The Edgeworth exchange formulation of bargaining models and market experiments*

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We construct Edgeworth exchange economies equivalent to demand and supply environments typically used in bargaining models and market experiments. This formulation clearly delineates environment, institution, and behavior for these models and experiments. To illustrate, we examine results by Gode and Sunder, who simulate random behavior in a double auction and argue that this institution leads to an efficient allocation, even in the absence of rationality. We use the Edgeworth exchange representation of their economic environment to demonstrate that they model individually rational behavior, and show that their model is a special case of theoretical results by Hurwicz, Radner, and Reiter.

Keywords: Double auction, Market experiment, Edgeworth exchange, Bounded rationality.

1 Introduction

Market experiments typically use the technique of induced supply and demand. Through use of this tool, the extensive literature on experimental markets has empirically established performance properties of several market institutions under a variety of economic environments and information structures.[1] Although induced supply and demand is an effective

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[1] Smith [1982] provides a thorough description of the theory and techniques of induced supply and demand. For extensive surveys of the literature and results on these issues, see Plott [1982] and Smith [1982].

experimental technique, it can lead to problems in interpretation, since many theoretical models of resource allocation are formulated in terms of utility functions and endowments. The objective of our paper is to develop a framework that facilitates interpretation of bargaining models and experiments by spanning this gap between theory and experiment.

We formulate market experiments as Edgeworth exchange economies by demonstrating that a buyer's induced demand schedule and a seller's induced supply schedule can each be represented as a quasi-linear utility function with an appropriate endowment, and show that the induced supply and demand formulation typical of market experiments is equivalent to an Edgeworth exchange economy. With this formulation we examine a result by Gode and Sunder [1993] (henceforth GS), who simulate bargaining in a double auction, and conclude that the structure of this market institution drives convergence to a Pareto optimum, even when agents do not seek profits and their bids are random. We demonstrate that when viewed as an Edgeworth exchange economy, agents in their model exhibit individual rationality, by which we mean that no agent attempts to take part in a trade that fails to increase, or at least leaves constant, his own utility.[2] The high efficiency observed in the GS simulations in fact results from this individual rationality coupled with the agents' quasi-linear utility functions. Furthermore, we show that the GS simulation model is a special case of an analytic model developed earlier by Hurwicz, Radner, and Reiter [1975] (henceforth HRR).

Our analysis in this paper is similar to the approach that Hurwicz [1995] takes in his critique of the "Coase theorem" (Coase [1960]). Hurwicz shows that Coase's result – which states roughly that efficiency in a market with externalities is independent of the assignment of property rights – holds only in the case of quasi-linear utility. Consequently, Hurwicz's recognition of the implicit preference structure underlying Coase's model has led to a reinterpretation of the *scope* of Coase's result. In this paper we reformulate supply and demand environments, commonly studied in market experiments, as Edgeworth exchange economies and use this formulation to reinterpret the *nature* and *scope* of the GS model.

2 The Edgeworth formulation of induced supply and demand

In this section, we describe induced supply and demand environments and formulate these environments as Edgeworth exchange economies. With this formulation, we develop an

[2] This definition follows Luce and Raiffa [1957], pp. 192-3.

example of the Edgeworth exchange representation of a supply and demand environment.

2.1 Relationship between induced demand and quasi-linear utility

In this subsection we describe an induced demand function, construct a quasi-linear utility function from the demand function, and then show that the demand function may be derived from constrained maximization of the quasi-linear utility function, provided the buyer's endowment is large enough to yield an interior solution to the maximization problem.

2.1.1 Induced demand and quasi-linear utility

Each buyer $j \in J$ has a vector of positive valuations $v_j = (v_j^1, v_j^2, \ldots, v_j^{n_j})$ for units of the commodity Y, where $v_j^1 \geq v_j^2 \geq v_j^3 \geq \cdots \geq v_j^{n_j} > 0$. The total redemption value to buyer j when she purchases y units is

$$r_j(y) = \begin{cases} 0, & y = 0; \\ \sum_{\gamma=1}^{y} v_j^{\gamma}, & y \in \{1, 2, \ldots, n_j\}; \\ \sum_{\gamma=1}^{n_j} v_j^{\gamma}, & y > n_j. \end{cases}$$

We use the buyer's redemption value function $r_j(\cdot)$ to develop the buyer's quasi-linear utility function. Define the consumption space of buyer j as $X \times Y$. Let $(x_j, y_j) \in X \times Y$ denote the number of units of the currency and commodity held by buyer j. Define the utility function for buyer j as

$$u_j(x, y) = x + r_j(y) + M_j \tag{1}$$

where M_j is a constant.[3] Equation (1) is linear in the currency (X) and additively separable in the currency and commodity, i.e., it is quasi-linear.[4]

The currency endowment $x_j^0 = \sum_{\gamma=1}^{n_j} v_j^{\gamma}$ is sufficient to guarantee that buyer j would be able to purchase each unit at any price at or below the value of the unit.

[3]The constant M_j has no theoretical implications but is relevant to experimental studies. If $M_j = -x_j^0$ then $u_j(x_j^0, y_j^0) = 0$ so that the autarky outcome has a payoff of 0.

[4] Our rationalization of the induced demand schedule as the solution to the constrained maximization of a quasi-linear utility function is similar to the construction by Smith [1982, p. 932]. Smith derives the induced demand curve by maximizing the utility function $u_j(x_j, y) = x_j + v_j(y_j)$ subject to the budget constraint $x_j + p\,y_j \leq 0$ where $x_j \leq 0$ and $y_j \geq 0$. In contrast, we define finite positive endowments of X for buyers and sellers that are consistent with the typical specification of consumer choice problems.

2.1.2 Derivation of induced demand from utility

Lemma 1 Buyer j's demand for Y – derived from maximization of equation (1) for a sufficiently large endowment – is characterized by v_j.

Proof The vector v_j of valuations is non-increasing, so that the total valuation function $v_j(y)$ is (weakly) concave for $y \in Y_j$. Therefore the utility function $u_j(x, y) = x + M_j + v_j(y)$ is (weakly) quasi-concave. The theorem of the maximum implies that for any given price p of good Y, the set of values that maximize $u_j(\cdot)$ is convex.

Let $y_j(p)$ be the demand of buyer j at price p, i.e., the solution to the maximization problem for $u_j(x, y)$. We complete the proof of the lemma by showing that the demand $y_j(p)$ has the same graph as the vector v_j of values. If $p = v_j^k$, then $y_j(p) \in \{k-1, k\}$. If $p \in (v_j^{k+1}, v_j^k)$, then $y_j(p) = k$, for a sufficiently large endowment. ∎

2.2 Relationship between induced supply and quasi-linear utility

In this subsection we describe an induced supply function, construct a quasi-linear utility function from this supply function, and then show that the supply function may be derived from constrained maximization of the quasi-linear utility function, with a commodity endowment of Y equal to the number of units for which the seller has a finite unit cost.

2.2.1 Induced supply schedule and quasi-linear utility

The supply function for seller i is given by his marginal cost schedule, which is represented by the vector $c_i = (c_i^1, c_i^2, c_i^3, \ldots, c_i^{m_i})$. Seller i has finite selling capacity, so that the marginal cost of any unit beyond m_i is infinite. Element c_i^k is interpreted as the marginal cost incurred by seller i when he produces his k^{th} unit. For $k = \{0, 1, 2, \ldots, m_i\}$ the redemption value for seller i when he sells k units is

$$r_i(k) = \begin{cases} 0, & k = 0; \\ -\sum_{\iota=1}^{k} c_i^{\iota}, & k \in \{1, 2, \ldots, m_i\}; \\ -\infty, & k > m_i. \end{cases}$$

We use the seller's redemption value function to develop the seller's quasi-linear utility function. Define the quasi-linear utility function for seller i as

$$u_i(x, y) = \begin{cases} x + r_i(m_i - y), & 0 \le y \le m_i; \\ x, & y > m_i. \end{cases} \tag{2}$$

Set the commodity endowment for seller i to $y_i^0 = m_i$.

2.2.2 Derivation of induced supply from utility

Lemma 2 Seller i's supply of Y, which is derived from maximization of equation (2), is characterized by c_i.

Proof The vector c_i of costs is non-decreasing, so that the total valuation function $r_i(k)$ is (weakly) concave for $k \in \{1, 2, \ldots, y_i^0\}$, as is $r_i(y_i^0 - y)$ for $y \in \{1, 2, \ldots, y_i^0\}$. Therefore the utility function $u_i(x, y) = x + r_i(y_i^0 - y)$ is (weakly) quasi-concave. The theorem of the maximum implies that for any given price p of good Y, the set of values that maximize $u_i(\cdot)$ is convex.

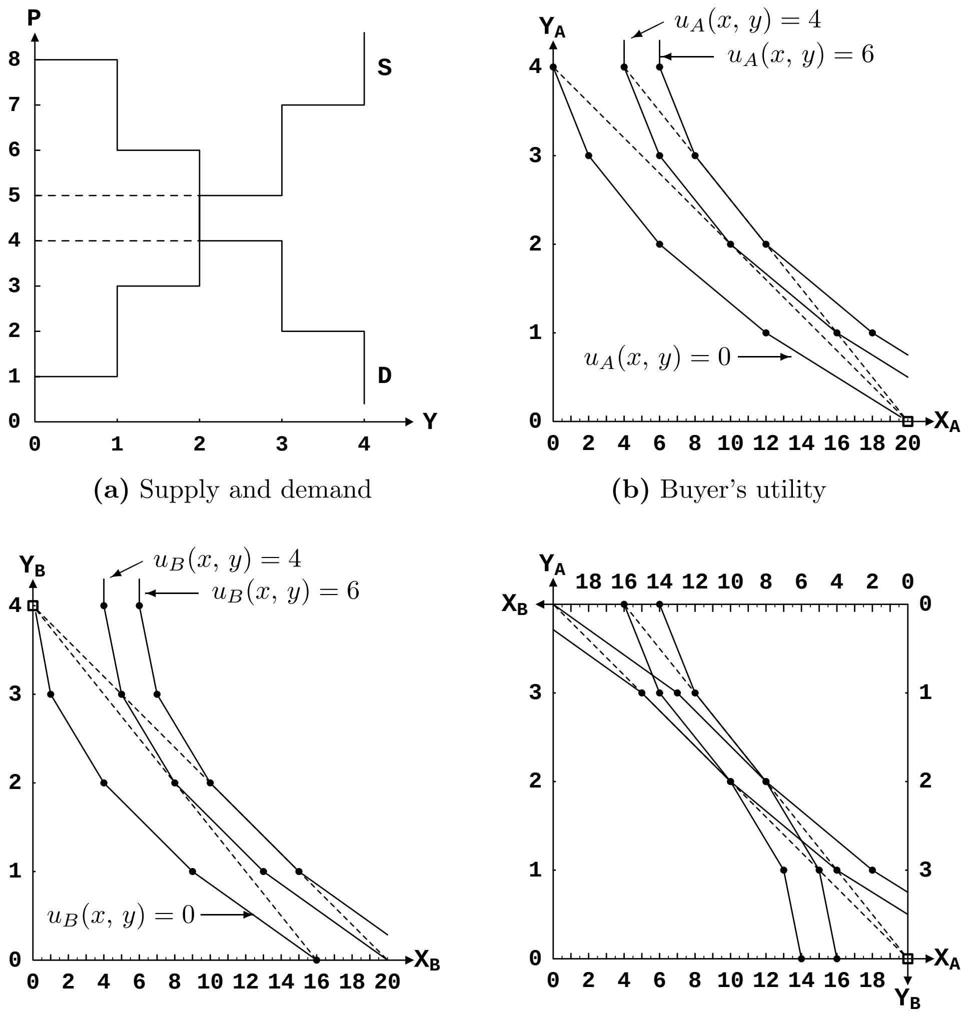
Let $y_i(p)$ be the supply of seller i at price p, i.e., the solution to the maximization problem for $u_i(x, y)$. We complete the proof of the lemma by showing that the supply $y_i(p)$ has the same graph as the vector c_i of costs. If $p = c_i^k$, then $y_i(p) \in \{y_i^0 - k, y_i^0 - k + 1\}$. If $p \in (c_i^k, c_i^{k+1})$, then $y_i(p) = y_i^0 - k$. ∎

2.3 Example

Figure 1 (a) shows a simple example of an induced supply and demand environment. In the example there is one seller with cost vector $c_i = (1, 3, 5, 7)$ and one buyer with the vector of values $v_j = (8, 6, 4, 2)$.

Figures 1 (b) and (c) show indifference curves for the buyer and for the seller. The buyer has endowment $(x^0, y^0) = (20, 0)$, the utility function dual to the vector of valuations $v_j = \{8, 6, 4, 2\}$, and the constant $M_j = -20$. Figure 1 (b) shows three indifference curves for the utility function $u_j(x, y)$, constructed using equation (1). The indifference curves $u_j(x, y) = 4$ and $u_j(x, y) = 6$ are horizontal translations of the indifference curve $u_j(x, y) = 0$, i.e., preferences are *quasi-linear*. Figure 1 (b) also shows the buyer's demand for two prices: $p = 4$, and $p = 5$. When the price is $p = 4$ (which is equal to v_j^3) the set of utility maximizing choices of the commodity (Y) is $y_j(4) \in \{2, 3\}$. If $p = 5$ then $p \in (4, 6) = (v_j^3, v_j^2)$, so the demand is $y_j(5) = 2$. The budget sets generated by these two prices are depicted in Figure 1 (b), along with the utility maximizing choice sets associated with these prices. Figure 1 (c), which shows several indifference curves for the seller with the utility function defined in equation (2), is interpreted analogously to Figure 1 (b). The buyer's and seller's utility functions are combined in the Edgeworth diagram in Figure 1 (d).

We can see in Figure 1 (d) that the range of competitive equilibrium prices in the Edgeworth exchange representation of this example is $p \in [4, 5]$, just as when the example is represented as an induced supply and demand environment in Figure 1 (a). At the lower price ratio, $p = \frac{p_y}{p_x} = 4$, which is the steeper price line in Figure 1 (d), the buyer is indifferent between 2 and 3 units and the seller would like to sell 2 units. This of course corresponds to the net demands for buyer and seller in Figure 1 (a). A similar observation holds for the high end of the equilibrium price range ($p = 5$).



(a) Supply and demand

(b) Buyer's utility

(c) Seller's utility

(d) Edgeworth diagram

Figure 1: Supply and demand, indifference curves, and Edgeworth diagram.

3 The "ZI" model, individual rationality, and the B-process

The framework developed in Section 2 can be used to clarify aspects of the model by Gode and Sunder [1993] of "zero-intelligence" (ZI) traders. Their model shows that the bargaining behavior of these traders, whose actions are random bids and offers in a double auction trading institution, leads to efficient outcomes. This is often considered surprising, since (according to GS) ZI agents "do not maximize or *seek* profits."[5] There are three conclusions that we draw from the Edgeworth exchange representation of the GS simulations. First, their agents exhibit individual rationality (according to the definition of Luce and Raiffa [1957]) and as a result, they do in fact seek profits. Secondly, we compare agent behavior and the market institution that GS employ in their simulations to the results in Hurwicz, Radner, and Reiter [1975] (HRR) and show that it is this profit seeking behavior that produces efficient outcomes in the GS model. Finally, we examine performance of ZI agents in environments with non-convexities and show that their result is not as general as the HRR result.

3.1 The "ZI" model reinterpreted

The DA market simulations in GS fit perfectly into the Edgeworth exchange representation developed in Section 2. GS report results of simulations with two primary treatment variables. We focus our attention on the treatment in which high allocative efficiency is observed: the treatment that they refer to as "budget constrained." In this treatment, each buyer has a positive valuation for a single unit, and each seller can sell a single unit at some positive marginal cost. Bargaining in the Gode and Sunder model takes place in a double auction. In this institution, those sellers who have not already sold their unit may submit an ask at any time, and those buyers who have not already purchased a unit may submit a bid at any time. When a bid and an ask cross, a trade occurs at a price that is equal to the bid if the bid precedes the ask or at the ask if the ask precedes the bid.

A buyer in the GS model submits bids that are drawn from a uniform distribution between 0 and her unit value; likewise, a seller submits asks that are randomly drawn between his unit cost and some upper bound. According to GS, "the market forbade traders to buy or sell at a loss because then they would not have been able to settle their accounts." This argument is inconsistent in the case of sellers, since each seller has an endowment

[5] See GS [1993, p. 120]. Emphasis added.

of a single unit of the commodity, and would therefore be able to settle any trade at a non-negative price. In the case of a buyer, a natural interpretation of her inability to settle her account is that she does not have sufficient currency endowment to purchase at the negotiated price. Viewed within the framework of Section 2, the constraint that GS impose implies that buyer i has a currency endowment x_i^0 equal to her unit value. Therefore, for each buyer the set of feasible trades[6] is identical to the set of individually rational trades. This ambiguity allows one to interpret "ZI" buyer behavior in either of two ways: (1) a buyer submits random bids from her feasible set of trades, or (2) a buyer submits bids that lie in her upper contour set. Since the first explanation is consistent only if each buyer's endowment is equal to the buyer's valuation, and since this is not generically true, we adopt the second explanation, and conclude that the "ZI" traders exhibit individual rationality.

We now have a complete description of a stochastic process and a microeconomic system in which performance can be evaluated. Their simulations generate high allocative efficiency in all markets. An efficiency loss can only occur when there is a trade of an extra-marginal unit (that is, a unit with a value below or a unit with a cost above the competitive equilibrium price). The double auction that GS consider prevents buyers from reselling a unit to another buyer with a higher valuation, and prevents a seller from purchasing from a seller with a lower unit cost.[7] When this result was first introduced, it was considered surprising. However, when viewed from the perspective of the equivalent Edgeworth exchange representation, it is apparent that the high allocative efficiency observed when agents randomly propose trades in their upper contour sets is a special case of the HRR model, as we describe in the next subsection.

3.2 The ZI model and the *B*-process

The *B*-process is a simple non-tatonnement trading institution. With a discrete commodity space, as in a market experiment, random sequences of proposed trades submitted from each agent result in a sequence of net trades. An element of the sequence of net trades is non-zero if submitted proposals include a compatible trade (i.e., there is at least one subset of trade proposals for which net trades sum to zero). HRR show under weak conditions

[6] 'Feasible' trade here means that a trade is feasible for both parties given their endowments and also that the trade is permissible under the rules of the institution.

[7] In a sequel to GS, Gode and Sunder [1997] present an analysis of the magnitude of expected efficiency losses in their simulations.

on preferences and technologies that if at every iteration of the bargaining process, each individual only submits trade proposals from their individually feasible and rational choice set, then the process converges to a Pareto optimal allocation in finite time.

This result applies to a wide class of environments that includes (but is not limited to) the one that GS consider. Recall that we earlier demonstrated that the GS “budget constrained” agents generate proposed trades randomly from their individually rational and feasible choice sets. The strong similarity between the HRR and the GS models generates optimism that the GS results are robust. For example, it would be interesting to know whether ZI behavior in a single unit sequential double auction generates Pareto optimal outcomes for any private good economy without externalities. Unfortunately this is not the case: there are many environments for which ZI behavior does not generate Pareto optimal outcomes in the DA. In fact, we have already seen that trades which include extra-marginal units, combined with the prohibition on retrading, can lead to non-Pareto optimal outcomes even in quasi-linear environments. Even if we dismiss this scenario as unrealistic since many markets allow agents to act as a buyer and seller, there are still classes of environments for which Pareto optimal outcomes are not guaranteed. We identify these classes of environments by examining differences between the double auction adopted by GS and the *B*-process.

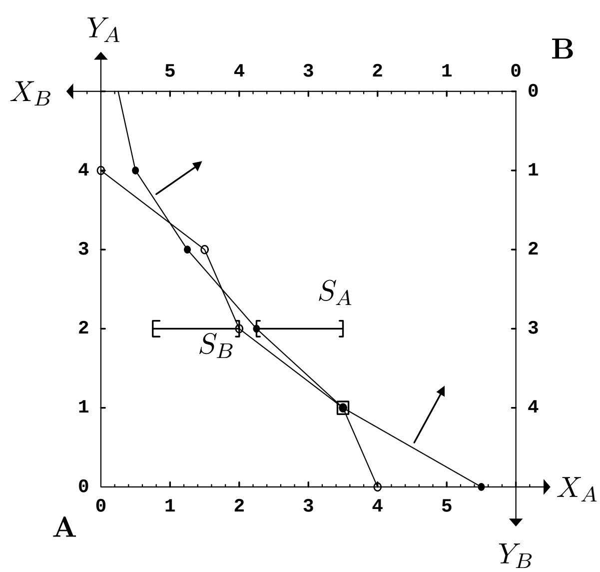


Figure 2: Edgeworth diagram with nonconvexity in preferences.

In the double auctions GS consider, each contract is for a fixed quantity of one unit of the commodity (Y); in the B-process contracts do not have a quantity restriction. The quantity restriction can prevent convergence to a Pareto optimal allocation. Figure 2 shows a simple Edgeworth exchange economy in which agent A (the buyer of commodity Y) has convex preferences, but agent B has a nonconvexity. The endowment point for A and B is shown as a square at $(3.5, 1)$ (from the perspective of A). The agents' indifference curves are presented for the endowments: A's indifference curve is marked by filled circles at its kinks and B's indifference curve is marked by the empty circles at its kinks. If A and B adopt "ZI" behavior in a single unit double auction, then the set of bids for one unit of commodity Y that would increase the utility of agent A are depicted by the set S_A and the set of offers that would increase the utility of agent B are represented by the set S_B. (In the representation for agent B, we assume, as in the GS model, that there is an upper bound on the offers that are made by a seller, although the seller would benefit from offers above this upper bound if the offer were accepted.) Since there is no overlap in these two supports no Pareto improving trade will be realized. However, a Pareto improving trade would occur if B could sell two units of Y to A for a price between $\frac{p_y}{p_x} = 1\frac{1}{8}$ and $\frac{p_y}{p_x} = 1$. The HRR model guarantees convergence to a Pareto optimum, even in the case of non-convexity of one of the agent's indifference curves, but we see easily in the Edgeworth representation that non-convexity can impede convergence with quantity restrictions.

4 Conclusions

The method of induced costs and values is a powerful and effective tool for *conducting* market experiments and defining bargaining models. However, our Edgeworth exchange formulation is a potent tool for *interpreting* and *understanding* these bargaining models and market experiments.

In Section 3 we demonstrate that the Edgeworth formulation of supply and demand environments clarifies the role of behavior and institution in the Gode and Sunder simulations of "Zero-intelligence" bargaining in the double auction. Once agents' objectives are represented as quasi-linear utility functions and these are combined to create an Edgeworth exchange economy, it is apparent that their behavior is individually rational and that the result of the ZI model is a special case of the B-process model by Hurwicz, Radner, and Reiter, who show analytically that individual rationality is sufficient to produce Pareto optimal outcomes for randomly generated bids in the B-process.

Since individual rationality in these mechanisms is sufficient to achieve Pareto optimal outcomes in an Edgeworth exchange economy with quasi-linear utility, it is natural to ask what behavior is sufficient to achieve competitive equilibrium (CE) outcomes in Edgeworth exchange. Representation of supply and demand environments as Edgeworth exchange environments helps address this issue, since we can regard models such as Rustichini, Satterthewaite, and Williams [1994] – who analyze convergence to CE in the k-double auction – or Gjerstad and Dickhaut [1998] – who simulate convergence to CE in a double auction – as convergent models of bargaining in Edgeworth exchange. This is important for two reasons. First, interpretation of these models as Edgeworth exchange economies in the special case of quasi-linear utility may provide insights that facilitate their extension to models of bargaining in Edgeworth exchange more generally. In addition, once extensions of bargaining models to Edgeworth exchange are formulated, their restriction to the quasi-linear case should be identical to their formulation in terms of supply and demand. Second, extension of bargaining models to general Edgeworth exchange economies is important because of the possible impact of restricting the models to the case of preferences that don't produce income effects. This issue has been demonstrated by Hurwicz, whose critique of the Coase theorem identifies the limitations of restricting attention exclusively to the case of quasi-linear utility.

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