DECOMPOSABLE PRINCIPAL-AGENT PROBLEMS

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Abstract: This paper investigates conditions under which the adverse selection principal-agent problem can be decomposed into a collection of pointwise maximization problems. The analysis uses an extension of the type assignment approach to optimal nonuniform pricing, pioneered by Goldman, Leland and Sibley (1984), to derive simple sufficient conditions under which such a decomposition is possible. These conditions do not preclude optimal bunching that arises because virtual surplus functions violate the single-crossing property or participation constraints bind at interior types.

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by Georg Nöldeke and Larry Samuelson

1 Introduction

This paper considers adverse selection principal-agent models with quasilinear utility functions and a one-dimensional decision variable (in addition to monetary transfers). The type of the agent is one dimensional and continuously distributed; the agent's preferences satisfy a single-crossing property.

The standard analysis of such models relies on the assumption that the agent's rent function is monotonic to transform the principal's maximization into a reduced problem, maximizing expected virtual surplus (i.e., surplus adjusted to account for the agent's informational rent (Myerson [21])) subject only to a monotonicity constraint. If, in addition, the virtual surplus function satisfies a single-crossing condition, then the monotonicity constraint does not bind and the problem can be decomposed into a collection of unconstrained pointwise maximization problems. The characterization of optimal contracts in such decomposable principal-agent problems is particularly simple and leads to rich economic insights. As emphasized by Rochet and Stole [26] in the context of multidimensional screening models, much of the difficulty in extending the analysis of principal-agent models beyond the standard setting is caused by the loss of the simple, recursive structure that allows for a decomposition of the problem.

Even within the one-dimensional framework we consider, there are many situations in which the standard method does not lead to a decomposable problem. First, single crossing of the virtual surplus may fail and hence the monotonicity constraint in the reduced problem may bind. To characterize the resulting optimal bunches, the standard approach resorts to ironing techniques (cf. Baron and Myerson [2], Guesnerie and Laffont [11], and Mussa and Rosen [23]). Second, and more fundamentally, the rent function may be non-monotonic. The participation constraint in the standard approach then becomes non-trivial, in the sense that the set of agents for which the constraint binds can no longer be identified a priori. As emphasized by Jullien [13], this implies that—in contrast to the case in which the rent function is monotonic—the virtual surplus function to be maximized is itself endogenously determined, undermining the simplicity of the reduced problem obtained in the first step of the standard method. Extensions of the standard method dealing with this difficulty have been developed by, for instance, Lewis and Sappington [15], [16], Maggi and Rodriguez-Clare [17], and Biais, Martimort and Rochet [3]. Jullien [13] offers a full-fledged characterization of optimal contracts. These papers rely on techniques from optimal control theory to characterize the optimal bunching that arises from the interaction between the participation constraint and the principal's objective of minimizing informational rents. In the process, the decomposability of the problem and the corresponding interpretation of the optimality conditions is lost.

This paper identifies circumstances under which principal-agent problems can be decomposed into a collection of unconstrained, pointwise maximization problems, even though optimal bunching may arise as a result of the virtual surplus function violating the single-crossing property or the participation constraint binding at interior (endogenously determined) types. The key observation underlying our analysis is that the conditions required for decomposability depend upon how one formulates the principal's problem.

The standard approach invokes the revelation principle to write the problem as a choice over functions attaching outcomes and rents to types of the agent. We obtain an equivalent formulation of the principal's problem as one of assigning types of the agents and tariffs to decisions, subject to constraints analogous to those of the standard formulation. This formulation of the principal's problem as an *assignment problem* is inspired by the analysis of optimal nonuniform pricing in Goldman, Leland and Sibley [9]. They consider a model in which the agent's utility is increasing in type and propose a formulation of the principal's problem which, given this monotonicity, is effectively equivalent to the assignment problem. Our work provides foundations for the analysis in Goldman, Leland and Sibley by formally demonstrating the equivalence of the assignment problem to the standard formulation of the principal's problem. At the same time, we extend their formulation to models in which the agent's utility function may be non-monotonic in type.

We then investigate the conditions under which the assignment problem can be reduced to a maximization subject only to the constraint that the assignment of types to decisions be increasing. In contrast to the standard approach, the issue here is whether one can identify the optimal tariff assigned to some decision (rather than the optimal rent of some type of the agent, namely one for whom the participation constraint binds). We introduce a condition on the agent's utility function, called the *minmax* property, generalizing the requirement that the agent's utility be monotonic in type, under which such an identification is possible. The class of models satisfying the minmax property is significantly broader than that in which the agent's utility is monotonic, including models of market making (cf. Biais, Martimort and Rochet [3] and Glosten [7, 8]) and countervailing incentives (cf. Lewis and Sappington [15, 16], Maggi and Rodriguez-Clare [17], and Feenstra and Lewis [5]).

The principal's objective in our reduced maximization problem can be interpreted as an integral over a marginal profit function. If, in addition to the minmax property, this marginal profit function satisfies a single-crossing property, then the assignment problem can be decomposed. The single-crossing condition on marginal profits does not imply the single-crossing condition on the virtual surplus function required for decomposition in the standard approach. Instead, it is equivalent to the assumption that virtual surpluses are strictly concave.

As noted above, our work is related to the analysis of optimal nonlinear prices in Goldman, Leland and Sibley [9]. It is also related to Wilson's demand profile approach [27]. Both approaches formulate the principal's problem as one of choosing an optimal marginal tariff schedule and both lead to the observation that the solution may be obtained by pointwise maximization, even when such a decomposition is not feasible in the standard approach. (See Rochet and Stole [26] for a discussion of this point in the context of the demand profile approach.) Neither Goldman, Leland and Sibley [9] nor Wilson [27] show the equivalence of their approaches to the standard approach at the level of generality that we prove equivalence to the assignment approach. In particular, there is no counterpart to our result establishing the decomposability of models satisfying the minmax property.

The following section introduces the model. Section 3 reviews the decomposition of the principal-agent problem under the standard approach. Section 4 presents the assignment approach. Section 5 introduces the minmax property and shows how it can be used to reduce the assignment approach. Section 6 establishes the decomposition results and uses them to present an alternative characterization of optimal bunching. Section 7 concludes.

2 The Model

The principal and the agent contract on a one-dimensional decision $x \in [\underline{x}, \overline{x}]$, referred to as a quantity, and a monetary transfer $m \in \mathbb{R}$. The agent's utility from trade depends on his type $\theta \in [\underline{\theta}, \overline{\theta}]$ and is given by $u(x, \theta) - m$. The principal's utility from trade may also depend on the type of the agent (i.e., we allow for common values) and is given by $v(x, \theta) + m$.

The agent knows his type. From the principal's perspective the agent's type is drawn from the interval $[\underline{\theta}, \overline{\theta}]$ according to the distribution function $F(\theta)$, with differentiable density $f(\theta) > 0$.

The functions $u(\cdot)$ and $v(\cdot)$ are assumed to be thrice continuously differentiable on $[\underline{x}, \overline{x}] \times [\underline{\theta}, \overline{\theta}]$. In addition, we assume throughout that the agent's utility function satisfies the strict single-crossing property (denoting partial derivatives by subscripts):

$$u_{x\theta}(x, \theta) > 0, \ \forall x, \theta. \tag{1}$$

The principal designs a contract to maximize his expected utility from trade, subject to an incentive compatibility constraint and a participation constraint that each type of agent receives at least his reservation utility, which we normalize to zero.[1] From the revelation principal there is no loss of generality in restricting the principal to truthful direct revelation mechanisms $(x(\cdot), m(\cdot))$, where $x : [\underline{\theta}, \overline{\theta}] \to [\underline{x}, \overline{x}]$ and $m : [\underline{\theta}, \overline{\theta}] \to I\!R$. The principal's problem is

$$\max_{x(\cdot), m(\cdot)} \int_{\underline{\theta}}^{\overline{\theta}} [v(x(\theta), \theta) + m(\theta)] \, f(\theta) d\theta$$

subject to the incentive constraint

$$u(x(\theta), \theta) - m(\theta) \geq u(x(\psi), \theta) - m(\psi), \quad \forall \theta, \psi$$

and the participation constraint

$$u(x(\theta), \theta) - m(\theta) \geq 0 \ \ \forall \theta.$$

Using the rent function $r : [\underline{\theta}, \overline{\theta}] \to I\!R$ defined by $r(\theta) = u(x(\theta), \theta) - m(\theta)$ to eliminate the transfers $m(\cdot)$ simplifies the maximization program. First, rewrite the incentive constraint as

$$r(\theta) - r(\psi) \geq u(x(\psi), \theta) - u(x(\psi), \psi), \ \forall \theta, \psi. \tag{2}$$

We refer to a pair $(x(\cdot), r(\cdot))$ as an *allocation*. An allocation is *incentive compatible* if it satisfies (2). The following characterization of incentive

[1] It is without loss of generality to assume that the reservation utility of all types of the agent is given by zero, as we can always interpret utilities as surpluses over a (possibly type dependent) reservation utility. Imposing full participation as a constraint is innocuous if, as in many applications, there is a "no-trade" quantity x that is equivalent to exclusion, i.e., such that $u(x, \theta) = v(x, \theta) = 0$ for all θ. See Jullien [13] for further discussion of the principal-agent problem with exclusion and conditions under which it is without loss of generality to impose full participation.

compatible allocations is familiar for the case in which additional smoothness conditions are imposed on the set of feasible allocations (e.g., Fudenberg and Tirole [6, ch. 7]). For the general case we consider here the result follows from Rochet [25] and Milgrom and Segal [20].

Lemma 1 *An allocation* $(x(\cdot), r(\cdot))$ *is incentive compatible if and only if* $x(\cdot)$ *is increasing (i.e.,* $x(\theta) \geq x(\psi)$ *for all* $\theta > \psi$*) and* $r(\theta) = r(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} u_\theta(x(\tilde{\theta}), \tilde{\theta}) d\tilde{\theta}$ *for all* θ*.*

Replacing the incentive constraint (2) with this characterization while using the rent function to rewrite the participation constraint as $r(\theta) \geq 0$ and the principal's payoff as

$$\Gamma(x(\cdot), r(\cdot)) \equiv \int_{\underline{\theta}}^{\overline{\theta}} \left[v(x(\theta), \theta) + u(x(\theta), \theta) - r(\theta) \right] f(\theta) d\theta,$$

one obtains the following formulation of the principal's problem:

$$\max_{x(\cdot), r(\cdot)} \Gamma(x(\cdot), r(\cdot)) \tag{3}$$

subject to

$$x(\cdot) \text{ is increasing,} \tag{4}$$

$$r(\theta) = r(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} u_\theta(x(\tilde{\theta}), \tilde{\theta}) d\tilde{\theta}, \ \forall \theta, \tag{5}$$

$$r(\theta) \geq 0, \ \forall \theta. \tag{6}$$

We refer to the program (3)–(6) as the *allocation problem.* An allocation is *feasible* if it satisfies (4)–(6) and *optimal* if it solves the allocation problem. We say that a quantity allocation or rent allocation is optimal if it is part of an optimal allocation. Jullien [13] shows that under the assumptions maintained here an optimal allocation exists.

Throughout the paper we will use the following example to illustrate the main points of our analysis.

Example. Let the agent's and principal's utility functions be given by

$$u(x, \theta) = \theta x - \frac{1}{2}\gamma x^2 \tag{7}$$

$$v(x, \theta) = -\alpha x \theta - \frac{1}{2}\beta x^2, \tag{8}$$

where $\alpha, \beta, \gamma \geq 0$, $\beta + \gamma > 0$, and $\underline{x} \leq 0 \leq \overline{x}$.

3 Decomposing the Allocation Problem

This section reviews the standard approach to the allocation problem, for the case in which the agent's utility function is monotonic in type. It is straightforward to compute the payoff-maximizing rent allocation as a function of the quantity allocation to be implemented. This leads to a *reduced problem*, determining an optimal quantity allocation, to which standard solution techniques can be applied if the agent's utility function is monotonic. If in addition the relevant virtual surplus function (cf. Myerson [21]) satisfies a single-crossing property, then the reduced problem can be decomposed into a collection of pointwise maximization problems.

3.1 Reduction

Suppose the agent's utility is increasing in type, i.e. $u_\theta(x, \theta) \geq 0$ for all x, θ. It then follows from (5) that an incentive compatible allocation satisfies the participation constraint (6) if and only if $r(\underline{\theta}) \geq 0$. Thus, for any given increasing quantity allocation $x(\cdot)$, the principal's payoff is maximized by setting $r(\underline{\theta}) = 0$ and $r(\theta) = \int_{\underline{\theta}}^{\theta} u_\theta(x(\tilde{\theta}), \tilde{\theta}) d\tilde{\theta}$. Substituting this expression for the rent function into the principal's objective function and integrating by parts we find that a quantity allocation is optimal if and only if it solves the reduced problem

$$\max_{x(\cdot) \text{ increasing}} \int_{\underline{\theta}}^{\overline{\theta}} \sigma(x(\theta), \theta) f(\theta) d\theta, \tag{9}$$

where

$$\sigma(x, \theta) = v(x, \theta) + u(x, \theta) - \frac{1 - F(\theta)}{f(\theta)} u_\theta(x, \theta)$$

is the virtual surplus function, which takes into account the rents that must be left to types higher than θ if quantity $x(\theta)$ is assigned to type θ.

The single-crossing property (1) ensures that the agent's utility will be increasing in type whenever $u_\theta(\underline{x}, \theta) \geq 0$ holds. In particular, the agent's utility is increasing in type in models of monopoly pricing (Goldman, Leland and Sibley [9], Maskin and Laffont [18], and Mussa and Rosen [23]), in which $\underline{x} = 0$ corresponds to the no-trade outcome (with $u(0, \theta) = 0$) and $c(x, \theta) = -v(x, \theta)$ corresponds to the principal's cost function (usually taken to be type independent).

An equivalent argument applies to the case in which the agent's utility function is decreasing in type, i.e. $u_\theta(x, \theta) \leq \theta$ for all x, θ. Condition

(5) then implies that an incentive compatible allocation satisfies the participation constraint (6) if and only if $r(\overline{\theta}) \geq 0$. Substituting the resulting expression for the payoff maximizing rent function $r(\theta) = \int_{\overline{\theta}}^{\theta} u_\theta(x(\tilde{\theta}), \tilde{\theta}) d\tilde{\theta}$ into the principal's objective function leads to the conclusion that a quantity allocation is optimal if and only if it solves the reduced problem

$$\max_{x(\cdot) \text{ increasing}} \int_{\underline{\theta}}^{\overline{\theta}} \beta(x(\theta), \theta) f(\theta) d\theta, \tag{10}$$

where

$$\beta(x, \theta) = v(x, \theta) + u(x, \theta) + \frac{F(\theta)}{f(\theta)} u_\theta(x, \theta)$$

is the virtual surplus function, which takes into account the rents that must be left to types below θ.

For an example in which the agent's utility function is decreasing in type, think of the agent as a regulated firm, as in Baron and Myerson [2] (see also Laffont and Tirole [14]). Let $\overline{x} = 0$ correspond to an outcome in which there is no trade, satisfying $u(\overline{x}, \theta) = 0$, and let $q = -x \geq 0$ correspond to the quantity produced by the regulated firm, at cost $c(q, \theta) = -u(x, \theta)$. The single-crossing property (1) then corresponds to the standard assumption that the firm's marginal production costs $c_q(q, \theta)$ are increasing in type. Because $u(\overline{x}, \theta) = 0$, the single-crossing property implies $u_\theta(x, \theta) < 0$ for $x < \overline{x}$.

3.2 Relaxation

The usual procedure for solving (9) and (10) is to relax the monotonicity constraint and maximize the objective functions pointwise, thus decomposing the problem into a sequence of independent maximization problems. This pointwise maximization yields a quantity allocation that is optimal *if* it is increasing. If the virtual surplus $\sigma(\cdot)$ satisfies the single-crossing condition $\sigma_{x\theta}(\cdot) \geq 0$, the pointwise maximization $\max_x \sigma(x, \theta)$ will indeed give a quantity allocation that is increasing in θ and will thus solve (9).[2] Similarly, if the virtual surplus $\beta(\cdot)$ satisfies the single-crossing condition $\beta_{x\theta}(\cdot) \geq 0$, then (10) can be solved by pointwise maximization.

Matters are more complicated if the pointwise maximization does not give an increasing solution. To ensure the monotonicity constraint is satisfied, the analysis must then resort to ironing techniques, as developed by Guesnerie and Laffont [11], Mussa and Rosen [23], and Myerson [21]).

[2] It follows from Theorem 4 in Milgrom and Shannon [19] that there exists an increasing selection from $\arg\max_x \sigma(x, \theta)$ if the single-crossing condition $\sigma_{x\theta}(\cdot) \geq 0$ holds.

The decomposition of the allocation problem thus rests on the twin assumptions that the agent's utility is monotonic in type and the virtual surpluses satisfy the single-crossing property. The assignment approach allows us to weaken the first assumption while replacing the second with the assumption that the virtual surpluses are strictly concave.

Example. In the example given by (7)–(8), the agent's utility function is increasing in type if and only if $\underline{x} = 0$, in which case an optimal quantity allocation is given by a solution to (9). The single-crossing condition $\sigma_{x\theta}(x, \theta) \geq 0$ is equivalent to

$$\frac{d}{d\theta}\left((1-\alpha)\theta - \frac{1-F(\theta)}{f(\theta)}\right) \geq 0, \tag{11}$$

which suffices to solve (9) by pointwise maximization. Note that whenever the pointwise solution is determined by the first order condition $\sigma_x(x, \theta) = 0$ for all θ, then (11) is also necessary for decomposing the allocation problem.

Similarly, the agent's utility will be decreasing in type if and only if $\overline{x} = 0$, in which case an optimal quantity allocation is given by a solution to the reduced problem (10). The single-crossing condition $\beta_{x\theta}(\cdot) \geq 0$ is then equivalent to

$$\frac{d}{d\theta}\left((1-\alpha)\theta + \frac{F(\theta)}{f(\theta)}\right) \geq 0 \tag{12}$$

and suffices to decompose the allocation problem. Condition (12) is also necessary if the solution to the pointwise maximization is interior for all θ.

4 The Assignment Problem

We replace the maximization with respect to allocations with an equivalent maximization with respect to *assignments* $(\theta(\cdot), t(\cdot))$. An assignment consists of a *type assignment* $\theta : [\underline{x}, \overline{x}] \to [\underline{\theta}, \overline{\theta}]$ and a *tariff assignment* $t : [\underline{x}, \overline{x}] \to \mathbb{R}$. We can view a type assignment as specifying the type of agent $\theta(x)$ who chooses quantity x, and a tariff assignment as a nonlinear pricing schedule specifying the accompanying transfer $t(x)$. Subsection 4.1 motivates the assignment approach and formulates the assignment problem, while Subsection 4.2 proves it is equivalent to the allocation problem.

4.1 Motivation

The taxation principle (see Guesnerie [10, chapter 1], Hammond [12], and Rochet [24]) asserts that an allocation $(x(\cdot), r(\cdot))$ is incentive compatible if and only if there exists a tariff assignment $t : [\underline{x}, \overline{x}] \to I\!R$ *implementing* the allocation, in the sense that $x(\theta) \in \arg\max_x[u(x,\theta) - t(x)]$ and $r(\theta) = \max_x[u(x,\theta) - t(x)]$ for all θ. In addition, it is clear that an allocation determined by these conditions will satisfy the participation constraint (6) if and only if the tariff assignment satisfies the constraint $\max_x[u(x,\theta) - t(x)] \geq 0$. This suggests that instead of viewing the principal as choosing a feasible allocation we may view the principal as choosing a tariff assignment subject to the constraint $\max_x[u(x,\theta) - t(x)] \geq 0$. The difficulty with this view is that the principal's payoff from choosing a tariff assignment $t(\cdot)$ is given by the payoff from the allocation $(x(\cdot), r(\cdot))$ implemented by $t(\cdot)$, which is characterized by the *global* (rather than local) optimality conditions $x(\theta) \in \arg\max_x[u(x,\theta) - t(x)]$ and $r(\theta) = \max_x[u(x,\theta) - t(x)]$.

We will demonstrate that, as suggested by Goldman, Leland and Sibley [9], this difficulty can be overcome by restricting the class of tariff assignments $t(\cdot)$ from which the principal can choose to those satisfying

$$t(x) = t(\underline{x}) + \int_{\underline{x}}^{x} u_x(\tilde{x}, \theta(\tilde{x}))d\tilde{x} \tag{13}$$

for some increasing type assignment $\theta(\cdot)$. To motivate (13), consider the case in which it holds for a type assignment $\theta(\cdot)$ which is continuous, strictly increasing and satisfies $\theta(\underline{x}) = \underline{\theta}$ and $\theta(\overline{x}) = \overline{\theta}$. The type assignment $\theta(\cdot)$ then has an inverse $x(\cdot)$. Because $t_x(x) = u_x(x, \theta(x))$ for all x, the inverse $x(\cdot)$ satisfies the first order condition $u_x(x(\theta), \theta) - t_x(x) = 0$ for all θ. Furthermore, using the single-crossing property (1) of the agent's utility function, this first order condition is sufficient to imply that $x(\theta)$ is the unique solution to the problem $\max_x[u(x,\theta) - t(x)]$ for all θ. Hence, the problem of identifying the allocation $(x(\cdot), r(\cdot))$ implemented by $t(\cdot)$ is solved: the quantity allocation $x(\cdot)$ is given by the inverse of the type assignment $\theta(\cdot)$. The rent allocation $r(\cdot)$ can then be determined from (5) and the initial condition $r(\underline{\theta}) = u(\underline{x}, \underline{\theta}) - t(\underline{\theta})$. The following subsection extends this argument to general increasing type assignments and, more fundamentally, proves that the taxation principle continues to hold when restricting attention to tariff assignments satisfying (13).

We associate with every tariff $t(\cdot)$ satisfying (13), for an increasing $\theta(\cdot)$, the payoff

$$\Pi(\theta(\cdot), t(\cdot)) \equiv \int_{\underline{x}}^{\overline{x}} s(x, \theta(x))dx + V(\underline{x}) + t(\underline{x}), \tag{14}$$

where

$$V(y) = \int_{\underline{\theta}}^{\overline{\theta}} v(y, \theta) f(\theta) d\theta \tag{15}$$

and

$$s(x,\theta) = \int_{\theta}^{\overline{\theta}} [u_x(x,\theta) + v_x(x,\tilde{\theta})] f(\tilde{\theta}) d\tilde{\theta}. \tag{16}$$

This extends Goldman, Leland and Sibley's [9] formulation and is related to the payoff expression in Wilson's [27] demand profile approach. To motivate (14), consider again the case in which $\theta(\cdot)$ is continuous, strictly increasing and satisfies the conditions $\theta(\underline{x}) = \underline{\theta}$ and $\theta(\overline{x}) = \overline{\theta}$, so that the quantity allocation $x(\cdot)$ implemented by $t(\cdot)$ is the inverse of $\theta(\cdot)$. If the principal assigns type θ to a quantity x, we may view the principal as providing type θ of the agent with the minimum quantity $\underline{x}$ and with the additional marginal units up to quantity x. The term $V(\underline{x}) + t(\underline{x})$ then reflects the principal's payoff from providing all types with $\underline{x}$ at tariff $t(\underline{x})$. The increment in the principal's payoff from providing the marginal unit x is given by $s(x,\theta)$: first, as $x(\cdot)$ is increasing, all types $\tilde{\theta} > \theta$ are provided with the marginal unit x, explaining the range of integration in (16). Second, the principal will provide the marginal unit x at a price (marginal tariff) equal to type θ's willingness to pay for the marginal unit, given by $u_x(x,\theta)$, whereas the direct effect on the principal's utility of providing type $\tilde{\theta}$ with the marginal unit x is given by $v_x(x,\tilde{\theta})$. In the following we will refer to $s(\cdot)$ as the principal's *marginal profit function*. Subsection 4.2 proves, for any increasing type assignment $\theta(\cdot)$ and corresponding tariff $t(\cdot)$ given by (13), that (14) is indeed the principal's payoff from any allocation implemented by $t(\cdot)$.

This discussion suggests the following reformulation of the principal's problem as choosing an assignment $(\theta(\cdot), t(\cdot))$ to solve:

$$\max_{\theta(\cdot),t(\cdot)} \Pi(\theta(\cdot), r(\cdot)) \tag{17}$$

subject to

$$\theta(\cdot) \text{ increasing,} \tag{18}$$

$$t(x) = t(\underline{x}) + \int_{\underline{x}}^{x} u_x(\tilde{x}, \theta(\tilde{x})) d\tilde{x}, \ \forall x, \tag{19}$$

and the participation constraint

$$\max_x [u(x,\theta) - t(x)] \geq 0, \ \forall \theta. \tag{20}$$

We refer to the program (17)–(20) as the *assignment problem.* Noting the formal analogy between the characterization of incentive compatible allocations in Lemma 1 and the requirement that (13) holds for an increasing type assignment, we offer

Definition 1 *An assignment* $(\theta(\cdot), t(\cdot))$ *is incentive compatible if (18)–(19) hold.*

An assignment is *feasible* if it satisfies (18)–(20). An assignment is *optimal* if it solves the assignment problem. A type assignment or tariff assignment is optimal if it is part of an optimal assignment.

In the following subsection we prove that the assignment problem is equivalent to the allocation problem.

4.2 Equivalence

We find it convenient to build on the intuition presented after the statement of (13): if $(\theta(\cdot), t(\cdot))$ is incentive compatible, then $t(\cdot)$ implements an allocation $(x(\cdot), r(\cdot))$ characterized by the conditions that $x(\cdot)$ is an inverse of $\theta(\cdot)$ and $r(\underline{x}) = u(\underline{x}, \underline{\theta}) - t(\underline{x})$. In making this intuition precise, we must accommodate the possibility that an increasing quantity allocation $x(\cdot)$ may have *gaps* (i.e., upward discontinuities) and *bunches* (i.e., intervals on which the function is constant), and may not satisfy the conditions $x(\underline{\theta}) = \underline{x}$ and $x(\overline{\theta}) = \overline{x}$. As a result, we must work with an appropriately generalized version of an inverse.

For every increasing quantity allocation $x(\cdot)$, let the correspondence $X : [\underline{\theta}, \overline{\theta}] \to [\underline{x}, \overline{x}]$ be given by

$$X(\theta) = [\lim_{\psi \uparrow \theta} x(\psi), \lim_{\psi \downarrow \theta} x(\psi)],$$

where we adopt the convention $\lim_{\psi \uparrow \underline{\theta}} x(\psi) = \underline{x}$ and $\lim_{\psi \downarrow \overline{\theta}} x(\psi) = \overline{x}$. We say that two quantity allocations are *equivalent* if they give rise to the same correspondence $X(\cdot)$: replacing $x(\theta)$ with any value from the interval $[\lim_{\psi \uparrow \theta} x(\psi), \lim_{\psi \downarrow \theta} x(\psi)]$ yields a different but equivalent quantity allocation.[3]

For any increasing type assignment $\theta(\cdot)$ let $\Theta : [\underline{x}, \overline{x}] \to [\underline{\theta}, \overline{\theta}]$ denote the correspondence defined by

$$\Theta(x) = [\lim_{y \uparrow x} \theta(y), \lim_{y \downarrow x} \theta(y)], \tag{21}$$

where we adopt the convention that $\lim_{y \uparrow \underline{x}} \theta(y) = \underline{\theta}$ and $\lim_{y \downarrow \overline{x}} \theta(y) = \overline{\theta}$. We say that two type assignments are equivalent if they give rise to the same correspondence $\Theta(\cdot)$.

[3] Note that if $(x(\cdot), r(\cdot))$ is incentive compatible then $(x'(\cdot), r(\cdot))$ is incentive compatible if and only if $x'(\cdot)$ is equivalent to $x(\cdot)$. Furthermore if $x'(\cdot)$ is equivalent to $x(\cdot)$ then the principal's payoffs from the incentive compatible allocations $(x(\cdot), r(\cdot))$ and $(x'(\cdot), r(\cdot))$ are identical.

Definition 2 *An increasing quantity allocation $x(\cdot)$ and an increasing type assignment $\theta(\cdot)$ are inverse if*

$$x \in X(\theta) \Leftrightarrow \theta \in \Theta(x).$$

An incentive compatible allocation $(x(\cdot), r(\cdot))$ and an incentive compatible assignment $(\theta(\cdot), t(\cdot))$ are consistent if $x(\cdot)$ and $\theta(\cdot)$ are inverse and

$$r(\underline{\theta}) = u(\underline{x}, \underline{\theta}) - t(\underline{x}). \tag{22}$$

Noticing that every increasing quantity allocation $x(\cdot)$ has an inverse, as does every increasing type assignment $\theta(\cdot)$, we obtain:

Lemma 2 *If the allocation $(x(\cdot), r(\cdot))$ is incentive compatible, then there exists an incentive compatible assignment $(\theta(\cdot), t(\cdot))$ such that $(x(\cdot), r(\cdot))$ and $(\theta(\cdot), t(\cdot))$ are consistent. If the assignment $(\theta(\cdot), t(\cdot))$ is incentive compatible, then there exists an incentive compatible allocation $(x(\cdot), r(\cdot))$ such that $(x(\cdot), r(\cdot))$ and $(\theta(\cdot), t(\cdot))$ are consistent.*

Different inverses of a given quantity allocation (resp. type assignment) must be equivalent. Hence, Lemma 2 implies that, up to equivalences, consistency establishes a bijection between the set of incentive compatible allocations and the set of incentive compatible assignments.

The following Lemma shows that if $(x(\cdot), r(\cdot))$ and $(\theta(\cdot), t(\cdot))$ are consistent then the tariff assignment $t(\cdot)$ implements the allocation $(x(\cdot), r(\cdot))$. In conjunction with the first sentence of Lemma 2, this establishes that the taxation principle holds when attention is restricted to tariff assignments satisfying (13) (for an increasing type assignment $\theta(\cdot)$).[4]

Lemma 3 *If $(x(\cdot), r(\cdot))$ and $(\theta(\cdot), t(\cdot))$ are consistent, then:*

$$X(\theta) = \arg\max_x \left[u(x, \theta) - t(x)\right], \quad \forall\theta \tag{23}$$

$$r(\theta) = \max_x [u(x, \theta) - t(x)], \quad \forall\theta. \tag{24}$$

In particular, $(x(\cdot), r(\cdot))$ is feasible in the allocation problem if and only if $(\theta(\cdot), t(\cdot))$ is feasible in the assignment problem.

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[4] We note that arguments analogous to the ones establishing Lemma 3 show that if $(x(\cdot), r(\cdot))$ and $(\theta(\cdot), t(\cdot))$ are consistent, then

$$t(x) = \max_\theta [u(x, \theta) - r(\theta)], \ \forall x.$$

For the special case in which $u(x, \theta) = x\theta$, functions $r(\theta)$ and $t(x)$ are then Fenchel conjugates of one another (Rockafellar [22, Section 12]).

It remains to show that if $(\theta(\cdot), t(\cdot))$ and $(x(\cdot), r(\cdot))$ are consistent, then the associated payoffs are identical. From Lemma 3, we can then conclude that if $(x(\cdot), r(\cdot))$ is implemented by $t(\cdot)$, then the former's payoff is $\Pi(\theta(\cdot), t(\cdot))$.

Lemma 4 *Let $(x(\cdot), r(\cdot))$ and $(\theta(\cdot), t(\cdot))$ be consistent. Then*

$$\Pi(\theta(\cdot), t(\cdot)) = \Gamma(x(\cdot), r(\cdot)).$$

We now have all the pieces required to establish the equivalence between the allocation and the assignment problem.

Proposition 1 *An allocation $(x^*(\cdot), r^*(\cdot))$ is optimal if and only if it is consistent with an optimal assignment $(\theta^*(\cdot), t^*(\cdot))$.*

5 Reducing the Assignment Problem

In this section we investigate the circumstances under which we can reduce the assignment problem to a maximization over type assignments, paralleling the reduction procedure for the allocation problem outlined in Section 3.1. The first subsection shows that this is possible whenever the agent's utility function satisfies a property we call the minmax property. The second subsection demonstrates that the class of principal-agent models satisfying this condition is significantly broader than the class of models in which the agent's utility function is monotonic in type.

5.1 The Minmax Property

To motivate the subsequent analysis, suppose we can find a quantity $y \in [\underline{x}, \overline{x}]$ and a transfer m such that an incentive compatible assignment is feasible in the assignment problem if and only if $t(y) \leq m$. For any given increasing type assignment $\theta(\cdot)$, it is in the principal's interest to maximize the expected transfer received from the agent, and thus (from (19)) the payoff maximizing tariff assignment should be given by $t(x) = m + \int_y^x u_x(\tilde{x}, \theta(\tilde{x}))d\tilde{x}$. This reduces the principal's problem to the choice of an increasing type assignment.

The following Lemma provides the representation of the principal's payoff required for the formulation of this reduced problem (the case $y = \underline{x}$ corresponds to (14)).

Lemma 5 *For every incentive compatible assignment* $(\theta(\cdot), t(\cdot))$ *and* y*:*

$$\Pi(\theta(\cdot), t(\cdot)) = \int_{\underline{x}}^{y} b(x, \theta(x))dx + \int_{y}^{\overline{x}} s(x, \theta(x))dx + V(y) + t(y) \qquad (25)$$

where $s(x, \theta)$ *is given by (16) and*

$$b(x, \theta) = -\int_{\underline{\theta}}^{\theta} \left[v_x(\tilde{\theta}, x) + u_x(\theta, x)\right] f(\tilde{\theta})d\tilde{\theta}. \qquad (26)$$

The function $b(\cdot)$ may again be interpreted as a marginal profit function, representing the principal's payoff of *obtaining* the marginal unit x from type θ and all lower types at a price corresponding to type $\theta's$ marginal willingness to pay.

To identify a candidate transfer m and quantity y for which an incentive compatible assignment is feasible if and only if $t(y) \leq m$, define

$$\hat{u}(x) = \min_{\theta} u(x, \theta).$$

Because $\max_x [u(x, \theta) - t(x)] \geq u(y, \theta) - t(y) \geq \hat{u}(y) - t(y)$ holds for all θ and y, for any choice of y the condition

$$t(y) \leq \hat{u}(y) \qquad (27)$$

is *sufficient* for an incentive compatible assignment $(\theta(\cdot), t(\cdot))$ to be feasible. Suppose now that (27) is also *necessary* for feasibility, that is every incentive compatible assignment satisfying (20) also satisfies (27). We then have the situation described in the opening paragraph of this subsection, with $m = \hat{u}(y)$.

We are thus led to investigate the circumstances under which, for some appropriate choice of y, every increasing type assignment must satisfy $t(y) \leq \hat{u}(y)$. The following definition identifies a property of a type assignment which is key for this purpose.

Definition 3 *An increasing type assignment* $\theta(\cdot)$ *satisfies the minmax property at* y *if there exists* $\theta \in \Theta(y)$ *such that* $u(y, \theta) = \hat{u}(y)$*.*

The significance of the minmax property (and its name) is due to the following result.

Lemma 6 *Let* $(\theta(\cdot), t(\cdot))$ *be incentive compatible and let* $\theta(\cdot)$ *satisfy the minmax property at* y*. Then*

$$\min_{\theta} \max_{x} [u(x, \theta) - t(x)] = \hat{u}(y) - t(y).$$

In particular, $(\theta(\cdot), t(\cdot))$ *is feasible if and only if* $t(y) \leq \hat{u}(y)$*.*

If every increasing type assignment satisfies the minmax property at the *same* value y^*, then Lemma 6 implies that every incentive compatible assignment satisfies the participation constraint if and only if $t(y^*) \leq \hat{u}(y^*)$. We thus obtain:

Proposition 2 *Suppose there exists y^* such that every increasing type assignment satisfies the minmax property at y^*. Then $\theta^*(\cdot)$ is optimal if and only if it solves* the reduced problem

$$\max_{\theta(\cdot) \text{ increasing}} \int_{\underline{x}}^{y^*} b(x, \theta(x))dx + \int_{y^*}^{\overline{x}} s(x, \theta(x)dx. \tag{28}$$

The corresponding optimal tariff assignment is given by

$$t^*(x) = \hat{u}(y^*) + \int_{y^*}^{x} u_x(\tilde{x}, \theta^*(\tilde{x}))d\tilde{x}. \tag{29}$$

Our next proposition replaces the assumption that every increasing type assignment satisfies the minmax property at the same y^* with the following weaker condition:

Assumption 1 *The agent's utility function satisfies the minmax property, in the sense that every increasing type assignment satisfies the minmax property at some y.*

Assumption 1 ensures that an optimal type assignment satisfies $t(y^*) \leq \hat{u}(y^*)$ for some value y^*. Solving the reduced problem (28) will still generate an optimal type assignment, as replacing the participation constraint (20) by $t(y^*) \leq \hat{u}(y^*)$ does not affect the feasibility of the optimal assignment (Lemma 6). The difficulty is that we must identify y^*. To do so, let

$$W(y) = \max_{\theta(\cdot) \text{ increasing}} \int_{\underline{x}}^{y} b(x, \theta(x))dx + \int_{y}^{\overline{x}} s(x, \theta(x))dx + V(y) + \hat{u}(y).$$

Then, from Lemma 5, $W(y)$ is the value of the assignment problem when its participation constraint is replaced by the constraint $t(y) \leq \hat{u}(y)$. Because the latter constraint is sufficient for the participation constraint, $W(y)$ cannot exceed the value of the assignment problem. But the value of the assignment problem must be $W(y^*)$, as an optimal type assignment solves the reduced problem given y^*. Hence, we can identify y^* as a maximizer of $W(y)$:

Proposition 3 *Let Assumption 1 hold. Then* $\theta^*(\cdot)$ *is an optimal type assignment if and only if there exists* $y^* \in \arg\max_y W(y)$ *such that* $\theta^*(\cdot)$ *solves*

$$\max_{\theta(\cdot) \text{ increasing}} \int_{\underline{x}}^{y^*} b(x, \theta(x))dx + \int_{y^*}^{\overline{x}} s(x, \theta(x))dx.$$

The corresponding optimal tariff assignment is given by (29).

The important implication of Assumption 1 is that *optimal* type assignments satisfy the minmax property at some y. Proposition 3 would hold with this weaker requirement. Notice in addition that, from Propositions 2 and 3, if every increasing type assignment satisfies the minmax property at y^*, then y^* maximizes $W(y)$.

5.2 Verifying the Minmax Property

This section presents cases in which the agent's utility function satisfies the minmax property. We begin with models with a value y^* at which every increasing type assignment satisfies the minmax property.

Suppose first that the agent's utility is increasing in type, as in monopoly pricing problems. It then follows that $\hat{u}(\underline{x}) = u(\underline{x}, \underline{\theta})$. Since $\underline{\theta} \in \Theta(\underline{x})$ holds for every increasing type assignment, this is sufficient to imply that every increasing type assignment satisfies the minmax property at $y^* = \underline{x}$. We may then apply Proposition 2 to conclude that an assignment is optimal if and only if it solves

$$\max_{\theta(\cdot) \text{ increasing}} \int_{\underline{x}}^{\overline{x}} s(x, \theta(x))dx.$$

The corresponding optimal tariff assignment can then be calculated from (29). This corresponds to the solution procedure proposed by Goldman, Leland and Sibley [9] to solve the nonlinear pricing problem of a monopolist. Note that reducing the assignment problem does not require the additional technical assumptions imposed by Goldman, Leland and Sibley.

The case in which the agent's utility function is decreasing in type, as in regulation problems, is similar. Here we have $\hat{u}(\overline{x}) = u(\overline{x}, \overline{\theta})$. Since every increasing type assignment satisfies $\overline{\theta} \in \Theta(\overline{x})$, it satisfies the minmax property at $y^* = \overline{x}$. Applying Proposition 2, an optimal type assignment solves the reduced problem

$$\max_{\theta(\cdot) \text{ increasing}} \int_{\underline{x}}^{\overline{x}} b(x, \theta(x))dx.$$

Proposition 2 is also applicable whenever the agent's utility function satisfies the condition that there exists $y \in [\underline{x}, \overline{x}]$ such that

$$u(y, \theta) = u(y, \psi) \text{ for all } \theta, \psi. \tag{30}$$

It is then immediate that $\hat{u}(y) = u(y, \theta)$ holds for all θ and, thus, every increasing type assignment satisfies the minmax property at $y^* = y$. When (30) holds for $y = \underline{x}$ or $y = \overline{x}$, we have the special case of a monopoly or regulation setting, where the agent's utility function is monotonic in type. The more interesting case is the one in which condition (30) holds for $y \in (\underline{x}, \overline{x})$, implying that the agent's utility function is non-monotonic in type. A simple example is provided by (7)–(8), with $\underline{x} < 0 = y < \overline{x}$. Biais, Martimort and Rochet [3] (see also Glosten [7, 8]) use a model with this property to examine market making. Here, the principal is a market maker who trades with the agent, either selling ($x > 0$) or buying ($x < 0$) quantity x of the good. The quantity $y = 0 \in (\underline{x}, \overline{x})$ corresponds to the no-trade outcome, satisfying $u(0, \theta) = 0$ for all θ and thus verifying condition (30) for an interior value of y. Condition (30) is also also satisfied in the model of "inflexible rules" considered by Lewis and Sappington [15].[5]

The model in Lewis and Sappington [15] is a special case of a model with countervailing incentives (Lewis and Sappington [16] and Maggi and Rodriguez-Clare [17]; also see Feenstra and Lewis [5]). As noted by Maggi and Rodriguez-Clare [17], the important structural property of Lewis and Sappington's model [16] of countervailing incentives is that the agent's utility function is quasi-convex in type. We can show that this implies the minmax property:

Lemma 7 *Suppose $u(x, \theta)$ is quasi-convex in θ. Then the agent's utility function satisfies the minmax property.*

The intuition for this result begins by noting that, because the utility function is quasi-convex in θ, the solution to $\min_\theta u(x, \theta)$ is a convex set that varies upper hemicontinuously in x. For any increasing type assignment $\theta(\cdot)$, the associated correspondence $\Theta(\cdot)$ similarly gives convex sets that vary upper hemicontinuously in x, with $\underline{\theta} \in \Theta(\underline{x})$ and $\overline{\theta} \in \Theta(\overline{x})$. The

[5]The agent's utility function in Lewis and Sappington [15] is given by $P - cQ + cK$, where P is a monetary transfer *received* by the agent, c is the type of the agent, $Q \geq 0$ is the production level and the constant $K \geq 0$ is the installed capacity. Setting $m = -P$, $c = \theta$, $x = -Q$, and $y = -K$ this corresponds to our formulation of the principal-agent problem with $u(x, \theta) = \theta(x - y)$, so that condition (30) is satisfied.

two correspondences must then intersect for some y, at which the minmax property holds.

It follows from Proposition 3 and Lemma 7 that when the agent's utility function is quasi-convex in θ, an assignment $\theta^*(\cdot)$ is optimal if and only if there exists y^* with $y^* \in \arg\max_y W(y)$ such that $\theta^*(\cdot)$ solves the reduced problem (28). Notice that in contrast to the other cases discussed in this subsection, determining the value y^* which maximizes the value function $W(y)$ is non-trivial, as (in general) different type assignments will satisfy the minmax property at different values of y. We return to the issue of maximizing $W(y)$ in Section 6.1, after having introduced an assumption sufficient to decompose the reduced problem.

Provided the agent's utility function is continuous and satisfies the single-crossing property (as we assume throughout), the proof of Lemma 7 shows that it satisfies the minmax property if the set $\arg\min_\theta u(x, \theta)$ is convex-valued for all $x \in (\underline{x}, \overline{x})$. It is not difficult to see that this condition is also necessary for Assumption 1 to hold. In particular, the agent's utility function violates the minmax property if it is strictly quasi-concave in θ and satisfies $u(\underline{x}, \underline{\theta}) > u(\underline{x}, \overline{\theta})$ and $u(\overline{x}, \overline{\theta}) > u(\overline{x}, \underline{\theta})$. Examples in which the agent's utility function is strictly quasi-concave in θ are discussed and solved by Maggi and Rodriguez-Clare [17] and Jullien [13]. In these examples optimal allocations have the property that quantity allocations are strictly increasing and the participation constraint binds for more than one type. It follows that not only Assumption 1, but also the characterization of optimal assignments in Proposition 3 (which only requires that optimal type assignments satisfy the minmax property), fails.[6]

6 Decomposing the Assignment Problem

This section first provides conditions under which the problem of determining an optimal assignment can be decomposed into a collection of pointwise maximization problems. We then show that if these conditions are satisfied, the assignment approach yields a simple characterization of optimal bunching.

[6]Suppose an assignment satisfies (29) for some y^* and that $\theta_1 < \theta_2$ satisfy $\max_x[u(x, \theta_1) - t(x)] = \max_x[u(x, \theta_2) - t(x)] = 0$. It then follows that y^* is an optimal choice for types θ_1 and θ_2. The strict single-crossing property (1) of the agent's utility function then implies y^* is the unique optimal choice for all types $\theta \in (\theta_1, \theta_2)$, so that any allocation implemented by the given assignment fails to be strictly increasing, a contradiction.

6.1 Pointwise Maximization

The monotonicity constraint in the reduced assignment problem (28) may be ignored if the pointwise maximization of the marginal profit functions $b(\cdot)$ and $s(\cdot)$ with respect to θ yields an increasing type assignment. To deal with the potential for multiple solutions of the pointwise maximization, we define[7]

$$\theta^s(x) = \max\{\arg\max_{\theta} s(x, \theta)\}$$

and

$$\theta^b(x) = \min\{\arg\max_{\theta} b(x, \theta)\}.$$

A sufficient condition for both $\theta^s(\cdot)$ and $\theta^b(\cdot)$ to be increasing is that the single-crossing conditions $s_{x\theta}(\cdot) \geq 0$ and $b_{x\theta}(\cdot) \geq 0$ are satisfied. To simplify the subsequent characterization of optimal assignments, we require strict single crossing.

Assumption 2 *The marginal profit functions $s(\cdot)$ and $b(\cdot)$ satisfy the strict single-crossing property, that is*

$$s_{x\theta}(x,\theta) = -[u_{xx}(x,\theta) + v_{xx}(x,\theta)]f(\theta) + [1 - F(\theta)]u_{xx\theta}(x,\theta) > 0, \quad (31)$$

$$b_{x\theta}(x,\theta) = -[u_{xx}(x,\theta) + v_{xx}(x,\theta)]f(\theta) - F(\theta)u_{xx\theta}(x,\theta) > 0, \quad (32)$$

holds for all x, θ.

Assumption 2 can be interpreted in terms of the virtual surplus functions. A straightforward calculation gives $s_{x\theta}(x,\theta) = -\sigma_{xx}(x,\theta)f(\theta)$, so that condition (31) is equivalent to $\sigma_{xx}(\cdot) > 0$, i.e. the virtual surplus $\sigma(\cdot)$ is strictly concave in x. Similarly, $b_{x\theta}(x,\theta) = -\beta_{xx}(x,\theta)f(\theta)$, again linking the strict single-crossing property of the marginal profit function $b(\cdot)$ to the strict concavity of the virtual surplus function $\beta(\cdot)$. Note that in the commonly studied case in which the agent's utility function satisfies $u_{xx\theta}(\cdot) = 0$, requiring $u_{xx}(\cdot) < 0$ and $v_{xx}(\cdot) < 0$ is sufficient for Assumption 2 (with no assumptions on the distribution of θ). More generally, the absolute value of $u_{xx\theta}(x,\theta)$ must not be too large. If $u_{xx}(\cdot) < 0$ and $v_{xx}(\cdot) < 0$, it suffices for Assumption 2 that

$$-\frac{f(\theta)}{F(\theta)} \leq \frac{u_{xx\theta}(x,\theta)}{v_{xx}(x,\theta) + u_{xx}(x,\theta)} \leq \frac{f(\theta)}{1 - F(\theta)}, \qquad \forall \theta.$$

[7] The maximum theorem implies that $\arg\max_\theta s(x,\theta)$ and $\arg\max_\theta s(x,\theta)$ are compact, ensuring that the following is well-defined. Notice also that any other selection from $\arg\max_\theta x(x,\theta)$ or $\arg\max_\theta b(x,\theta)$ would yield an equivalent type assignment.

Differentiating the marginal profit functions (16) and (26), we have

$$\begin{aligned} s_\theta(x,\theta) &= -[u_{xx}(x,\theta)+v_{xx}(x,\theta)]-(1-F(\theta))u_{x\theta}(x,\theta) \\ b_\theta(x,\theta) &= -[u_{xx}(x,\theta)+v_{xx}(x,\theta)]+F(\theta)u_{x\theta}(x,\theta), \end{aligned}$$

and thus $s_\theta(y,\theta)-b_\theta(y,\theta)>0$ (since $u_{x\theta}(y,\theta)>0$). It follows that $\theta^s(y)\geq\theta^b(y)$ for all x, with strict inequality unless both values lie on the boundary. Hence, the reduced problem (28) can be decomposed if $\theta^b(\cdot)$ and $\theta^s(\cdot)$ are both increasing, which follows from Assumption 2. In addition, the strict inequalities in Assumption 2 imply that (up to equivalence) the optimal type assignment $\theta^*(\cdot)$ is unique:

Proposition 4 *Suppose Assumptions 1 and 2 are satisfied and let $y^*\in\arg\max_y W(y)$. Then a type assignment $\theta(\cdot)$ is optimal if and only if it is equivalent to*

$$\theta^*(x)=\begin{cases}\theta^b(x), & \text{if } x\leq y^* \\ \theta^s(x), & \text{if } x>y^*\end{cases}. \tag{33}$$

Our assumptions do not preclude the possibility that $W(y)$ has multiple maximizers. However, Proposition 4 implies that all the assignments constructed according to (33) for some $y^*\in\arg\max_y W(y)$ are equivalent, so that the choice of maximizer is irrelevant in identifying the optimal type assignment.

If all type assignments are known to satisfy the minmax property at the same y^*, then y^* maximizes $W(y)$ (cf. Propositions 2 and 3) and the optimal type assignment has been identified.[8] In general, a two-step procedure allows us to identify y^* and hence an optimal type assignment. First, determine $\theta^b(\cdot)$ and $\theta^s(\cdot)$ and obtain (cf. the proof of Proposition 4):

$$W(y)=\int_{\underline{x}}^{y} b(x,\theta^b(x))dx+\int_{y}^{\overline{x}} s(x,\theta^s(x))dx+V(y)+\hat{u}(y).$$

This allows us to calculate the derivative $W_y(\cdot)$ (when it exists). Second, our next result demonstrates that under Assumption 2, a suitable generalization of the first order condition $W_y(y^*)=0$ always characterizes the maxima of the value function, even though $W(\cdot)$ need not be concave. The generalization is required because $W(\cdot)$ need not be differentiable and the maximization may have boundary solutions.

[8]Note that in this case, requiring (31) only for $x\geq y^*$ and (32) only for $x\leq y^*$ instead of Assumption 2 suffices for Proposition 4. In particular, if $u(\cdot)$ is increasing (resp. decreasing), in type then (31) (resp. (32)) is sufficient.

To state the result, let the type assignment $\hat{\theta}(\cdot)$ satisfy $u(y, \hat{\theta}(y)) = \hat{u}(y)$ for all y. Due to the single-crossing property of the agent's utility function, the type assignment $\hat{\theta}(\cdot)$ is *decreasing* (in contrast to all the other type assignment considered in this paper). In analogy with (21), which applies to increasing type assignments, let $\hat{\Theta} : [\underline{x}, \overline{x}] \to [\underline{\theta}, \overline{\theta}]$ denote the correspondence defined by

$$\hat{\Theta}(x) = [\lim_{y \downarrow x} \hat{\theta}(y), \lim_{y \uparrow x} \hat{\theta}(y)],$$

where we adopt the convention that $\lim_{y \downarrow \overline{x}} \hat{\theta}(y) = \underline{\theta}$ and $\lim_{y \uparrow \underline{x}} \hat{\theta}(y) = \overline{\theta}$.

Proposition 5 *Suppose Assumption 2 is satisfied. Then* $y^* \in \arg\max W(y)$ *if and only if there exists* $\psi \in \hat{\Theta}(y^*)$ *such that*

$$\left[b(y^*, \theta^b(y^*)) - b(y^*, \psi)\right] - [s(y^*, \theta^s(y^*)) - s(y^*, \psi)] = 0. \tag{34}$$

To interpret this result, consider $y \in (\underline{x}, \overline{x})$ and suppose that the solution of the problem $\min_\theta u(y, \theta)$ is unique and thus given by $\hat{\theta}(y)$. Then $W(\cdot)$ is differentiable at y with derivative

$$W_y(y) = \left[b(y, \theta^b(y) - s(y, \theta^s(y))\right] + V_x(y) + u_x(y, \hat{\theta}(y),$$

which can be rewritten (cf. the proof of Proposition 5) as

$$W_y(y) = \left[b(y, \theta^b(y)) - b(y, \hat{\theta}(y))\right] - \left[s(y, \theta^s(y)) - s(y, \hat{\theta}(y))\right].$$

Condition (34) thus generalizes the first order condition $W_y(y^*) = 0$ by replacing $\hat{\theta}(y)$ with a value $\theta \in \hat{\Theta}(y)$ when the former is not unique.

Example. Continuing with the example in which utility functions are given by (7)–(8), we have already noted in Section 5.2 that every increasing type assignment satisfies the minmax property at 0. This allows us to reduce the assignment problem to

$$\max_{\theta(\cdot) \text{ increasing}} \int_{\underline{x}}^{0} b(x, \theta(x))dx + \int_{0}^{\overline{x}} s(x, \theta(x))dx.$$

Notice that this contrasts with the standard approach based on the analysis of the allocation problem, where we needed $\underline{x} = 0$ or $\overline{x} = 0$ to do the equivalent reduction (cf. Section 3.1). Because

$$s_{x\theta}(x, \theta) = b_{x\theta}(x, \theta) = (\gamma + \beta) f(\theta) > 0,$$

Assumption 2 is satisfied, so we can proceed with the pointwise maximization, without (again in contrast to the standard approach) requiring the distribution function to satisfy conditions (11) and (12). ||

6.2 Optimal Bunching

Bunching refers to a situation in which the same quantity is allocated to multiple types or, equivalently, multiple types are assigned to the same quantity:

Definition 4 *An increasing type assignment $\theta(\cdot)$ has the bunch $\Theta(x)$ at x if $\Theta(x) \neq \{\theta(x)\}$.*

Throughout this subsection we impose Assumptions 1 and 2. Under these assumptions, the characterization of optimal bunches is immediate from the characterization of optimal type assignments in Propositions 4 and 5. Three kinds of bunches may arise in the optimal type assignment $\theta^*(\cdot)$ given by (33).

First, there may be (trivial) bunches at the boundaries $\underline{x}$ or $\overline{x}$, occurring whenever it is optimal for the principal to assign an interior type to these boundaries, i.e. $\theta^*(\underline{x}) > \underline{\theta}$ or $\theta^*(\overline{x}) < \overline{\theta}$.

Second, there may be bunches at interior quantities $x \in (\underline{x}, \overline{x})$ with $x \neq y^*$. Since the correspondences defined by $\max_\theta s(x, \theta)$ and $\max_\theta b(x, \theta)$, are upper hemi-continuous, such bunches occur if and only if, at quantity x, the relevant marginal profit function $s(x, \cdot)$ (if $x > y^*$) or $b(x, \cdot)$ (if $x < y^*$) has multiple maximizers, causing an upwards discontinuity in $\theta^s(\cdot)$ or $\theta^b(\cdot)$ at x. In particular, such bunches do not arise if the marginal profit functions are strictly quasiconcave in θ. Similarly, bunches at interior quantities $x \neq y^*$ are also excluded if the virtual surplus functions $\sigma(\cdot)$ and $\beta(\cdot)$ satisfy the strict single-crossing property.[9]

Third, if $y^* \in (\underline{x}, \overline{x})$, then there is an optimal bunch at y^*, given by $[\theta^b(y^*), \theta^s(y^*)]$, if and only if $\theta^b(y^*) < \theta^s(y^*)$. Because $s_\theta(x, \theta) - b_\theta(x, \theta) = u_{x\theta}(x, \theta) > 0$, the case $\theta^b(y^*) = \theta^s(y^*) \in (\underline{\theta}, \overline{\theta})$ cannot arise and there *must* be a bunch at y^* unless $\theta^b(y^*) = \theta^s(y^*) = \underline{\theta}$ or $\theta^b(y^*) = \theta^s(y^*) = \overline{\theta}$.[10] A simple sufficient for the existence of an optimal bunch at y^* is then given by $s_\theta(y^*, \overline{\theta}) \leq 0 \leq b_\theta(y^*, \underline{\theta})$ or, equivalently,

$$u_x(y^*, \overline{\theta}) + v_x(y^*, \overline{\theta}) \quad \geq \quad 0 \quad \geq \quad u_x(y^*, \underline{\theta}) + v_x(y^*, \underline{\theta}).$$

[9] Suppose $\sigma_{x\theta}(\cdot) > 0$. Then for every x there is at most one solution to the first order condition $s_\theta(x, \theta) = -\sigma_x(x, \theta) f(\theta) = 0$ and, if such a solution exists, it is a maximizer of $s(x, \theta)$. Consequently the solution to the problem $\max_\theta s(x, \theta)$ is unique. An equivalent argument applies to $b(\cdot)$.

[10] In the first of these cases, the optimal type assignment as given by (33) is identical to $\theta^s(\cdot)$ (because $\theta^b(x) = \theta^s(x) = \underline{x}$ holds for all $x \leq y^*$). In the second of these cases it is identical to $\theta^b(\cdot)$ (because $\theta^s(x) = \theta^b(x) = \overline{\theta}$ holds for all $x \geq y^*$). Vice versa, whenever there exists an optimal type assignment which is not equivalent to $\theta^b(\cdot)$ or $\theta^s(\cdot)$, it must have a bunch at y^* (which then must be the unique maximizer of $W(\cdot)$, as two type assignments satisfying (33) for two different values of y^* can then not be equivalent).

This condition can be interpreted in terms of the first-best quantity allocation $x^{FB}(\cdot)$ given by a solution of $\max_x[u(x,\theta)+v(x,\theta)]$. Because Assumption 2 implies that total surplus $u(x,\theta)+v(x,\theta)$ is strictly concave in x for $\theta=\underline{\theta}$ and $\theta=\overline{\theta}$, the optimal quantity assignment must have a bunch at y^* whenever $x^{FB}(\underline{\theta}) \leq y^* \leq x^{FB}(\overline{\theta})$.

Example. To illustrate our characterization of optimal bunching at y^*, we continue with our running example. Suppose $\underline{x} < 0 < \overline{x}$ and, as in the model of market making from Biais, Martimort and Rochet [3], $\underline{\theta} < 0 < \overline{\theta}$. Assume, in addition, $\alpha < 1$. Straightforward calculation yields

$$\begin{aligned} s_\theta(0,\theta) &= -(1-\alpha)\theta f(\theta) + (1-F(\theta)), \\ b_\theta(0,\theta) &= -(1-\alpha)\theta f(\theta) - F(\theta). \end{aligned}$$

Noting that $s_\theta(0,\theta) > 0$ for all $\theta \leq 0$ and $s_\theta(0,\overline{\theta}) < 0$, it follows that $\theta^s(0) \in (0,\overline{\theta})$. Similarly, it follows that $\theta^b(0) \in (\underline{\theta},0)$. In particular, we have $\theta^b(0) < \theta^s(0)$ and the optimal type assignment thus has a bunch at $y^*=0$. This bunch contains type $\theta=0$, but never consumes all types. Interpreting $y^*=0$ as the no-trade outcome, implementing the no-trade allocation is not an optimal choice for the principal (in contrast to some of the examples considered in Glosten [8]). ||

A standard result from the application of optimal control techniques to solve the allocation problem is that optimal bunches at interior quantities satisfy the condition that the average of the marginal virtual surpluses over a bunch must be equal to zero (see, for instance, Fudenberg and Tirole [6, Chapter 7, Appendix]). Obtaining optimal bunches as the solution to pointwise, unconstrained maximiation problems provides a simple, alternative interpretation of this optimality condition. Consider first the case of an optimal bunch $\Theta^*(x)=[\theta_1,\theta_2]$ at $x \in (y^*,\overline{x})$. As both θ_1 and θ_2 are maximizers of $s(x,\theta)$, we have $s(x,\theta_1)=s(x,\theta_2)$ and thus

$$0=\int_{\theta_1}^{\theta_2} s_\theta(x,\theta)d\theta = \int_{\theta_1}^{\theta_2} \sigma_x(x,\theta)f(\theta)d\theta,$$

where the second equality uses the identity $s_\theta(x,\theta) = -\sigma_x(x,\theta)f(\theta)$. Similarly, a bunch $\Theta^*(x)=[\theta_1,\theta_2]$ at $x \in (\underline{x},y^*)$ must satisfy

$$0=\int_{\theta_1}^{\theta_2} b_\theta(x,\theta)d\theta = \int_{\theta_1}^{\theta_2} \beta_x(x,\theta)f(\theta)d\theta.$$

In each case, the average of the marginal virtual surpluses is zero.

For the case of an optimal bunch at $y^* \in (\underline{x}, \overline{x})$, given by $\Theta^*(y^*) = [\theta^b(y^*), \theta^s(y^*)]$, a corresponding characterization follows from Proposition 5. Using integration by parts, we have

$$\begin{aligned} \left[b(y, \theta^b(y)) - b(y, \psi)\right] &= \int_{\theta^b(y)}^{\psi} \beta_x(y, \theta) f(\theta) d\theta \\ \left[s(y, \theta^s(y)) - s(y, \psi)\right] &= -\int_{\psi}^{\theta^s(y)} \sigma_x(y, \theta) f(\theta) d\theta. \end{aligned}$$

Hence, we may rewrite the generalized first order condition (34) in terms of the virtual surplus functions

$$\int_{\theta^b(y^*)}^{\psi} \beta_x(y^*, \theta) f(\theta) d\theta + \int_{\psi}^{\theta^s(y^*)} \sigma_x(y^*, \theta) f(\theta) d\theta = 0,$$

again corresponding to the condition that—for an appropriate choice of $\psi \in \hat{\Theta}(y^*)$—the average of the marginal virtual surpluses over the types in the bunch is equal to zero. Note that the issue of choosing ψ is moot if the agent's utility function is strictly quasi-concave in θ (as in the model of countervailing incentives from Lewis and Sappington [16]). Then the solution to $\min_\theta u(y^*, \theta)$ is unique and thus $\hat{\Theta}(y^*) = \{\hat{\theta}(y^*)\}$, with the above condition holding for $\psi = \hat{\theta}(y^*)$.

7 Discussion

We have identified a class of decomposable principal-agent models, in which a solution can be obtained from a collection of unconstrained pointwise maximization problems. Our approach relies on first reformulating the principal's problem as an equivalent assignment problem. We then demonstrate that this assignment problem can be decomposed if the agent's utility function satisfies the minmax property and the marginal profit functions satisfy the strict single-crossing property, assumptions that are satisfied for many interesting specifications of the principal-agent model. The ability to solve such models by pointwise maximization leads to a simple and intuitive interpretation of the conditions characterizing optimal bunching.

Throughout the paper, we have restricted attention to one-dimensional, quasi-linear principal-agent models in which the agent's utility function satisfies the strict single-crossing condition. There appears to be little prospect for decomposing more complicated problems. Rochet and Stole [26] discuss the difficulties that arise in multidimensional models, while Araujo and

Moreira [1] work without the single-crossing property. At the same time, the heart of the assignment approach—the transformation of the problem into a maximization over assignments, with its accompanying intuition—will generalize. For example, as suggested by the analysis in Goldman, Leland and Sibley [9], there is good reason to believe that the assignment approach should prove useful in examining models without quasi-linear utilities.

Our requirement that every increasing type assignment satisfy the minmax property can be trivially weakened to the requirement that the property hold for optimal type assignments. Because the minmax property plays a central role in our elimination of the participation constraint from the principal's problem, there appears to be little scope for further weakening.

The exposition is simplified considerably by requiring strict single crossing in Assumption 2, but much of the analysis carries over to the case of weak single crossing. The type assignment identified in Proposition 4 remains optimal and Proposition 5 remains unchanged. If the optimal type assignment is unique (see Jullien [13, Theorem 4] for sufficient conditions for uniqueness), then we also have the necessary implication of Proposition 4. Finally, the optimality of the type assignment given by Proposition 4 obtains whenever the relevant type assignments are increasing. This will hold under alternative conditions, such as the strict quasi-concavity of both the virtual surplus and marginal profit functions.

Our goal has been to identify conditions under which the principal-agent problem is simple, in the sense that it is decomposable. Whereas it is standard that the single-crossing property of the agent's utility function suffices to replace global constraints by local ones, we end up with no constraints at all. It is thus not surprising that the class of principal-agent problems we identify as simple must satisfy additional conditions. What is surprising is that these conditions are much less stringent than the ones that have been previously identified in the literature.

8 Appendix: Proofs

Proof of Lemma 1. Proposition 1 in Rochet [25] shows that there exists $r(\cdot)$ such that the allocation $(x(\cdot), r(\cdot))$ is incentive compatible if and only if $x(\cdot)$ is increasing. The characterization of $r(\cdot)$ is from Theorem 2, in conjunction with footnote 10, in Milgrom and Segal [20]. ||

Proof of Lemma 2. Let $(x(\cdot), r(\cdot))$ be incentive compatible. Then $x(\cdot)$ is increasing (Lemma 1). Let $\Phi : [\underline{x}, \overline{x}] \to [\underline{\theta}, \overline{\theta}]$ be the correspondence defined by

$$\Phi(x) = \{\theta \mid x \in X(\theta)\}, \ \forall x.$$

By construction $X(\cdot)$ satisfies $\cup_\theta X(\theta) = [\underline{x}, \overline{x}]$, and hence $\Phi(x)$ is non-empty for all x. In addition, $\Phi(x)$ is convex-valued. It is also increasing (i.e., $\theta_1 \in \Phi(x_1)$ and $\theta_2 \in \Phi(x_2)$ implies $\theta_1 \leq \theta_2$ if $x_1 < x_2$): otherwise, there exists $x_1 > x_2$ with $\Phi(x_1) \ni \theta_1 < \theta_2 \in \Phi(x_2)$, and hence with $x_1 \in X(\theta_1)$ and $x_2 \in X(\theta_2)$, contradicting the fact that $x(\cdot)$ is increasing. Let $\theta(\cdot)$ be a selection from $\Phi(\cdot)$ (and hence increasing), and note that any such selection gives $\Theta(\cdot) = \Phi(\cdot)$. (Because $\Phi(\cdot)$ is increasing, we have $\lim_{y\uparrow x} \theta(y) = \lim_{y\uparrow x} \theta'(y)$ for any selections $\theta(\cdot)$ and $\theta'(\cdot)$, with a similar equality for $\lim_{y\downarrow x} \theta(\cdot)$, which combines with the convex-valuedness of $\Phi(\cdot)$ to suffice for the result.) Hence, $x(\cdot)$ and $\theta(\cdot)$ are inverses. Letting $t(\cdot)$ be given by (13) with $t(\underline{x}) = u(\underline{x}, \theta(\underline{x})) - r(\underline{x})$, the assignment $(\theta(\cdot), t(\cdot))$ is incentive compatible, and $(x(\cdot), r(\cdot))$ and $(\theta(\cdot), t(\cdot))$ are consistent. An analogous argument establishes the second statement in the lemma. ||

Proof of Lemma 3. Let $(x(\cdot), r(\cdot))$ and $(\theta(\cdot), t(\cdot))$ be consistent. Then $(\theta(\cdot), t(\cdot))$ is incentive compatible which, as we show below, implies

$$y \in \arg\max_x \,[u(x, \theta) - t(x)] \ \Leftrightarrow \ \theta \in \Theta(y). \tag{35}$$

Because $x(\cdot)$ and $\theta(\cdot)$ are inverse, (35) implies (23). Let the rent allocation $\rho(\cdot)$ be given by

$$\rho(\theta) = u(x(\theta), \theta) - t(x(\theta)).$$

From (23) we have

$$\rho(\theta) = \max_x [u(x, \theta) - t(x)] \geq u(x(\psi), \theta) - t(x(\psi)), \ \ \forall \theta, \psi.$$

Using the identity $t(x(\psi)) = u(x(\psi), \psi) - \rho(\psi)$, it follows that the allocation $(x(\cdot), \rho(\cdot))$ satisfies (2) and thus is incentive compatible. From Lemma 1, it then suffices to show $\rho(\underline{\theta}) = r(\underline{\theta})$ to conclude $\rho(\cdot) = r(\cdot)$, thus yielding (24). As $\underline{\theta} \in \Theta(\underline{x})$, (35) implies $\rho(\underline{\theta}) = u(\underline{x}, \underline{\theta}) - t(\underline{x})$. Thus, the consistency condition (22) yields $\rho(\underline{\theta}) = r(\underline{\theta})$, as desired. Because $(x(\cdot), r(\cdot))$ is incentive compatible, it is feasible in the allocation problem if and only if (6) holds. Similarly, because $(\theta(\cdot), t(\cdot))$ is incentive compatible it is feasible in the assignment problem if and only if (20) holds. From (24) the participation constraints (6) and (20) are equivalent, implying the statement in the last sentence of the Lemma.

It remains to establish (35). Using (13) and the identity $u(y, \theta) - u(x, \theta) = \int_x^y u_x(\tilde{x}, \theta) d\tilde{x}$ we have

$$\begin{aligned} [u(y,\theta) - u(x,\theta)] - [t(y) - t(x)] &= \int_x^y [u_x(\tilde{x}, \theta) - u_x(\tilde{x}, \theta(\tilde{x}))] \, d\tilde{x} \\ &= \int_x^y \left[\int_{\theta(\tilde{x})}^{\theta} u_{x\theta}(\tilde{x}, \tilde{\theta}) d\tilde{\theta} \right] d\tilde{x}. \end{aligned} \tag{36}$$

As $\theta(\cdot)$ is increasing, the strict single-crossing property (1) implies that for $x < y$ the right side of (36) is positive if and only if $\theta(x) \leq \theta$, whereas for $x > y$ the right side of (36) is positive if and only if $\theta(x) \geq y$. Thus

$$[u(y,\theta) - u(x,\theta)] - [t(y) - t(x)] \geq 0, \ \forall x \ \ \Leftrightarrow \ \ \lim_{x \to y} \theta(x) \leq y \leq lim_{x \downarrow y} \theta(x). \tag{37}$$

Noting that the right side of (37) is equivalent to $y \in \arg\max_x [u(x, \theta) - t(x)]$ and the left side is equivalent to $\theta \in \Theta(y)$, (35) follows. ||

Proof of Lemma 4. Let $(x(\cdot), r(\cdot))$ and $(\theta(\cdot), t(\cdot))$ be consistent. Because $(x(\cdot), r(\cdot))$ satisfies (5) we have

$$\Gamma(x(\cdot), r(\cdot)) = \int_{\underline{x}}^{\overline{x}} \left[v(x(\theta), \theta) + u(x(\theta), \theta) - \left[\int_{\underline{\theta}}^{\theta} u_\theta x(\tilde{\theta}), \tilde{\theta}) d\tilde{\theta} \right] \right] f(\theta) d\theta - r(\underline{\theta}).$$

Using integration by parts (for the first equality) and a straightforward calculation (for the second equality) we have

$$\begin{aligned} \Gamma(x(\cdot), r(\cdot)) &= \int_{\underline{\theta}}^{\overline{\theta}} \left[\int_{\underline{x}}^{x(\theta)} \sigma_x(x, \theta) f(\theta) dx + \sigma(\underline{x}, \theta) f(\theta) \right] d\theta - r(\underline{\theta}) \\ &= \int_{\underline{\theta}}^{\overline{\theta}} \left[\int_{\underline{x}}^{x(\theta)} \sigma_x(x, \theta) f(\theta) dx \right] d\theta + V(\underline{x}) + u(\underline{x}, \underline{\theta}) - r(\underline{\theta}). \end{aligned}$$

Because $x(\cdot)$ and $\theta(\cdot)$ are inverses, we can apply Fubini's theorem to the double integral to obtain

$$\Gamma(x(\cdot), r(\cdot)) = \int_{\underline{x}}^{\overline{x}} \left[\int_{\theta(x)}^{\overline{\theta}} \sigma_x(x,\theta) f(\theta) d\theta \right] dx + V(\underline{x}) + u(\underline{x}, \underline{\theta}) - r(\underline{\theta}).$$

Using integration by parts, we have

$$s(x,\theta) = \int_{\theta}^{\overline{\theta}} \sigma_x(x,\tilde{\theta}) f(\tilde{\theta}) d\tilde{\theta}$$

and thus

$$\Gamma(x(\cdot), r(\cdot)) = \int_{\underline{x}}^{\overline{x}} s(x, \theta(x)) dx + u(\underline{x}, \underline{\theta}) - r(\underline{\theta}) = \Pi(\theta(\cdot), t(\cdot)),$$

where the final equality uses $t(\underline{x}) = u(\underline{x}, \underline{\theta}) - r(\underline{\theta})$. ||

Proof of Proposition 1. Let $(x^*(\cdot), r^*(\cdot))$ be an optimal allocation and let $(\theta^*(\cdot), t^*(\cdot))$ be consistent with it (the existence of such an assignment is ensured by Lemma 2). From Lemma 3 the assignment $(\theta^*(\cdot), t^*(\cdot))$ is feasible. Let $\theta(\cdot), t(\cdot))$ be any feasible assignment. By Lemma 2 there exists an allocation $(x(\cdot), r(\cdot))$ consistent with it. From Lemma 3 the allocation $(x(\cdot), r(\cdot))$ is feasible. From the optimality of $(x^*(\cdot), r^*(\cdot))$ we then have $\Gamma(x^*(\cdot), r^*(\cdot)) \geq \Gamma(x(\cdot), r(\cdot))$. Applying Lemma 4 this implies $\Pi(\theta^*(\cdot), t^*(\cdot)) \geq \Pi(\theta(\cdot), t(\cdot))$. Hence, $(\theta^*(\cdot), t^*(\cdot))$ is optimal. The argument for the reverse implication is analogous. ||

Proof of Lemma 5. If $(\theta(\cdot), t(\cdot))$ is incentive compatible then

$$\begin{aligned}
\Pi(\theta(\cdot), t(\cdot)) &= \int_{\underline{x}}^{\overline{x}} s(x, \theta(x)) dx + V(\underline{x}) + t(\underline{x}) \\
&= \int_{\underline{x}}^{\overline{x}} s(x, \theta(x)) dx + \left[V(y) - \int_{\underline{x}}^{y} V_x(x) \right] + \left[t(y) - \int_{\underline{x}}^{y} u_x(x, \theta(x)) dx \right] \\
&= \int_{\underline{x}}^{y} \left[s(x, \theta(x)) - V_x(x) - u_x(x, \theta(x)) \right] dx + \int_{y}^{\overline{x}} s(x, \theta(x)) dx + V(y) + t(y) \\
&= \int_{\underline{x}}^{y} b(x, \theta(x)) dx + \int_{y}^{\overline{x}} s(x, \theta(x)) dx + V(y) + t(y),
\end{aligned}$$

where the first equality reproduces (14), the second applies (13) and the fourth is by straightforward calculation using the definitions (15), (16) and (26). ||

Proof of Lemma 6. Let $(\theta(\cdot), t(\cdot))$ be incentive compatible and let $\theta(\cdot)$ satisfy the minmax property at y. We then have (cf. (35) in the proof of Lemma 3)

$$y \in \arg\max_x [u(x,\theta) - t(x)], \forall \theta \in \Theta(y). \tag{38}$$

Now, let $\hat{\theta} \in \Theta(y)$ be a type such that $\hat{u}(y) = u(y, \hat{\theta})$. From (38) we then have

$$\hat{u}(y) - t(y) = \max_x \left[u(x,\hat{\theta}) - t(x)\right] \geq \min_\theta \max_x [u(x,\theta) - t(x)] \tag{39}$$

and from $\hat{u}(x) = \min_\theta u(x,\theta)$ we have

$$\hat{u}(y) - t(y) = \min_\theta [u(y,\theta) - t(y)] \leq \min_\theta \max_x [u(x,\theta) - t(x)]. \tag{40}$$

Combining the inequalities in (39) and (40) yields $\min_\theta \max_x [u(x,\theta) - t(x)] = \hat{u}(y) - t(y)$. Because the function $\max_x [u(x,\theta) - t(x)]$ is continuous in θ, the participation constraint (20) is equivalent to

$$\min_\theta \max_x [u(x,\theta) - t(x)] \geq 0,$$

implying the equivalence in the last sentence of the statement of the Lemma. ||

Proof of Proposition 2. If every increasing type assignment satisfies the minmax property at y^*, it follows from Lemma 6 that an incentive compatible assignment $(\theta(\cdot), t(\cdot))$ satisfies (20) if and only if $t(y^*) \leq \hat{u}(y^*)$. Hence, using (25) to rewrite the objective function of the assignment problem, an incentive compatible assignment $(\theta^*(\cdot), t^*(\cdot))$ is optimal if and only if it solves

$$\max_{\theta(\cdot),t(\cdot)} \int_{\underline{x}}^{y^*} b(x,\theta(x))dx + \int_{y^*}^{\overline{x}} s(x,\theta(x)dx + V(y^*) + t(y^*)$$

subject to (18)–(19) and $t(y^*) \leq \hat{u}(y^*)$. The result is then immediate upon noticing that in a solution to this problem the constraint $t(y^*) \leq \hat{u}(y^*)$ must be satisfied with equality. ||

Proof of Proposition 3. Let $(\theta^*(\cdot), t^*(\cdot))$ be an optimal assignment. Then $(\theta^*(\cdot), t^*(\cdot))$ is incentive compatible and thus $\theta^*(\cdot)$ is increasing and, by Assumption 1, satisfies the minmax property at some y^*. As the constraint $t(y^*) \leq \hat{u}(y^*)$ implies (20), it follows that $\theta^*(\cdot)$ must solve (28) and that $t^*(\cdot)$ is given by (29). Furthermore, we have $\Pi(\theta^*(\cdot), t^*(\cdot)) = W(y^*)$. Now suppose $y^* \notin \arg\max W(y)$. Then there exists $\hat{y}$ such that $W(\hat{y}) > W(y^*)$. Let $\hat{\theta}(\cdot)$ be a type assignment solving the problem in the definition of $W(\hat{y})$ and let

$$\hat{t}(x) = \hat{u}(\hat{y}) + \int_{\hat{y}}^{x} u_x(\tilde{x}, \hat{\theta}(\tilde{x}))d\tilde{x}.$$

Then $(\hat{\theta}(\cdot), \hat{t}(\cdot))$ is incentive compatible. As $(\hat{\theta}(\cdot), \hat{t}(\cdot))$ satisfies $t(\hat{y}) \leq \hat{u}(y)$ it also satisfies (20) and thus is feasible in the assignment problem. As $\Pi(\hat{\theta}(\cdot)), \hat{t}(\cdot) = W(\hat{y}) > W(y^*)$ this contradicts the optimality of $(\theta^*(\cdot), t^*(\cdot))$, implying the "only if" in the statement of the proposition.

Let $y^* \in \arg\max W(y)$, let $\theta^*(\cdot)$ solve (28) and let $t^*(\cdot)$ be given by (29). By the same arguments as the ones just given for the assignment $(\hat{\theta}(\cdot), \hat{t}(\cdot))$, it follows that $(\theta^*(\cdot), t^*(\cdot))$ is feasible in the assignment problem and satisfies $\Pi(\theta^*(\cdot), t^*(\cdot)) = W(y^*)$. Now consider any other incentive compatible assignment $(\theta(\cdot), t(\cdot))$ satisfying (20). As $\theta(\cdot)$ is increasing there exists y such that $\theta(\cdot)$ satisfies the minmax property at y. Thus $(\theta(\cdot), t(\cdot))$ satisfies (18)– (19) and $t(y) \leq \hat{u}(y)$, implying $\Pi(\theta(\cdot), t(\cdot)) \leq W(y) \leq W(y^*)$. Consequently, $(\theta^*(\cdot), t^*(\cdot))$ is optimal. ||

Proof of Lemma 7. Let $\theta(\cdot)$ be an increasing type assignment. Define the correspondence $F : [\underline{x}, \overline{x}] \to [\underline{\theta}, \overline{\theta}]$ by $F(x) = \{\phi - \psi : \phi \in \Theta(x), \psi \in \arg\min_\theta u(x, \theta)\}$. Then $F(x)$ is convex-valued (because $\Theta(x)$ is convex and the quasiconvexity of $u(x, \theta)$ ensures that $\arg\min_\theta(u(x, \theta)$ is convex), upper hemicontinuous and compact (because $\Theta(\cdot)$ is upper hemicontinuous and compact-valued and, by the maximum theorem (Berge [4, Theorem 12.1]), so is $\arg\min_{\theta \in [\underline{\theta}, \overline{\theta}]} u(x, \theta))$. In addition, $\min F(\underline{x}) \leq 0$ (because $\underline{\theta} \in \Theta(\underline{x})$) and $\max F(\overline{x}) \geq 0$ (because $\overline{\theta} \in \Theta(\overline{x})$). Let $\max F(\underline{x}) < 0$ and $\min F(\overline{x}) > 0$, since otherwise we immediately have that the minmax property holds at either $\underline{x}$ or $\overline{x}$. Then the correspondence $G(x)$ defined on $[\underline{x} - 1, \overline{x} + 1]$ by

$$G(x) = \left\{ x - \frac{z}{\overline{\theta} - \underline{\theta}} : z \in F(x) \right\}$$

if $x \in [\underline{x}, \overline{x}]$ and by

$$G(x) = \begin{cases} G(\overline{x}) & \text{if } x > \overline{x} \\ G(\underline{x}) & \text{if } x < \underline{x} \end{cases}$$

is a nonempty, compact and convex-valued upper hemicontinuous correspondence from $[\underline{x}-1, \overline{x}+1]$ into itself,[11] and hence has a fixed point [4, Corollary

[11] Note that $z \in F(x)$ ensures $z/(\overline{\theta} - \underline{\theta}) \in [-1, 1]$. By assumption, $\max F(\underline{x}) < 0$ and hence $G(\underline{x}) > \underline{x}$, and $\min F(\overline{x}) < 0$ and hence $G(\overline{x}) > \overline{x}$.

15.3 (Kakutani)]. By construction, such a fixed point must occur at some $y \in (\underline{x}, \overline{x})$ for which $0 \in F(y)$, and hence for which the minmax property holds. ||

Proof of Proposition 4. We first show that, under Assumption 2 and for any y^*, the type assignment $\theta^*(\cdot)$ given by (33) solves

$$\max_{\substack{\theta(\cdot) \\ \text{increasing}}} \int_{\underline{x}}^{y^*} b(x, \theta(x)) + \int_{y^*}^{\overline{x}} s(x, \theta(x)). \tag{41}$$

In particular, as every increasing type assignment $\theta(\cdot)$ satisfies

$$\int_{\underline{x}}^{y^*} b(x, \theta^b(x)) = \int_{\underline{x}}^{y^*} \max_{\theta} b(x, \theta(x)) \geq \int_{\underline{x}}^{y^*} b(x, \theta(x))$$
$$\int_{y^*}^{\overline{x}} s(x, \theta^s(x)) = \int_{y^*}^{\overline{x}} \max_{\theta} s(x, \theta(x)) \geq \int_{y^*}^{\overline{x}} s(x, \theta(x)),$$

it follows that $\theta^*(\cdot)$ solves (41) if it is increasing. From Theorem 4 in Milgrom and Shannon ([19]), Assumption 2 implies that the functions $\theta^b(x)$ and $\theta^s(x)$ are increasing. Hence, $\theta^*(\cdot)$ is increasing (because, as argued in the text, $\theta^b(y^*) \leq \theta^s(y^*)$).

This establishes that for any y^*, the type assignment $\theta^*(\cdot)$ given by (33) solves (41). That (33) is optimal, given Assumption 1 and $y^* \in \arg\max_y W(y)$, then follows from Proposition 3, establishing the "if" portion of Proposition 4.

Assumption 2 implies that the virtual surplus functions $\sigma(\cdot)$ and $\beta(\cdot)$ are strictly concave in x (cf. the discussion following the statement of the Assumption). It then follows from the proof of Theorem 4 in Jullien [13] that (up to equivalence) the solution to the allocation problem (3)–(6) is unique. From Proposition 1, the same is then true for the solution to the assignment problem. Hence, if $\theta(\cdot)$ is an optimal type assignment it is equivalent to $\theta^*(\cdot)$. ||

Proof of Proposition 5.

[Step 1] Let $y \in [\underline{x}, \overline{x}]$. From the opening argument in the proof of Proposition 4, Assumption 2 implies that the type assignment $\theta(\cdot)$ given by

$$\theta(x) = \begin{cases} \theta^b(x), & \text{if } x \leq y \\ \theta^s(x), & \text{if } x > y \end{cases}$$

solves the problem

$$\max_{\theta(\cdot)\text{ increasing}} \int_{\underline{x}}^{y} b(x,\theta(x))dx + \int_{y}^{\overline{x}} s(x,\theta(x))dx.$$

Hence

$$W(y) = \int_{\underline{x}}^{y} (b(x,\theta^b(x))dx + \int_{y}^{\overline{x}} s(x,\theta^s(x))dx + V(y) + \hat{u}(y)$$

holds for all y. The first three summands in this expression for $W(y)$ are continuously differentiable with derivative

$$b(y,\theta^b(y)) - s(y,\theta^s(y)) + V_x(y)$$

(where the continuity of the first two terms is from Berge's maximum theorem [4, p. 64]). From Theorem 2 in Milgrom and Segal [20] we have

$$\hat{u}(y) = \hat{u}(\underline{x}) + \int_{\underline{x}}^{y} u_x(x,\hat{\theta}(x))dx.$$

This term is then also absolutely continuous, ensuring that $W(y)$ is an indefinite integral:

$$W(y) = W(\underline{x}) + \int_{\underline{x}}^{y} m(s,\hat{\theta}(s))ds, \tag{42}$$

where

$$m(x,\theta) = b(x,\theta^b(x)) - s(x,\theta^s(x)) + V_x(x) + u_x(x,\theta).$$

[Step 2] We now show that the function $m(\cdot)$ satisfies

$$m(y,\theta) \geq 0 \;\;\Rightarrow\;\; m(x,\theta) \geq 0,\ \forall x < y, \tag{43}$$

$$m(y,\theta) \leq 0 \;\;\Rightarrow\;\; m(x,\theta) \leq 0,\ \forall x > y. \tag{44}$$

To prove (43) – (44) we begin by noting that, as $b(x,\theta^b(x)) \geq b(x,\theta^s(x))$ and $s(x,\theta^s(x)) \geq s(x,\theta^b(x))$ we have

$$b(x,\theta^b(x)) - s(x,\theta^b(x)) \geq m(x,\theta) - V_x(x) - u_x(x,\theta) \geq b(x,\theta^s(x)) - s(x,\theta^s(x)).$$

Using the identity

$$b(x,\theta) - s(x,\theta) = -V_x(x) - u_x(x,\theta),\ \forall x,\theta, \tag{45}$$

this implies

$$u_x(x,\theta) - u_x(x,\theta^b(x)) \;\geq\; m(x,\theta) \;\geq\; u_x(x,\theta) - u_x(x,\theta^s(x)).$$

Using the single-crossing property of the agent's utility function we then have

$$\theta \geq \theta^s(x) \quad \Rightarrow \quad m(x,\theta) \geq 0, \tag{46}$$
$$\theta \leq \theta^b(x) \quad \Rightarrow \quad m(x,\theta) \leq 0. \tag{47}$$

As $\theta^s(\cdot)$ is increasing, $\theta \geq \theta^s(y)$ implies $\theta \geq \theta^s(x)$ for all $x < y$. Provided that the condition $\theta \geq \theta^s(y)$ is satisfied, (43) then follows from (46). Similarly, (47) implies (44) provided the condition $\theta \leq \theta^b(y)$ is satisfied.

As $\theta^b(x) \leq \theta^s(x)$ holds for all x, it remains to establish (43) and (44) for the case in which $\theta^b(y) < \theta < \theta^s(y)$. From (45) we have

$$m(x,\theta) = B(x,\theta) - S(x,\theta), \tag{48}$$

where

$$B(x,\theta) = \Big[b(x,\theta^b(x)) - b(x,\theta)\Big]$$

and

$$S(x,\theta) = \left[s(x,\theta^s(x)) - s(x,\theta)\right].$$

Now suppose $\theta^b(y) < \theta < \theta^s(y)$ and $m(y,\theta) \geq 0$. Let $x < y$. If $\theta^s(x) \leq \theta$ then (43) follows from (46), so suppose $\theta^s(x) > \theta$. Then

$$B(x,\theta) \geq \Big[b(x,\theta^b(y)) - b(x,\theta)\Big] > B(y,\theta),$$

where the second inequality is from the single-crossing property (32) of $b(\cdot)$ and the inequality $\theta^b(y) < \theta$. Similarly,

$$S(y,\theta) \geq \left[s(y,\theta^s(x)) - s(y,\theta)\right] > S(x,\theta),$$

where the second inequality is from the single-crossing property (31) of $s(\cdot)$ and the inequality $\theta^s(x) > \theta$. Combining these inequalities, it follows from (48) that $m(x,\theta) > m(y,\theta) \geq 0$, thus establishing (43). The argument establishing (44) for the case $\theta^b(y) < \theta < \theta^s(y)$ and $m(y) \leq 0$ is analogous.

[Step 3] Because $\hat{\theta}(\cdot)$ is decreasing and $m_\theta(y,\theta) = u_{x\theta}(y,\theta) > 0$, (43)–(44) imply

$$m(y,\psi) \geq 0 \quad \Rightarrow \quad m(x,\hat{\theta}(x)) \geq 0, \ \forall x < y, \psi \in \hat{\Theta}(y) \tag{49}$$
$$m(y,\psi) \leq 0 \quad \Rightarrow \quad m(x,\hat{\theta}(x)) \leq 0, \ \forall x > y, \psi \in \hat{\Theta}(y). \tag{50}$$

Notice that, using (48), condition (34) in the statement of the proposition is equivalent to the condition $m(y,\psi) = 0$. Hence, to finish the proof it suffices to show that y^* maximizes $W(y)$ if and only if there exists $\psi \in \hat{\Theta}(y^*)$ satisfying $m(y^*, \psi) = 0$.

Suppose there exists $\psi \in \hat{\Theta}(y^*)$ such that $m(y^*, \psi) = 0$. Let $y < y^*$. Using (42) and (49) we have

$$W(y^*) - W(y) = \int_y^{y^*} m(x, \hat{\theta}(x))dx \geq 0.$$

Similarly, using (50) instead of (49), for $y > y^*$ we have

$$W(y) - W(y^*) = \int_{y^*}^{y} m(x, \hat{\theta}(x)dx \leq 0.$$

Hence, $y^* \in \arg\max_y W(y)$.

Conversely, suppose $y^* \in \arg\max_y W(y)$. If $m(y^*, \hat{\theta}(y^*)) = 0$ there is nothing to prove, so suppose $m(y^*, \hat{\theta}(y^*)) > 0$ (the case $m(y^*, \hat{\theta}(y^*)) < 0$ is analogous). Let $\psi_+(y^*) = \lim_{y \downarrow y^*} \hat{\theta}(y)$. We show that $m(y^*, \psi_+(y^*)) \leq 0$, at which point it follows from the continuity of $m(y, \theta)$ in θ that there exists $\psi \in \hat{\Theta}(y^*)$ satisfying $m(y^*, \psi) = 0$. Suppose first that $y^* = \overline{x}$. Then $\psi_+(y^*) = \underline{\theta}$ and $m(y^*, \psi_+(y^*)) \leq 0$ follows from (47). Next, suppose $y^* < \overline{x}$ and suppose $m(y^*, \psi_+(y^*)) > 0$. Then (from (49) and the fact that $\hat{x}$ is decreasing) there exists $y > y^*$ such that $m(x, \hat{\theta}(x)) > 0$ holds for all $x \in (y^*, y)$. From (42) it then follows that $W(y) > W(y^*)$, contradicting the optimality of y^*. Consequently, $m(y^*, \psi_+(y^*)) \leq 0$ holds in this case, too, finishing the proof. ||

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