The Stability and Efficiency of Economic and Social Networks

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Abstract

This paper studies the formation of networks among individuals. The focus is on the compatibility of overall societal welfare with individual incentives to form and sever links. The paper reviews and synthesizes some previous results on the subject, and also provides new results on the existence of pairwise-stable networks and the relationship between pairwise stable and efficient networks in a variety of contexts and under several definitions of efficiency.

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1 Introduction

Many interactions, both economic and social, involve network relationships. Most importantly, in many interactions the specifics of the network structure are important in determining the outcome. The most basic example is the exchange of information. For instance, personal contacts play critical roles in obtaining information about job opportunities (e.g., Boorman (1975), Montgomery (1991), Topa (1996), Arrow and Borzekowski (2000), Calvo-Armengol (2000), Calvo-Armengol and Jackson (2001)). Networks also play important roles in the trade and exchange of goods in non-centralized markets (e.g., Tesfatsion (1997, 1998), Weisbuch, Kirman and Herreiner (1995)), and in providing mutual insurance in developing countries (e.g., Fafchamps and Lund (1997)).

Although it is clear that network structures are of fundamental importance in determining outcomes of a wide variety of social and economic interactions, far beyond those mentioned above, we are only beginning to develop theoretical models that are useful in a systematic analysis of how such network structures form and what their characteristics are likely to be. This paper outlines such an area of research on network formation. The aim is to develop a systematic analysis of how incentives of individuals to form networks align with social efficiency. That is, when do the private incentives of individuals to form ties with one another lead to network structures that maximize some appropriate measure of social efficiency?

This paper synthesizes and reviews some results from the previous literature on this issue,[1] and also presents some new results and insights into circumstances under private incentives to form networks align with social efficiency.

The paper is structured as follows. The next section provides some basic definitions

[1]There is a large and growing literature on network interactions, and this paper does not attempt to survey it. Instead, the focus here is on a strand of the economics literature that uses game theoretic models to study the formation and efficiency of networks. Let me offer just a few tips on where to start discovering the other portions of the literature on social and economic networks. There is an enormous "social networks" literature in sociology that is almost entirely complementary to the literature that has developed in economics. An excellent and broad introductory text to the social networks literature is Wasserman and Faust (1994). Within that literature there is a branch which has used game theoretic tools (e.g., studying exchange through cooperative game theoretic concepts). A good starting reference for that branch is Bienenstock and Bonacich (1997). There is also a game theory literature that studies communication structures in cooperative games. That literature is a bit closer to that covered here, and the seminal reference is Myerson (1977) which is discussed in various pieces here. A nice overview of that literature is provided by Slikker (2000).

and a few simple stylized examples of network settings that have been explored in the recent literature. Next, three definitions of efficiency of networks are presented. These correspond to three perspectives on societal welfare which differ based on the degree to which intervention and transfers of value are possible. The first is the usual notion of Pareto efficiency, where a network is Pareto efficient if no other network leads to better payoffs for all individuals of the society. The second is the much stronger notion of efficiency, where a network is efficient if it maximizes the sum of payoffs of the individuals of the society. This stronger notion is essentially one that considers value to be arbitrarily transferable across individuals in the society. The third is an intermediate notion of efficiency that allows for a natural, but limited class of transfers to be made across individuals of the society. With these definitions of efficiency in hand, the paper turns its focus on the existence and properties of pairwise stable networks, i.e., those where individuals have no incentives to form any new links or sever any existing links. Finally, the compatibility of the different efficiency notions and pairwise stability is studied from a series of different angles.

2 Definitions

Networks[2]

A set $N = \{1, \ldots, n\}$ of individuals are connected in a network relationship. These may be people, firms, or other entities depending on the application.

The network relationships are reciprocal and the network is thus modeled as a non-directed graph. Individuals are the nodes in the graph and links indicate bilateral relationships between the individuals.[3] Thus, a network g is simply a list of which pairs of individuals are linked to each other. If we are considering a pair of individuals i and j, then $\{i, j\} \in g$ indicates that i and j are linked under the network g.

There are many variations on networks which can be considered and are appropriate for different sorts of applications.[4] Here it is important that links are bilateral. This is

[2]The notation and basic definitions follow Jackson and Wolinsky (1996) when convenient.

[3]The word "link" follows Myerson's (1977) usage. The literature in economics and game theory has largely followed that terminology. In the social networks literature in sociology, the term "tie" is standard. Of course, in the graph theory literature the terms vertices and edges (or arcs) are standard. I will try to keep a uniform usage of individual and link in this paper, with the appropriate translations applying.

[4]A nice overview appears in Wasserman and Faust (1994).

appropriate, for instance, in modeling many social ties such as marriage, friendship, as well as a variety of economic relationships such as alliances, exchange, and insurance, among others. The key important feature is that it takes the consent of *both* parties in order for a link to form. If one party does not consent, then the relationship cannot exist. There are other situations where the relationships may be unilateral: for instance advertising or links to web sites. Those relationships are more appropriately modeled by directed networks.[5] As some degree of mutual consent is the more commonly applicable case, I focus attention here on non-directed networks.

An important restriction of such a simple graph model of networks is that links are either present or not, and there is no variation in intensity. This does not distinguish, for instance, between strong and weak ties which has been an important area of research.[6] Nevertheless, the simple graph model of networks is a good first approximation to many economic and social interactions and a remarkably rich one.

For simplicity, write ij to represent the link $\{i,j\}$, and so $ij \in g$ indicates that i and j are linked under the network g.

More formally, let g^N be the set of all subsets of N of size 2. $G = \{g \subset g^N\}$ denotes the set of all possible networks or graphs on N, with g^N being the full or complete network.

For instance, if $N = \{1,2,3\}$ then $g = \{12, 23\}$ is the network where there is a link between individuals 1 and 2, a link between individuals 2 and 3, but no link between individuals 1 and 3.

The network obtained by adding link ij to an existing network g is denoted by $g+ij$ and the network obtained by deleting link ij from an existing network g is denoted $g - ij$.

For any network g, let $N(g)$ be the set of individuals who have at least one link in the network g. That is, $N(g) = \{i \mid \exists j \text{ s.t. } ij \in g\}$.

Paths and Components

Given a network $g \in G$, a *path* in g between i an j is a sequence of individuals $i_1, \ldots, i_K$ such that $i_k i_{k+1} \in g$ for each $k \in \{1, \ldots, K-1\}$, with $i_1 = i$ and $i_K = j$.

A (connected) *component* of a network g, is a nonempty subnetwork $g' \subset g$, such that

[5]For some analysis of the formation and efficiency of such networks see Bala and Goyal (2000) and Dutta and Jackson (2000).

[6]For some early references in that literature, see Granovetter (1973) and Boorman (1975).

- if $i \in N(g')$ and $j \in N(g')$ where $j \neq i$, then there exists a path in g' between i and j, and
- if $i \in N(g')$ and $j \notin N(g')$ then there does not exist a path in g between i and j.

Thus, the components of a network are the distinct connected subgraphs of a network.

The set of components of g is denoted $C(g)$. Note that $g = \cup_{g' \in C(g)} g'$.

Value Functions

Different network configurations lead to different values of overall production or overall utility to a society. These various possible valuations are represented via a value function.

A *value function* is a function $v : G \to I\!R$.

I maintain the normalization that $v(\emptyset) = 0$.

The set of all possible value functions is denoted $\mathcal{V}$.

Note that different networks that connect the same individuals may lead to different values. This makes a value function a much richer object than a characteristic function used in cooperative game theory. For instance, a soceity $N = \{1, 2, 3\}$ may have a different value depending on whether it is connected via the network $g = \{12, 23\}$ or the network $g^N = \{12, 23, 13\}$.

The special case where the value function depends only on the groups of agents that are connected, but not how they are connected, corresponds to the communication networks considered by Myerson (1977).[7] In most applications, however, there may be some cost to links and thus some difference in total value across networks even if they connect the same sets of players, and so this more general and flexible formulation is more powerful and encompasses many more applications.

It is also important to note that the value function can incorporate costs to links as well as benefits. It allows for arbitrary ways in which costs and benefits may vary

[7] To be precise, Myerson started with a transferable utility cooperative game in characteristic function form, and layered on top of that network structures that indicated which agents could communicate. A coalition could only generate value if its members were connected via paths in the network. But, the particular structure of the network did not matter, as long as the coalition's members were connected somehow. In the approach taken here (following Jackson and Wolinsky (1996)), the value is a function that is allowed to depend on the specific network structure. A special case is where $v(g)$ only depends on the coalitions induced by the component structure of g, which corresponds to the communication games.

across networks. This means that a value function allows for externalities both within and across components of a network.

Allocation Rules

A value function only keeps track of how the total societal value varies across different networks. We also wish to keep track of how that value is allocated or distributed among the individuals forming a network.

An *allocation rule* is a function $Y : G \times \mathcal{V} \to I\!R^N$ such that $\sum_i Y_i(g, v) = v(g)$ for all v and g.[8]

It is important to note that an allocation rule depends on both g and v. This allows an allocation rule to take full account of an individual i's role in the network. This includes not only what the network configuration is, but also and how the value generated depends on the overall network structure. For instance, consider a network $g = \{12, 23\}$ in a situation where $v(g) = 1$. Individual 2's allocation might be very different on what the value of other networks are. For instance, if $v(\{12, 23, 13\}) = 0 = v(\{13\})$, then in a sense 2 is essential to the network and may receive a large allocation. If on the other hand $v(g') = 1$ for all networks, then 2's role is not particularly special. This information can be relevant, which is why the allocation rule is allowed (but not required) to depend on it.

There are two different perspectives on allocation rules that will be important in different contexts. First, an allocation rule may simply represent the natural payoff to different individuals depending on their role in the network. This could include bargaining among the individuals, or any form of interaction. This might be viewed as the "naturally arising allocation rule" and is illustrated in the examples below. Second, an allocation rule can be an object of economic design, i.e., representing net payoffs resulting from natural payoffs coupled with some intervention via transfers, taxes, or subsidies. In what follows we will be interested in when the natural underlying payoffs lead individuals to form efficient networks, as well as when intervention can help lead to efficient networks.

Before turning to that analysis, let us consider some examples of models of social and economic networks and the corresponding value functions and allocation rules that describe them.

Some Illustrative Examples

[8] This definition builds balance ($\sum_i Y_i(g, v) = v(g)$) into the definition of allocation rule. This is without loss of generality for the discussion in this paper, but there may be contexts in which imbalanced allocation rules are of interest.

Example 1 *The Connections Model* (Jackson and Wolinsky (1996))

The basic connections model is described as follows. Links represent social relationships between individuals; for instance friendships. These relationships offer benefits in terms of favors, information, etc., and also involve some costs. Moreover, individuals also benefit from indirect relationships. A "friend of a friend" also results in some benefits, although of a lesser value than a "friend," as do "friends of a friend of a friend" and so forth. The benefit deteriorates in the "distance" of the relationship. For instance, in the network $g = \{12, 23, 34\}$ individual 1 gets a benefit δ from the direct connection with individual 2, an indirect benefit δ^2 from the indirect connection with individual 3, and an indirect benefit δ^3 from the indirect connection with individual 4. For $\delta < 1$ this leads to a lower benefit from an indirect connection than a direct one. Individuals only pay costs, however, for maintaining their direct relationships. These payoffs and benefits may be relation specific, and so are indexed by ij.

Formally, the payoff player i receives from network g is

$$Y_i(g) = \sum_{j \neq i} \delta_{ij}^{t(ij)} - \sum_{j: ij \in g} c_{ij},$$

where $t(ij)$ is the number of links in the shortest path between i and j (setting $t(ij) = \infty$ if there is no path between i and j).[9] The value function in the connections model of a network g is simply $v(g) = \sum_i Y_i(g)$.

Some special cases are of particular interest. The first is the "symmetric connections model" where there are common δ and c such that $\delta_{ij} = \delta$ and $c_{ij} = c$ for all i and j. This case is studied extensively in Jackson and Wolinsky (1996).

The second is one with spatial costs, where there is a geography to locations and c_{ij} is related to distance (for instance, if individuals are spaced equally on a line then costs are proportional to $|i - j|$). This is studied extensively in Johnson and Gilles (2000).

Example 2 *The Co-Author Model* (Jackson and Wolinsky (1996))

The co-author model is described as follows. Each individual is a researcher who spends time working on research projects. If two researchers are connected, then they are working on a project together. The amount of time researcher i spends on a

[9] $t(ij)$ is sometimes referred to as the geodesic.

given project is inversely related to the number of projects, n_i, that he is involved in. Formally, i's payoff is represented by

$$Y_i(g) = \sum_{j:ij \in g} \frac{1}{n_i} + \frac{1}{n_j} + \frac{1}{n_i n_j}$$

for $n_i > 0$, and $Y_i(g) = 0$ if $n_i = 0$.[10] The total value is $v(g) = \sum_i Y_i(g)$.

Note that in the co-author model there are no directly modeled costs to links. Costs come indirectly in terms of diluted synergy in interaction with co-authors.

Example **3** *A Bilateral Bargaining Model* (Corominas-Bosch (1999))

Corominas-Bosch (1999) considers a bargaining model where buyers and sellers bargain over prices for trade. A link is necessary between a buyer and seller for a transaction to occur, but if an individual has several links then there are several possibilities as to whom they might transact with. Thus, the network structure essentially determines bargaining power of various buyers and sellers.

More specifically, each seller has a single unit of an indivisible good to sell which has no value to the seller. Buyers have a valuation of 1 for a single unit of the good. If a buyer and seller exchange at a price p, then the buyer receives a payoff of $1 - p$ and the seller a payoff of p. A link in the network represents the opportunity for a buyer and seller to bargain and potentially exchange a good.[11]

Corominas-Bosch models bargaining via the following variation on a Rubinstein bargaining protocol. In the first period sellers simultaneously each call out a price. A buyer can only select from the prices that she has heard called out by the sellers to whom she is linked. Buyers simultaneously respond by either choosing to accept some single price offer they received, or to reject all price offers they received. [12] If there are several sellers who have called out the same price and/or several buyers who have accepted the same price, and there is any discretion under the given network connections as to which trades should occur, then there is a careful protocol for determining

[10] It might also make sense to set $Y_i(g) = 1$ when an individual has no links, as the person can still produce reseach! This is not in keeping with the normalization of $v(\emptyset) = 0$, but it is easy to simply subtract 1 from all payoffs and then view Y as the extra benefits above working alone.

[11] In the Corominas-Bosch framework links can only form between buyers and sellers. One can fit this into the more general setting where links can form between any individuals, by having the value function and allocation rule ignore any links except those between buyers and sellers.

[12] So buyers accept or reject price offers, rather than accepting or rejecting the offer of some specific seller.

which trades occur (which is essentially designed to maximize the number of eventual transactions).

At the end of the period, trades are made and buyers and sellers who have traded are cleared from the market. In the next period the situation reverses and buyers call out prices. These are then either accepted or rejected by the sellers connected to them in the same way as described above. Each period the role of proposer and responder switches and this process repeats itself indefinitely, until all remaining buyers and sellers are not linked to each other.

Buyers and sellers are impatient and discount according to a common discount factor $0 < \delta < 1$. So a transaction at price p in period t is worth only $\delta^t p$ to a seller and $\delta^t(1-p)$ to a buyer.

Corominas-Bosch outlines a subgame perfect equilibrium of the above game, and this equilibrium has a very nice interpretation as the discount factor approaches 1.

Some easy special cases are as follows. First, consider a seller linked to each of two buyers, who are only linked to that seller. Competition between the buyers to accept the price will lead to an equilibrium price of 1. So the payoff to the seller in such a network will be 1 while the payoff to the buyers will be 0. This is reversed for a single buyer linked to two sellers. Next, consider a single seller linked to a single buyer. That corresponds to Rubinstein bargaining, and so the price (in the limit as $\delta \to 1$) is $1/2$, as are the payoffs to the buyer and seller.

More generally, which side of the market outnumbers the other is a bit tricky to determine as it depends on the overall link structure which can be much more complicated than that described above. Quite cleverly, Corominas-Bosch describes an algorithm[13] for subdividing any network into three types of sub-networks: those where a set of sellers are collectively linked to a larger set of buyers and sellers get payoffs of 1 and buyers 0, those where the collective set of sellers is linked to a same-sized collective set of buyers and each get payoff of $1/2$, and those where sellers outnumber

[13]The decomposition is based on Hall's (marriage) Theorem, and works roughly as follows. Start by identifying groups of two or more sellers who are all linked only to the same buyer. Regardless of that buyer's other connections, take that set of sellers and buyer out as a subgraph where that buyer gets a payoff of 1 and the sellers all get payoffs of 0. Proceed, inductively in k, to identify subnetworks where some collection of more than k sellers are collectively linked to k or fewer buyers. Next reverse the process and progressively in k look for at least k buyers collectively linked to fewer than k sellers, removing such subgraphs and assigning those sellers payoffs of 1 and buyers payoffs of 0. When all such subgraphs are removed, the remaining subgraphs all have "even" connections and earn payoffs of $1/2$.

buyers and sellers get payoffs of 0 and buyers 1. This is illustrated in Figure 1 for a few networks.

[Insert Figure 1 here]

While the algorithm prevents us from providing a simple formula for the allocation rule in this model, the important characteristics of the allocation rule for our purposes can be summarized as follows.

(i) if a buyer gets a payoff of 1, then some seller linked to that buyer must get a payoff of 0, and similarly if the roles are reversed,

(ii) a buyer and seller who are only linked to each other get payoffs of 1/2, and

(iii) a connected component is such that all buyers and all sellers get payoffs of 1/2 if and only if any subgroup of k buyers in the component can be matched with at least k distinct sellers and vice versa.

In what follows, I will augment the Corominas-Bosch model to consider a cost to each link of c_s for sellers and c_b for buyers. So the payoff to an individual is their payoff from any trade via the bargaining on the network, less the cost of maintaining any links that they are involved with.

EXAMPLE **4** *A Model of Buyer-Seller Networks* (Kranton and Minehart (1998))

The Kranton and Minehart model of buyer-seller networks is similar to the Corominas-Bosch model described above except that the valuations of the buyers for a good are random and the determination of prices is made through an auction rather than alternating offers bargaining.

The Kranton and Minehart model is described as follows. Again, each seller has an indivisible object for sale. Buyers have independently and identically distributed utilities for the object, denoted u_i. Each buyer knows her own valuation, but only the distribution over other buyers' valuations, and similarly sellers know only the distribution of buyers' valuations.

Again, link patterns represent the potential transactions, however, the transactions and prices are determined by an auction rather than bargaining.In particular, prices rise simultaneously across all sellers. Buyers drop out when the price exceeds their valuation (as they would in an English or ascending oral auction). As buyers drop out, there emerge sets of sellers for whom the remaining buyers still linked to those sellers is no larger than the set of sellers. Those sellers transact with the buyers still linked to them.[14] The exact matching of whom trades with whom given the link pattern is done carefully to maximize the number of transactions. Those sellers and buyers are cleared from the market, and the prices continue to rise among remaining sellers, and the process repeats itself.

For each link pattern every individual has a well-defined expected payoff from the above described process (from an ex-ante perspective before buyers know their u_i's). From this expected payoff can be deducted costs of links to both buyers and sellers.[15]

This leads to well-defined allocation rules Y_i's and a well-defined value function v. The main intuitions behind the Kranton and Minehart model are easily seen in a simple case, as follows.

Consider a situation with one seller and n buyers. Let the u_i's be uniformly and independently distributed on $[0,1]$. In this case the auction simplifies to a standard second-price auction. If k is the number of buyers linked to the seller, the expected revenue to the seller is the second order statistic out of k, which is $\frac{k-1}{k+1}$ for a uniform distribution. The corresponding expected payoff to a bidder is $\frac{1}{k(k+1)}$.[16]

For a cost per link of c_s to the seller and c_b to the buyer, the allocation rule for any network g with $k \geq 1$ links between the buyers and seller is[17]

$$Y_i(g) = \begin{cases} \frac{1}{k(k+1)} - c_b & \text{if } i \text{ is a linked buyer} \\ \frac{k-1}{k+1} - kc_s & \text{if } i \text{ is the seller} \\ 0 & \text{if } i \text{ is a buyer without any links.} \end{cases} \tag{1}$$

The value function is then

$$v(g) = \sum_i Y_i(g) = \frac{k}{k+1} - k(c_s + c_b).$$

[14] It is possible, that several buyers drop out at once and so one or more of the buyers dropping out will be selected to transact at that price.

[15] Kranton and Minehart (1998) only consider costs of links to buyers. They also consider potential investment costs to sellers of producing a good for sale, but sellers do not incur any cost per link. Here, I will consider links as being costly to sellers as well as buyers.

[16] Each bidder has a $\frac{1}{k}$ chance of being the highest valued bidder. The expected valuation of the highest bidder for k draws from a uniform distribution on [0,1] is $\frac{k}{k+1}$, and the expected price is the expected second highest valuation which is $\frac{k-1}{k+1}$. Putting these together, the ex-ante expected payoff to any single bidder is $\frac{1}{k}\left(\frac{k}{k+1} - \frac{k-1}{k+1}\right) = \frac{1}{k(k+1)}$.

[17] For larger numbers of sellers, the Y_i's correspond to the V_i^b and V_i^s's in Kranton and Minehart (1999) (despite their footnote 16) with the subtraction here of a cost per link for sellers.

Thus, the total value of the network is simply the expected value of the good to the highest valued buyer less the cost of links.

Similar calculations can be done for larger numbers of sellers and more general network structures.

Some Basic Properties of Value and Allocation Functions

Component Additivity

A value function is *component additive* if $v(g) = \sum_{g' \in C(g)} v(g')$ for all $g \in G$.

Component additive value functions are ones for which the value of a network is simply the sum of the value of its components. This implies that the value of one component does not depend on the structure of other components. This condition is satisfied in Examples 1–4, and is satisfied in many economic and social situations. It still allows for arbitrary ways in value can depend on the network configuration within a component. Thus, it allows for externalities among individuals within a component.

An example where component additivity is violated is that of alliances among competing firms (e.g., see Goyal and Joshi (2000)), where the payoff to one set of interconnected firms may depend on how other competing firms are interconnected. So, what component additivity rules out is externalities *across* components of a network, but it still permits them within components.

Component Balance

When a value function is component additive, the value generated by any component will often naturally be allocated to the individuals among that component. This is captured in the following definition.

An allocation rule Y is component balanced if for any component additive v, $g \in G$, and $g' \in C(g)$

$$\sum_{i \in N(g')} Y_i(g', v) = v(g').$$

Note that component balance only makes requirements on Y for v's that are component additive, and Y can be arbitrary otherwise. If v is not component additive, then requiring component balance of an allocation rule $Y(\cdot, v)$ would necessarily violate balance.

Component balance is satisfied in situations where Y represents the value naturally accruing in terms of utility or production, as the members of a given component have no incentive to distribute productive value to members outside of their component, given that there are no externalities across components (i.e., a component balanced v). This is the case in Examples 1–4, as in many other contexts.

Component balance may also be thought of as a normative property that one wishes to respect if Y includes some intervention by a government or outside authority - as it requires that that value generated by a given component be allocated among the members of that component. An important thing to note is that if Y violates component balance, then there will be some component receiving less than its net productive value. That component could improve the standing of all its members by seceding. Thus, one justification for the condition is as a component based participation constraint.[18]

Anonymity and Equal Treatment

Given a permutation of individuals π (a bijection from N to N) and any $g \in G$, let $g^\pi = \{\pi(i)\pi(j)|ij \in g\}$. Thus, g^π is a network that shares the same architecture as g but with the specific individuals permuted.

A value function is *anonymous* if for any permutation π and any $g \in G$, $v(g^\pi) = v(g)$.

Anonymous value functions are those such that the architecture of a network matters, but not the labels of individuals.

Given a permutation π, let v^π be defined by $v^\pi(g) = v(g^{\pi^{-1}})$ for each $g \in G$.

An allocation rule Y is *anonymous* if for any v, $g \in G$, and permutation π, $Y_{\pi(i)}(g^\pi, v^\pi) = Y_i(g, v)$.

Anonymity of an allocation rule requires that if all that has changed is the labels of the agents and the value generated by networks has changed in an exactly corresponding fashion, then the allocation only change according to the relabeling. Of course, anonymity is a type of fairness condition that has a rich axiomatic history, and also naturally arises situations where Y represents the utility or productive value coming directly from some social network.

Note that anonymity allows for asymmetries in the ways that allocation rules operate even in completely symmetric networks. For instance, anonymity does *not* require that each individual in a complete network get the same allocation. That would be

[18]This is a bit different from a standard individual rationality type of constraint given some outside option, as it may be that the value generated by a component is negative.

true only in the case where v was in fact anonymous. Generally, an allocation rule can respond to different roles or powers of individuals and still be anonymous.

An allocation rule Y satisfies *equal treatment of equals* if for any anonymous $v \in \mathcal{V}$, $g \in G$, $i \in N$, and permutation π such that $g^\pi = g$, $Y_{\pi(i)}(g, v) = Y_i(g, v)$.

Equal treatment of equals says that all allocation rule should give the same payoff to individuals who play exactly the same role in terms of symmetric position in a network under a value function that depends only on the structure of a network. This is implied by anonymity, which is seen by noting that $(g^\pi, v^\pi) = (g, v)$ for any anonymous v and a π as described in the definition of equal treatment of equals. Equal treatment of equals is more of a symmetry condition that anonymity, and again is a condition that has a rich background in the axiomatic literature.

Some Prominent Allocation Rules

There are several allocation rules that are of particular interest that I now discuss. The first naturally arises in situations where the allocation rule comes from some bargaining (or other process) where the benefits that accrue to the individuals involved in a link are split equally among those two individuals.

Equal Bargaining Power and the Myerson Value

An allocation rule satisfies equal bargaining power if for any component additive v and $g \in G$

$$Y_i(g) - Y_i(g - ij) = Y_j(g) - Y_j(g - ij).$$

Note that equal bargaining power does *not* require that individuals split the marginal value of a link. It just requires that they equally benefit or suffer from its addition. It is possible (and generally the case) that $Y_i(g) - Y_i(g - ij) + Y_j(g) - Y_j(g - ij) \neq v(g) - v(g - ij)$.

It was first shown by Myerson (1977), in the context of communication games, that such a condition leads to an allocation that is a variation on the Shapley value. This rule was subsequently referred to as the Myerson value (e.g., see Aumann and Myerson (1988)).

The Myerson value also has a corresponding allocation rule in the context of networks beyond communication games, as shown by Jackson and Wolinsky (1996). That allocation rule is expressed as follows.

Let

$$g|_S = \{ij \ : \ ij \in g \text{ and } i \in S, j \in S\}.$$

Thus $g|_S$ is the network found deleting all links except those that are between individuals in S.

$$Y_i^{MV}(g,v) = \sum_{S \subset N \setminus \{i\}} (v(g|_{S \cup i}) - v(g|_S)) \left(\frac{\#S!(n - \#S - 1)!}{n!} \right) \tag{2}$$

The following proposition from Jackson and Wolinsky (1996) is an extension of Myerson's (1977) result from the communication game setting to the network setting.

Proposition 1 (Myerson (1977), Jackson and Wolinsky (1996))[19] *Y satisfies component balance and equal bargaining power if and only if $Y(g,v) = Y^{MV}(g,v)$ for all $g \in G$ and any component additive v.*

The surprising aspect of equal bargaining power is that it has such strong implications for the structuring of the allocation rule.

Egalitarian Rules

Two other allocation rules that are of particular interest are the egalitarian and component-wise egalitarian rule.

The *egalitarian* allocation rule Y^e is defined by

$$Y_i^e(g,v) = \frac{v(g)}{n}$$

for all i and g.

The egalitarian allocation rule splits the value of a network equally among all members of a society regardless of what their role in the network is. It is clear that the egalitarian allocation rule will have very nice properties in terms of aligning individual incentives with efficiency.

However, the egalitarian rule violates component balance. The following modification of the egalitarian rule respects component balance.

The *component-wise egalitarian* allocation rule Y^{ce} is defined as follows for component additive v's and any g.

$$Y_i^{ce}(g,v) = \begin{cases} \frac{v(h)}{|N(h)|} & \text{if there exists } h \in C(g) \text{ such that } i \in h, \\ 0 & \text{otherwise.} \end{cases}$$

For any v that is not component additive, set $Y^{ce}(\cdot, v) = Y^e(\cdot, v)$.

[19] Dutta and Mutuswami (1997) extend the characterization to allow for weighted bargaining power, and show that one obtains a version of a weighted Shapley (Myerson) value.

The component-wise egalitarian splits the value of a component network equally among all members of that component, but makes no transfers across components.

The component-wise egalitarian rule has some nice properties in terms of aligning individual incentives with efficiency, although not quite to the extent that the egalitarian rule does.[20]

3 Defining Efficiency

In evaluating societal welfare, we may take various perspectives. The basic notion used is that of Pareto efficiency - so that a network is inefficient if there is some other network that leads to higher payoffs for all members of the society. The differences in perspective derive from the degree to which transfers can be made between individuals in determining what the payoffs are.

One perspective is to see how well society functions on its own with no outside intervention (i.e., where Y arises naturally from the network interactions). We may also consider how the society fares when some intervention in the forms of redistribution takes place (i.e., where Y also incorporates some transfers). Depending on whether we allow arbitrary transfers or we require that such intervention satisfy conditions like anonymity and component balance, we end up with different degrees to which value can be redistributed. Thus, considering these various alternatives, we are led to several different definitions of efficiency of a network, depending on the perspective taken. Let us examine these in detail. I begin with the weakest notion.

Pareto Efficiency

A network g is *Pareto efficient* relative to v and Y if there does not exist any $g' \in G$ such that $Y_i(g', v) \geq Y_i(g, v)$ for all i with strict inequality for some i.

This definition of efficiency of a network takes Y as fixed, and hence can be thought of as applying to situations where no intervention is possible.

Next, let us consider the strongest notion of efficiency.[21]

Efficiency

A network g is *efficient* relative to v if $v(g) \geq v(g')$ for all $g' \in G$.

[20] See Jackson and Wolinsky (1996) Section 4 for some detailed analysis of the properties of the egalitarian and component-wise egalitarian rules.

[21] This notion of efficiency was called strong efficiency in Jackson and Wolinsky (1996).

This is a strong notion of efficiency as it takes the perspective that value is fully transferable. This applies in situations where unlimited intervention is possible, so that any naturally arising Y can be redistributed in arbitrary ways.

Another way to express efficiency is to say that g is efficient relative to v if it is Pareto efficient relative to v and Y *for all* Y. Thus, we see directly that this notion is appropriate in situations where one believes that arbitrary reallocations of value are possible.

Constrained Efficiency

The third notion of efficiency falls between the other two notions. Rather than allowing for arbitrary reallocations of value as in efficiency, or no reallocations of value as in Pareto efficiency, it allows for reallocations that are anonymous and component balanced.

A network g is *constrained efficient* relative to v if there does not exist any $g' \in G$ and a component balanced and anonymous Y such that $Y_i(g', v) \geq Y_i(g, v)$ for all i with strict inequality for some i.

Note that g is constrained efficient relative to v if and only if it is Pareto efficient relative to v and Y for every component balanced and anonymous Y.

There exist definitions of constrained efficiency for any class of allocation rules that one wishes to consider. For instance, one might also consider that class of component balanced allocation rules satisfying equal treatment of equals, or any other class that is appropriate in some context.

The relationship between the three definitions of efficiency we consider here is as follows. Let $PE(v, Y)$ denote the Pareto efficient networks relative to v and Y, and similarly let $CE(v)$ and $E(v)$ denote the constrained efficient and efficient networks relative to v, respective.

Remark: If Y is component balanced and anonymous, then $E(v) \subset CE(v) \subset PE(v, Y)$.

Given that there always exists an efficient network (any network that maximizes v, and such a network exists as G is finite), it follows that there also exist constrained efficient and Pareto efficient networks.

Let us also check that these definitions are distinct.

Example **5** $E(v) \neq CE(v)$

Let $n = 5$ and consider an anonymous and component additive v such that the complete network g^N has value 10, a component consisting of pair of individuals with one link between them has value 2, and a completely connected component among three individuals has value 9. All other networks have value 0.

The only efficient networks are those consisting of two components: one component consisting of a pair of individuals with one link and the other component consisting of a completely connected triad (set of three individuals). However, the completely connected network is constrained efficient.

To see that the completely connected network is constrained efficient even though it is not efficient, first note that any anonymous allocation rule must give each individual a payoff of 2 in the complete network. Next, note that the only network that could possibly give a higher allocation to all individuals is an efficient one consisting of two components: one dyad and one completely connected triad. Any component balanced and anonymous allocation rule must allocate payoffs of 3 to each individual in the triad, and 1 to each individual in the dyad. So, the individuals in the dyad are worse off than they were under the complete network. Thus, the fully connected network is Pareto efficient under every Y that is anonymous and component balanced. This implies that the fully connected network is constrained efficient even though it is not efficient. This is pictured in Figure 2.

[Insert Figure 2 here.]

Example **6** $CE(v) \neq PE(v, Y)$

Let $n = 3$. Consider an anonymous v where the complete network has a value of 9, a network with two links has a value of 8, and a network of a single link network has any value.

Consider a component balanced and anonymous Y that allocates 3 to each individual in the complete network, and in any network with two links allocates 2 to each of the individuals with just one link and 4 to the individual with two links (and splits value equally among the two individuals in a link if there is just one link). The network $g = \{12, 23\}$ is Pareto efficient relative to v and Y, since any other network results in a lower payoff to at least one of the players (for instance, $Y_2(g, v) = 4$, while $Y_2(g^N, v) = 3$). The network g is not constrained efficient, since under the component balanced and anonymous rule $\overline{Y}$ such that $\overline{Y}_1(g, v) = \overline{Y}_2(g, v) = \overline{Y}_3(g, v) = 8/3$, all individuals prefer to be in the complete network g^N where they receive payoffs of 3. See Figure 3.

[Insert Figure 3 here.]

4 Modeling Network Formation

A simple, tractable, and natural way to analyze the networks that one might expect to emerge in the long run is to examine a sort of equilibrium requirement that individuals not benefit from altering the structure of the network. A weak version of such a condition is the following pairwise stability notion defined by Jackson and Wolinsky (1996).

Pairwise Stability

A network g is *pairwise stable* with respect to allocation rule Y and value function v if

(i) for all $ij \in g$, $Y_i(g,v) \geq Y_i(g-ij,v)$ and $Y_j(g,v) \geq Y_j(g-ij,v)$, and

(ii) for all $ij \notin g$, if $Y_i(g+ij,v) > Y_i(g,v)$ then $Y_j(g+ij,v) < Y_j(g,v)$.

Let us say that g' is *adjacent* to g if $g' = g + ij$ or $g' = g - ij$ for some ij.

A network g' *defeats* g if either $g' = g - ij$ and $Y_i(g',v) > Y_i(g',v)$, or if $g' = g + ij$ with $Y_i(g',v) \geq Y_i(g',v)$ and $Y_i(g',v) \geq Y_i(g',v)$ with at least one inequality holding strictly.

Pairwise stability is equivalent to saying that a network is pairwise stable if it is not defeated by another (necessarily adjacent) network.

There are several aspects of pairwise stability that deserve discussion.

First, it is a very weak notion in that it only considers deviations on a single link at a time. If other sorts of deviations are viable and attractive, then pairwise stability may be too weak a concept. For instance, it could be that an individual would not benefit from severing any single link but would benefit from severing several links simultaneously, and yet the network would still be pairwise stable. Second, pairwise stability considers only deviations by at most a pair of individuals at a time. It might be that some group of individuals could all be made better off by some more complicated reorganization of their links, which is not accounted for under pairwise stability.

In both of these regards, pairwise stability might be thought of as a necessary but not sufficient requirement for a network to be stable over time. Nevertheless, we will see that pairwise stability already significantly narrows the class of networks to the point where efficiency and pairwise stability are already in tension at times.

There are alternative approaches to modeling network stability. One is to explicitly model a game by which links form and then to solve for an equilibrium of that game. Aumann and Myerson (1988) take such an approach in the context of communication games, where individuals sequentially propose links which are then accepted or rejected. Such an approach has the advantage that it allows one to use off-the-shelf game theoretic tools. However, such an approach also has the disadvantage that the game is necessarily ad hoc and fine details of the protocol (e.g., the ordering of who proposes links when, whether or not the game has a finite horizon, individuals are impatient, etc.) may matter. Pairwise stability can be thought of as a condition identifies networks that are the only ones that could emerge at the end of any well defined game where players where the process does not artificially end, but only ends when no player(s) wish to make further changes to the network.

Dutta and Mutuswami (1997) analyze the equilibria of a link formation game under various solution concepts and outline the relationship between pairwise stability and equilibria of that game. The game is one first discussed by Myerson (1991). Individuals simultaneously announce all the links they wish to be involved in. Links form if both individuals involved have announced that link. While such games have a multiplicity of unappealing Nash equilibria (e.g., nobody announces any links), using strong equilibrium and coalition-proof Nash equilibrium, and variations on strong equilibrium where only pairs of individuals might deviate, lead to nicer classes of equilibria. The networks arising in variations of the strong equilibrium are in fact subsets of the pairwise stable networks.[22]

Finally, there is another aspect of network formation that deserves attention. The above definitions (including some of the game theoretic approaches) are both static and myopic. Individuals do not forecast how others might react to their actions. For instance, the adding or severing of one link might lead to the subsequent addition or severing of another link. Dynamic (but still myopic) network formation processes are studied by Watts (2001) and Jackson and Watts (1998), but a fully dynamic and forward looking analysis of network formation is still missing. [23]

[22]See Jackson and van den Nouweland (2000) for additional discussion of coalitional stability notions and the relationship to core based solutions.

[23]The approach of Aumann and Myerson (1988) is a sequential game and so forward thinking is incorporated to an extent. However, the finite termination of their game provides an artificial way by which one can put a limit on how far forward players have to look. This permits a solution of the

Myopic considerations on the part of the individuals in a network are natural in large situations where individuals might be faced with the consideration of adding or severing a given link, but might have difficulty in forecasting the reactions to this. For instance, in deciding whether or not a firm wishes to connect its computer system to the internet, the firm might not forecast the impact of that decision on the future evolution of the internet. Likewise in forming a business contact or friendship, an individual might not forecast the impact of that new link on the future evolution of the network. Nevertheless, there are other situations, such as strategic alliances among airlines, where individuals might be very forward looking in forecasting how others will react to the decision. Such forward looking behavior has been analyzed in various contexts in the coalition formation literature (e.g., see Chwe (1994)), but is still an important issue for further consideration in the network formation literature.[24]

Existence of Pairwise Stable Networks

In some situations, there may not exist any pairwise stable network. It may be that each network is defeated by some adjacent network, and that these "improving paths" form cycles with no undefeated networks existing.[25]

An *improving path* is a sequence of networks $\{g_1, g_2, \ldots, g_K\}$ where each network g_k is defeated by the subsequent (adjacent) network g_{k+1}.

A network is pairwise stable if and only if it has no improving paths emanating from it. Given the finite number of networks, it then directly follows that if there does not exist any pairwise stable network, then there must exist at least one cycle, i.e., an improving path $\{g_1, g_2, \ldots, g_K\}$ where $g_1 = g_K$. The possibility of cycles and non-existence of a pairwise stable network is illustrated in the following example.

Example **7** *Exchange Networks – Non-existence of a Pairwise Stable Network* (Jackson and Watts (1998))

game via backward induction, but does not seem to provide an adequate basis for a study of such forward thinking behavior. A more truly dynamic setting, where a network stays in place only if no player(s) wish to change it given their forecasts of what would happen subsequently, has not been analyzed.

[24] It is possible that with some forward looking aspects to behavior, situations are plausible where a network that is not pairwise stable emerges. For instance, individuals might not add a link that appears valuable to them given the current network, as that might in turn lead to the formation of other links and ultimately lower the payoffs of the original individuals. This is an important consideration that needs to be examined.

[25] Improving paths are defined by Jackson and Watts (1998, 2002), who provide some additional results on existence of pairwise stable networks.

The society consists of $n \geq 4$ individuals who get value from trading goods with each other. In particular, there are two consumption goods and individuals all have the same utility function for the two goods which is Cobb-Douglas, $u(x, y) = xy$. Individuals have a random endowment, which is independently and identically distributed. A individual's endowment is either (1,0) or (0,1), each with probability 1/2.

Individuals can trade with any of the other individuals in the same component of the network. For instance, in a network $g = \{12, 23, 45\}$, individuals 1, 2 and 3 can trade with each other and individuals 4 and 5 can trade with each other, but there is no trade between 123 and 45. Trade flows without friction along any path and each connected component trades to a Walrasian equilibrium. This means, for instance, that the networks $\{12, 23\}$ and $\{12, 23, 13\}$ lead to the same expected trades, but lead to different costs of links.

The network $g = \{12\}$ leads to the following payoffs. There is a $\frac{1}{2}$ probability that one individual has an endowment of (1,0) and the other has an endowment of (0,1). They then trade to the Walrasian allocation of $(\frac{1}{2}, \frac{1}{2})$ each and so their utility is $\frac{1}{4}$ each. There is also a $\frac{1}{2}$ probability that the individuals have the same endowment and then there are no gains from trade and they each get a utility of 0. Expecting over these two situations leads to an expected utility of $\frac{1}{8}$. Thus, $Y_1(\{12\}) = Y_2(\{12\}) = \frac{1}{8} - c$, where c is the cost (in utils) of maintaining a link. One can do similar calculations for a network $\{12, 23\}$ and so forth.

Let the cost of a link $c = \frac{5}{96}$ (to each individual in the link).

Let us check that there does not exist a pairwise stable network. The utility of being alone is 0. Not accounting for the cost of links, the expected utility for a individual of being connected to one other is $\frac{1}{8}$. The expected utility for a individual of being connected (directly or indirectly) to two other individuals is $\frac{1}{6}$; and of being connected to three other individuals is $\frac{3}{16}$. It is easily checked that the expected utility of a individual is increasing and strictly concave in the number of other individuals that she is directly or indirectly connected to, ignoring the cost of links.

Now let us account for the cost of links and argue that there cannot exist any pairwise stable network. Any component in a pairwise stable network that connects k individuals must have exactly $k - 1$ links, as some additional link could be severed without changing the expected trades but saving the cost of the link. Also, any component in a pairwise stable network that involves 3 or more individuals cannot contain a individual who has just one link. This follows from the fact that a individual connected to some individual who is not connected to anyone else, loses at most $\frac{1}{6} - \frac{1}{8} = \frac{1}{24}$ in expected utility from trades by severing the link, but saves the cost of $\frac{5}{96}$ and so should sever this link. These two observations imply that a pairwise stable network must consist of pairs of connected individuals (as two completely unconnected individuals benefit from forming a link), with one unconnected individual if n is odd. However, such a network is not pairwise stable, since any two individuals in different pairs gain from forming a link (their utility goes from $\frac{1}{8}-\frac{5}{96}$ to $\frac{3}{16}-\frac{5}{96}$). Thus, there is no pairwise stable network. This is illustrated in Figure 4.

[Insert Figure 4 here.]

A cycle in this example is $\{12, 34\}$ is defeated by $\{12, 23, 34\}$ which is defeated by $\{12, 23\}$ which is defeated by $\{12\}$ which is defeated by $\{12, 34\}$.

Existence of Pairwise Stable Networks under the Myerson Value

While the above example shows that pairwise stable networks may not exist in some settings for some allocation rules, there are interesting allocation rules for which pairwise stable networks always exist.

Existence of pairwise stable networks is straightforward for the egalitarian and component-wise egalitarian allocation rules. Under the egalitarian rule, any efficient network will be pairwise stable. Under the component-wise egalitarian rule, one can also always find a pairwise stable network. An algorithm is as follows:[26] find a component h that maximizes the payoff $Y_i^{ce}(h, v)$ over i and h. Next, do the same on the remaining population $N \setminus N(h)$, and so on. The collection of resulting components forms the network.[27]

What is less transparent, is that the Myerson value allocation rule also has very nice existence properties. Under the Myerson value allocation rule there always exists a pairwise stable network, all improving paths lead to pairwise stable networks, and there are no cycles. This is shown in the following Proposition.

PROPOSITION **2** *There exists a pairwise stable network relative to Y^{MV} for every $v \in \mathcal{V}$. Moreover, all improving paths (relative to Y^{MV}) emanating from any network (under any $v \in \mathcal{V}$) lead to pairwise stable networks. Thus, there are no cycles under the Myerson value allocation rule.*

[26]This is specified for component additive v's. For any other v, Y^e and Y^{ce} coincide.

[27]This follows the same argument as existence of core-stable coalition structures under the weak top coalition property in Banerjee, Konishi and Sönmez (2001). However, these networks are not necessarily stable in a stronger sense (against coalitional deviations). A characterization for when such strongly stable networks exist appears in Jackson and van den Nouweland (2001).

Proof of Proposition 2: Let

$$F(g) = \sum_{T \subset N} v(g|_T) \left(\frac{(|T|-1)!(n-|T|)!}{n!} \right).$$

Straightforward calculations that are left to the reader verify that for any g, i and $ij \in g$ [28]

$$Y_i^{MV}(g,v) - Y_i^{MV}(g-ij,v) = F(g) - F(g-ij). \tag{3}$$

Let g^* maximize $F(\cdot)$. Thus $0 \geq F(g^*+ij) - F(g^*)$ and likewise $0 \geq F(g^*-ij) - F(g^*)$ for all ij. It follows from (3) that g^* is pairwise stable.

To see the second part of the proposition, note that (3) implies that along any improving path F must be increasing. Such an increasing path in F must lead to g which is a local maximizer (among adjacent networks) of F. By (3) it follows that g is pairwise stable.[29] ▮

5 The Compatibility of Efficiency and Stability

Let us now turn to the central question of the relationship between stability and efficiency of networks.

As mentioned briefly above, if one has complete control over the allocation rule and does not wish to respect component balance, then it is easy to guarantee that all efficient networks are pairwise stable: simply use the egalitarian allocation rule Y^e. While this is partly reassuring, we are also interested in knowing whether it is generally the case that some efficient network is pairwise stable without intervention, or with intervention that respects component balance. The following proposition shows that there is no component balanced and anonymous allocation rule for which it is always the case that some constrained efficient network is pairwise stable.

[28] It helps in these calculations to note that if $i \notin T$ then $g|_T = g - ij|_T$. Note that F is what is known as a potential function (see Monderer and Shapley (1996)). Based on some results in Monderer and Shapley (1996) (see also Quin (1996)), potential functions and the Shapley value have a special relationship; and it may be that there is a limited converse to Proposition 2.

[29] Jackson and Watts (1998, working paper version) show that for any Y and v there exist no cycles (and thus there exist pairwise stable networks and all improving paths lead to pairwise stable networks) if and only if there exists a function $F : G \to I\!R$ such that g defeats g' if and only if $F(g) > F(g')$. Thus, the existence of the F satisfying (3) in this proof is actually a necessary condition for such nicely behaved improving paths.

Proposition 3 *There does not exist any component balanced and anonymous allocation rule (or even a component balanced rule satisfying equal treatment of equals) such that for every v there exists a constrained efficient network that is pairwise stable.*

Proposition 3 strengthens Theorem 1 in Jackson and Wolinsky (1996) in two ways: first it holds under equal treatment of equals rather than anonymity, and second it applies to constrained efficiency rather than efficiency. Most importantly, the consideration of constrained efficiency is more natural that the consideration of the stronger efficiency notion, given that it applies to component balanced and anonymous allocation rules.

The proof of Proposition 3 shows that there is a particular v such that for every component balanced and anonymous allocation rule none of the constrained efficient networks are pairwise stable. It uses the same value function as Jackson and Wolinsky (1996) used to prove a similar proposition for efficient networks rather than constrained efficient networks. The main complication in the proof is showing that there is a unique constrained efficient architecture and that it coincides with the efficient architecture. As the structure of the value function is quite simple and natural, and the difficulty also holds for many variations on it, the proposition is disturbing. The proof appears in the appendix.

Proposition 3 is tight. If we drop component balance, then as mentioned above the egalitarian rule leads to $E(v) \subset PS(Y^e, v)$ for all v. If we drop anonymity (or equal treatment of equals), then a careful and clever construction of Y by Dutta and Mutuswami (1997) ensures that $E(v) \cap PS(Y, v) \neq \emptyset$ for a class of v. This is stated in the following proposition.

Let $\mathcal{V}^* = \{v \in \mathcal{V} \mid g \neq \emptyset \Rightarrow v(g) > 0\}$

Proposition 4 (Dutta and Mutuswami (1997)) *There exists a component balanced Y such that $E(v) \cap PS(Y, v) \neq \emptyset$ for all $v \in \mathcal{V}^*$. Moreover, Y is anonymous on some networks in $E(v) \cap PS(Y, v)$.*[30] [31]

[30] The statement that Y is anonymous on some networks that are efficient and pairwise stable means that one needs to consider some other networks to verify the failure of anonymity.

[31] Dutta and Mutuswami actually work with a notion called strong stability, that is (almost) a stronger requirement than pairwise stability in that it allows for deviations by coalitions of individuals. They show that the strongly stable networks are a subset of the efficient ones. Strong stability is not quite a strengthening of pairwise stability, as it only considers one network to defeat another if there is a deviation by a coalitions that makes all of its members strictly better off; while pairwise stability

This proposition shows that if one can design an allocation rule, and only wishes to satisfy anonymity on stable networks, then efficiency and stability are compatible.

While Proposition 4 shows that if we are willing to sacrifice anonymity, then we can reconcile stability with efficiency, there are also many situations where we need not go so far. That is, there are value functions for which there do exist component balanced and anonymous allocation rules for which some efficient networks are pairwise stable.

The Role of "Loose-Ends" in the Tension between Stability and Efficiency

The following proposition identifies a very particular feature of the problem between efficiency and stability. It shows that if efficient networks are such that each individual has at least two links, then there is no tension. So, problems arise only in situations where efficient networks involve individuals who may be thought of as "loose ends."

A network g has *no loose ends* if for any $i \in N(g)$, $|\{j|ij \in g\}| \geq 2$.

PROPOSITION **5** *There exists an anonymous and component balanced Y such that if v is anonymous and such that there exists $g^* \in E(v)$ with no loose ends, then $E(v) \cap PS(Y, v) \neq \emptyset$.*

The proof of Proposition 5 appears in the appendix. In a network with no loose ends individuals can alter the component structure by adding or severing links, but they cannot decrease the total number of individuals who are involved in the network by severing a link. This limitation on the ways in which individuals can change a network is enough to ensure the existence of a component balanced and anonymous allocation rule for which such an efficient network is stable, and is critical to the proof.

The proof of Proposition 5 turns out to be more complicated that one might guess. For instance, one might guess that the component wise egalitarian allocation rule Y^{ce} would satisfy the demands of the proposition.[32] However, this is not the case as the following example illustrates.

EXAMPLE **8**

Let $n = 7$. Consider a component additive and anonymous v such that the value of a ring of three individuals is 6, the value of a ring of 4 individuals is 20, and the

allows one of the two individuals adding a link to be indifferent. However, one can check that the construction of Dutta and Mutuswami extends to pairwise stability as well.

[32]See the discussion of critical link monotonicity in Jackson and Wolinsky (1996) for a complete characterization of when Y^{ce} provides for efficient networks that are pairwise stable.

value of a network where a ring of three individuals with a single bridge to a ring of four individuals (e.g., $g^* = \{12, 23, 13, 14, 45, 56, 67, 47\}$) is 28. Let the value of other components be 0. The efficient network structure is g^*. Under the component wise egalitarian rule each individual gets a payoff of 4 under g^*, and yet if 4 severs the link 14, then 4 would get a payoff of 5 under any anonymous rule or one satisfying equal treatment of equals. Thus g^* would not be stable under the component-wise egalitarian rule. See Figure 5.

[insert Figure 5 here]

Thus, a Y that guarantees the pairwise stability of g^* will have to recognize that individual 4 can get a payoff of 5 by severing the link 14. This involves a carefully defined allocation rule, as provided in the appendix.

Taking the Allocation Rule as Given

As we have seen, efficiency and even constrained efficiency are only compatible with pairwise stability under certain allocation rules and for certain settings. Sometimes this involves quite careful design of the allocation rule, as under Propositions 4 and 5.

While there are situations where the allocation rule is an object of design, we are also interested in understanding when naturally arising allocation rules lead to pairwise stable networks that are (Pareto) efficient.

Let us examine some of some of the examples discussed previously to get a feeling for this.

EXAMPLE **9** *Pareto Inefficiency in the Symmetric Connections Model.*

In the symmetric connections model (Example 1) efficient networks fall into three categories:

- empty networks when there are high costs to links,
- star networks ($n-1$ individuals all having 1 link to the n-th individual) when there are middle costs to links, and
- complete networks when there are low costs to links.

For high and low costs to links, these coincide with the pairwise stable networks.[33] The problematic case is for middle costs to links.

[33] The compatibility of pairwise stability and efficiency in the symmetric connections model is fully characterized in Jackson and Wolinsky (1996). The relationship with Pareto efficient networks is not noted.

For instance, consider a situation where $n = 4$ and $\delta < c < \delta + \frac{\delta^2}{2}$. In this case, the only pairwise stable networks is the empty network. To see this, note that since $c > \delta$ an individual gets a positive payoff from a link only if it also offers an indirect connection. Thus, each individual must have at least two links in a pairwise stable network, as if i only had a link to j, then j would want to sever that link. Also an individual maintains at most 2 links, since the payoff to an individual with three links (given $n = 4$) is less than 0 since $c > \delta$. So, a pairwise stable network where each individual has two links would have to be a ring (e.g., $\{12, 23, 34, 14\}$). However, such a network is not pairwise stable since, the payoff to any player is increased by severing a link. For instance, 1's payoff in the ring is $2\delta + \delta^2 - 2c$, while severing the link 14 leads to $\delta + \delta^2 + \delta^3 - c$ which is higher since $c > \delta$.

Although the empty network is the unique pairwise stable network, it is not even Pareto efficient. The empty network is Pareto dominated by a line (e.g., $g = \{12, 23, 34\}$). To see this, not that under the line, the payoff to the end individuals (1 and 4) is $\delta + \delta^2 + \delta^3 - c$ which is greater than 0, and to the middle two individuals (2 and 3) the payoff is $2\delta + \delta^2 - 2c$ which is also greater than 0 since $c < \delta + \frac{\delta^2}{2}$.

Thus, there exist cost ranges under the symmetric connections model for which all pairwise stable networks are *Pareto* inefficient, and other cost ranges where all pairwise stable networks are efficient. There are also some cost ranges where some pairwise stable networks are efficient and some other pairwise stable networks are not even Pareto efficient.

Example **10** *Pareto Inefficiency in the Co-Author Model.*

Generally, the co-author model results in Pareto inefficient networks. To see this, consider a simple setting where $n = 4$. Here the only pairwise stable network is the complete network, as the reader can check with some straightforward calculations. The complete network leads to a payoff of 2.5 to each player. However, a network of two distinct linked pairs (e.g., $g = \{12, 34\}$) leads to payoffs of 3 for each individual. Thus, the unique pairwise stable network is Pareto inefficient.

Example **11** *Efficiency in the Corominas-Bosch Bargaining Networks*

While incentives to form networks do not always lead to efficiency in the connections model, the news is better in the bargaining model of Corominas-Bosch (Example 3). In that model the set of pairwise stable networks is often exactly the set of efficient networks, as it outlined in the following Proposition.

Proposition **6** *In the Corominas-Bosch model as outlined in Example 3, with costs to links $1/2 > c_s > 0$ and $1/2 > c_b > 0$, the pairwise stable networks are exactly the set of efficient networks.*[34] *The same is true if $c_s > 1/2$ and/or $c_b > 1/2$ and $c_s + c_b \geq 1$. If $c_s > 1/2$ and $1 > c_s + c_b$, or $c_b > 1/2$ and $1 > c_s + c_b$, then the only pairwise stable network is inefficient, but Pareto efficient.*

The proof of Proposition 6 appears in the appendix. The intuition for the result is fairly straightforward. Individuals get payoffs of either 0, 1/2 or 1 from the bargaining, ignoring the costs of links. An individual getting a payoff of 0 would never want to maintain any links, as they cost something but offer no payoff in bargaining. So, it is easy to show that all individuals who have links must get payoffs of 1/2. Then, one can show that if there are extra links in such a network (relative to the efficient network which is just linked pairs) that some particular links could be severed without changing the bargaining payoffs and thus saving link costs.

The optimistic conclusion in the bargaining networks is dependent on the simple set of potential payoffs to individuals. That is, either all linked individuals get payoffs of 1/2, or for every individual getting a payoff of 1 there is some linked individual getting a payoff of 0. The low payoffs to such individuals prohibit them from wanting to maintain such links. This would not be the case, if such individuals got some positive payoff. We see this next in the next example.

Example **12** *Pareto Inefficiency in Kranton and Minehart's Buyer-Seller Networks*

Despite the superficial similarities between the Corominas-Bosch and Kranton and Minehart models, the conclusions regarding efficiency are quite different. This difference stems from the fact that there is a possible heterogeneity in buyers' valuations in the Kranton and Minehart model, and so efficient networks are more complicated

[34]Corominas-Bosch (1999) considers a different definition of pairwise stability, where a cost is incurred for creating a link, but none is saved for severing a link. Such a definition can clearly lead to over-connections, and thus a more pessimistic conclusion than that of Proposition 6 here. She also considers a game where links can be formed unilaterally and the cost of a link is incurred only by the individual adding the link. In such a setting, a buyer (say when there are more sellers than buyers) getting a payoff of 1/2 or less has an incentive to add a link to some seller who is earning a payoff of 0, which will then increase the buyer's payoff. As long as this costs the seller nothing, the seller is indifferent to the addition of the link. So again, Corominas-Bosch obtains an over-connection result. It seems that the more reasonable case is one that involves some cost for and consent of both individuals, which is the case treated in Proposition 6 here.

than in the simpler bargaining setting of Corominas-Bosch. It is generally the case that these more complicated networks are not pairwise stable.

Before showing that all pairwise stable networks may fail to be Pareto efficient, let us first show that they may fail to be efficient as this is a bit easier to see.

Consider Example 4, where there is one seller and up to n buyers.

The efficient network in this setting is one where $\frac{k}{k+1} - k(c_s + c_b)$ is maximized. This occurs where[35]

$$\frac{1}{k(k+1)} \geq c_s + c_b \geq \frac{1}{(k+1)(k+2)}.$$

Let us examine the pairwise stable networks. From (1) it follows that the seller gains from adding a new link to a network of with k links as long as

$$\frac{2}{(k+1)(k+2)} > c_s.$$

Also from (1) it follows that a buyer wishes to add a new link to a network of k links as long as

$$\frac{1}{k(k+1)} > c_b.$$

If we are in a situation where $c_s = 0$, then the incentives of the buyers lead to exactly the right social incentives: and the pairwise stable networks are exactly the efficient ones.[36] This result for $c_s = 0$ extends to situations with more than one seller and to general distributions over signals, and is a main result of Kranton and Minehart (1998).

However, let us also consider situations where $c_s > 0$, and for instance $c_b = c_s$. In this case, the incentives are not so well behaved.[37] For instance, if $c_s = 1/100 = c_b$, then any efficient network has six buyers linked to the seller ($k = 6$). However, buyers will be happy to add new links until $k = 10$, while sellers wish to add new links until $k = 13$. Thus, in this situation the pairwise stable networks would have 10 links, while networks with only 6 links are the efficient ones.

To see the intuition for the inefficiency in this example note that the increase in expected price to sellers from adding a link can be thought of as coming from two

[35] Or at n if such a $k > n$.

[36] Sellers always gain from adding links if $c_s = 0$ and so it is the buyers' incentives that limit the number of links added.

[37] See Kranton and Minehart (1998) for discussion of how a costly investment decision of the seller might lead to inefficiency. Although it is not the same as a cost per link, it has some similar consequences.

sources. One source is the expected increase in willingness to pay of the winning bidder due to an expectation that the winner will have a higher valuation as we see more draws from the same distribution. This increase is of social value, as it means that the good is going to someone who values it more. The other source of price increase to the seller from connecting to more buyers comes from the increased competition among the bidders in the auction. There is a greater number of bidders competing for a single object. This source of price increase is not of social value since it only increases the proportion of value which is transferred to the seller. Buyers' incentives are distorted relative to social efficiency since although they properly see the change in social value, they only bear part of the increase in total cost of adding a link.

While the pairwise stable networks in this example are not efficient (or even constrained efficient), they are Pareto efficient, and this is easily seen to be generally true when there is a single seller as then disconnected buyers get a payoff of 0. This is not true with more sellers as we now see.

Let us now show that it is possible for (non-trivial) pairwise stable networks in the Kranton-Minehart model to be Pareto inefficient. For this we need more than one seller.

Consider a population with 2 sellers and 4 buyers. Let individuals 1 and 2 be the sellers and 3,4,5,6, be the buyers. Let the cost of a link to a seller be $c_s = \frac{5}{60}$ and the cost of a link to a buyer be $c_b = \frac{1}{60}$.

Some straightforward (but tedious) calculations lead to the following payoffs to individuals in various networks:

$g^a = \{13\}$: $Y_1(g^a) = -\frac{5}{60}$ and $Y_1(g^a) = \frac{29}{60}$.

$g^b = \{13, 14\}$: $Y_1(g^b) = \frac{10}{60}$ and $Y_3 = Y_4(g^b) = \frac{9}{60}$.

$g^c = \{13, 14, 15\}$: $Y_1(g^c) = \frac{15}{60}$ and $Y_3 = Y_4 = Y_5(g^c) = \frac{4}{60}$.

$g^d = \{13, 14, 15, 16\}$: $Y_1(g^d) = \frac{16}{60}$ and $Y_3 = Y_4 = Y_5(g^d) = \frac{2}{60}$.

$g^e = \{13, 14, 25, 26\}$: $Y_1 = Y_2(g^e) = \frac{10}{60}$ and $Y_3 = Y_4 = Y_5 = Y_6(g^e) = \frac{9}{60}$.

$g^f = \{13, 14, 15, 25, 26\}$: $Y_1(g^f) = \frac{13}{60}$, $Y_2(g^f) = \frac{8}{60}$, and $Y_3 = Y_4(g^f) = \frac{6}{60}$, while $Y_5(g^f) = \frac{10}{60}$ and $Y_6(g^f) = \frac{11}{60}$.

$g^g = \{13, 14, 15, 24, 25, 26\}$: $Y_1 = Y_2(g^g) = \frac{9}{60}$ and $Y_3 = Y_4 = Y_5 = Y_6(g^g) = \frac{8}{60}$.

There are three types of pairwise stable networks here: the empty network, networks that look like g^d, and networks that look like g^g.[38] Both the empty network and g^g are not Pareto efficient, while g^d is. In particular, g^g is Pareto dominated by g^e. Also, g^d

[38]The reader is left to check networks that are not listed here.

is not efficient nor is it constrained efficient.[39] In this example, one might hope that g^e would turn out to be pairwise stable, but as we see 1 and 5 then have an incentive to add a link; and then 2 and 4 which takes us to g^g. Thus, individuals have an incentive to over-connect as it increases their individual payoffs even when it is decreasing overall value.

It is not clear whether there are examples where all pairwise stable networks are Pareto inefficient in this model, as there are generally pairwise stable networks like g^d where only one seller is active and gets his or her maximum payoff. But this is an open question, as with many buyers this may be Pareto dominated by networks where there are several active sellers. And as we see here, it is possible for active sellers to want to link to each others' buyers to an extent that is inefficient.

Pareto Inefficiency under the Myerson Value

As we have seen in the above examples, efficiency and Pareto efficiency are properties that sometimes but not always satisfied by pairwise stable networks. To get a fuller picture of this, and to understand some sources of inefficiency, let us look at an allocation rule that will arise naturally in many applications. As equal bargaining power is a condition that may naturally arise in a variety of settings, the Myerson value allocation rule that is worthy of serious attention. Unfortunately, although it has nice properties with respect to the existence of pairwise stable networks, the pairwise stable networks are not always Pareto efficient networks.

The intuition behind the (Pareto) inefficiency under the Myerson value is that individuals can have an incentive to over-connect as it improves their bargaining position. This can lead to overall *Pareto* inefficiency. To see this in some detail, it is useful to separate costs and benefits arising from the network.

Let us write $v(g) = b(g) - c(g)$ where $b(\cdot)$ represents benefits and $c(\cdot)$ costs and both functions take on nonnegative values and have some natural properties.

$b(g)$ is *monotone* if

- $b(g) \geq b(g')$ if $g' \subset g$, and
- $b(\{ij\}) > 0$ for any ij.

$b(g)$ is *strictly monotone* if $b(g) > b(g')$ whenever $g' \subset g$.

Similar definitions apply to a cost function c.

[39] To see constrained inefficiency, consider an allocation rule that divides payoffs equally among buyers in a component and gives 0 to sellers. Under such a rule, g^e Pareto dominates g^d.

Proposition 7 *For any monotone and anonymous benefit function b there exists a strictly monotone and anonymous cost function c such that all pairwise stable networks relative to Y^{MV} and $v = b - c$ are Pareto inefficient. In fact, the pairwise stable networks are over-connected in the sense that each pairwise stable network has some subnetwork that Pareto dominates it.*

Proposition 7 is a fairly negative result, saying that for any of a wide class of benefit functions there is some cost function for which individuals have incentives to over-connect the network, as they each try to improve their bargaining position and hence payoff.

Proposition 7 is actually proven using the following result, which applies to a narrower class of benefit functions but is more specific in terms of the cost functions.

Proposition 8 *Consider a monotone benefit function b for which there is some efficient network g^* relative to b ($g^* \in E(b)$) such that $g^* \neq g^N$. There exists $\overline{c} > 0$ such that for any cost function c such that $\overline{c} \geq c(g)$ for all $g \in G$, the pairwise stable networks relative to Y^{MV} and $v = b - c$ are all inefficient. Moreover, if b is anonymous and g^* is symmetric,*[40] *then each pairwise stable networks is Pareto dominated by some subnetwork.*

Proposition 8 says that for any monotone benefit function that has at least one efficient network under the benefit function that is not fully connected, if costs to links are low enough, then all pairwise stable networks will be over-connected relative to the efficient networks. Moreover, if the efficient network under the benefit function is symmetric does not involve too many connections, then all pairwise stable networks will be Pareto inefficient.

Proposition 8 is somewhat limited, since it requires that the benefit function have some network smaller than the complete network which is efficient. However, as there are many b's and c's that sum to the same v, this condition actually comes without much loss of generality, which is the idea behind the proof of Proposition 7. The proof of Propositions 7 and 8 appear in the appendix.

[40] A network g is symmetric if for every i and j there exists a permutation π such that $g = g^\pi$ and $\pi(j) = i$.

6 Discussion

The analysis and overview presented here shows that the relationship between the stability and efficiency of networks is context dependent. Results show that they are not always compatible, but are compatible for certain classes of value functions and allocation rules. Looking at some specific examples, we see a variety of different relationships even as one varies parameters within models.

The fact that there can be a variety of different relationships between stable and efficient networks depending on context, seems to be a somewhat negative finding for the hopes of developing a systematic theoretical understanding of the relationship between stability and efficiency that cuts across applications. However, there are several things to note in this regard. First, a result such as Proposition 5 is reassuring, since it shows that some systematic positive results can be found. Second, there is hope of tying incompatibility between individual incentives and efficiency to a couple of ideas that cut across applications. Let me outline this in more detail.

One reason why individual incentives might not lead to overall efficiency is one that economists are very familiar with: that of externalities. This comes out quite clearly in the failure exhibited in the symmetric connections model in Example 9. By maintaining a link an individual not only receives the benefits of that link (and its indirect connections) him or herself, but also provides indirect benefits to other individuals to whom he or she is linked. For instance, 2's decision of whether or not to maintain a link to 3 in a network $\{12, 23\}$ has payoff consequences for individual 1. The absence of a proper incentive for 2 to evaluate 1's well being when deciding on whether to add or delete the link 23 is a classic externality problem. If the link 23 has a positive benefit for 1 (as in the connections model) it can lead to under-connection relative to what is efficient, and if the link 23 has a negative effect on 1 (as in the co-author model) it can lead to over-connection.

Power-Based Inefficiencies

There is also a second, quite different reason for inefficiency that is evident in some of the examples and allocation rules discussed here. It is what we might call a "power-based inefficiency". The idea is that in many applications, especially those related to bargaining or trade, an individual's payoff depends on where they sit in the network and not only what value the network generates. For instance, individual 2 in a network $\{12, 23\}$ is critical in allowing any value to accrue to the network, as deleting all of 2's links leaves an empty network. Under the Myerson value allocation rule, and many others, 2's payoff will be higher than that of 1 and 3; as individual 2 is rewarded well for the role that he or she plays. Consider the incentives of individuals 1 and 3 in such a situation. Adding the link 13 might lower the overall value of the network, but it would also put the individuals into equal roles in the network, thereby decreasing individual 2's importance in the network and resulting bargaining power. Thus, individual 1 and 3's bargaining positions can improve and their payoffs under the Myerson value can increase; even if the new network is less productive than the previous one. This leads 1 and 3 to over-connect the network relative to what is efficient. This is effectively the intuition behind the results in Propositions 7 and 8, which says that this is a problem which arises systematically under the Myerson value.

The inefficiency arising here comes not so much from an externality, as it does from individuals trying to position themselves well in the network to affect their relative power and resulting allocation of the payoffs. A similar effect is seen in Example 12, where sellers add links to new buyers not only for the potential increase in value of the object to the highest valued buyer, but also because it increases the competition among buyers and increases the proportion of the value that goes to the seller rather than staying in the buyers' hands.[41]

An interesting topic for further research is to see whether inefficiencies in network formation can always be traced to either externalities or power-based incentives, and whether there are features of settings which indicate when one, and which one, of these difficulties might be present.

Some other issues for further study

There are other areas that deserve significant attention in further efforts to model the formation of networks.

First, as discussed near the definition of pairwise stability, it would be useful to develop a notion of network stability that incorporates farsighted and dynamic behavior. Judging from such efforts in the coalition formation literature, this is a formidable and potentially ad hoc task. Nevertheless, it is an important one if one wants to apply network models to things like strategic trade alliances.

Second, in the modeling here, allocation rules are taken as being separate from the

[41] Such a source of inefficiency is not unique to network settings, but are also observed in, for example, search problems and bargaining problems more generally (e.g., see Stole and Zwiebel (1996) on intra-firm bargaining and hiring decisions). The point here is that this power-based source of inefficiency is one that will be particularly prevalent in network formation situations, and so it deserves particular attention in network analyses.

network formation process. However, in many applications, one can see bargaining over allocation of value happening simultaneously with the formation of links. Intuitively, this should help in the attainment of efficiency. In fact, in some contexts it does, as shown by Currarini and Morelli (2000) and Mutuswami and Winter (2000). The contexts explored in those models use given (finite horizon) orderings over individual proposals of links, and so it would be interesting to see how robust such intuition is to the specification of bargaining protocol.

Third, game theory has developed many powerful tools to study evolutionary pressures on societies of players, as well as learning by players. Such tools can be very valuable in studying the dynamics of networks over time. A recent literature has grown around these issues, studying how various random perturbations to and evolutionary pressures on networks affects the long run emergence of different networks structures (e.g., Jackson and Watts (1998, 1999), Goyal and Vega-Redondo (1999), Skyrms and Pemantle (2000), and Droste, Gilles and Johnson (2000)). One sees from this preliminary work on the subject that network formation naturally lends itself to such modeling, and that such models can lead to predictions not only about network structure but also about the interaction that takes place between linked individuals. Still, there is much to be understood about individual choices, interaction, and network structure depend on various dynamic and stochastic effects.

Finally, experimental tools are becoming more powerful and well-refined, and can be brought to bear on network formation problems, and there is also a rich set of areas where network formation can be empirically estimated and some models tested. Experimental and empirical analyses of networks are well-founded in the sociology literature (e.g., see the review of experiments on exchange networks by Bienenstock and Bonacich (1993)), but is only beginning in the context of some of the recent network formation models developed in economics (e.g., see Corbae and Duffy (2000) and Charness and Corominas-Bosch (2000)). As these incentives-based network formation models have become richer and have many pointed predictions for wide sets of applications, there is a significant opportunity for experimental and empirical testing of various aspects of the models. For instance, the hypothesis presented above, that one should expect to see over-connection of networks due to the power-based inefficiencies under equal bargaining power and low costs to links, provides specific predictions that are testable and have implications for trade in decentralized markets.

In closing, let me say that the future for research on models of network formation is quite bright. The multitude of important issues that arise from a wide variety of applications provides a wide open landscape. At the same time the modeling proves to be quite tractable and interesting, and has the potential to provide new explanations, predictions and insights regarding a host of social and economic settings and behaviors.

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Appendix

Proof of Proposition 3: The proof uses the same value function as Jackson and Wolinsky (1996), and is also easily extended to more individuals. The main complication is showing that the constrained efficient and efficient networks coincide. Let $n = 3$ and the value of the complete network be 12, the value of a single link 12, and the value of a network with two links 13.

Let us show that the set of constrained efficient networks is exactly the set of networks with two links. First consider the complete network. Under any component balanced Y satisfying equal treatment of equals (and thus anonymity), each individual must get a payoff of 4. Consider the component balanced and anonymous Y which gives each individual in a two link network 13/3. Then $g = \{12, 23\}$ offers each individual a higher payoff than g^N, and so the complete network is not constrained efficient. The empty network is similarly ruled out as being constrained efficient. Next consider the network $g' = \{12\}$ (similar arguments hold for any permutation of it). Under any component balanced and Y satisfying equal treatment of equals, $Y_1(g', v) = Y_2(g', v) = 6$. Consider $g'' = \{13, 23\}$ and a component balanced and anonymous Y such that $Y_1(g'', v) = Y_2(g'', v) = 6.25$ and $Y_3(g'', v) = .5$. All three individuals are better off under g'' than g' and so g' is not constrained efficient. The only remaining networks are those with two links, which are clearly efficient and thus constrained efficient.

To complete the proof, we need to show that any component balanced Y satisfying equal treatment of equals results in none of the two link networks being pairwise stable.

As noted above, under any component balanced Y satisfying equal treatment of equals, each individual in the complete network gets a payoff of 4, and the two individuals with connections in the single link network each get a payoff of 6. So consider the network $g = \{12, 23\}$ (or any permutation of it) and let us argue that it cannot be pairwise stable. In order for individual 2 not to want to sever a link, 2's payoff must be at least 6. In order for individuals 1 and 3 not to both wish to form a link (given equal treatment of equals) their payoffs must be at least 4. Thus, in order to have g be pairwise stable it must be that $Y_1(g, v) + Y_2(g, v) + Y_3(g, v) \geq 14$, which is not feasible. ∎

Proof of Proposition 5: Let $N^*(g) = |C(g)| + n - |N(g)|$. Thus, $N^*(g)$ counts the components of g, and also counts individuals with no connections. So if we let a component* be either a component or isolated individual, then N^* counts component*'s. For instance, under this counting the empty network has one more component* than the network with a single link.

Let

$$B(g) = \{i | \exists j \text{ s.t. } |N^*(g - ij)| > |N^*(g)|\}.$$

Thus $B(g)$ is the set of individuals who form bridges under g, i.e., those individuals who by severing a link can alter the component structure of g. Let[42]

$$SB(g) = \{i | \exists j \text{ s.t. } |N^*(g-ij)| > |N^*(g)| \text{ and } i \in N(h_i), h_i \in C(g-ij), h_i \text{ is symmetric}\}.$$

$SB(g)$ identifies the individuals who form bridges and who by severing the bridge end up in a symmetric component.

Claim 1: If g is connected ($|C(g)| = 1$) and has no loose ends, then $i \in SB(g)$ implies that i has at most one bridge in g. Also, for any such g, $|N(g)|/3 \geq |SB(g)|$, and if $\{i, j\} \subset SB(g)$ and $ij \in g$, then $\{i, j\} = B(g)$.

Proof of claim: Since there are no loose ends under g, each $i \in N(g)$ has at least two links. This implies that if $i \in SB(g)$ severs a link and ends up in a symmetric component h of $g - ij$, that h will have at least three individuals since each must have at least two links. Also $N(h) \cap SB(g) = \{i\}$. To see this note that if not, then there exists some $k \neq i$, $k \in N(h)$, such that k has a bridge under h. However, given the symmetry of h and the fact that each individual has at least two links, there are at least two distinct paths connecting any two individuals in the component, which rules out any bridges. Note this implies that i has at most one bridge. As we have shown that for each $i \in SB(g)$ there are at least two other individuals in $N(g^*) \setminus SB(g)$ and so $|N(g)|/3 \geq |SB(g)|$. If $\{i, j\} \subset SB(g)$ and $ij \in g$, then given the symmetry of the component from severing a bridge, it must be that ij is the bridge for both i and j and that severing this results in two symmetric components with not bridges. This completes the claim.

Pick g^* to be efficient under v and have no loose ends. Also, choose g^* so that if $h^* \in C(g^*)$ then $v(h^*) > 0$. (Simply replace any $h^* \in C(g^*)$ such that $0 \geq v(h^*)$ with an empty component, which preserves efficiency.)

Consider any i that is non-isolated under g^* and the component $h_i^* \in C(g^*)$ with $i \in N(h_i^*)$. Define $\widehat{Y}(h_i^*, v)$ as follows.

$$\widehat{Y}_i(h_i^*, v) = \begin{cases} \max[Y^{ce}(g^*, v), Y_i^{ce}(h_i, v)] & \text{if } i \in SB(h) \text{, where } h_i \text{ is the symmetric} \\ & \text{component when } i \text{ severs his bridge} \\ \frac{v(h_i^*) - \sum_{k \in SB(h)} \widehat{Y}_k(h_i^*, v)}{|N(h_i^*) \setminus SB(h_i^*)|} & \text{if } i \in N(h_i^*) \setminus SB(h_i^*). \end{cases}$$

[42]Recall that a network g is symmetric if for every i and j there exists a permutation pi such that $g = g^\pi$ and $\pi(j) = i$.

Let $\widehat{Y}(g^*, v)$ be the component balanced allocation rule defined on g^* from $\widehat{Y}$ defined above.

Claim 2: $\widehat{Y}_i(g^*, v) > 0$ for all $i \in N(g^*)$.

This is clear for $i \in SB(h_i^*)$ since i gets at least $Y_i^{ce}(h_i^*, v) > 0$. Consider $i \in N(h^*) \setminus SB(h_i^*)$. From the definition of $\widehat{Y}$, we need only show that $v(h^*) > \sum_{k \in SB(h^*)} \widehat{Y}_k(h^*, v)$. Given that by Claim 1 we know $|N(h^*)|/3 \geq |SB(h^*)|$, it is sufficient to show that $\frac{2v(h^*)}{|N(h^*)|} \geq \widehat{Y}_k(h^*, v)$ for any $k \in SB(h^*)$. Let h_k be the symmetric component obtained when k severs his bridge. By efficiency of g^* and anonymity of v

$$v(h^*) \geq v(h_k) \left(\frac{|N(h^*)|}{|N(h_k)|} \right)^{-}$$

where $(\cdot)^{-}$ rounds down.

$$\frac{v(h^*)}{|N(h_k)| \left(\frac{|N(h^*)|}{|N(h_k)|} \right)^{-}} \geq \frac{v(h_k)}{|N(h_k)|}.$$

Also note that $|N(h_k)| \left(\frac{|N(h^*)|}{|N(h_k)|} \right)^{-} \geq \frac{1}{2}$. Thus,

$$\frac{2v(h^*)}{|N(h^*)|} \geq \frac{v(h^*)}{|N(h_k)| \left(\frac{|N(h^*)|}{|N(h_k)|} \right)^{-}} \geq \frac{v(h_k)}{|N(h_k)|}.$$

So, from the definition of $\widehat{Y}$, we know that for any $k \in SB(h^*)$ that $\frac{2v(h^*)}{|N(h^*)|} \geq \widehat{Y}_k(h^*, v)$. As argued above, this completes the proof of the claim.

Now let us define $\widehat{Y}$ on other networks to satisfy the Proposition.

For a component of a network h let the symmetry groups be coarsest partition of $N(h)$, such that if i and j are in the same symmetry group, then there exists a permutation π with $\pi(i) = j$ and $h^\pi = h$. Thus, individuals in the same symmetry group are those who perform the same role in a network architecture and must be given the same allocation under an anonymous allocation rule when faced with an anonymous v.

For g adjacent to g^*, so that $g = g^* + ij$ or $g = g^* - ij$ for some ij, set $\widehat{Y}$ as follows.

Consider $h \in C(g)$

Case 1. There exists $k \in N(h)$ such that k is not in the symmetry group of either i nor j under g: split $v(h)$ equally among the members of k's symmetry group within h, and 0 to other members of $N(h)$.

Case 2. Otherwise, set $\widehat{Y}(h, v) = Y^{ce}(h, v)$.

For anonymous permutations of g^* and its adjacent networks define $\widehat{Y}$ according to the corresponding permutations of $\widehat{Y}$ defined above. For any other g let $\widehat{Y} = Y^{ce}$.

Let us verify that g^* is pairwise stable under $\widehat{Y}$.

Consider any $ij \in g^*$ and $g = g^* - ij$. Consider $h_i \in C(g)$ such that $i \in N(h_i)$. We show that i (and hence also j since the labels are arbitrary) cannot be better off.

If h_i falls under Case 1 above, then i gets 0 which by Claim 2 cannot be improving.

Next consider case where h_i has a single symmetry group. If $N(h_i) \cap SB(g^*) = \emptyset$, then ij could not have been a bride and so $N(h_i)$ was the same group of individuals i was connected to under g^* ($N(h_i) = N(h_i^*)$). Thus i got $Y_i^{ce}(g^*, v)$ under g^* and now gets $Y_i^{ce}(g, v)$, and so by efficiency this cannot be improving since i is still connected to the same group of individuals. If $N(h_i) \cap SB(g^*) \neq \emptyset$, then it must be that $i \in SB(g^*)$ and ij was i's bridge. In this case it follows from the definition of $\widehat{Y}_i(g^*, v)$ that the deviation could not be improving.

The remaining case is where $N(h_i) \subset N_i \cup N_j$, where N_i and N_j are the symmetry groups of i and j under g, and $N_i \cap N_j = \emptyset$. If i and j are both in $N(h_i)$ it must be that $N(h_i) = N(h_i^*)$, and that $N(h_i) \cap SB(g^*) = \emptyset$. [To see this suppose the contrary. ij could not be a bridge since i and j are both in $N(h_i)$. Thus, there is some $k \notin \{i, j\}$ with $k \in SB(g^*)$. But then there is no path from i to j that passes through k. Thus i and j are in the same component when k severs a bridge, which is either the component of k - which cannot be since then k must be in a different symmetry group from i and j under g - or in the other component. But then $k \in SB(g)$. This implies that either $i \in SB(g)$ or $j \in SB(g)$ but not both. Take $i \in SB(g)$. By severing i's bridge under g, i's component must be symmetric and include j (or else j also has a bridge under g and there must be more than two symmetry groups which would be a contradiction). There is some $l \neq j$ connected to i who is not i's bridge. But l and j cannot be in the same symmetry group under g since l is connected to some $i \in SB(g)$ and j cannot be (by claim 1) as $ij \notin g$. Also, l is not in i's symmetry group (again the proof of claim 1), and so his is a contraction.] Thus i got $Y_i^{ce}(g^*, v)$ under g^* and now gets $Y_i^{ce}(g, v)$, and so by efficiency this cannot be improving since i is still connected to the same group of individuals. If i and j are in different components under g, then it must be that they are in identical architectures given that $N(h_i) \subset N_i \cup N_j$. In this case ij was a bridge and since h_i (and h_j) are not symmetric and $N(h_i) \subset N_i \cup N_j$, it follows the component of g^* containing i and j had no members of $SB(g^*)$. Thus $\widehat{Y}_i(g^*, v) = Y_i^{ce}(g^*, v)$ and also $\widehat{Y}_i(g, v) = Y_i^{ce}(g, v)$. Since the two components that are obtained when ij is severed are identical, by efficiency it follows that the payoffs to i

(and j) are at least as high under g^* as under g.

Next, consider any $ij \in g^*$ and $g = g^* + ij$. Consider $h_i \in C(g)$ such that $i \in N(h_i)$. We show that if i is better off, then j must be worse off.

If h_i falls under Case 1 above, then i gets 0 which by Claim 2 makes i no better off.

Next consider case where h_i has a single symmetry group. Then since ij was added, and each individual had two links to begin with, it follows that $N(h_i) \cap SB(g^*) = \emptyset$. Moreover, it must be that $N(h_i) = N(h_i^*)$, where h_i^* is i's component under g^*. This implies that i got $Y_i^{ce}(g^*, v)$ under g^* and now gets $Y_i^{ce}(g, v)$. By efficiency, this cannot be improving for i.

The remaining case is where h_i is not symmetric and $N(h_i) \subset N_i \cup N_j$, where N_i and N_j are the symmetry groups of i and j under g, and $N_i \cap N_j = \emptyset$. As argued below, $N(h_i) \cap SB(g^*) = \emptyset$. Also, it follows again that $N(h_i) = N(h_i^*)$, and so the argument from the case above applies again. So to complete the proof we need only show that $N(h_i) \cap SB(g^*) = \emptyset$. First, note that ij cannot be a bridge as by the arguments of claim 1 there must be some $l \notin B(g)$, which would then put l is a different symmetry group than either i or j which would be a contradiction of this case. Consider the case where $B(g) = B(g^*)$. Then it must be that either $i \in SB(g^*)$ or $j \in B(g^*)$, but not both (given only two symmetry groups under g). Take $i \in SB(g^*)$. Then by severing i's bridge, the resulting component (given the addition of ij under g) is not symmetric. But this means there is some l in that component not in j's symmetry class, and also not in $B(g)$ and so l is in a third symmetry class which is a contradiction. Thus $B(g) \neq B(g^*)$. This means that ij is a link that connects two components that were only connected via some other link kl under g^*. Given there are only two symmetry classes N_i and N_j under h_i, then it must be that every individual is involved in such a duplicate bridge and that the duplicate ij was not present in g^*, which contradicts the fact that some individual in $N(h_i)$ is in $SB(g^*)$. ∎

Proof of Proposition 6: Under (i) from Example 3, it follows that any buyer (or seller) who gets a payoff of 0 from the bargaining would gain by severing any link, as the payoff from the bargaining would still be at least 0, but at a lower cost. Thus, in any pairwise stable network g all individuals who have any links must get payoffs of 1/2. Thus, from (iii) from Example 3, it follows that there is some number $K \geq 0$ such that there are exactly K buyers collectively linked to exactly K sellers and that we can find some subgraph g' with exactly K links linking all buyers to all sellers. Let us show that it must be that $g = g'$. Consider any buyer or seller in $N(g)$. Suppose that buyer (seller) has two or more links. Consider a link for that buyer (seller) in $g \setminus g'$.

If that buyer (seller) severs that link, the resulting network will still be such that any subgroup of k buyers in the component can be matched with at least k distinct sellers and vice versa, since g' is still a subset of the resulting network. Thus, under (iii) that buyer (seller) would still get a payoff of 1/2 from the trading under the new network, but would save a cost c_b (or c_s) from severing the link, and so g cannot be pairwise stable.

Thus, we have shown that all pairwise stable networks consist of $K \geq 0$ links connecting exactly K sellers to K buyers, and where all individuals who have a link get a payoff of 1/2.

To complete the proof, note that if there is any pair of buyer and seller who each have no link and each cost is less than 1/2, then both would benefit from adding a link, and so that cannot be pairwise stable. Without loss of generality assume that the number of buyers is at least the number of sellers. We have shown that any pairwise stable network is such that each seller is connected to exactly one buyer, and each seller to a different buyer. It is easily checked (by similar arguments) that any such network is pairwise stable. Since this is exactly the set of efficient networks for these cost parameters, the first claim in the Proposition follows.

The remaining two claims in the proposition follow from noting that in the case where $c_s > 1/2$ or $c_b > 1/2$, then K must be 0. Thus, the empty network is the only pairwise stable network in those cases. It is always Pareto efficent in these cases since someone must get a payoff less than 0 in any other network in this case. It is only efficient if $c_s + c_b \geq 1$. ∎

Proof of Proposition 8: The linearity of the Shapley value operator, and hence the Myerson value allocation rule,[43] implies that $Y_i(v,g) = Y_i(b,g) - Y_i(c,g)$. It follows directly from (2) that for monotone b and c, that $Y_i(b,g) \geq 0$ and likewise $Y_i(c,g) \geq 0$. Since $\sum_i Y_i(b,g) = b(g)$, and each $Y_i(b,g)$ is nonnegative it also follows that $b(g) \geq Y_i(b,g) \geq 0$ and likewise that $c(g) \geq Y_i(c,g) \geq 0$.

Let us show that for any monotone b and small enough $\overline{c} \geq c(\cdot)$, that the *unique* pairwise stable network is the complete network ($PS(Y^{MV}, v = b - c) = \{g^N\}$). We first show that for any network $g \in G$, if $ij \notin g$, then

$$Y_i(g + ij, b) \geq Y_i(g, b) + \frac{2b(\{ij\})}{n(n-1)(n-2)} \tag{4}$$

[43] This linearity is also easily checked directly from (2).

From (2) it follows that

$$Y_i^{MV}(g,b) - Y_i(g-ij,b) = \sum_{S\subset N\setminus\{i\}:j\in S} (b(g+ij|_{S\cup i}) - b(g|_{S\cup i}))\frac{\#S!(n-\#S-1)!}{n!}.$$

Since b is monotone, it follows that $b(g+ij|_{S\cup i}) - b(g|_{S\cup i}) \geq 0$ for every S. Thus,

$$Y_i^{MV}(g,b) - Y_i(g-ij,b) \geq (b(g+ij|_{\{i,j\}}) - b(g|_{\{i,j\}}))\frac{\#2!(n-3)!}{n!}.$$

Since $b(g+ij|_{S\cup i}) - b(g|_{S\cup i}) = b(\{ij\}) > 0$, (4) follows directly.

Let $\overline{c} < \min_{ij} \frac{2b(\{ij\})}{n(n-1)(n-2)}$. (Note that for a monotone b, $b(\{ij\}) > 0$ for all ij.) Then from (4)

$$Y_i(g+ij,v) - Y_i(g,v) \geq \frac{2b(\{ij\})}{n(n-1)(n-2)} - (Y_i(g+ij,c) - Y_i(g,c)).$$

Note that since $\overline{c} \geq c(g) \geq Y_i(c,g) \geq 0$ for all g', it follows that $\overline{c} \geq Y_i(g+ij,c) - Y_i(g,c)$. Hence, from our choice of $\overline{c}$ it follows that $Y_i(g+ij,v) - Y_i(g,v)$ for all g and $ij \notin g$. This directly implies that the only pairwise stable network is the complete network.

Given that $g^* \neq g^N$ is efficient under b and c is strictly monotone, then it follows that the complete network is not efficient under v. This establishes the first claim of the proposition.

If b is such that $g^* \subset g \subset g^N$ for some symmetric $g \neq g^N$, then given that b is monotone it follows that g is also efficient for b. Also, the symmetry of g and anonymity of Y^{MV} implies that $Y_i(g,b) = Y_j(g,b)$ for all i and j. Since this is also true of g^N, it follows that $Y_i(g,b) \geq Y_i(g^N,b)$ for all i. For a strictly monotone c, this implies that $Y_i(g,b-c) > Y_i(g^N,b-c)$ for all i. Thus, g^N is Pareto dominated by g. Since g^N is the unique pairwise stable network, this implies the claim that $PS(Y^{MV},v) \cap PE(Y^{MV},v) = \emptyset$. ▮

Proof of Proposition 7: Consider b that is anonymous and monotone. Consider a symmetric g such that $C(g) = g$ and $N(g) = N$ and $g \neq g^N$. Let $b'(g') = \min[b(g'), b(g)]$. Note that b' is monotone and that g is efficient for b'. Find a strictly monotone c' according to Proposition 8, for which the unique pairwise stable network under $b' - c'$ is the complete network while the Pareto efficient networks are incomplete. Let $c = c' + b - b'$. It follows that c is strictly monotone. Also, $v = b - c = b' - c'$ and so the unique pairwise stable network under $b' - c'$ is the complete network while the Pareto efficient networks are incomplete. ▮