Maximality with or without binariness: transfer-type characterizations

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Abstract.

We provide the first necessary and sufficient conditions in the literature for the existence of maximal elements in a non-binary choice framework. As an application, the characterization of the existence of maximal elements for acyclic binary relations obtained in Rodríguez-Palmero and García-Lapresta, 2002, Mathematical Social Sciences 43, 55-60, is deduced as a Corollary. Further characterizations in different settings given by Tian and Zhou, 1995, Journal of Mathematical Economics 24, 281-303, follow as well. Analogous characterization problems in the k-acyclic binary cases are solved too.

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1 Introduction

The pioneering work of Nehring (1996) represented the first contribution to the problem of ensuring the existence of maximal elements for non-binary choice functions. Among the motivations that made this study relevant, we cite (a) the existence of common situations where an agent is not able to resolve his/her preference (see the Introductions in Nehring, 1996 and 1997), and (b) the increasing relevance of non-classical choice mechanisms (cf. Aizerman and Malishevski, 1981). Now it is known that it is possible to take advantage of that general approach to optimization in order to develop other kind of sensible models in totally different settings. The recent Alcantud (2002a) exemplifies such possibility, by introducing a solution to the problem under new assumptions that permit direct applications e.g. to game theory, as in Alcantud and Alós-Ferrer (2002). All this accounts for the richness of the general problem proposed by Nehring.

Focusing on the Nehring's model, we emphasize some issues. The condition that an element is chosen in a set whenever it is chosen in all two-element situations extracted from it is abandoned. What replaces it is the so-called "finitariness" condition. The setting is completed by requesting a weak consistency axiom and also by postulating that any finite subset has a non-empty choice. According to his Remark 1, this model is relatively close to that of a choice correspondence that can be deduced by optimizing an acyclic binary relation. Owing to this, Nehring's contribution generalized the extensively used Bergstrom-Walker theorem: he provides a continuity condition that ensures non-empty choices on compact sets. That possibility was further exploited in Llinares and Sánchez (1999), where a yet weaker consistency axiom is used.

Our purpose is to complete the study by presenting *necessary and sufficient* conditions for a choice funtion under axioms weaker than those of the Nehring's model to have non-empty choices on compact sets. The conditions we propose are of transfer type. This kind of properties have provided milestones in the literature on the maximization of binary relations. Their use gave complete solutions to the question of the existence of maximal elements for complete preorders and for interval orders -cf. Tian and Zhou (1995)- and for acyclic and only irreflexive binary relations -cf. Rodríguez-Palmero and García-Lapresta (2002). Hence, we here show that a similar achievement is available in fairly general non-binary models of choice. That is accomplished in Section 2. As an application, the characterization of the existence of maximal elements for acyclic binary relations proven in Rodríguez-Palmero and García-Lapresta (2002) is deduced from our results in Section

3. Each of these solutions encompassed the cases tackled by Tian and Zhou (1995) which, therefore, must follow from our characterization results as well. Some further discussion on variations of our results and open questions put an end to our exposition in Section 4. In particular, we close some remaining gaps of the binary literature on the existence of maximal elements, namely, the study of the k-acyclic case. Some relationships to the approach initiated in Alcantud (2002b) complete our analysis.

2 A necessary and sufficient condition for non-binary maximization

We begin by describing our general framework.

Unless otherwise stated, X will denote a compact topological space. Let $\mathcal{D}$ be a domain of non-empty subsets of X, that represents all the choice situations to which the agent has been or could conceivably be faced. Denote by $C : \mathcal{D} \longrightarrow X$ a correspondence such that $C(S) \subseteq S$ for all $S \in \mathcal{D}$. As in Nehring (1996), it is henceforth assumed that all finite subsets of X belong to $\mathcal{D}$. $\mathcal{F}(S)$ will denote the set of all non-empty finite subsets of the choice situation S.

We stick to the notation and terminology of Nehring (1996) with regard to the following axioms:

Non-emptiness. If $S \in \mathcal{D}$ is finite then $C(S) \neq \varnothing$.

Contraction consistency or Chernoff condition. For all $S, T \in \mathcal{D} : T \subseteq S$ implies $C(S) \cap T \subseteq C(T)$ if S is finite.

Finitariness. For all $S \in \mathcal{D}$, if $x \in S$ satisfies that for all $T \in \mathcal{F}(S)$, $x \in T$ implies $x \in C(T)$, then $x \in C(S)$.

Contraction consistency says that an element x chosen in a set S is also chosen in any smaller set T containing x. Finitariness says that if for a given available set S, there is an element x which is always chosen in every finite subset of S which contains it, then that element has to be chosen in A. In Section 3 we shall relate these conditions to the case where an underlying binary relation exists. Complementarily, we also address the reader to Remark 1 in Nehring (1996).

Nehring (1996) proves that, under these three independent axioms, the choice correspondence C will assign a non-empty choice to any compact set belonging to $\mathcal{D}$ if a certain continuity condition is fulfilled. Llinares and Sánchez (1999) have proven a similar result which weakens the Chernoff condition and enlarges the class of subsets for which non-empty choice is guaranteed.

Our technique will require to make use of further definitions.

Definitions. The choice function C is said to satisfy *non-binary transfer continuity* (respectively: non-binary k-transfer continuity) if for all $x \in X$ such that there is S_x finite (respectively: with $|S_x| < k$) with $x \notin C(S_x \cup \{x\})$, there exists a neighborhood $N(x)$ of x and a $y_x \in X$ such that: for all S finite (respectively: with $|S| < k$) with $y_x \notin C(S \cup \{y_x\})$ it is also true that $z \notin C(S \cup \{z\})$ whenever $z \in N(x)$. We abbreviate these definitions by NBTC and NBkTC respectively.

Also, the choice function C is said to satisfy the *weak Chernoff condition* (resp. k-weak Chernoff condition) if for all $S \subseteq X$ with $C(S) \neq \varnothing$, there exists $x \in C(S)$ such that for all $T \subseteq S$ finite (resp. with $|T| < k$) it is true that $x \in C(T \cup \{x\})$.

For k-finitariness we shall mean: for all $S \in \mathcal{D}$, if $x \in S$ satisfies that for all $T \subseteq S$ with $x \in T$ and $|T| \leqslant k$ it is true $x \in C(T)$, then $x \in C(S)$.

Theorem 1. *Suppose that C satisfies Non-emptiness, k-Finitariness, and the k-weak Chernoff condition. Then, $C(X)$ is nonempty if and only if C satisfies NBkTC*

Remark. The case k=1 is useless to any purpose, since it contains one single choice correspondence for each ($\mathcal{D}$ associated with) X. In fact, 1-finitariness plus $C(x) \neq \varnothing$ for all $x \in X$ together already force $C(A) = A$ whenever $A \in \mathcal{D}$. Also, 2-finitariness is the binariness property that is stated as A5 in Nehring (1996). Section 3 will take advantage of this particular case.

Proof. *Necessity.* Being $C(X) \neq \varnothing$, the k-weak Chernoff condition provides an element y such that for all S with $|S| < k$ it is true that $y \in C(S \cup \{y\})$. Thus, for all $x \in X$ and any neighborhood $N(x)$ of x, $y_x = y$ satisfies vacuously the condition required by NBkTC.

Sufficiency. We distinguish two incompatible cases.

Case 1: If there is $x \in X$ for which $x \in C(S \cup \{x\})$ whenever $|S| < k$, then k-finitariness says that $x \in C(X)$.

Case 2: Suppose Case 1 does not hold. We show that a contradiction arises.

Due to NBkTC, with each $x \in X$ we can associate a neighborhood $N(x)$ of x and a $y_x \in X$ such that: for all $S \subseteq X$ with $|S| < k$ and $y_x \notin C(S \cup \{y_x\})$ it is also true that $z \notin C(S \cup \{z\})$ whenever $z \in N(x)$. Because $X = \bigcup_{x \in X} N(x)$, compactness yields the existence of a finite number of elements $x_1, ..., x_n$ such that $X = \bigcup_{i=1,...,n} N(x_i)$. In order to alleviate the notation, we shall denote y_i instead of y_{x_i} henceforth. The fact that Case 1 has been rejected ensures the existence of S_i such that $|S_i| < k$ and $y_i \notin C(S_i \cup \{y_i\})$, for each $i = 1, ..., n$. Hence, due to the NBkTC condition we can guarantee that

$$z \notin C(S_i \cup \{z\}) \text{ whenever } z \in N(x_i)$$

Define $S = S_1 \cup .... \cup S_n \in \mathcal{D}$, being a finite subset of X. Non-emptiness ensures the existence of $a \in C(S) \subseteq S$, thus $a \in C(S \cup \{a\})$. However, the k-weak Chernoff condition provides $b \in C(S) \subseteq S$ such that $b \in C(T)$ for every $T \subseteq S$ with $|T| \leqslant k$ and $b \in T$. By construction, there is i for which $b \in N(x_i)$. Then we get $b \in C(S_i \cup \{b\})$ because $S_i \cup \{b\} \subseteq S$ and $|S_i \cup \{b\}| \leqslant k$, a contradiction. *Q.E.D.*

The next result is a variation on the model of the Theorem above. It appeals to a quite remarkable framework, since it encompasses the motivating Nehring's model. Observe that k-finitariness implies both $(k+1)$-finitariness and finitariness, and also that the weak Chernoff condition is stronger than the k-weak Chernoff condition. The proof of Theorem 2 is virtually identical to that provided for Theorem 1 and is omitted.

Theorem 2. *Suppose that C satisfies Non-emptiness, Finitariness, and the weak Chernoff condition. Then, $C(X)$ is nonempty if and only if C satisfies NBTC*

3 Application: the characterization in the acyclic binary case

Rodríguez-Palmero and García-Lapresta (2002) have put forward conditions that characterize of the existence of maximal elements for acyclic binary relations defined on compact spaces. Actually, they obtained such result as an application of the characterization given for only irreflexive relations. We proceed to deduce their contribution for the acyclic case from Theorem 2. Because interval orders are acyclic (and irreflexive), the aforementioned cases studied by Tian and Zhou (1995) are encompassed in the characterization for the acyclic case and thus they must follow from our Theorem 2 as well.

For any P binary relation defined on the set X, define $C(T) = \{x \in T : yPx$ is false, $\forall y \in T\}$ whenever $T \subseteq X$. It is clear that it satisfies finitariness -actually, it satisfies 2-finitariness or binariness- as well as the Chernoff condition. It is well known and straightforward -see Theorem 2.5 in Aleskerov and Monjardet (2002) for an illustrative proof in the case where X finite- that: P is acyclic if and only if $C(T) \neq \varnothing$ for every finite subset T of X. This amounts to saying: C fulfils non-emptiness if and only if P is acyclic. We observe that P is irreflexive if and only if $C(\{x\}) \neq \varnothing$ for all $x \in X$.

We let $P(x) = \{y \in X : yPx\}$, for each $x \in X$. Thus, an element x will be maximal of P in X if and only if $P(x) = \varnothing$. The transitive closure of P will be denoted by P^∞. Rodríguez-Palmero and García-Lapresta (2002) introduced the concept of *TALC* (transfer acyclic lower continuity). It applies to P if and only if: whenever $P(x) \neq \varnothing$, there is $y \in X$ and a neighborhood of x, $N(x)$, such that $z\, P^\infty\, y$ entails $z\, P^\infty\, a$, for each $a \in N(x)$.

Besides, it is trivial that P is acyclic if and only if P^∞ is irreflexive. Obviously, in such case P^∞ is acyclic (as well as transitive).

Corollary 1 (Rodríguez-Palmero and García-Lapresta). *Let P be an acyclic binary relation on X compact topological space. Then: P has a maximal element on X if and only if P is TALC.*

Proof. The necessity if the condition is plain. Conversely, assume that P is TALC. Define $C(T) = \{x \in T : yP^\infty x$ is false, $\forall y \in T\}$. We have argued that our Theorem 2 applies to it: C satisfies Non-emptiness, Finitariness, and the (weak) Chernoff condition. In order to conclude, we only need to justify that C fulfils NBTC, because $C(X) \neq \varnothing$ implies that P has a maximal element.

Take $x \in X$ such that there is a finite $S_x \subseteq X$ with $x \notin C(S_x \cup \{x\})$. We need to produce a neighborhood $N(x)$ of x and a $y(x) \in X$ such that: for all $S \in \mathcal{F}(X)$ with $y(x) \notin C(S \cup \{y(x)\})$ it is also true that $z \notin C(S \cup \{z\})$ whenever $z \in N(x)$. Because P is TALC and $P(x) \neq \varnothing$, there is $y_x \in X$ and a neighborhood of x, N_x, such that $z\, P^\infty\, y$ entails $z\, P^\infty\, a$ for each $a \in N_x$. We check that these serve to our purpose. Indeed, if $S \in \mathcal{F}(X)$ displays $y_x \notin C(S \cup \{y_x\})$ there must be $z \in S$ with $z\, P^\infty\, y_x$ by the construction of C, which entails $z\, P^\infty\, a$, for each $a \in N_x$. This ends the proof. *Q.E.D.*

4 Comments, questions for further research, and the characterization in the k-acyclic binary case

We conclude with some comments and questions that seem interesting to us.

1.- It makes sense to reconsider our problem for settings that are different from ours, according to one's interests or necessities. In particular: what condition must replace NBkTC (respectively: NBTC) in our Theorem 1 (respectively: in Theorem 2) in the case that we replace non-emptiness by the much weaker condition $C(\{x\}) \neq \varnothing$ for all $x \in X$? As we argued above, the latter result would provide a natural generalization of the characterization of the existence of maximal elements for irreflexive binary relations available in Rodríguez-Palmero and García-Lapresta (2002). This could be motivated out of interest in understanding the foundations of optimization at the individual level that precede equlibrium results such as e.g. Gale and Mas-Colell (1975) or Shafer and Sonnenschenin (1975).

2.- Equally, and in a related line of thought, one might be interested in studying frameworks where non-emptiness is replaced by the intermediate requirements: $C(S) \neq \varnothing$ for all $S \subseteq X$ with cardinality at most k. These models have plain interpretations in terms of myopia and are related to generalizations of *k-acyclic binary optimization* ($k = 1, 2, ...$), which seem to lack a specific study in terms of explicit characterizations. We here close such gap of the binary literature by an appeal to the technique used in Rodríguez-Palmero and García-Lapresta (2002), who did give an explicit characterization in the 1-acyclic case, as we proceed to recall.

In general, for k-acyclicity we mean: $x_1 \, P \, x_2 \, P \, ... P \, x_l$ implies $x_l \, P \, x_1$ false, if $l \leqslant k$ (in case $l = 1$ the implication is satisfied vacuously by convention, so k-acyclicity implies irreflexivity and irreflexivity implies 1-acyclicity). P^k denotes the following k-transitive closure: $x_1 \, P^k \, y$ if and only if there is $l \leqslant k$ and $x_1, ..., x_l$ with $x_1 \, P \, x_2 \, P \, ... P \, x_l \, P \, y$ (the case $l = 1$ is to be interpreted $x_1 \, P \, y$, thus P^k extends P and they coincide for $k = 1$). Clearly, P is k-acyclic if and only if P^k is irreflexive. This entails necessary and sufficient conditions for the existence of maximal elements for k-acyclic binary relations by virtue of the 1-acyclic case as trivially as in Theorem 2 in Rodríguez-Palmero and García-Lapresta (2002). We just need to translate their TILC condition when P^k is used:

Definitions. An irreflexive binary relation P on X is *TILC* (transfer irreflexive lower continuity) if and only if: whenever $P(x) \neq \varnothing$, there is $y \in X$ and a neighborhood of x, $N(x)$, such that $z\, P^{\infty}\, y$ entails $z\, P\, a$, for each $a \in N(x)$. An irreflexive binary relation P on X is *TkALC* (transfer k-acyclic lower continuity) if and only if P^k is TILC if and only if whenever $P(x) \neq \varnothing$, there is $y \in X$ and a neighborhood of x, $N(x)$, such that $z\, P^{\infty}\, y$ entails $z\, P^k\, a$, for each $a \in N(x)$. Note that $P(x) \neq \varnothing$ is equivalent to $P^k(x) \neq \varnothing$, and also that the transitive closure of P^k is P^{∞} too.

Corollary 2. *Let P be a k-acyclic binary relation on X compact topological space. Then: P has a maximal element on X if and only if P is TkALC.*

Proof. Apply Theorem 1 in Rodríguez-Palmero and García-Lapresta (2002) to the associated P^k, that is irreflexive. This result states that P^k has a maximal element if and only if P^k is TILC, that is, if and only if P is TkALC. *Q.E.D.*

Observe that P is 2-acyclic if and only if P is asymmetric, which gives raise to a particular characterization in an interesting case not yet explicited, and that P is k-acyclic for every $k = 1, 2, ...$ if and only if P is acyclic.

As we mentioned before, and because P is k-acyclic if and only if every subset of X with cardinality at most k has a maximal element, maximality in the k-acyclic binary case addresses to the variation of any of our Theorems 1 or 2 where non-emptiness is replaced by the weaker axiom: $C(S) \neq \varnothing$ for all $S \subseteq X$ with cardinality at most k.

3.- The fact that a binary relation has a maximal element is not of topological nature: it is simply set-theoretical. This is even more obvious in the approach by choice correspondences: the fact that $C(X)$ is either empty or not is none but a set-theoretical statement. In view of these plain facts, Alcantud (2002b) has proposed that a different way to state the results on maximality could be more appropriate and illustrative. It is argued that fixing a topology *a priori* may seem comfortable, but it is nonetheless true that the (transfer) continuity issues that are typically to be ellucidated in order to ensure the existence of maximal elements may be much easier to check if a different topology had been employed. Consequently, Theorems 4 and 5 in Alcantud (2002b) constituted the first characterizations of the existence of maximal elements of acyclic relations in the literature. We can apply the technique developed there and provide what, in our view, is a more complete and procedurally correct statement of Theorems 1 and 2 above.

Theorem 1 (Alternative Statement). *Let X be a set and $\mathcal{D}$ a domain of non-empty subsets of X such that $\mathcal{F}(X) \subseteq \mathcal{D}$. Denote by $C : \mathcal{D} \longrightarrow X$ a correspondence such that $C(S) \subseteq S$ for all $S \in \mathcal{D}$.*

Suppose that C satisfies Non-emptiness, k-Finitariness, and the k-weak Chernoff condition. The following conditions are equivalent:

(a) $C(X)$ is nonempty

(b) there is a topology on X for which X is compact and C satisfies NBkTC

Proof. Theorem 1 says that (b) implies (a).

Assume that (a) holds. By the k-weak Chernoff condition, there must be an element z such that for all $S \subseteq X$ with $|S| < k$ it is true that $z \in C(S \cup \{z\})$. Endow X with the topology on X whose non-trivial open sets are all the subsets of X that do not contain z. The set X is compact, since every open cover of X includes X itself, and thus $\{X\}$ is a finite subcover. Also, C satisfies NBkTC. Indeed, for all $x \in X$ such that there is S_x with $|S_x| < k$ with $x \notin C(S_x \cup \{x\})$, we just take any neighborhood $N(x)$ of x and $y_x = z$. Then, the condition that for all S with $|S| < k$ and $y_x \notin C(S \cup \{y_x\})$ it is also true that $z \notin C(S \cup \{z\})$ whenever $z \in N(x)$ is satisfied vacuously. *Q.E.D.*

That being proven, the details of the proof of the next statement are left to the reader:

Theorem 2 (Alternative Statement). *Let X be a set and $\mathcal{D}$ a domain of non-empty subsets of X such that $\mathcal{F}(X) \subseteq \mathcal{D}$. Denote by $C : \mathcal{D} \longrightarrow X$ a correspondence such that $C(S) \subseteq S$ for all $S \in \mathcal{D}$.*

Suppose that C satisfies Non-emptiness, Finitariness, and the weak Chernoff condition. The following conditions are equivalent:

(a) $C(X)$ is nonempty

(b) there is a topology on X for which X is compact and C satisfies NBTC

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