On the Concentration of Allocations and Comparisons of Auctions in Large Economies*

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Abstract

We analyze competitive pressures in a sequence of auctions with a growing number of bidders, in a model that includes private and common valuations as special cases. We show that the key determinant of bidders' surplus (and implicitly auction revenue) is how the goods are distributed. In any setting and sequence of auctions where the allocation of good(s) is concentrated among a shrinking proportion of the population, the winning bidders enjoy no surplus in the limit. If instead the good(s) are allocated in a dispersed manner so that a non-vanishing proportion of the bidders obtain objects, then in any of a wide class of auctions bidders enjoy a surplus that is bounded away from zero. Moreover, under dispersed allocations, the format of the auction matters. If bidders have constant marginal utilities for objects up to some limit, then uniform price auctions lead to higher revenue than discriminatory auctions. If agents have decreasing marginal utilities for objects, then uniform price auctions are asymptotically efficient, while discriminatory auctions are asymptotically *in*efficient. Finally, we show that in some cases where dispersed allocations are efficient, revenue may increase by bundling goods at the expense of efficiency.

Keywords: Auction, Competition, Mechanism, Asymptotic Efficiency, Revenue Equivalence.

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1 Introduction

In many markets, including treasury auctions, IPO's, security markets, and internet based markets, large numbers of agents compete for a limited supply of resources. Moreover, these markets use a variety of rules for setting prices including uniform price auctions, pay-your-bid auctions, hybrids of these, as well as many other rules. A fundamental question arises as to when competitive pressures render the choice of auction mechanism irrelevant.

The mechanism design literature provides a partial answer to this question in the revenue equivalence theorem (Myerson (1981), Riley and Samuelson (1981), Ausubel and Cramton (1998), Reny (1999), Jehiel and Moldovanu (2001b), Krishna and Maenner (2001)), which states that if a unit of a good is to be allocated and bidders have independent information and satisfy some regularity conditions on interdependencies,[1] then the choice of auction formats in a given class is irrelevant. Although revenue equivalence extends to multiple units,[2] it does not hold when types are correlated or affiliated. With such correlation in information, auctions resulting in the same allocation of goods can result in different revenues (e.g., see Milgrom and Weber (1982)).

While the independent case is of some interest, there are many important applications, including most of those mentioned above, that involve some correlation or affiliation of information. The main focus of this paper is to look beyond independent types and understand when and to what degree the choice of an auction mechanism matters in large economies; and in particular how this is related to the way that goods are being allocated.

One might guess that for some standard auction formats where the previous literature gives us some guide to the structure of equilibria we could make direct calculations of the limiting outcome. However, even in the simplest cases this is an illusion. For example, even with a first price auction in a pure common value setting, where there is a known closed form solution for equilibrium strategies, it is difficult to directly compute the bidders' surplus in the limit. Hence, we follow the more general framework of mechanism design and work with what are commonly referred to as direct revelation mechanisms. As it turns out, working with this more abstract structure provides much more direct insight and allows us to prove some general results about limiting bidders' surplus and auction revenue. Moreover, in many cases of interest, one can then go back and apply these results to deduce what will happen standard auctions, as we will discuss.

A main insight of the paper is that the concentration or dispersion of the allocation of goods is the critical determinant of whether bidders in auctions enjoy any surplus, and whether (and how) the auction format impacts revenue and efficiency.

To be more specific, we examine sequences of auctions with growing numbers of bidders, in a model (building on that of Milgrom (1981)) including private and common valuations as special cases. We consider the allocation of some quantity of a good that might vary with population size. The key distinction that we uncover is between two cases. One case is what we call concentrated allocations: where the equilibrium allocation of the good is concentrated

[1] See Ausubel and Cramton (1998) and Reny (1999) for versions that cover classes of interdependent types under independence.

[2] See Weber (1983), Englebrecht-Wiggans (1988), and Krishna and Perry (1997). It also extends, in an approximate sense, to allow for uncertainty about supply and multiple unit demands as shown by Swinkels (2001).

among a vanishing fraction of bidders. The other case is what we call dispersed allocations: where the equilibrium allocation of the good is dispersed among a non-trivial proportion of the bidders.

Before proceeding any further, let us point out several important aspects of these definitions of concentrated or dispersed allocations.

First, these definitions apply to the allocations that result in equilibrium. The characterization of auctions in terms of equilibrium allocations allows us to make statements that do not depend directly on specific features of the auctions such as whether there is a reserve price, how winning bidders are determined, how prices are determined, etc.. This has the advantage of making the intuition clear and direct, and makes for a sharp and tight characterization. It has a disadvantage as well: one has to have knowledge of some properties of an auction's equilibrium allocations, and clearly the allocation is endogenous. Nevertheless, the classification of whether or not the allocation turns out to be concentrated or dispersed is often straightforward. For instance, as we shall discuss, a concentrated allocation will necessarily result if the quantity of good to be allocated is a vanishing fraction of n, or when there is some private component to the valuation and agents have no limit on the amount they desire. A dispersed allocation will necessarily result if bidders have a finite bound on the amount of the good that they desire and the amount of the good grows in proportion to n.

Second, these definitions apply to sequences of auctions and concern the limiting allocations, thus all of our results are about asymptotics. We present some results on rates of convergence so that we have a sense of how "large" an economy has to be in order for the results to apply.

Third, the relevant factor of the distinction between concentrated and dispersed allocations is the relative holdings of the good, and so these conditions are not logically related to straight identifications of numbers of objects. For instance, a sequence of auctions of only one object, but such that the object is simply randomly given away (ignoring bids) so that each bidder has an equal chance of getting the object would be a dispersed allocation. In contrast, a sequence of auctions of increasing numbers of goods, but such that all goods go to the single highest bidder would be a concentrated allocation, even though the number of goods is increasing. We discuss this in detail in what follows.

Fourth, the definitions of concentrated and dispersed allocations do not keep track of what the efficient allocation is; rather just what the equilibrium allocation is. At times these will be related, and we discuss efficiency at several points in the paper. As we remark at several points, however, most of the results in the paper apply directly to the equilibrium allocations, and only indirectly to what might be efficient.

We establish this distinction between concentrated and dispersed allocations through a series of results that can be briefly described as follows.

(1) **Concentrated Allocations-**

   (a) In any sequence of auctions with concentrated allocations the per- unit surplus enjoyed by the bidders goes to zero.[3] The intuition behind this is that under

[3] The surplus of a bidder is the total utility obtained from objects won in the auction less payments in the auction.

concentrated allocations there is a strong competition for *all* goods. For any unit of the good that is obtained by some bidder, there is a bidder with nearly the same information and preferences who has a low probability of getting a unit of the good. This results in a competition that drives away all of the surplus that goes to the winning bidders.

(b) The above implies an "asymptotic revenue equivalence result" for concentrated allocations: If two sequences of auctions lead to approximately the same concentrated allocations, then they lead to the same limiting revenue.

(c) Another corollary is that any sequence of auctions that leads to an approximately efficient and concentrated allocation, leads to the optimal revenue in the limit. Hence, a variety of standard auctions (first, second, English) will provide full revenue extraction in the limit.

(d) We can also establish rates of convergence for standard auctions when the item is indivisible. For instance, mechanisms that award the entire good to the highest bidder lead to a per-unit surplus to bidders that is of the order $1/n$ where n is the number of bidders. The rate of convergence of both bidders' surplus and revenue applies to many standard auctions such as the first, second price, and English auctions. We show how this can be used to study endogenous (costly) entry decisions regardless of the auction format.

(e) For the case where the efficient allocation is concentrated, we describe simple mechanisms that extract all revenue (as if the auctioneer was fully informed) in the limit.

(2) **Dispersed Allocations-**

(a) In any sequence of auctions that results in dispersed allocations, if there is any private aspect to bidders' valuations and an individual rationality constraint is satisfied, then bidders enjoy a per-unit surplus that is bounded away from 0. This is the counterpoint to (1-a) and has the following intuition. Here the allocation of goods is such that there are some goods and corresponding winning bidders for whom any other bidder with similar information and preferences also expects to obtain some goods. For these goods there is less competitive pressure and so at least some surplus is enjoyed by the bidders in such sequences of auctions.

Under dispersed allocations, with any correlation among the information observed by bidders, the choice of auction format matters in some systematic ways. In some cases where efficient allocations are dispersed, we compare some standard auction formats to find that:

(b) If bidders have "flat" demand curves,[4] then uniform price auctions result in revenues that are higher, by an amount bounded below, than discriminatory auctions, even when resulting in exactly the same allocations.

[4]That is, a bidder values objects equally up to some number of items that they wish to purchase. This includes, for instance, single unit demands.

(c) With downward sloping demand function and private (possibly correlated) values, the uniform auction is asymptotically efficient, while in contrast, discriminatory auctions are necessarily asymptotically inefficient.

(d) Efficient auction mechanisms can be dominated in terms of revenues by auctions that inefficiently bundle objects together for sale.

(e) If the efficient allocation is dispersed, then any sequence of mechanisms which extract full revenue in the limit must violate an individual rationality constraint.[5] This together with (1-e) shows how optimal mechanism design depends on the structure of the efficient allocation.

The fact that the choice of auction matters under dispersed allocations contrasts with what we saw under concentrated allocations. We note, however, that this contrast is not an obvious implication of other differences between concentrated and dispersed allocations: for instance, bidders' surplus depends on whether the allocation is concentrated or dispersed. That is, it is conceivable that even though dispersed allocations lead to some bidders' surplus, the surplus would be the same across any auctions that led to similar allocations. However, this turns out not to be true. The details depend on how bidders must behave to be sure they get an object when they have high values, and we discuss this in some detail.

1.1 Contributions and Relation to the Literature

Two closely related papers in terms of examining the asymptotics of revenue across auction formats are Kremer (2002) and Bali and Jackson (2002). Both papers consider auctions with growing numbers of bidders and a single unit of a good for sale. Kremer shows that in some common values settings the expected revenues of first price, second price and English auctions all converge to the expected value of the object.[6] Bali and Jackson show that such convergence holds in a across a wide class of auctions and information settings. The intuition is that in a large population the bidder observing the highest signal (and winning the object) faces competition from bidders who have nearby signals and hence almost the same information. Such competing bidders can act as if they had a slightly higher signal no matter what the payment mechanism, and so the winning bidder's surplus will be competed away. This ties down the revenue of the auction simply through incentive compatibility.

The results we show here broaden our understanding in several directions. First, we show that the key characteristic determining whether or not mechanisms matter is the allocation is dispersed or concentrated. So, the asymptotic revenue equivalence and full extraction result is not restricted to single object auctions, but extends provided the allocation is concentrated among a shrinking set of bidders. Moreover, we show that the size of the surplus going to winning bidders as a function of the population is of the order $\frac{1}{n}$. This tight bound is useful, for example, in characterizing endogenous entry. Third, and perhaps most

[5]The type of individual rationality constraint in question is what we call 'safety'. It is stronger than the standard interim constraint, but weaker than an ex-post constraint. It is equivalent to interim individual rationality under independent signals, but slightly stronger otherwise.

[6]See Goeree and Offerman (1999) for a related result.

importantly, we show that under dispersed allocations competition no longer ties down the asymptotic revenue and the specifics of the auction make a significant difference, both in terms of revenue and efficiency. Moreover, we show that under dispersed allocations (with some minimal private value component to valuations), bidders enjoy non-vanishing surplus. Thus, we develop an explicit understanding of how the way in which goods are allocated determines the extent to which competitive forces dictate price formation.

We make two other small remarks before proceeding.

Our work also has some side implications for how prices can aggregate information in large economies and how that depends on the price setting mechanism. Pesendorfer and Swinkels (1997) examined purely common value settings, and showed that whether or not price converges to value in large uniform price auctions depends on whether or not both the number of bidders getting objects and the number of bidders not getting an object go to infinity. Since we show that different auction formats lead to different revenues with dispersed allocations, we can deduce that this nice property of information aggregation that is enjoyed by uniform price auctions with large numbers of objects is not exhibited by other prominent auctions. In particular, for discriminatory price auctions, not even the average price (nor the max, min, or any order statistic) converges to value. We discuss this in more detail in what follows.

Finally, our work also points out some situations where simple mechanisms extract nearly all of the revenue. Crémer and McLean (1985, 1988) (see also McAfee and Reny (1992)) have shown that with some correlation in information a seller can extract the full value of an object in auctions of a single object.[7] However, such mechanisms are quite complicated and depend critically on knowledge of the underlying distribution of information. As our asymptotic revenue equivalence results show, the complicated mechanisms required for full revenue extraction are not needed when there are large numbers of bidders and concentrated allocations are efficient. In these cases any sequence of auctions that is approximately efficient will also fully extract revenue in the limit. Most importantly, one can use standard auction formats that are independent of the distribution of information in the society (and even when information is independent).

2 Information and Preferences

Economies

A sequence of economies is indexed by n, the number of agents in the economy. A non-random quantity Q^n of a good is to be sold in economy n. It may be fully divisible or may come in indivisible units. Through randomization in the allocation, indivisible units may in effect be divided. We will be addressing conditions relating to Q^n that identify when and how differences in auction formats appear.

Information

[7] The auctions we consider in obtaining the *lower bound* of $\frac{1}{n}$ on the surplus going to bidders place an upper limit on the price paid and only have payments made by winning bidders. These conditions are violated by the Crémer-McLean style mechanisms, explaining why their full extraction results do not contradict our convergence rates.

Information is described by a framework that we borrow from Milgrom (1981), as described in detail below. Note, however, that through much of the paper we do not assume the monotone likelihood ratio property. This information structure is particularly suited to the discussion of growing sequences of economies, all based on the same underlying information structure. We use upper case letters to denote random variables and lower case letters to denote realizations. We use f to denote a density or conditional density of a random variable and F to denote a distribution. In some cases when it may be unclear to which random variables we refer, we use subscripts such as $f_X(\cdot)$, while when it is clear we will omit the subscripts.

Each agent $i \in \{1, \ldots, n\}$ in economy n observes a private signal S_i that takes on values in $[0, 1]$. There is also an underlying random variable X taking on values in $[0, 1]$. The S_i's are independently and identically distributed conditional on X. This conditional distribution of S_i given X is described by the density function $f(s_i|x)$.[8] We assume that the unconditional (marginal) density of each S_i, $f(s_i)$, is positive for all s_i.

Let S denote the vector of signals $S_1, \ldots, S_n$ and let S_{-i} denote the vector of signals omitting S_i. Let $Y(k)$ denote the k-th order statistic of the signals S and let $Y_{-i}(k)$ denote the k-th order statistic of the signals S_{-i}.

We also assume that:

(A1) There exists $\alpha > 0$ such that for almost every x the density of X conditional on S_i satisfies

$$|f(x|s_i) - f(x|s_i')| < \alpha |s_i - s_i'| f(x|s_i').$$

(A1) is a Lipschitz condition that implies uniform continuity in signals across x.[9] The important implication of this condition is that two nearby signals provide similar information about the realization of X.

Preferences

Agent i's valuation for the good is described by $v : [0, 1]^2 \to [0, 1]$, where $v(s_i, x)$ is i's valuation given the realizations (s_i, x) of i's signal and of the state variable. In this setting, X represents an objective quality of the good or part that is common to all bidders; and the signal S_i has a dual role: it contains information regarding X and also represents a personal taste (see Milgrom (1981)). This framework includes as special cases settings of pure private values, where $v(s_i, x) = s_i$, and pure common values, where $v(s_i, x) = x$.

The following condition requires that at least one of the variables be important, and that there is some upper bound on the derivative with respect to the private signal.

(A2) $v(s_i, x)$ is differentiable and non-decreasing in both variables. Moreover, there exist $\rho > 0$ and $\gamma > 0$ such that $\frac{dv}{ds_i} + \frac{dv}{dx} > \rho$ and $\frac{dv}{ds_i} < \gamma$.

Condition (A2) is important in implying that signals have some importance, either directly in terms of private values, or in providing some information about preferences through

[8] The assumption that the random variables have continuous distributions is made to simplify the exposition, but is not critical to the results.

[9] For most of our results it is sufficient to assume only uniform continuity.

the common component. The upper bound also puts some limit on how sensitive preferences are to information.

A Private Value Component

We sometimes refer to situations in which there is some private value component to the valuation. This does not have to be a case of pure private values, but is captured by the following definition.

A good has a *private value component* if there exists $\tau > 0$ such that $\frac{dv}{ds_i}(s_i, x) > \tau$ for any (s_i, x).

In the first part of our analysis, bidders have "flat demand curves." That is, the payoff to a bidder from consuming an amount q_i is simply

$$v(s_i, x)q_i.$$

This assumption provides for a more straightforward exposition in the first part of the paper, without much effect on the qualitative results.[10] Once we get to the case of dispersed allocations, however, this assumption starts to have important consequences. Thus, in Section 4.4 we explicitly account for how utility depends on quantity consumed.

Mechanisms

Invoking the well-known revelation principle, we restrict attention to direct mechanisms.[11] A mechanism in the n-th economy is a pair of functions (q^n, t^n), where

1. $q^n : [0,1]^n \to I\!R^n$ is an allocation rule that assigns quantities to bidders as a function of the profile of announced signals s, such that $\sum_{i=1}^{n} q_i^n(s) \leq Q^n$, and
2. $t^n : [0,1]^n \to I\!R_+^n$ is a payment function that specifies the payment each bidder makes as a function of the profile of announced signals s, where $t_i(s)$ denotes the payment of bidder i.

We make three remarks about the allocation and payment functions. First, the allocation function does not necessarily allocate all of Q^n. This allows for the incorporation of reserve prices into the auctions considered. Second, t_i^n can be positive even when q_i^n is not. Thus, the specification allows for bidders to pay even when they do not receive any allocation, and so it allows for features such as entry fees and "all-pay" requirements. Third, q_i^n and t_i^n can be thought of as expected allocations and payments so that randomization is permitted.

[10] Under concentrated allocations, assuming bidders who see similar signals have similar demands, competition still drives their surplus to zero. Under dispersed allocations, bidders still enjoy some surplus and mechanisms matter.

[11] Regarding existence: under our information and preference assumptions, there exists a (symmetric) equilibrium for a wide variety of auction formats (including all the standard ones) if the tie-breaking is allowed to be endogenous (see Jackson, Simon, Swinkels, and Zame (2002)), and even with fixed tie-breaking for the case of private and possibly correlated values (see Jackson and Swinkels (2001)). Regardless of whether one deals with a pure or mixed strategy equilibrium or the nature of the tie-breaking, the corresponding direct mechanism is handled by our approach in this paper, and so the results here apply.

In some of the examples we consider mechanisms that treat bidders symmetrically. Most of the results, however, apply to asymmetric mechanisms. We will be explicit in noting when symmetry is assumed.

Payoffs and Incentive Compatibility

Bidders are risk neutral, and so under a mechanism (q^n, t^n) the expected surplus (or payoff) obtained by bidder i who has a signal s_i and declares s_i' is

$$Sur_i^n (s_i, s_i') = E\left[q_i^n (s_i', S_{-i}) v(s_i, X) - t_i^n (s_i', S_{-i}) |S_i = s_i\right].$$

Note that Sur_i^n is a function of the mechanism (q^n, t^n), but we suppress this notation as the mechanisms will usually be given.

Incentive compatibility is written as

$$Sur_i^n(s_i, s_i) \geq Sur_i^n (s_i, s_i')$$

for each i, s_i, and s_i'.[12]

Individual Rationality

In what follows we refer to different forms of participation constraints, depending on the timing with respect to which they are applied. We state a familiar one here and defer the other definitions until they are needed.

A mechanism (q^n, t^n) is *interim individual rational* if

$$Sur_i^n (s_i, s_i) \geq 0$$

for each i and s_i.

2.1 Concentrated versus Dispersed Allocations

One of the main insights in this paper is the difference between concentrated and dispersed allocations.

Let $\overline{q}_n \equiv \frac{Q_n}{n}$ denote the per-capita supply of objects for sale, and P be the (unconditional) probability measure over S_i.

Concentrated Allocations:

A sequence of allocation functions $\{q^n\}$ is *concentrated* if for every $b > 0$ there exists an n' such that for $n > n'$ and every i

$$P\left(\left\{E\left[\frac{q_i^n(S)}{\bar{q}_n}|S_i\right] \geq b\right\}\right) < b.$$

Dispersed Allocations:

[12] As usual, all conditions are required to hold only almost surely and we omit such mention in what follows.

A sequence of allocation functions $\{q^n\}$ is *dispersed* if there exists $b > 0$ such that for infinitely many n

$$P\left(\left\{E\left[\frac{q_i^n(S)}{\overline{q}_n}|S_i\right] \geq b\right\}\right) \geq b,$$

for a number of agents i that is at least bn.

In what follows, we refer to a sequence of mechanisms as being concentrated or dispersed if their corresponding allocation functions are.

The intuition behind these definitions is that a completely evenly dispersed allocation would give $q_i^n = \overline{q}_n$ to each bidder, so that $\frac{q_i^n}{\overline{q}_n}$ would be 1. If this expression is going to zero for almost all bidders, then the allocation is concentrated in the hands of just a small proportion of the bidders (i.e., those who saw certain signals), while if it is not vanishing for some non-trivial proportion of bidders (and so one can expect to get objects conditional on seeing a non-trivial range of signals) then there is reasonable dispersion.

Let us make a few remarks about the details of the definitions.

First, they are defined relative to the per-capita supply of the good. It is possible to have a dispersed allocation even if $\overline{q}_n \to 0$ and each bidder's allocation is actually going to 0. Similarly, it is possible to have a concentrated allocation even when $\overline{q}_n \to \infty$ and where every bidder is getting an arbitrarily large allocation in the limit, but the highest signal bidders are getting the lion's share.[13] So the important intuition regarding competition that emerges here is that it is the *relative* disparities in allocations that determine whether or not surplus is competed away in large auctions.

Second, the definitions allow for asymmetric mechanisms, so that different bidders might have different expectations under the mechanisms in question. However, it is important to note that concentration requires a uniformity in the convergence across bidders i. Without this, it would be possible, for example, to have non-vanishing fractions of bidders expecting to get significant fractions of the objects at any date. For example, suppose that objects are simply randomly given to agents with labels between $n/2$ and n in auction n. Here, any given bidder eventually expects to get no objects at all, and yet the allocation is clearly not what one would want to call "concentrated." Thus, the uniformity in convergence rates across bidders under the definition of concentration is critical.

Third, there is a gap between the definitions of concentrated and dispersed allocations. Some sequences of mechanisms do not fall into either category. This gap is in fact necessary given that we wish to account for asymmetric mechanisms. For instance, consider the following situation. There are n bidders and the object is always simply given to bidder 1 at a price of 0.[14] This clearly fails to be a concentrated sequence under the definition. This is important, because the results that are claimed for sequences of concentrated mechanisms (e.g., bidders' surplus going to 0) would not be true for this particular sequence of

[13] These features make the conditions different, for instance, from checking whether Q^n and $n - Q^n$ are getting large as in the double-largeness condition of Pesendorfer and Swinkels (1997). In fact, $n - Q^n$ does not play any role in our analysis, and Q^n only plays a role in the denominator in determining relative allocations. This means that there are some differences between conditions that ensure information aggregation, and those which correspond to surplus extraction.

[14] This is not such a silly mechanism, as note that it corresponds to an asymmetric equilibrium in a second price auction where bidder 1 always bids 1 regardless of her signal and all other bidders always bid 0.

asymmetric mechanisms. Note also, that it would not make sense to call this a "dispersed allocation," as the objects are always going to one bidder. The reason that our results do not apply is that the particular asymmetry in the mechanism has eliminated all competition for one bidder. Thus, allowing for asymmetric mechanisms requires some gap between the definitions. Note, however, that if one restricts attention to symmetric mechanisms, then the definitions are essentially complementary.

Finally, whether the allocation is concentrated or dispersed depends on the setting and the equilibrium that will result in a given sequence of auctions. Looking at things through the lens of the allocation allows us to extract the general insight regarding how competitive forces work in large economies and how this depends on the distribution of goods. Nevertheless, it is important to be able to tell which type of allocation applies in different situations. The classification of whether or not the allocation turns out to be concentrated or dispersed is often straightforward. That is, many situations can be categorized into general classes where it is clear which type of allocation will result under most standard auction formats. A simple classification is as follows.

- A concentrated allocation will necessarily result if: the quantity of good to be allocated (Q^n) is a vanishing fraction of n, and the good is (asymptotically and approximately) efficiently allocated and there is some private component to the valuation.
- A dispersed allocation will necessarily result if: Bidders have a finite bound on the amount of the good that they desire and the amount of the good grows in proportion to n.

The above only provides a rough classification, but still covers many of the cases of interest. Auctions of limited numbers of objects (e.g., an art auction) will generally fall into the first case and have a concentrated allocation, while auctions of many objects (e.g., treasury auctions) will often fall into the second case and have dispersed allocations. Given the variety of mechanisms and settings admitted in the model, a fuller characterization of when concentrated versus dispersed allocations result would be quite complicated, without adding much insight. We provide a fuller treatment of two prominent auction formats (uniform and discriminatory) in what follows. We now turn to analyzing auctions under the two types of allocations.

3 Concentrated Allocations

We first examine sequences of mechanisms that result in concentrated allocations. In concentrated allocations, for any bidder who gets a significant proportion of the good we can find another bidder who receives a nearby signal (and hence has nearby beliefs and valuation), but only gets a relatively small amount of the good. The competition from such nearby bidders eliminates the surplus enjoyed by all bidders.

The following "continuity" lemma is useful in establishing this result and some others that follow. The lemma states that the surplus obtained by a given type is nearly obtainable by a nearby type who pretends to be of the given type.

Lemma 1 *If (A1) and (A2) are satisfied, then in any sequence of interim individually rational mechanisms and for any n*

$$\left|Sur_i^n\left(s_i, s_i'\right) - Sur_i^n\left(s_i', s_i'\right)\right| \leq (2\alpha + \gamma)\left|s_i - s_i'\right| E^n\left[q_i^n\left(S\right) \mid S_i = s_i'\right],$$

where α and γ are as defined in (A1) and (A2).

Lemma 1 follows from the continuity assumptions and the structure of information we outlined. The intuition is straightforward: altering an agent's signal slightly, but not their report, leads to nearly the same beliefs, preferences, and expectations of the allocation. Under individual rationality, the payments cannot vary much more drastically than the allocation. So the surplus cannot vary by much relative to the total expected allocation.

Once we couple Lemma 1 with incentive compatibility, we deduce that nearby signals must lead to similar expected surpluses. An implication of the above Lemma is that if the allocation sequence is concentrated, then agents compete away their surplus, which is stated as follows.

Theorem 1 *If (A1) and (A2) hold, then for any sequence of concentrated, interim individually rational, and incentive compatible mechanisms, $\frac{\sum_i Sur_i^n(S_i,S_i)}{Q^n}$ converges to 0 in probability.*[15]

The intuition behind the theorem is as follows. Under a concentrated allocation, the circumstances in which a bidder can expect to win non-trivial amounts of objects (in per-capita terms) is shrinking. That is, the set of signals under which a bidder expects to win objects is a shrinking set. Nearby signals must lead to expectations of no surplus. Then given incentive compatibility, and the continuity noted in Lemma 1, since nearby signals expect a low surplus, the winning signals must also expect a low surplus. In terms of more traditional language of competition: the objects are being concentrated in the hands of just a few winning bidders. As the economy grows, there will also be many other bidders who have very similar information and preferences to those who end up winning. The competition between these bidders eliminates the surplus.

3.1 Revenue Equivalence

Theorem 1 implies that the total revenue in an a sequence of auctions with concentrated allocation functions is the approximate full (expected) valuation of the objects to the winning bidders. As we have not specified the allocation functions beyond being concentrated, this does not necessarily imply full revenue extraction. For instance, it could be that the mechanisms never give any objects away and do not result in any revenue.

However, Theorem 1 still provides a revenue equivalence result, in that any two sequences with similar allocation functions must result in similar revenues. This is stated in the following corollary.

[15] We actually prove that the *expected* per unit surplus converges to 0, which implies convergence in probability since this is a nonnegative random variable.

Corollary 1 *Let (A1),(A2) hold and consider two sequences of incentive compatible and interim individually rational mechanisms, $\{(q^n, t^n)\}$ and $\{(\overline{q}^n, \overline{t}^n)\}$, with concentrated allocation functions. If the allocation functions are approximately the same, i.e.,*

$$\frac{E\left[\sum_i q_i^n(S)v(S_i, X) - \sum_i \overline{q}_i^n(S)v(S_i, X)\right]}{Q^n} \to 0,$$

then they lead to approximately the same expected revenues:

$$\frac{E\left[\sum_i t_i^n(S) - \sum_i \overline{t}_i^n(S)\right]}{Q^n} \to 0.$$

Corollary 1 provides a fairly general asymptotic revenue equivalence theorem, as it applies with correlated values and/or common values, and the sale of more than one object.[16]

As we shall see, it is critical to the above result that the mechanisms be concentrated. Otherwise, mechanisms with identical allocation functions can lead to very different revenues, even in the limit.

3.2 Optimal Mechanisms

Let us note another important implication of Theorem 1: any sequence of auctions that results in concentrated and efficient allocations provides full revenue extraction in the limit. One implication of this is that with large numbers of bidders, auction formats that lead to efficient and concentrated allocations also lead to approximately full revenue and one does not need to resort to the complicated and parametric types of mechanisms identified by Crémer and McLean (1988) and McAfee and Reny (1992). Moreover, this holds in a variety of settings, including correlated private values, common values, as well as under complete independence (where Crémer and McLean mechanisms fail to work).

To be careful, we have to argue that there will exist some mechanisms that achieve efficiency (at least approximately when efficient allocations are concentrated) in order for the above statements to be non-vacuous.[17] Indeed, there exist mechanisms, even symmetric ones, that will achieve an approximately efficient allocation in a variety of situations, and so full revenue extraction is feasible. This mechanism even satisfies ex-post individual rationality constraints and works without the correlation structure inherent in the Crémer-McLean approach. Without giving a formal argument, let us heuristically describe such mechanisms in the case where the efficient allocation involves awarding all objects to one bidder.[18],[19] Pick some subset of agents and ask them their signals. If symmetry is desired, randomly pick the agents. Keep this set of surveyed agents of size $\sqrt{n}$, so that it grows with n, but is negligible

[16]Corollary 1 generalizes the main result of Bali and Jackson (1999), in that it applies to the auctioning of more than one good, and also allows for entry fees. However, it requires more structure on information (the mineral rights setting) and on mechanisms than the results of Bali and Jackson (1999).

[17]We thank a referee for pointing this out.

[18]With flat demands it is efficient to treat the supply as indivisible. This can be modified as long as the efficient allocation is concentrated.

[19]Variations on this sort of "folk" mechanism appear in a number of places. For an auctions version, where interdependencies in valuations are present, see Jackson (1998). For versions satisfying strategy-proofness in private values settings, see Cordoba and Hammond (1998) and Kovalenkov (2002).

in the limit. These agents will not get any of the allocation, so it is incentive compatible for them to reveal their information. Based on their announcements, estimate X, and then $v(1,X)$. Randomly order the remaining agents, and make them take it or leave it offers at the price of $v(1,X)-\varepsilon$, until some agent agrees to buy the objects. This will happen with very high probability for large enough n, and the object(s) will end up in the hands of an agent who values them at nearly the maximal possible level.

3.3 Indivisible Goods

A case of concentrated allocations that is of particular interest is where an indivisible good is to be auctioned to the highest bidder. This covers, for instance, first price, second price, and English auctions. We focus on this case to get some insight into rates of convergence to the competitive outcome. We first show that the surplus to the winning bidder decreases at a faster rate than n^{a-1} for any $a>0$. To develop a tight bound on surplus, we also consider the following condition on the information structure.

(A3) There exists $\beta>0$ such that $\beta > f(s_i \mid x) > \frac{1}{\beta}$ for every s_i and x.

(A3) bounds the likelihood of any signal conditional on a given X both above and below, thus limiting the informativeness of any given signal and implying some diversity in the signals observed.

Theorem 2 *Let (A1)-(A3) hold and consider a sequence of incentive compatible and interim individually rational mechanisms, $\{(q^n,t^n)\}$, which award the entire Q^n to a highest signal observer. The bidders' surplus (per unit) converges to zero at a rate faster than n^a for any $a<1$. That is for any $a<1$, $n^a\frac{\sum_i Sur_i^n(S_i,S_i)}{Q^n}$ converges to 0 in probability.*

The bound in Theorem 2 comes from bounding the conditional probability of winning for an observer of a given signal. That probability goes to 0 at an exponential rate, even for signals going to 1 at a rate n^a $(a<1)$. This bounds the surplus that can be expected for high signals. The Lipschitz continuity of information then implies that this surplus is approximately the same as is enjoyed by the highest signal. The complete proof appears in the appendix.

We now explore the tightness of this bound. We show that the surplus going to bidders is at least the order of $\frac{1}{n}$ in any case where the good has some private value component, and so the bounds established in Theorem 2 are tight for a standard class of auctions.

Theorem 3 *Let (A1)-(A3) hold and consider a good that has a private value component and a sequence of incentive compatible mechanisms q^n,t^n such that all of the good is given to a single bidder who has the highest signal, and payments $(t_i^n(s))$ never exceed 1 and are only made conditional on receiving the object ($t_i^n(s)>0$ implies $q_i^n(s)>0$).*[20] *There exists*

[20] This "losers do not pay" condition has been studied by Lopomo (2001) in analyzing conditions under which the English auction is optimal. See Lopomo (2001) for additional discussion of this condition, and optimal mechanisms subject to it.

$\phi > 0$ *such that the total surplus per unit to the bidders is at least* $\frac{\phi}{n}$ *for all* n*. That is, there exists* $\phi > 0$ *such that for any* n

$$\frac{E\left[\sum_i Sur_i^n(S_i, S_i)\right]}{Q^n} > \frac{\phi}{n}.$$

Theorem 3 shows that the bound established in Theorem 2 is tight. It is proven by showing that the winner expects a distance between her signal and the next highest signal that is on the order of $\frac{1}{n}$. This implies, given the private value component, that the winner expects to have a valuation that is higher than the second highest by an amount that is of the order of $\frac{1}{n}$. Then, regardless of the particular payment format, incentive compatibility implies that winner must get a surplus of the order of $\frac{1}{n}$.

If we allow for arbitrary mechanisms, then with correlation among the signals there is a possibility of extracting full surplus from the bidders, as shown by Crémer and McLean (1988)[21]. The full extraction mechanisms are ruled out under Theorem 3 as payments never exceed the maximum possible value and are only made conditional on receiving the object. Neither of these conditions are met by the Crémer-McLean style mechanisms, as such mechanisms require occasionally large payments and payments even by bidders who do not obtain the object. These features of Crémer-McLean style mechanisms are not exhibited by many standard auction formats (e.g., first price, second price, English auctions, etc.) which satisfy the condition of Theorem 3.

3.4 An Application to Endogenous Entry

We now show that the results regarding the convergence rate of bidders' surplus are not simply a technical curiosity, but can be used to provide insight into auctions where the entry decision is endogenous and costly.[22]

Suppose that a quantity of the good Q is sold as an indivisible good, and bidders must pay an (ex-ante) entry fee of c.[23] Let us examine the number of entrants and the markdown in prices as a function of Q and entry cost, c. The novelty is that we establish these relations without relying on a specific mechanism. We only assume that mechanism does not charge a bidder unless they get the good (excluding the entry cost), payment never exceeds the upper bound on the good's value, and the good has some private value component (we also assume the information assumptions from the last section).

Theorem 3 tells us that there exists some $\phi > 0$ such that total surplus that goes to bidders exceeds $\frac{\phi Q}{n}$ for any n. Hence, in order for it to be an equilibrium for n and not $n+1$ bidders to enter (at an ex ante stage before signals are observed) we know that

$$c \geq \frac{1}{n+1}\left(\frac{\phi Q}{n+1}\right)$$

[21]McAfee and Reny (1992) show that this is also true for the continuous signal case, to an arbitrary approximation.

[22]For other examples of usefulness of such convergence rates in auctions and bargaining see Satterthwaite, Rustichini, and Williams (1994) and Neeman (1999).

[23]We are considering a two stage process where bidders first decide whether to enter or not and then observe their signals and participate in the auction if they have paid the entry fee.

or that

$$n \geq \sqrt{\frac{\phi Q}{c}} - 1.$$

This gives us a lower bound on n. Next, let us explore an upper bound. If $\frac{Q}{c}$ (and hence n) is large, Theorem 2 bounds the total surplus to be below $\frac{Q}{n^a}$ for any $a < 1$. Thus, for n bidders to be willing to enter we must have

$$\left(\frac{1}{n}\right) \frac{Q}{n^a} \geq c,$$

or

$$\left(\frac{Q}{c}\right)^{\frac{1}{1+a}} \geq n.$$

Putting these lower and (approximate) upper bounds together leads to

$$\left(\frac{Q}{c}\right)^{\frac{1}{1+a}} \geq n \geq \sqrt{\frac{\phi Q}{c}} - 1,$$

for any $a < 1$. So, we have obtained an approximation on the number of bidders who will enter an auction:[24]

$$n \propto \sqrt{\frac{Q}{c}}$$

Since the expected surplus excluding entry costs that goes to the bidders is on the order of $\frac{Q}{n}$ (Theorems 2 and 3), substituting from the approximation for n we find that the expected surplus going to bidders in the auction is approximately proportional to $\sqrt{Qc}$. This in turn implies that the average markdown in price per unit (compared to the winner's valuation) is approximately proportional to $\sqrt{\frac{c}{Q}}$.

4 Dispersed Allocations

We now turn our attention to sequences of auctions with dispersed allocations. First we use the general mechanism design approach to provide some impossibility result for surplus extraction. We then turn our attention to explicit mechanisms: the uniform and discriminatory auctions.

4.1 The Impossibility of Full Surplus Extraction

Individual Rationality and Safety

In what follows we consider a strengthening of interim individual rationality. It is useful to compare it to a standard strengthening is the following condition, which is the following.

A mechanism (q^n, t^n) is *ex-post individual rational* if

$$q_i^n(s) E[v(s_i, x) | S = s] - t_i^n(s) \geq 0$$

[24]The approximation, of course, is only valid for large n, and so is more accurate if the total value of goods to be auctioned relative to the entry cost ($\frac{Q}{c}$) is large.

for each i and s.

Ex post individual rationality requires that agents do not over-pay when conditioning on all signals. While this condition holds for some standard mechanisms in a private values (possibly correlated) setup, it is often violated when there is any common value component.

We introduce a concept that is intermediate to interim and ex-post individual rationality, which captures the idea that bidders should not expect to over pay conditional on their own information - but not independent of the realization of the other signals.

A mechanism (q^n, t^n) is *safe* if

$$E\left[q_i^n\left(s_i, S_{-i}\right) v\left(s_i, X\right) - t_i^n\left(s_i, S_{-i}\right) | S_i = s_i'\right] \geq 0$$

for each i, s_i, and s_i' such that $s_i' \geq s_i$.

This condition of safety looks like interim individual rationality, except that beliefs are taken relative to any signal $s_i' \geq s_i$ rather than s_i; and hence it is a stronger condition. It requires that a given type s_i' does not expect lower types to over-pay. It is equivalent to interim individual rationality if signals are independent, and is always implied by ex-post individual rationality. If the often-assumed monotone likelihood ratio property holds, then safety is satisfied by many standard auctions, even under common values. This is because a higher type estimates the common value component to be higher.

There is a major difference in behavior between mechanisms with dispersed versus concentrated allocations. The following theorem shows that the (approximate) full-extraction of revenue that occurs with concentrated allocations will generally not hold with dispersed allocations.

Theorem 4 *Let (A1) and (A2) hold and consider a good that has a private value component, and a sequence of incentive compatible and safe mechanisms that are dispersed. There exists* $\phi > 0$ *such that the expected total surplus per unit obtained by the bidders in the auction is at least* ϕQ^n *for any* n*. That is, there exists* $\phi > 0$ *such that for every* n

$$E\left[\sum_i Sur_i^n(S_i, S_i)\right] \geq \phi Q^n.$$

Theorem 4 tells us that under dispersion, if there is any private component to the valuation structure, then bidders will capture some rents.

The theorem itself follows from a simple intuition. We state the intuition for the case where allocations are to high bidders, but such monotonicity is not essential (see the proof for details). Under dispersed allocations, a bidder with a high signal could pretend to have a slightly lower signal and still expect with some non-trivial probability to obtain some of the object. As long as (i) the high signal bidder does not expect to pay more than what would be fair for a bidder for the lower signal (the role of the safety condition), and (ii) according to the high type's belief a lower type should expect to get a non-trivial fraction of the good (the role of dispersion); it follows that the high signal bidder could obtain a positive expected surplus by pretending to have observed the lower signal. By incentive compatibility, the high-signal observing bidder must get at least this surplus under truthful announcement of his signal.

4.2 Non-Existence of Approximately Optimal Mechanisms

An implication of Theorem 4 is that the possibility of designing approximately optimal mechanisms in terms of extracting full revenue is precluded for the case where the efficient allocation is dispersed, at least if one wants to respect safety. With independent signals safety is no stronger than the standard incentive constraint. Hence, there does not exist a mechanism that achieves first best. With correlated signals, in order to extract full revenue, especially from high-valued signal observers, one can still resort to methods of à la Crémer and McLean (1988). The necessary violation of safety, however, means that extracting full revenues, even in a limiting sense, must involve very sensitive use of the correlation structure.

4.3 Different Mechanisms–Different Revenues

While Theorem 4 shows that the full revenue extraction that held with concentrated allocations fails under dispersed allocations, it does not tell us whether the particulars of the auction format matter. We now show that under dispersed allocations, the auction format matters significantly, and this represents a further departure from the results of concentrated allocations.

For the following result, we concentrate on a particular, but still prominent and interesting class of dispersed allocations. We return to a more general setting in the next section.

The class we examine here is one such that a quantity Q^n is auctioned in indivisible units and any bidder is awarded (or values) at most one unit. In particular, we look at mechanisms for which, in equilibrium, the goods are allocated to the Q^n bidders with the highest signals. In this setting, the allocation is dispersed.

To obtain a comparison of revenues of auctions, we work under the familiar strict monotone likelihood ratio property. To simplify the exposition we also assume continuity:

(A4) $F(s_i|x)$ and $f(x)$ are continuous in x and the Strict Monotone Likelihood Ratio Property (henceforth, MLRP) holds:

$$\frac{f\left(s_i \mid x\right)}{f\left(s_i' \mid x\right)} > \frac{f\left(s_i \mid x'\right)}{f\left(s_i' \mid x'\right)} \text{ for all } s_i > s_i' \text{ and } x > x'.$$

We now show that there is an asymptotic revenue difference between two standard mechanisms that have dispersed equilibrium allocations when the number of objects for sale Q^n is proportional to n. The mechanisms we examine are:

- Discriminatory price (pay-your-bid) auction: each bidder submits a bid for a single unit and pays his bid upon winning.
- Uniform price (pay the highest losing bid) auction: the Q^n highest bidders each get a single unit and pay the highest losing bid.

Note that under (A4) there exist (symmetric) equilibria for both mechanisms and that they support the same efficient allocation, where the Q^n highest signal holders obtain the objects.[25] We denote the corresponding expected payments by $t^{n,d}$ and $t^{n,u}$, respectively.

Theorem 5 *Let (A2) and (A4) and hold and* $\frac{Q^n}{n} \to b$ *where* $1 > b > 0$*. The uniform price auction yields higher expected revenue per capita (and per-unit) than a discriminatory price auction, by an amount that is bounded below as* $n \to \infty$*.*

We prove the theorem (in the appendix) using the following technique. Using the logic of Milgrom and Weber (1982), we can show that the expected payment of any given signal holder in a uniform auction is at least that of his clone in a discriminatory auction. We then argue that above a certain signal (one that is approximately sure to receive an object for large n) the payment schedule for a discriminatory auction flattens out, as higher signal holders can always bid as if they had this lower signal and will still be approximately certain to get an object. In the uniform auction, however, this payment schedule does not flatten out, as under strict MLRP observers of higher signals expect higher market clearing prices. Through this we establish a bound on the difference in revenues.

We provide two simple examples that illustrate Theorem 5. We consider two cases, a pure private value case and a pure common value case. In both cases the revenue in the uniform price auction is approximately 16% higher than that of the discriminatory auction, even as n becomes large.

The following describes the information structure for these examples.

- X is distributed uniformly on $[0, 1]$.
- Signals are distributed uniformly on $[0, x]$, that is $(S_i|X = x) \sim U[0, x]$.
- $Q^n = \frac{n}{2}$.

Example 1 *Private Values.*

First, consider the case of private values where $v(s_i, x) = s_i$.

We begin by analyzing the discriminatory (pay-your-bid) auction format. Let $F_{Y_{n/2}|S_i}$ and $f_{Y_{n/2}|S_i}$ denote the distribution and density functions of the median signal conditional on S_i. A similar argument to that of Milgrom and Weber (1982) shows that a monotonic symmetric pure strategy equilibrium bidding function in the n-th economy, b^n, is the solution to the following differential equation:

$$b^{n\prime}(s_i) F_{Y_{n/2}|S_i}(s_i|s_i) + b^n(s_i) f_{Y_{n/2}|S_i}(s_i|s_i) = s_i f_{Y_{n/2}|S_i}(s_i|s_i),$$

which satisfies the boundary condition $b^n(0) = 0$.This equation has a closed form solution that implies $b^n(s_i) \to b(s_i)$, where $b(s_i)$ satisfies the following equation for $s_i < 0.5$.[26]

$$b'(s_i) \frac{s_i}{1 - s_i} + b(s_i) \frac{2}{1 - s_i} = s_i \frac{2}{1 - s_i}, \tag{1}$$

[25] Milgrom (1981) provides the unique symmetric equilibrium strategy in the uniform price auction (see also Pesendorfer and Swinkels (1997)). An extension of the equilibrium of the first price auction described in Milgrom and Weber (1982) constitutes a symmetric equilibrium in a multiple unit discriminatory mechanism. It is the unique equilibrium in the class of monotonic symmetric equilibria in which bidders use pure strategies, but its uniqueness properties are more generally difficult to ascertain.

[26] Note that $F_{Y_{n/2}|S_i}(t|t) \to \min[1, \frac{t}{1-t}]$.

and $b'(s_i) = 0$ for $s_i > 0.5$. Together with the boundary condition, $b(0) = 0$, we get a unique expression for the limit of the bidding functions, b.

$$b(s_i) = \begin{cases} \frac{2s_i}{3} & \text{if } s_i < .5\text{, and} \\ \frac{1}{3} & \text{if } s_i \geq .5. \end{cases}$$

Conditional on $X = x$, we compute expected revenues by computing the average bid over winning signals. For large n, the distribution of 'winning' signals is approximately uniform over $[x/2, x]$. Hence, for $x < 0.5$ we get an approximate average winning bid of[27]

$$\frac{1}{0.5x} \int_{x/2}^{x} \frac{2s_i}{3} ds_i = \frac{x}{2}.$$

For $x \geq 0.5$, the average winning bid converges to:

$$\frac{1}{0.5x} \left(\int_{x/2}^{0.5} \frac{2s}{3} ds + \int_{0.5}^{x} \frac{1}{3} ds \right) = -\frac{x}{6} + \frac{2}{3} - \frac{1}{6x}.$$

Taking expectations over X, average revenue per unit converges to

$$\int_0^{0.5} \frac{x}{2} dx + \int_{0.5}^{1} \left(-\frac{x}{6} + \frac{2}{3} - \frac{1}{6x} \right) dx = 0.215.$$

Let us compare this to the revenue in a uniform price auction. There the Q^n highest bidders get objects and pay the $Q^n + 1$ highest bid. There is a symmetric equilibrium (which involves unique dominant strategies) where each bidder bids his personal valuation, that is, $b^n(s_i) = s_i$. This implies that the price is set to the valuation of the agent who has the $Q^n + 1$ highest signal. The price in a uniform auction thus converges to $\frac{x}{2}$ as $n \to \infty$. When taking expecations over x, we find that in the uniform auction the average revenue per unit converges to

$$\int_0^1 \frac{x}{2} dx = 0.25.$$

Thus, the revenue in large uniform price auctions is approximately 16% more than that of discriminatory auctions.

Also note that as Theorem 5 predicts, both mechanisms fail to extract all the surplus from the bidders even in the limit. The expected average value of the goods to winning bidders is $E[\frac{3X}{4}] = .325$.

Example 2 *Common Values.*

Next, consider the case of common values where $v(s_i, x) = x$.

A similar argument to the one used in the private value setup implies that the symmetric monotonic equilibrium bidding strategy in the discriminatory auction converges to the solution of the following differential equation:

[27] This and other calculations below follow from the Dominated Convergence Theorem and the point-wise convergence of b^n to b.

$$b'(s_i)\frac{2}{1-s_i}+b(s_i)\frac{s_i}{1-s_i}=\frac{4s_i}{1-s_i} \text{ for } s_i<0.5$$

and $b'(s_i)=0$ for $s_i>0.5$. Using the boundary condition of $b(0)=0$ we get the following characterization for the limiting bid function.

$$b(s_i)=\begin{cases}\frac{4s_i}{3} & \text{if } s_i<.5\text{, and}\\ \frac{2}{3} & \text{if } s_i\geq .5.\end{cases}$$

Note that this is twice the bidding function that we saw in the private values case, and hence expected revenue per unit converges to .43, which is twice that of the private value case.

In the uniform price auction, it is an equilibrium for an agent with a signal s_i to bid the expected value of X conditional on $Y_{\frac{n}{2}}=Y_{\frac{n}{2}+1}=s_i$ (see Milgrom (1981)). Straightforward calculations lead to

$$b^n(s_i)\rightarrow b(s_i)=2s_i.$$

Price per unit is given by the marginal bid $b\left(Y_{\frac{n}{2}+1}\right)$. This implies that conditional on $X=x$, the price converges to the true value of the good, x. When averaging over x we get revenues per unit of 0.5.

Again, the uniform price auction leads to revenue that is 16% more than that of a discriminatory auction.

4.3.1 Information Aggregation and Efficiency Under Flat Demands

The results that different auction formats lead to different revenues also has a side implication for the ability of different auction formats to aggregate the information of bidders.

Pesendorfer and Swinkels (1997) examine uniform price auctions in a common value setting which has features similar to the previous section. They address the issue of information aggregation in competitive markets. They argue that while the condition for information aggregation in the case of a single object identified by Milgrom (1979, 1981) (see also Wilson (1977)) is very strong, prices aggregate information in cases in which the number of goods increases with the number of bidders. Specifically, they show that if a double largeness condition holds: $Q^n\rightarrow\infty$ and $(n-Q^n)\rightarrow\infty$, then price converges to value almost surely.[28]

Our analysis shows that the results in Pesendorfer and Swinkels (1997) do not extend to discriminatory auctions.[29] Of course, in discriminatory auctions, there is no single price to identify. Nevertheless, one might expect that in a discriminatory price auction the average price paid would converge to the good's value. As Theorem 5 and Example 2 show, not only is the average price not a consistent estimator for the good's value, it is a biased estimator: the average price is biased downward even in the limit. This implies that the minimal price

[28] See Hong and Shun (2000) for a detailed look at how the rate of information aggregation in common value uniform auctions compares to that of standard single object auctions.

[29] Again, the concepts of dispersed and concentrated allocations consider the relative distribution of objects, and so do not correspond to the double largeness condition of Pesendorfer and Swinkels. However, there are still allocations that are both dispersed and doubly large.

being paid is also biased downward, and as the example shows, so is the median price. It also shows that the highest price does not converge to the good's value.

Finally we note the fact that in a discriminatory auction the average price does not converge to the asset's value may lead to efficiency loss. This occurs if the seller has a reservation value or has a cost of producing the items. In the next section we demonstrate that the discriminatory auction results in efficiency loss even when the selling decision is taken as given.

4.4 Efficiency in Uniform and Discriminatory Auctions

In the previous section we examined uniform and discriminatory auctions when agents have single unit demands. While that case is of interest, we now a develop a wider understanding of the behavior of uniform and discriminatory auctions under dispersion.

In order to develop a deeper understanding, we specialize to the case of private (but possibly correlated or affiliated) valuations. At the same time, however, we also generalize the previous setting in another direction. We allow bidders to have different valuations for different quantities.[30] In particular, a bidder may have a decreasing marginal valuation for additional objects.[31] [32] Each agent values m units. The value of the j-th unit for an agent who has a signal s_i is given by $v_j(s_i)$. We modify (A2) to:

(A2') Agents have private values and decreasing marginal utilities, that is, $v_k(s_i) \geq v_j(s_i)$ for any s_i and $j \geq k$ and $v_1(s_i) > v_m(s_i)$. $\{v_j(s_i)\}_{j=1}^{m}$ are differentiable and increasing in s_i. We normalize $v_1(s_i)$ to be equal to s_i.

The utility of an agent i who is awarded k units and pays t_i is

$$\sum_{j=1}^{k} v_j(s_i) - t_i$$

In uniform auctions with multi-unit demands, the impact a bidder has on price can affect equilibrium behavior.[33] This can persist even with large numbers of bidders. However, if there is a small uncertainty about the number of active bidders, then the problem disappears, as shown by Swinkels (2001). So, we follow Swinkels (2001) (see his definition 3) in assuming that

(A5) There is some probability $\tau > 0$ that each bidder is inactive. The inactivity is independent across bidders and is independent of X.

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[30] The previous analysis was essentially one of a flat demand, where a bidder's valuation was constant up to some limit and then zero thereafter. Although we assumed that the marginal valuation was constant, it is straightforward to extend the analysis to allow for a zero valuation after some limit.

[31] This is similar to the setting of Swinkels (2001), except that we maintain the structure of affiliated signals while Swinkels (2001) examines the case of independent signals. This results in some critical differences in behavior of the auctions.

[32] See Jackson and Swinkels (2001) for a proof of existence of equilibria for a wide range of auction formats in such private value settings.

[33] As is well-known, the fine details of how the price is set matters. For instance, the true Vickrey-Groves auction form does not encounter such difficulties, while a uniform auction format that is often used in practice does.

(A5) can be thought as adding an atom in the distribution of signals at $S_i = 0$. It is also equivalent to having a random number of participants. The randomness however is quite mild. Laws of large numbers imply that in the limit there are approximately $(1 - \tau)\, n$ active bidders. But, as mentioned above, this slight randomness helps in ruling out persistent price manipulation in large economies.

We consider uniform price and discriminatory auctions. In each auction each bidder submits m bids; we denote the j-th bid of bidder i by $b_{ij}(s_i)$. We order bids so that $b_{ij}(s_i)$ is non-increasing in j. In either auction format, the Q^n highest bids are each awarded a unit of the good. In the discriminatory auction bidders pay the sum of their winning bids. In a uniform price mechanism all bidders pay the same price per unit, which is the $Q^n + 1$-st highest bid.

It is important to note that in both cases our results will apply to *all* equilibria, including asymmetric ones or those in mixed strategies. The existence of a pure-strategy equilibrium or a monotonic equilibrium are important open problems in our setup. However, this existence does not affect our conclusions.

4.4.1 Allocations and Efficiency

We break our analysis into two parts. First we analyze and compare the auctions with regards to the allocations they induce and their asymptotic efficiency properties. After that, we return to the question of revenue comparisons.

Let

$$u(s_i, q_i) = \sum_{j=1}^{q_i} v_j(s_i).$$

This is the utility of agent i observing signal s_i and obtaining a number of objects q_i.

Asymptotic Efficiency

A sequence of allocations is said to be *asymptotically efficient* if in the limit the per unit loss of ex-ante total surplus compared to the efficient surplus converges to zero. That is, letting $q^{*n}(s)$ be an efficient allocation, q^n is asymptotically efficient if

$$E\left[\frac{\sum_i u_i(q_i^{*n}(S), S_i)}{Q^n} - \frac{\sum_i u_i(q_i^n(S), S_i)}{Q^n}\right] \to 0.$$

We focus on the case in which $\frac{Q^n}{n(1-\tau)} \to a < m$, where $a > 0$. This implies that the allocation will be dispersed, and that a non-trivial fraction of bidders receive less than their full demand.[34] We maintain this assumption for the remainder of Section 4.

[34] The case where $a \geq m$ is a trivial one where all (active) agents can simultaneously be satiated at a bid of zero. At the other extreme where $Q^n/n \to 0$, if the monotone likelihood ratio property holds, then under either of the auction formats the allocation is approximately efficient and concentrated. In that situation the full surplus is extracted and the choice of mechanism does not matter in determining either the allocation or revenue.

4.4.2 Efficient Allocations in Uniform Price Auctions

We first argue that:

Lemma 2 *Consider a setting satisfying (A1), (A2'), and (A5), and a sequence of equilibria of uniform price auctions. For any* $\varepsilon > 0$, *there exists a large enough* n *such that conditional on any state* X *the probability that any bidder* i *can influence the price by more than an* ε *is smaller than* ε.

Lemma 2 implies that in the limit the outcome of a uniform price auction is competitive. Price converges to the price that would occur if there was no asymmetric information and the allocation becomes efficient. Let p_c^n denote this price (the $Q^n + 1$-st highest valuation in the population) which we term the "competitive" price.

Theorem 6 *Consider a setting satisfying (A1), (A2'), and (A5), and any sequence equilibria of uniform price auctions. The equilibrium allocations are asymptotically efficient and the corresponding equilibrium prices converge to the competitive price in probability; that is,* $p^n - p_c^n \to 0$ *in probability (where* p^n *is the equilibrium price, the* $Q_n + 1$ *highest bid).*

Note that this applies to any sequence of equilibria, and not just symmetric ones. The key is that under (A5), the asymmetric strategies where some bidders bid 1 and others bid 0, for instance, are not equilibria. The fact that some bidders may be inactive give all bidders some chance of winning objects in equilibrium.

This theorem tells us that uniform auctions are well behaved in this dispersed allocation setting, providing efficient allocations and competitive prices. We now turn to the more muddied analysis of discriminatory auctions.

4.4.3 Inefficient Allocations in Discriminatory Auctions

The analysis of discriminatory auctions is trickier. To get some intuition as to why, note that in a sense the discriminatory auction is like an asymmetric auction (we discuss this in more detail below). For instance, with $n = m = 2 = Q^n$ each bidder's high bid competes with the other bidder's low bid, and vice versa. With a uniform auction, even with this sort of asymmetry, the incentives are reasonably straightforward as ones bid is unlikely to affect the price. However, with a discriminatory auction, one's bid always affects the price paid (if an object is won). This asymmetry means that bids are no longer monotonic in value when compared across bidders. For example a value of 1/2 on a second unit corresponds to a different signal and hence information about the potential bids of others, than a value of 1/2 on a first unit. This loss of monotonicity across bidders is the key reason why the discriminatory auction is inefficient even in the limit.

To simplify the exposition we add two new assumptions. Let M^n denote the efficient cutoff, i.e., the Q^n-th highest valuation. Let

$$min(s_i) = \sup \left\{ v \mid lim_n \text{Prob} \left(M^n \geq v | S_i = s_i \right) = 1 \right\}.$$

So, $\min(s_i)$ is the minimum of the support of M^n under the limiting distribution conditional on $S_i = s_i$.[35]

The MLRP and continuity assumptions imply that $\min(s_i)$ is continuous and non-decreasing.

(A6) In an efficient allocation, at least one type sees a positive probability of obtaining m units. That is, $\min(s_i) < v_m(s_i)$ for some s_i.

Since $\min(s_i)$ and $v_m(s_i)$ are continuous, and $\min(s_i)$ lies above $v_m(s_i)$ at $s_i = 0$ and below $v_m(s_i)$ for some s_i under (A6), it follows that there exists s^* so that $\min(s^*) = v_m(s^*)$.

We assume that

(A7) $\min(v_m(s^*)) < \min(s^*)$, for some s^* such that $\min(s^*) = v_m(s^*)$.

While (A6) is fairly mild, (A7) has a bit more to it. (A6) is essentially without loss of generality, as otherwise we can reset m to simply cover units that might be obtained. (A7) requires that $\min(s)$ is increasing between $v_m(s^*)$ and s^* (or at least comparing the endpoints). This means that signals convey some information about the support of the efficient cutoff value M^n, at least in the limit.

Theorem 7 states that under the above assumptions, the discriminatory auction always yields inefficient outcomes even in the limit.

Theorem 7 *Under assumptions (A1), (A2'), (A6) and (A7), any sequence of equilibria of discriminatory auctions fails to be asymptotically efficient.*[36]

The intuition behind the theorem is quite straightforward. If the allocation were to be efficient, some of the marginal bids would be coming from bidders who are bidding on their first unit (with value s_i), while others would be coming from bidders who are bidding on their last unit (with value $v_m(s_i)$). Under (A7), these two classes of bidders near the margin for an efficient allocation have different beliefs about the cutoff value. This provides for different bidding behavior of the same valuations. The resulting bids are not monotonic in values and the allocation fails to be efficient, even approximately and asymptotically.

We conclude this subsection with an example.

Example 3 *Inefficiency in Discriminatory Auctions.*

- $X \sim U[0.5, 1]$
- $S_i \sim U[0, X]$
- $Q^n = n$ and $m = 2$
- $v_2(s_i) = \alpha s_i$, where $0 < \alpha < 1$.

[35] Note that this may (and generally will) be above 0 even if M^n has full support for each n.

[36] This theorem also holds if (A5) is added, with effectively no changes to the proof, and so can be compared to Theorem 6 under (A6) and (A7).

In the limit there is a fraction $\frac{X-v}{X}$ of the bidders with a value of the first unit exceeding some value v, and a fraction $\frac{\alpha X-v}{\alpha X}$ of bidders who have a value for their second unit that exceeds v. The limit of the cutoff M^n, denoted M is the solution of

$$\frac{X-M}{X}+\frac{\alpha X-M}{\alpha X}=1\Rightarrow M \;=\; \frac{\alpha X}{1+\alpha}.$$

It follows that

$$\min\left(s_i\right)=\max\left\{\frac{\alpha}{2\left(1+\alpha\right)\right)},\frac{s_i\alpha}{1+\alpha}\right\}.$$

Assumption (A6) is satisfied since $\min\left(1\right)=\frac{\alpha}{\alpha+1}<\alpha$ together with the fact that $\min\left(0.5\right)=\frac{\alpha}{2(1+\alpha))}>\alpha/2$. We conclude that there exists a unique $s^*\in[0.5,1]$ for which $\min\left(s^*\right)=\alpha s^*$. Since $\min\left(s_i\right)=\frac{s_i\alpha}{1+\alpha}$ on $[0.5,1]$, it follows that $\min\left(\alpha s^*\right)<\min\left(s^*\right)$ and that assumption (A7) also holds.

5 Concluding Remarks

We have shown that whether or not the auction format matters in large societies is related to whether or not the allocation of objects is concentrated or dispersed. In addition, for certain cases we are able to provide tight bounds on the revenues raised in concentrated allocations, and discuss at length how allocations and revenue may differ under dispersed allocations. While this work points out the importance of the distinction between concentrated and dispersed allocations, it also points to important questions for future research, of which we now mention some obvious ones.

5.1 Comparison of Revenues in the Two Auctions

Although we have allowed for asymmetric mechanisms, we have worked under an assumption of some symmetry in information and preferences across bidders. While it is clear that removing this assumption will not impact the basic properties of concentrated and dispersed allocations, the symmetric setting was critical to results such as Theorem 5, which shows that with flat demands the uniform price auction leads to higher revenue per unit than discriminatory auctions by an amount that is bounded below. As is evident from an example of Maskin and Riley (2000), with asymmetries among bidders these revenue ranking can be reversed. As this question is an important one for a number of applications,[37] it will be important to untangle how asymmetries affect the relative performance of various auction formats under dispersed allocations.[38]

Also, let us also mention that the revenue ranking between the auctions depends in some other ways on the setting considered. For instance, it is natural to conjecture that

[37]For example, this has been an important issue for treasury auctions for many years. See Bikhchandani and Huang (1993) for an overview of some of the debate over use of uniform versus discriminatory auctions; and Binmore and Swiezbinski (2000) and Hortascu (2000) for some recent empirical investigations.

[38]Recent work by Pekec and Tsetlin (2002) shows that uncertain participation by bidders can be another important factor in ranking auctions. We have not faced this issue, as we have been working with large numbers. But clearly, understanding when large numbers of agents will participate is an issue of importance.

the asymptotic revenue for the discriminatory auction would be no higher than that of the uniform price auction in symmetric settings. The reason that this seems natural is that the uniform price auction leads to an asymptotically efficient allocation and the competitive price in the limit, while the discriminatory auction can lead to an inefficient allocation, and one might guess, that it then leads to a correspondingly lower price. This would dovetail nicely with the analysis of Section 4. The following example illustrates, however, that there are some additional issues to think about.

Example 4 *Higher Revenue from Discriminatory Auctions.*

Reconsider Example 1 with some modifications.

The information structure is the same: Each agent observes S_i uniform on $[0, X]$, where X is uniformly distributed on [0,1]. However, agents value two objects. The value for the first object is $K + s_i$ and the value for the second is s_i, where K is to be defined below. There are $3n/2$ objects for sale. So, when $K \geq 1$, the efficient allocation is that each bidder gets at least one object, and the $n/2$ bidders having the highest s_i's each get two objects.

The asymptotic expected revenue per object in the Vickrey auction is easy: it converges to $X/2$ and in expectation is .25.

In the discriminatory auction, for large enough K, the following is the limit of a sequence of equilibria: bidders' first bid is always 1/3; and the bidders' second bid is as described in Example 1.

This sequence of equilibria is asymptotically efficient[39] and also gives higher revenue in the limit than the Vickrey (or the uniform) auction. The revenue is as follows: in the limit 2/3 of the objects are sold at a price of 1/3 and 1/3 of them are sold at a price of .215; so this is an average price of .294. The average price in the Vickrey auction is .25.

Without providing full detail, let us sketch why this is the limit of a sequence of equilibria. If all the first bids are above the support of the low bids, then the second bids are still the limit of a sequence of equilibria since the n bidders end up bidding for the remaining $n/2$ items, exactly as in Example 1. So the argument is that for some large enough K and n, all bidders place their first bid at the top of the support of the lower bids. This simply requires that when looking conditional on any s_i, there is some minimum (bounded away from zero) of the conditional density of the cutoff bid falling near the top of the support (which must happen in an equilibrium). This means that even if a bidders sees a low s_i, that bidder still places some chance on high X's. Then for large K, lowering a bid by some ε below the top of the support of the expected cutoff will lower the payment by ε when winning, but loses a value of at least K with probability ε times the marginal probability lost which is bounded below.

This example shows, when combined with Theorem 5, that one cannot generally rank the auctions in terms of the asymptotic revenue they generate. Here, the downward sloping demands introduce large enough additional asymmetries between first and second object valuations to result in some interesting behavior and reversal in revenue ranking from what we saw before.

[39]The example can easily be modified to be inefficient, and so the ranking of revenues has no general relationship with asymptotic efficiency.

This example also provides a comment on Theorem 7 regarding the inefficiency of the discriminatory auctions. Here K is large enough so that first objects are effectively not competitive with second objects. It is almost as if there are two separate auctions going on, and this results in efficient allocations. This is a rather special case, but points out the importance of an assumption behind Theorem 7 - that some signals tell a bidder that he will not be getting any objects. That fails in the above example, where even bidders seeing the lowest signals are sure that they should get at least one object.

5.2 Revenue versus Efficiency

It is often natural to think about standard auctions where bidders can enter any number of bids and objects are allocated to the highest bids. We now point out, however, that even when this results in an efficient allocation, there may be other auction designs that lead to higher revenues, and hence there can be a very fundamental tension between efficiency and revenue maximization.

In auction design, there are several tensions between efficiency and revenue that have been noted in the literature. First, the commitment to a reservation fee can raise expected revenues while decreasing efficiency (e.g., see Myerson (1981)), as sometimes an object is not sold when it would be efficient to do so. Second, with asymmetric distributions of information, awarding the object to the bidder with the highest virtual utility (which maximizes revenue) may conflict with awarding the object to the bidder with the highest utility (which is efficient), as shown by Myerson (1981). Third, with heterogeneous objects, a seller may have an incentive to bundle objects together (Palfrey (1983), Jehiel and Moldavanu (2001)).

The example below points out that such an incentive to inefficiently allocate objects (in particular to bundle them) arises in situations where dispersed allocations are efficient, even in a case with homogeneous objects and independent symmetric type distributions. In particular we show that any mechanism which results in an allocation which is approximately efficient is dominated in terms of revenue by one that bundles goods and sells them in an inefficient manner. The intuition for this follows closely from the optimal nonlinear pricing literature (e.g., see Wilson (1990)) where there is often a tension between a monopolist's profit maximization and efficiency. This shows that even Vickrey mechanisms, or any other approximately efficient variation, lead to lower revenue than from bundling the goods and auctioning them in pairs. The benefits of inefficient bundling of goods is different from the benefits of inefficiency arising from use of a reservation price.

The following example is one where dispersed allocations are efficient, and any mechanism leading to an (approximately) efficient allocation provides less revenue than one which allocates the goods in a dispersed, but inefficiently bundled manner.[40]

Example 5 *Bundling*

S_i is distributed uniformly on $[0,1]$. There are $\frac{n}{2}$ indivisible objects to be allocated. Bidders have a value of s_i for a first object, a value of $\frac{s_i}{2}$ for a second object, and no value for any additional objects.

[40] As shown by Ausubel and Cramton (1999), a perfect resale market for goods leads to a revenue maximizing mechanism being an efficient one. So, it is important that perfect (costless) resale is not possible in this example.

First, consider any (approximately) efficient allocation, which for large numbers corresponds (approximately) to giving one object to each of the $\frac{n}{2}$ highest signal observers. Given the independent signals, revenue equivalence among individually rational mechanisms holds in this world (see Ausubel and Cramton (1995)), and so the expected revenue of any mechanism that results in this allocation converges to $\frac{1}{2}$ per object.

Next, consider the following inefficient allocation. Objects are bundled and only sold in pairs. The pairs of objects are awarded to the $\frac{n}{4}$ highest signal holders via a Vickrey auction. In this case, the price setting bidder for large n will have a signal of approximately $\frac{3}{4}$. That bidder's valuation for a pair of objects will be $\frac{3}{4} + \left(\frac{1}{2}\right)\frac{3}{4} = \frac{9}{8}$, and the revenue per object converges to $\frac{9}{16}$.

Although bundling leads to an inefficient allocation, it lead to an increase in revenue of over 6% compared to mechanisms leading to the efficient allocation.

In the case of dispersed allocations we know that the mechanism matters. We have made some progress here in comparing uniform and discriminatory auctions under different scenarios. However, as the last example (Example 5) shows, sellers might prefer auctions which bundle or allocate objects in ways beyond these two standard auctions. Moreover, this is not due to some externalities across objects, so that a combinatorial auction is of value. This is more directly tied to incentive compatibility.

Obtaining a better understanding of optimal mechanisms in such situations, as well as the tension between efficiency and revenue maximization, is a challenging but important open problem.

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6 Appendix

Proof of Lemma 1: Write

$$I^n + II^n \geq |Sur_i^n(s_i, s_i') - Sur_i^n(s_i', s_i')|$$

where I^n is the difference in the utility from the good received:

$$I^n = \left| \int q_i^n(s_i', s_{-i}) v(s_i, x) dF^n(s_{-i}, x|s_i) - \int q_i^n(s_i', s_{-i}) v(s_i', x) dF^n(s_{-i}, x|s_i') \right|$$

and II^n is the difference in expected payment

$$II^n = \left| \int t_i^n(s_i', s_{-i}) dF^n(s_{-i}|s_i) - \int t_i^n(s_i', s_{-i}) dF^n(s_{-i}|s_i') \right|.$$

Step 1: $I^n < (\alpha + \gamma)|s_i - s_i'| E[q_i^n(S) \mid S_i = s_i']$
(A2) ($\frac{dv}{ds_i} < \gamma$) implies that for any n

$$\left| \int q_i^n(s_i', s_{-i}) [v(s_i, x) - v(s_i', x)] dF^n(s_{-i}, x|s_i') \right| \leq \gamma |s_i - s_i'| E[q_i^n(S) \mid S_i = s_i']. \quad (2)$$

(A1) implies that:

$$|dF^n(s_{-i}|s_i) - dF^n(s_{-i}|s_i')| = \left| \int_x f^n(s_{-i}|x)(f(x|s_i) - f(x|s_i'))dx \right| \leq \alpha|s_i - s_i'| dF^n(s_{-i}|s_i').$$

Then since $1 \geq |v(s, x)|$, we deduce that:

$$\begin{aligned} &\left| \int q_i^n(s_i', s_{-i}) v(s_i, x) dF(s_{-i}, x|s_i) - \int q_i^n(s_i', s_{-i}) v(s_i, x) dF^n(s_{-i}, x|s_i') \right| \quad (3) \\ &\leq \alpha|s_i - s_i'| E^n[q_i^n(S) \mid S_i = s_i']. \end{aligned}$$

Hence, the claim in Step 1 follows from (2) and (3).
Step 2: $II^n \leq \alpha|s_i - s_i'| E^n[q_i^n(S) \mid S_i = s_i']$
Again using (A1), it follows that:

$$\left| \int t_i^n(s_i', s_{-i}) dF^n(s_{-i}|s_i) - \int t_i^n(s_i', s_{-i}) dF^n(s_{-i}|s_i') \right| \leq \alpha|s_i - s_i'| E^n[t_i^n(S)|S_i = s_i'],$$

and so $II^n < \alpha|s_i - s_i'| E^n[t_i^n(S)|S_i = s_i']$. From the interim individual rationality constraint and the fact that $1 \geq v(s_i, x)$, it follows that:

$$E^n[q_i^n(S)|S_i = s_i'] \geq E^n[t_i^n(S)|S_i = s_i'].$$

which concludes the argument. ∎
Proof of Theorem 1: We actually prove that

$$\frac{E\left(\sum_i Sur_i^n(S_i, S_i)\right)}{Q^n} = \sum_i \int_{[0,1]} \frac{Sur_i^n(s, s)}{Q_n} dF(s),$$

converges to 0, which implies convergence in probability since $Sur_i^n(S_i,S_i)$ is a nonnegative random variable.
Fix any small ε and by Lemma 1 find a δ such that $|s-s^*|<\delta$ implies that for all i and n

$$|Sur_i^n(s^*,s)-Sur_i^n(s,s)|\leq \varepsilon E^n\left[q_i^n(S)\,|S_i=s\right].$$

For the given ε and any i, let $A_i^n(\varepsilon)$ denote the set of types who expect to receive supply less than $\varepsilon\bar{q}_n$, that is:

$$A_i^n(\varepsilon)=\{s_i \text{ s.t. } E\left[\frac{q_i^n(S)}{\bar{q}_n}|S_i=s_i\right]<\varepsilon\}.$$

Pick $n>n'$ (where n' is defined by concentration) so that for any s that is in the support of f, there exists $s_i^n(s)\in A_i^n(\varepsilon)$ such that $|s-s_i^n(s)|<\delta$.[41] Incentive compatibility and Lemma 1 then imply that:

$$Sur_i^n(s,s)\leq Sur_i^n(s_i^n(s),s_i^n(s))+\varepsilon E^n\left[q_i^n(S)|S_i=s\right]. \tag{4}$$

By the definition of $A_i^n(\varepsilon)$ and $\overline{q}_n$, we can bound the surplus of types $s_i^n(s)\in A_i^n(\varepsilon)$:

$$\sum_i \frac{Sur_i^n(s_i^n(s),s_i^n(s))}{Q_n}\leq \sum_i E\left[\frac{q_i^n(S)}{Q_n}|S_i=s_i^n(s)\right]<\varepsilon. \tag{5}$$

Thus, from (4) and (5) it follows that for large enough n

$$\sum_i\int_{[0,1]}\frac{Sur_i^n(s,s)}{Q_n}dF(s)\leq \varepsilon+\sum_i\int_{[0,1]}\varepsilon\frac{E^n\left[q_i^n(S)|S_i=s\right]}{Q_n}dF(s). \tag{6}$$

Since

$$Q_n\geq \sum_i\int_{[0,1]}E^n\left[q_i^n(S)|S_i=s\right]dF(s),$$

it follows that

$$1\geq \sum_i\int_{[0,1]}\frac{E^n\left[q_i^n(S)|S_i=s\right]}{Q_n}dF(s). \tag{7}$$

(6) and (7) then imply that for large enough n

$$\sum_i\int_{[0,1]}\frac{Sur_i^n(s,s)}{Q_n}dF(s)\leq 2\varepsilon.$$

∎

The following lemma is useful in the proof of Theorem 2.

Lemma 3 *Let $s^n=1-n^{a-1}$. If (A3) is satisfied, then there exists some $b>0$ and some N such that $F_{Y_{-i}^n(1)|S_i}(s^n|s_n^i)<e^{-bn^a}$ for all $n>N$.*

[41]Note the following simple claim. Consider any density f on [0,1]. For any $\delta>0$, there exists $\gamma<1$ such that for every s in the support of f and every set $A\subset[0,1]$ with measure at least γ under f, the distance of s to A is less than δ. (To see this, simply subdivide the interval into into $\delta/2$ sized pieces, and set γ to be one minus the smallest positive probability given to a subinterval under f.)

Proof of Lemma 3: The claim is clear if $a \geq 1$, so consider $a < 1$. Write

$$F_{Y^n_{-i}(1)|S_i}(s^n|s^i_n) = \int_x F_{Y^n_{-i}(1)|X}(s^n|x)f(x|s^n_i)dx.$$

By (A3) it follows that for any x

$$F_{Y^n_{-i}(1)|X}(s^n|x) < (1 - \frac{n^{a-1}}{\beta})^{n-1}.$$

Thus,

$$F_{Y^n_{-i}(1)|S_i}(s^n|s^i_n) < (1 - \frac{n^{a-1}}{\beta})^{n-1}.$$

Since $(1 - \frac{n^{a-1}}{\beta})^{n-1} \to e^{-\frac{n^a}{\beta}}$, the claim follows. ▮

Proof of Theorem 2: First, we show that for any $a > 0$, there exists N' such that for all s_i and all $n > N'$

$$Sur^n\left(s_i, s_i\right) < 2\alpha n^{a-1}Q^n, \tag{8}$$

where α is identified in (A1).

Let $s^n = 1 - n^{a-1}$, and identify b and N from Lemma 3, such that $F_{Y^n_{-i}(1)|S_i}(s^n|s^i_n) < e^{-bn^a}$ for all $n > N$.

If $s_i < s^n$ for some $n > N$, then it follows that $Sur^n\left(s_i, s_i\right) < Q^n e^{-bn}$. Taking, N'' to be large enough so that $e^{-bn^a} < \alpha n^{a-1}$, we know that (8) holds for any $s_i < s^n$ when $n > N' = \max\{N, N''\}$.

Next, consider any $s_i \geq s^n$ for some $n > N'$. By Lemma 1

$$\left|Sur^n\left(s^n, s_i\right) - Sur^n\left(s_i, s_i\right)\right| < \alpha n^{a-1}Q^n.$$

Since $e^{-bn}Q^n \geq Sur^n\left(s^n, s^n\right) \geq Sur^n\left(s^n, s_i\right)$, this implies that

$$Sur^n\left(s_i, s_i\right) < e^{-bn}Q^n + \alpha n^{a-1}Q^n.$$

Since $n > N'$ we know that $e^{-bn^a} < \alpha n^{a-1}$ (recall the definitions of N'' and N'), and so (8) holds for any $s_i \geq s^n$. Thus, we have established (8).

So, let us now argue that for any $a > 0$, $\frac{n^{1-a}\sum_i Sur^n_i(S_i,S_i)}{Q^n}$ converges to 0 in probability.

Since Q^n goes to just one bidder,[42]

$$\int Sur^n(s_i, s_i)dF_{Y^n(1)}(s_i) \geq E\left[\sum_i Sur^n(S_i, S_i)\right].$$

So, from (8) it follows that for any $a > 0$ there exists N such that for any $n > N$

$$E\left[\sum_i Sur^n(S_i, S_i)\right] < 2\alpha n^{a-1}Q^n. \tag{9}$$

[42]This inequality needs not hold with equality, since it may be that payments are made by losing bidders.

Let us verify that this implies the theorem. First, we show that $\frac{n^{1-a}E\left[\sum_i Sur_i^n(S_i,S_i)\right]}{Q^n}$ converges to 0. Suppose the contrary. Then there exists $a' > 0$ and $\delta > 0$ such that

$$E\left[\sum_i Sur^n(S_i, S_i)\right] > \delta n^{a'-1}Q^n$$

for infinitely many n. Taking $a < a'$, this violates (9) for some large enough n. Thus, our supposition was incorrect and so $\frac{n^{1-a}E\left[\sum_i Sur_i^n(S_i,S_i)\right]}{Q^n}$ converges to 0. Since $Sur_i^n(S_i, S_i) \geq 0$, it follows that for any $a > 0$, $\frac{n^{1-a}\sum_i Sur_i^n(S_i,S_i)}{Q^n}$ converges to 0 in probability. ∎

Proof of Theorem 3

We bound $nSur^n(s_i, s_i')/Q^n$ from below (across n) for a bidder observing some $s_i > 1-\frac{a}{n}$ and reporting $s_i' = 1 - \frac{2a}{n}$, for some $a > 0$. By incentive compatibility, this gives a lower bound on $nSur^n(s_i, s_i)/Q^n$. We then show that there is a probability bounded from below that a winning bidder observes such a signal, which then implies that $nE\left[\sum_i Sur^n(S_i, S_i')\right]/Q^n$ is bounded below.

So, let us show that $nSur^n(s_i, s_i')/Q^n$ is bounded below for a bidder observing some $s_i > 1-\frac{a}{n}$ and reporting $s_i' = 1 - \frac{2a}{n}$, for some $a > 0$.

$$Sur(s_i, s_i') = \int q_i^n\left(s_i', s_{-i}\right) v\left(s_i, x\right) - t_i^n\left(s_i', s_{-i}\right) dF^n\left(x, s_{-i}|s_i\right) = I^n + II^n$$

where,

$$I^n = \int q_i^n\left(s_i', s_{-i}\right)\left[v\left(s_i, x\right) - v\left(s_i', x\right)\right] dF^n\left(s_{-i}, x|s_i\right)$$

and

$$II^n = \int \left[q_i^n\left(s_i', s_{-i}\right) v\left(s_i', x\right) - t_i\left(s_i', s_{-i}\right)\right] dF^n\left(x|s_i\right)$$

We first examine I^n. Since the good has a private value component, $v\left(s_i, x\right) - v\left(s_i', x\right) > \frac{\tau a}{2n}$. This implies that:

$$I^n > Q_n \frac{\tau a}{2n} F_{Y_{-i}^n(1)|S_i}\left(s_i' \mid s_i\right). \tag{10}$$

Given the state $X = x$ signals are independent, hence:

$$F_{Y_{-i}^n(1)|S_i,X}\left(s_i' \mid s_i, x\right) = F_{Y_{-i}^n(1)|X}\left(s_i' \mid x\right) = F_{S_i|X}\left(s_i' \mid x\right)^{n-1}$$

Using assumption (A3), we conclude that $F_{S_i|X}\left(s_i'|x\right) > 1 - \frac{2a\beta}{n}$ for all x which implies that that there exists some $\alpha^* > 0$ so that[43]

$$F_{Y_{-i}^n(1)|X}\left(s_i' \mid x\right) > \alpha^*$$

Thus, since $F_{Y_{-i}^n(1)|S_i}\left(s_i' \mid s_i\right) = \int_x F_{Y_{-i}^n(1)|X}\left(s_i' \mid x\right) df(x|s_i)$, it follows from (10) that $I^n/\frac{Q^n}{n}$ is bounded below.

[43]The expression $(1 - \frac{2a\beta}{n})^{n-1}$ converges to $e^{-2a\beta}$.

So, if $II^n \geq 0$, then the claim that $nSur^n(s_i, s_i')/Q^n$ is bounded below follows. So, we need only consider the case where $II^n < 0$. We establish the claim by showing that in this case there exists some $\tau' > 0$ so that $II^n > -\frac{\tau'}{n^2}$. From assumption (A1), we know that:

$$\left(1 - \frac{2\alpha a}{n}\right) f\left(x, s_{-i} \mid s'\right) < f\left(x, s_{-i}|s\right) < \left(1 + \frac{2\alpha a}{n}\right) f\left(x, s_{-i}|s_i'\right)$$

Hence we conclude that:

$$II^n > Sur^n\left(s_i', s_i'\right) - \frac{2\alpha a}{n} \int \left|q_i^n\left(s_i', s_{-i}\right) v\left(s', x\right) - t_i^n\left(s_i', s_{-i}\right)\right| f\left(x, s_{-i}|s_i'\right) d(x, s_{-i})$$

Individual rationality implies that $Sur^n\left(s_i', s_i'\right) > 0$, and so

$$II^n > -\frac{2\alpha a}{n} \int \left|q_i^n\left(s_i', s_{-i}\right) v\left(s', x\right) - t_i^n\left(s_i', s_{-i}\right)\right| f^n\left(x, s_{-i}|s_i'\right) d(x, s_{-i}) \tag{11}$$

We also know that:

$$\int \left|q_i^n\left(s_i', s_{-i}\right) v\left(s_i', x\right) - t_i^n\left(s_i', s_{-i}\right)\right| f^n\left(x, s_{-i}|s_i'\right) d(x, s_{-i})$$

$$= Sur^n\left(s_i', s_i'\right) + 2 \int \left\{q_i^n\left(s_i', s_{-i}\right) v\left(s_i', x\right) - t_i^n\left(s_i', s_{-i}\right)\right]\right\}^- f^n\left(x, s_{-i}|s_i'\right) d(x, s_{-i})$$

where Z^- equals $-Z$ when Z is negative and zero otherwise. Given that we are in the case where $II^n < 0$, and the fact that $\left|q_i^n\left(s_i', s_{-i}\right) v\left(s_i', x\right) - t_i^n\left(s_i', s_{-i}\right)\right| < Q^n$ we conclude that $Sur^n\left(s_i', s_i'\right) < \frac{2\alpha}{n}$. Since $s_i' = 1 - \frac{2a}{n}$ and payment per unit is bounded by 1, we conclude that:

$$\int \left\{q_i^n\left(s_i', s_{-i}\right) s' - t_i^n\left(s_i', s_{-i}\right)\right\}^- f^n\left(x, s_{-i}|s_i'\right) d(x, s_{-i}) < \frac{2a}{n} Q^n$$

Hence the claim follows from (11).

To complete the proof, we need to show that the probability that the winning signal is larger than $1 - \frac{a}{n}$ is bounded below. Again, (A3) implies that $F_{S_i|X}\left(1 - \frac{a}{n}|x\right) < 1 - \frac{a}{\beta n}$ for all x and $s_i > 1 - \frac{a}{n}$. Thus, $F_{Y^n(1)|X}\left(1 - \frac{a}{n}|x\right) < (1 - \frac{a}{\beta n})^n$, which converges to $e^{-a/\beta}$. So, there is a probability bounded below that the winning signal exceeds $1 - \frac{a}{n}$. ∎

Proof of Theorem 4: We need only prove the theorem for symmetric mechanisms. The extension to asymmetric mechanisms is then seen rather simply.[44] Suppose to the contrary that some sequence of dispersed asymmetric mechanisms leads to an expected surplus heading to zero. Construct a sequence of symmetric mechanisms by randomly labeling the agents in the n-th mechanism. This must lead to the same expected total surplus, and is still incentive compatible, safe, and dispersed. But this would contradict the fact that the result holds for symmetric mechanisms.

By dispersion, there exists $\varepsilon > 0$, $a > 0$, and for each n (taking a subsequence if necessary) a signal $s^n < 1 - 3\varepsilon$ such that

$$E(q_i^n(s^n, S_{-i})|S_i = s^n) > \frac{a}{n} Q_n.$$

[44] We thank an anonymous referee for pointing this out.

Since, by (A1) $\left|\frac{f(x|s_i)}{f(x|s_i')} - 1\right| < \alpha|s_i - s^n|$, it follows that $\left|\frac{f(s_{-i}|s_i)}{f(s_{-i}|s^n)} - 1\right| < \alpha|s_i - s^n|$. Thus, there exists ε such that for any $s_i \in [s^n + \varepsilon, s^n + 2\varepsilon]$

$$E(q_i^n(s^n, s_{-i})|S_i = s_i) > \frac{a}{2n}Q_n.$$

In a *safe* mechanism,

$$E[q_i^n(s^n, s_{-i})v(s^n, x) - t_i^n(s^n, s_{-i})|s_i] \geq 0, \tag{12}$$

for any $s_i \in [s^n + \varepsilon, s^n + 2\varepsilon]$. Since the good has a private value component, and by (A2) preferences are non-decreasing in signal, we know that there exists $\tau > 0$ such that $v(s_i, x) - v(s_i^n, x) > \tau\varepsilon$. Thus,

$$E[q_i^n(s^n, s_{-i})\{v(s_i, x) - v(s^n, x)\}|s_i] \geq \tau\varepsilon\frac{a}{2n}Q_n \tag{13}$$

Since

$$Sur^n(s_i, s^n) = E[q_i^n(s^n, s_{-i})v(s_i, x) - t_i^n(s^n, s_{-i})|s_i],$$

(12) and (13) imply that for any $s_i \in [s^n + \varepsilon, s^n + 2\varepsilon]$

$$Sur^n(s_i, s^n) \geq \tau\varepsilon\frac{a}{2n}Q_n.$$

By incentive compatibility,

$$Sur^n(s_i, s_i) \geq \tau\varepsilon\frac{a}{2n}Q_n$$

This shows that conditional on getting a signal in $[s_i^n + \varepsilon, s_i^n + 2\varepsilon]$ any agent expects a surplus that is bounded away from zero (relative to the per-capita supply of objects $\frac{Q^n}{n}$). The positive density of $f(s_i)$ implies that there is a minimum positive probability that signals fall in $[s^n + \varepsilon, s^n + 2\varepsilon]$ regardless of the choice of s^n, and hence the claim follows. ▮

Proof of Theorem 5: Let $s^*(x)$ be the solution to $F(s_i \mid x) = 1 - b$, where $\frac{Q^n}{n} \to b$. Note that under (A4), $s^*(x)$ is increasing and continuous in x.

Let $\overline{t}^{n,d}(s_i), \overline{t}^{n,u}(s_i)$ be the expected payment of a bidder conditional on observing signal $S_i = s_i$, in the discriminatory and the uniform price mechanisms respectively. The expected revenues in the respective mechanisms are

$$n\int \overline{t}^{n,l}(s_i)\,dF(s_i) \text{ for } l = u, d. \tag{14}$$

Given (A2) and (A4), an argument similar to that underlying Theorem 15 in Milgrom and Weber (1982) implies that

$$\overline{t}^{n,u}(s_i) \geq \overline{t}^{n,d}(s_i) \text{ for every } n \text{ and } s_i.$$

This implies that the expected revenue in the uniform price mechanism is no lower than in the discriminatory one. However, it does not guarantee that there exists a positive difference that is bounded below as we increase the number of bidders. To show that such gap exists, we argue that there is an interval in which $\overline{t}^{n,d}(s_i)$ becomes flat as n increases and that on the same interval, $\overline{t}^{n,u}(s_i)$ is increasing. Specifically, we show this for the interval $[s^*,1]$, where s^* is the max of the support of the random variable $s^*(X)$. Thus, $[s^*,1]$ is the interval in which $1-b \geq F(s_i|x)$ for almost every x.

Let us first show that $\overline{t}^{n,d}(s_i)$ becomes flat as n increases on the interval $[s^*,1]$. Note that $s^* < 1$ since the distribution of S_i conditional on $X = 1$ is described by a density function and $b < 1$ and so $1 > s^*(1) \geq s^*$. Also, since $f(s_i) > 0$ for all s_i, there is a positive (unconditional) probability that $S_i \in [s^*,1]$. For any $s_i \in (s^*,1]$, there is a large enough n so that a bidder observing $S_i = s_i$ has an arbitrarily high (prespecified) probability of observing one of the highest Q^n signals. This implies that an agent who has a signal $s_i > s^*$ can pretend to have a lower signal with a minimal effect on the probability of getting an object, for some large enough n. Hence, we conclude that for any $s_i > s_i' > s^*$

$$\overline{t}^{n,d}(s_i) - \overline{t}^{n,d}(s_i') \to 0. \tag{15}$$

This implies that while $\overline{t}^{n,d}(s_i)$ is increasing in s_i,[45] it converges to being flat over the interval $(s^*,1]$ as n increases.

Before we show that the sequence $\overline{t}^{n,u}$ converges pointwise on $(s^*,1]$ to a function t^* that is increasing over $(s^*,1]$, let us check that this together with the fact that $\overline{t}^{n,d}(s_i)$ converges to a constant function on the interval $[s^*,1]$ imply the theorem.

To do this, we first show that for any $\overline{s}_i \in (s^*,1]$ there exists an N such that for all $n > N$

$$\int_{s^*}^{1} \overline{t}^{n,d}(s_i)dF(s_i) < t^*(\overline{s}_i)\left(1 - F(s^*)\right). \tag{16}$$

Pick some $s_i' \in (s^*,\overline{s}_i]$ and find N such that

$$|\overline{t}^{n,u}(s_i') - t^*(s_i')| < (t^*(\overline{s}_i) - t^*(s_i'))/3$$

for all $n > N$. Find N' such that for any $n > N'$

$$|\overline{t}^{n,d}(1) - \overline{t}^{n,d}(s_i')| < (t^*(\overline{s}_i) - t^*(s_i'))/3.$$

It then follows that since $\overline{t}^{n,u}(s_i') \geq \overline{t}^{n,d}(s_i')$ that if $n > \max[N,N']$ then for any $s_i \in [s_i',1]$

$$\overline{t}^{n,d}(s_i) < t^*(\overline{s}_i) - (t^*(\overline{s}_i) - t^*(s_i'))/3.$$

(16) then follows since $\overline{t}^{n,d}$ is increasing.

Next, note that since t^* is increasing we can find $\varepsilon > 0$ and $\overline{s}_i \in (s^*,1]$ so that

$$t^*(\overline{s}_i)(1 - F(s^*)) < \varepsilon + \int_{s^*}^{1} t^*(s_i)dF(s_i).$$

Thus, by (16) it follows that there exists $\varepsilon > 0$ and N such that for all $n > N$

$$\int_{s^*}^{1} \overline{t}^{n,d}(s_i)dF(s_i) < \varepsilon + \int_{s^*}^{1} t^*(s_i)dF(s_i). \tag{17}$$

[45]This follows from incentive compatibility, (A2), and the strict MLRP-(A4).

By the Dominated Convergence Theorem

$$\int_{s^*}^{1} \overline{t}^{n,u}(s_i)dF(s_i) \to \int_{s^*}^{1} t^*(s_i)dF(s_i).$$

This coupled with (17) implies the theorem.

We now complete the proof of the theorem by showing that $\overline{t}^{n,u}$ converges pointwise on $(s^*, 1]$ to a function t^* that is increasing over $(s^*, 1]$. This is established through the following lemmas.

Lemma 4 *If $s_i > s^*$, then*

$$\overline{t}^{n,u}(s_i) \to t^*(s_i) = E(v(s^*(X), X) | S_i = s_i).$$

Lemma 5 *$\overline{t}^*(s_i)$ is increasing on $(s^*, 1]$.*

Proof of Lemma 4:
Step 1: For any s_i in the support of $s^*(X)$,

$$b^n(s_i) \to b^*(s_i) = v(s_i, x^*(s_i)),$$

where b^n is the equilibrium bidding function and $x^*(s_i)$ is the inverse of $s^*(x)$.[46]
For any s_i in the support of $s^*(X)$, X conditioned on $S_i = s_i$ and $Y_{-i}^n(Q^n) = s_i$ converges in probability to $x^*(s_i)$. Thus, since (see Milgrom (1981))

$$b^n(s_i) = E\left[v(s_i, X) | S_i = s_i, Y_{-i}^n(Q^n) = s_i\right], \tag{18}$$

the claim follows from the convergence in probability and the continuity and boundedness of v in x.
Step 2: For almost any x

$$E\left(b^n\left(Y_{-i}^n(Q^n)\right) | X = x\right) \to v(s^*(x), x).$$

For any x, $Y_{-i}^n(Q^n)$ conditioned on $X = x$ converges in probability to $s^*(x)$. By the continuity of $f(x)$, almost every x is in the interior of the support of X. Thus, given strict MLRP and the continuity of $F(s_i|x)$ in x, it follows that for almost every x, $s^*(x)$ is in the interior of the support of $s^*(X)$. So, from Step 1 it follows that for almost every x there is a neighborhood B of $s^*(x)$ such that $b^n(s_i) \to b^*(s_i)$ for all $s_i \in B$. Given that $Y_{-i}^n(Q^n)$ conditioned on $X = x$ converges in probability to $s^*(x)$, the probability of $Y_{-i}^n(Q^n)$ conditioned on $X = x$ has probability approaching 1 placed on B. Then from the dominated convergence theorem, $E\left(b^n\left(Y_{-i}^n(Q^n)\right) | X = x\right) \to E\left(b^*\left(Y_{-i}^n(Q^n)\right) | X = x\right)$, and given the fact that $b^*(s_i) = v(s_i, x^*(s_i))$ is bounded and continuous, the claim follows since $Y_{-i}^n(Q^n)$ conditioned on $X = x$ converges in probability to $s^*(x)$.
Step 3 For $s_i > s^*$, $\overline{t}^{n,u}(s_i) \to t^*(s_i)$.
We know that

$$\overline{t}^{n,u}(s_i) = E\left[I_{s_i \geq Y_{-i}^n(Q^n)} b^n\left(Y_{-i}^n(Q^n)\right) | S_i = s_i\right].$$

[46] By the strict MLRP, s^* is increasing in x and so x^* is well-defined on the support of $s^*(X)$.

For $s_i > s^*$, $I_{s_i \geq Y^n_{-i}(Q^n)}$ goes to 1, and so

$$\bar{t}^{n,u}(s_i) \to E\left[b^n\left(Y^n_{-i}\left(Q^n\right)\right)|S_i = s_i\right].$$

Then, given the conditional independence of signals conditional on X, we can write

$$\bar{t}^{n,u}(s_i) \to E\left[E\left(b^n\left(Y^n_{-i}\left(Q^n\right)\right)|X\right)|S_i = s_i\right].$$

So, from Step 2,

$$\bar{t}^{n,u}(s_i) \to E\left[v\left(s^*\left(X\right), X\right)|S_i = s_i\right],$$

which is the desired conclusion. ▮

Proof of Lemma 5:
By (A2) and since $s^*(x)$ is continuous and increasing, it follows that $v(s^*(x), x)$ is continuous and increasing. Assumption (A4) implies that the distribution of X conditional on $S_i = s_i$ is stochastically dominated by the distribution of X conditional on $S_i = s_i'$, where $s_i' > s_i$. The result then follows from the stochastic dominance. ▮

This concludes the proof of Theorem 5. ▮

Proof of Lemma 2: We first measure the influence a bidder has on prices by looking at the event in which he is able to push the price above some threshold $y \in [0, 1]$, given that the price would be below y in the absence of the bidder's bids. Let $Y_{-i}(l)$ denote the l-th highest bid excluding i's bids. Using this notation for bidder i to be able to push the price above y it must be that $Y_{-i}(Q^n - m) > y$ and $Y_{-i}(Q^n) < y$. Laws of large numbers imply that this happens with low probability. Specifically, the argument used in Lemma 9.2 in Swinkels (2001) shows that for any $\varepsilon > 0$ there exists some N_ε so that $n > N_\varepsilon$ implies that for any state X and bidder i:

$$Pr\left(Y_{-i}\left(Q^n - m\right) > y \text{ and } Y_{-i}\left(Q^n\right) < y|X\right) < \varepsilon$$

A sketch of the argument is as follows. There are $(n-1)m$ bids of bidders besides i. One can define random variables $\{Z_j\}_{j=1}^{(n-1)}$ that give the number of bids above y that are submitted by bidder j. The probability that i pushes the price above y when starting below is then bounded by $pr\left(\sum_{j=1}^{(n-1)} Z_j \in [Q^n - m, Q^n]\right)$. Conditional on X bids are independent and hence so are the $\{Z_j\}$. This implies that this probability is negligible for large n since m is finite.
The above inequality implies that if we fix a positive integer J, then for any $j \in \{1, \ldots, J\}$ there exists N_j such that for $n > N_j$

$$Pr\left(Y_{-i}\left(Q_n - m\right) > \frac{j}{J} \text{ and } Y_{-i}\left(Q_n\right) < \frac{j}{J}|X\right) < \frac{1}{J^2}.$$

Letting $N^* = \max_j N_j$, it follows that for all $n > N^*$

$$Pr\left(\exists j : Y_{-i}\left(Q_n - m\right) > \frac{j}{J} \text{ and } Y_{-i}\left(Q_n\right) < \frac{j}{J}|X\right) < \frac{J}{J^2} = \frac{1}{J}$$

Let p_1 denote the price if bidder i submits the maximal possible bid on all of his units and let p_0 denote the price if i bids zero on all of his units. The above inequality implies that for any J there exists N_J such that $Pr\left(p_1 - p_0 > \frac{2}{J}\right) < \frac{1}{J}$ for $n > N_J$. To conclude the Lemma let $J = 2/\varepsilon$. ∎

Proof of Theorem 6: First, let us show that the price converges to the competitive price. That is, for any $\delta > 0$, for all high enough n, $Pr\left(|p_c^n - p^n| > \delta\right) < \delta$.
Suppose that this is not the case, so that there exists $\delta > 0$ such that for all n (taking a subsequence if necessary), $Pr\left(p_c^n - p^n > \delta\right) > \delta$. [The case where p^n exceeds p_c^n is analogous.] Since both prices and values are bounded in $[0, 1]$ there exists some $\delta^* > 0$ and some interval $[a, a + \delta^*]$ such that for all n[47]

$$Pr\left(p^n < a \text{ and } p_c^n > a + \delta^*\right) > \delta^*.$$

Hence, for large enough n there is a probability bounded away from 0 that some bidder who values a unit by more than $a + \frac{\delta^*}{2}$ does not obtain that object and the price is less than a. Consider a deviation for such bidders to bid truthfully instead. This increases the expected number of units such a bidder gets, but may affect the clearing price. However, by Lemma 2 the price implications for such a deviation become negligible for large n. This yields a contradiction as it guarantees an extra unit at a profit bounded away from zero for such a bidder for large enough n, with negligible price impact.
Next, note that a similar argument to that above (again invoking Lemma 2) implies that although bidders may place some bids above or below their corresponding values in equilibrium, this can only be in cases where for large enough n, changing those bids to be equal to the corresponding values would have a negligible probability of affecting the equilibrium allocation.
The approximate efficiency follows from the convergence of price to the competitive one, and bidders bidding as if they bid their values. ∎

Proof of Theorem 7: The following lemma is useful.
Consider an n bidder discriminatory auction. Let p^n denote the minimal price paid in equilibrium, i.e., the $Q^n - th$ highest bid, and let $b_j^n(s_i)$ denote the bid in the n-th auction for a j-th object by a bidder observing s_i. Also, for any $1 \leq j \leq m$ and $\delta > 0$ let

$$A_{j\delta}^n = \left\{s_i | Pr\left(v_j\left(s_i\right) > M^n + \delta \text{ and } p^n > b_j^n\left(s_i\right)\right) > \delta\right\}$$

$$B_{j\delta}^n = \left\{s_i | Pr\left(v_j\left(s_i\right) < M^n - \delta \text{ and } p^n < b_j^n\left(s_i\right)\right) > \delta\right\}.$$

Lemma 6 *If for some j and δ it is true that for any N there exists $n > N$ such that either $Pr\left(A_{j\delta}^n\right) > \delta$ or $Pr\left(B_{j\delta}^n\right) > \delta$, then the discriminatory price auction is not asymptotically efficient.*

Proof: Suppose that $Pr\left(A_{j\delta}^n\right) > \delta$ for arbitrary large n; the case where $Pr\left(B_{j\delta}^n\right) > \delta$ is similar. Any signal that belongs to this set results in an efficiency loss of at least δ^2. The fact

[47] Partition $[0, 1]$ into $2/\delta$ intervals of size $\delta/2$ each, $\{[a_i, a_{i+1}]\}_{i=1}^{1/2\delta}$. The case in which $p^n < p_c^n - \delta$ implies that we can find some a_i so that $p^n < a_i$ and $p_c^n > a_{i+1}$. This implies that there exists some interval for which $Pr\left(p^n < a_i \text{ and } p_c^n > a_{i+1}\right) > \delta^2/2$. Hence, we let $\delta^* = \delta^2/2$.

that $Pr\left(A^n_{j\delta}\right) > \delta$ implies (appealing to the law of large numbers) that there is an ex-ante efficiency loss that is bounded away from zero when summing across bidders. ▮

Assume by contradiction that the outcome is asymptotically efficient.
First, let us argue that there exists δ_1 such that $b^n_1(s_i) < v_m(s^*) - \delta_1$ for all signals $s_i \in [v_m(s^*), v_m(s^*) + \delta_1]$ for large enough n. Using the continuity of $\min(s)$ and assumptions (A6)-(A7) we conclude that there exists an ε such that agents with a signal $s_i \in [v_m(s^*), v_m(s^*) + \varepsilon]$ can expect to win their first unit with a probability of at least ε by bidding $v_m(s^*) - \varepsilon$ for large enough n. This follows as prices never exceed M^n. In equilibrium, these bidders make an expected profit on their first unit of at least ε^2 and hence they must be bidding below their value by at least ε^2. If we let $\delta_1 = \varepsilon^2$ we get that $b^n_1(s_i) < v_m(s^*) - \delta_1$ for all signals $s_i \in [v_m(s^*), v_m(s^*) + \delta_1]$.
Next, note that there exists some $\delta_2 > 0$ so that $s_i \in [s^*, s^* + \delta_2]$ implies that $b^n_m(s_i) > v_m(s^*) - \delta_1$. This follows since bidders with signals close to (but above) s^* have a low but positive probability of winning their $m-th$ object in an efficient allocation, but that probability goes to zero in n if they underbid by any fixed amount. As a result they bid close to their reservation value on their m-th object for large n. Hence, there exists some $\delta_2 > 0$ so that $s_i \in [s^*, s^* + \delta_2]$ implies that $b^n_m(s_i) > v_m(s^*) - \delta_1$.
If we let $\delta^* = \min(\delta_1, \delta_2)$ we conclude that agents with signal $s_i \in [v_m(s^*), v_m(s^*) + \delta_1]$ bid on their first unit no more than $v_m(s^*) - \delta^*$ while agents with signal $s_i \in [s^*, s^* + \delta^*]$ bid on their $m-th$ unit at least this amount. Consider now the event that:

$$M^n \in (v_m(s^*) + 2\delta^*/5, v_m(s^*) + 3\delta^*/5)$$

Since we assume that the distribution of signals has a full support, this event occurs with some positive probability δ^{**}.
Consider two cases:
(i) $p^n \geq v_m(s^*) - \delta^*$: This event maps to A in Lemma 6. Bidders with types $s_i \in [v_m(s^*) + 4\delta^*/5, v_m(s^*) + \delta^*]$ are not awarded their first object despite the fact that their valuation exceeds the cutoff by more than $\delta^*/5$.
(ii) $p^n < v_m(s^*) - \delta^*$: This event maps to B in Lemma 6. Bidders with types $s_i \in [v_m(s^*), v_m(s^*) + \delta^*/5]$ are awarded their $m-th$ unit despite the fact that their valuation is lower than the cutoff by more than $\delta^*/5$.
Since either (i) or (ii) occurs with probability of at least $\delta^{**}/2$ for large enough n, Lemma 6 implies a contradiction. ▮