.* [11]). Imagine that when z is not available the decision maker chooses the greatest element according to the ordering B_1 given by $(x,y) \in B_1$, while when z is available he chooses the greatest element according to the ordering B_2 given by $(y,x),(x,z) \in B_2$. This yields the sequentially non-rationalizable choice function. On the other hand, if the same procedure was followed but the ordering B_2 was given by $(y,z),(z,x) \in B_2$ (which is also in the spirit of Luce and Raiffa's example), it would be possible to sequentially rationalize the choice function by applying first $P_1 = \{(z,x),(y,z)\}$ and then $P_2 = \{(x,y)\}$.

4.2 Rational Shortlist Methods and Choice over Time

Throughout the paper we have focused on general violations of rationality. However, we believe that RSMs can prove very useful to explain other choice anomalies in specific contexts, in which certain rationales can suggest themselves. Here we consider an application to choice over time.

The standard model of choice over time is the exponential discounting model (EDM). It has been observed that actual choices in experimental settings violate consistently its predictions. The most notable violation is possibly preference reversal. Let P_γ refer to observed pairwise choices over date outcome pairs $(x,t) \in X \times T$, where X is a set of monetary outcomes and T is a set of dates. In this context, preference reversal is the shorthand for the following situation: $(x,t_x)\,P_\gamma\,(y,t_y)$ and $(y,t_y+t)\,P_\gamma\,(x,t_x+t)$. This violates stationarity of time preferences, a premise on which the EDM model is constructed.

This choice pattern can be easily accounted for by interpreting γ as a RSM with rationales P_1 and P_2 defined as follows. For some function $u : X \times T \to \Re$ and number $\sigma > 0$, $(x,t_x)\,P_1\,(y,t_y)$ if and only if $u(x,t_x) > u(y,t_y)+\sigma$, and $(x,t_x)\,P_2\,(y,t_y)$ if and only if $u(y,t_y) \leq u(x,t_x) \leq u(y,t_y)+\sigma$, and either $x > y$, or $x = y$ and $t_x < t_y$. That is, the decision maker looks first at discounted value, and chooses one alternative over the other if it exceeds the discounted value of the latter by an amount of at least σ. Otherwise he looks first at the outcome dimension and if this is not decisive at the time dimension.

This is compatible with preference reversal even with an exponential discounting type of u function. Let $x < y$, $t_x < t_y$ and $u(x, t_x) = x\delta^{t_x}$ for $\delta \in (0, 1)$. Suppose that $x\delta^{t_x} > y\delta^{t_y} + \sigma$ so that (x, t_x) is chosen over (y, t_y) by application of P_1. Given σ, if t is sufficiently large it will be $x\delta^{t_x+t} < y\delta^{t_y+t} + \sigma$, so that the two date outcome pairs $(x, t_x + t)$ and $(y, t_y + t)$ are not comparable via P_1. However, the application of P_2 yields the choice of $(y, t_y + t)$ over $(x, t_x + t)$, thus 'reversing the (revealed) preference'.

Obviously, P_γ could also be sequentially rationalized by using three rationales, where the outcome and time dimension comparisons are used in two separate P_i.

The same model can explain cyclical intertemporal choices and other 'anomalies' (see Manzini and Mariotti [15] and bibliography therein).

5 Concluding Remarks

We have proposed an economic, 'revealed preference' approach to the type of decision making procedures which are often promoted by psychologists. For example Gigerenzer and Todd [9] in their work on 'fast and frugal' heuristics observe that "One way to select a single option from multiple alternatives is to follow the simple principle of elimination: successive cues are used to eliminate more and more alternatives and thereby reduce the set of remaining options, until a single option can be decided upon". Such heuristics focus mostly on the simplicity of cues used to narrow down possible candidates for choice. Simplicity is an essential virtue in a world in which time is pressing. An overarching preference relation - let alone a utility function - is not a cognitively simple object, and as a consequence these authors stress the difference from heuristics based reasoning and the "unlimited demonic or supernatural reasoning"[15] relied upon in economics. Yet in this paper we have shown that the standard tools, concepts and properties of revealed preference theory can be used to formalize and infer the use of such heuristics. A seemingly limited form of menu-dependence (encapsulated in our WARP* property) is equivalent to the use of a two-stage procedure that may generate economically 'irrational' choice behavior.

Our way of incorporating bounded rationality is to translate the psychological notion of 'cues' into a set of *not necessarily complete* binary relations. Rationality for us is the *consistent* application of a sequence of rationales. The order in which they are applied may be hardwired and may depend on the specific context and on the type of decision maker[16], but it should be the same in a relevant class of decision problems. Each single

[15] See Gigerenzer and Todd [9].

[16] For example, in order to 'choose' whether to stay or flee in the presence of a bird, a rabbit may use as its first rationale the fact that bird is gliding, which would identify a predator. Conversely, a human decision maker may well look first at size or shape in order to recognize the bird.

rationale in itself needs not exhibit any other strong property, such as completeness and transitivity.

The usefulness of elimination heuristics in practical decision making is self-evident[17] and widely spread in disparate fields, from clinical medicine[18] to marketing and management. In this perspective the *sequentiality* in the application of rationales, which lies at the core of our analysis, is an appealing feature of our rationalization results. Our approach may be contrasted with the recent contribution by Kalai *et al.* [11] and Apesteguia and Ballester [2]. They use multiple rationales to explain choices, but each rationale is applied to a subset of the domain of choice. This results in all choices being rationalizable and the focus becomes that of 'counting' the minimum number of rationales necessary to explain choices. Finally we should mention the work by Ok [17] who characterizes the choice correspondences satisfying Independence of Irrelevant Alternatives by means of a two-stage procedure. Unlike this paper, in the second stage of Ok's procedure elimination of alternatives does not occur on the basis of a relation, but rather on the information contained in the entire feasible set.

Appendix A

Proof of Theorem 2:[19] *Necessity:* Let γ be an RSM on X and let P_1 and P_2 be the rationales.

(i) Expansion. Let $x = \gamma(S) = \gamma(T)$ for $S, T \in \mathcal{P}(X)$. We show that for any $y \in S \cup T$ it cannot be $(y, x) \in P_1$, and for any $y \in M_1(S \cup T)$ it cannot be $(y, x) \in P_2$. If $(y, x) \in P_1$, this would immediately contradict $x = \gamma(S)$ or $x = \gamma(T)$ and γ being rationalized. Suppose now that for some $y \in M_1(S \cup T)$ we had $(y, x) \in P_2$. Since $M_1(S \cup T) \subseteq M_1(S) \cup M_1(T)$, we have $y \in M_1(S)$ or $y \in M_1(T)$, contradicting $x \in M_2(S)$ or $x \in M_2(T)$.

Therefore x survives both rounds of elimination and we can conclude that $x = \gamma(S \cup T)$.[20]

[17] As put very effectively by Gigerenzer and Todd [9] "If we can decide quickly and with few cues whether an approaching person or bear is interested in fighting, playing, or courting, we will have more time to prepare and act accordingly (though in the case of the bear all three intentions may be equally unappealing)".

[18] As an example, the online self-help guide of the UK National Health Service (http://www.nhsdirect.nhs.uk/SelfHelp/symptoms/) helps users to recognize an ailment by giving yes/no answers along a sequence of symptoms. This presumably formalizes the mental process of a trained doctor.

[19] We are grateful to Ariel Rubinstein for suggesting the simple construction in the Sufficiency part of the proof, which replaces a previous more complicated argument.

[20] Note that this argument cannot be iterated further in the case of more than two rationales

(ii) WARP*. Let $x = \gamma(xy) = \gamma(S)$, $y \in S$. Then $x = \gamma(xy)$ implies that $(x,y) \in P_1 \cup P_2$. If $(x,y) \in P_1$, then the desired conclusion follows immediately. Suppose then that $(x,y) \in P_2$. The fact that $x = \gamma(S)$ implies that for all $z \in S$ it is the case that $(z,x) \notin P_1$. Therefore $x \in M_1(R)$ for all $R \subset S$ for which $x \in R$. Since $(x,y) \in P_2$ then $y \notin M_2(R)$ for all such R, and thus $y \neq \gamma(R)$.

Sufficiency: Suppose that γ satisfies the axioms. We construct the rationales explicitly. Define

$$P_1 = \{(x,y) \in X \times X \mid \text{there exists no } S \in \mathcal{P}(X) \text{ such that } y = \gamma(S) \text{ and } x \in S\}$$

Define $(x,y) \in P_2$ if and only if $x = \gamma(xy)$.

Observe that P_1 and P_2 are asymmetric: if xP_1y and yP_1x then in particular $\gamma(xy) \neq x, y$ which is not possible; and P_2 is consistent with binary choices.

To check that P_1 and P_2 rationalize γ take any $S \in \mathcal{P}(X)$ and let $x = \gamma(S)$. All $z \in S$ such that $z = \gamma(xz)$ are eliminated by P_1. For suppose not: this means that for all $y \in S \backslash z$ there exists $T_{yz} \ni y, z$ such that $z = \gamma(T_{yz})$. Then by Expansion $z = \gamma\left(\bigcup_{y \in S \backslash z} T_{yz}\right)$. Since $S \subseteq \bigcup_{y \in S \backslash z} T_{yz}$, by WARP* $x \neq \gamma(S)$, a contradiction.

Clearly x is not eliminated by either P_1 or P_2: for $y \in S$, if yP_1x then it could not be $x = \gamma(S)$, whereas if yP_2x by the argument in the previous paragraph y would have been eliminated by the application of P_1 before P_2 can be applied.

Finally all $z \in S$ such that $x = \gamma(xz)$ are eliminated by P_2. ■

Appendix B

We establish by means of examples that the set of axioms in Theorem 2is tight. In order to describe choice functions compactly in examples we use the following notation: given $x \in X$, let $C_\gamma(x) = \{S \in \mathcal{P}(X) \mid x = \gamma(S)\}$.[21]

Example 7 ***Expansion but not WARP*:***

$X = \{x, y, w, z\}$

$C_\gamma(w) = \{wx\}$

$C_\gamma(x) = \{xy, xz, xyz, wxy, wxyz\}$

$C_\gamma(y) = \{wy, yz, wyz\}$

---

since it is not necessarily true that $M_2(S \cup T) \subseteq M_2(S) \cup M_2(T)$. There could in fact be $y \in (M_1(S) \cup M_1(T)) \backslash M_1(S \cup T)$ such that $(y,z) \in P_2$ for some $z \in M_1(S) \cup M_1(T)$ while for all $y' \in M_1(S \cup T)$ it is the case that $(y', z) \notin P_2$. So if it were $(z,x) \in P_3$, x could not be chosen from $S \cup T$.

[21] In this notation , the Expansion axiom says that for all $x \in X : C_\gamma(x)$ is closed under set union.

$C_\gamma(z) = \{wz, wxz\}$

Binary choices are visualized in figure 2, where $a \to b$ stands for $a = \gamma(ab)$. It is straightforward to verify that this choice function satisfies Expansion, but not WARP* (e.g. $x = \gamma(X)$ and $x = \gamma(xz)$ but $z = \gamma(wxz)$). This choice function is not an RSM. To see this, suppose $(w,x) \in P_1$. Then $x = \gamma(X)$ cannot be rationalized. Suppose then that $(w,x) \in P_2$. Then $z = \gamma(wxz)$ cannot be rationalized, for x will eliminate z regardless of whether $(x,z) \in P_2$ or $(x,z) \in P_1$.

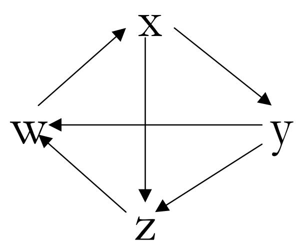

Figure 2: The base relation for example 7.

Example 8 ***WARP* but not Expansion:***

$X = \{x, y, z\}$

$C_\gamma(x) = \{xy, xz\}$

$C_\gamma(y) = \{yz, xyz\}$

$C_\gamma(z) = \{\varnothing\}$

Binary choices are visualized in figure 3. While this choice function satisfies WARP* (trivially, as the premise of WARP* does not apply), it fails Expansion. This choice function is not an RSM. Indeed, it is not sequentially rationalizable. As before, for any two alternatives $a, b \in X$, let $i(a,b)$ be the smallest i such that P_i relates a and b. Suppose by contradiction that γ were sequentially rationalizable by $P_1, ..., P_K$. Since $x = \gamma(xy)$ it must be $(x,y) \in P_{i(x,y)}$. Given this, $y = \gamma(xyz)$ can only hold if $(z,x) \in P_{i(x,z)}$, which contradicts $x = \gamma(xz)$.

The above examples can be used to make two additional points. First, there are choice functions which are not RSMs but are sequentially rationalizable. Namely, γ in example 7 is rationalized by $P_1 = \{(y,w)\}$, $P_2 = \{(z,w), (w,x), (x,y)\}$ and $P_3 = \{(x,z), (y,z)\}$. Second, the notion of sequential rationalizability is not vacuous, in the sense that there exist choice functions which are not sequentially rationalizable (example 8).

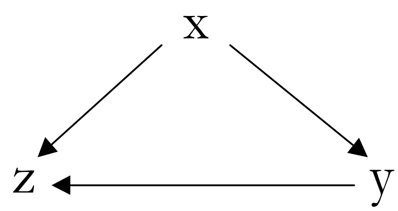

Figure 3: The base relation for example 8.

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