An Analysis of Advertising Wars*

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Abstract

Comparative advertising by one brand against another showcases its merits versus the demerits of the other. In a two-stage game with finitely many firms, firms decide first how much to advertise against whom. In the second stage, given the advertising configuration, firms compete as Cournot oligopolists. In the symmetric case, equilibrium advertising constitutes a clear welfare loss. Equilibrium advertising levels and advertising expenditures decline with rising advertising costs. Whereas equilibrium advertising levels decrease in the number of firms, aggregate advertising expenditures increase. In the asymmetric case, a variety of outcomes are possible in equilibrium depending on parameter values. We further relate effectiveness of advertising to proximity in product space. With two firms, comparative advertising and quality choice have similar effects. In a three-stage game, where firms choose first location (variety), then advertising levels (quality), and then quantities, we observe maximum horizontal product differentiation and minimum vertical product differentiation.

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1 Introduction

In the present paper, we address a fairly common phenomenon in today's advertising world that has been largely ignored in the literature, namely advertising wars where one brand compares itself favorably with a competing brand in various kinds of media, especially television. The reader is reminded of the cola wars and similar episodes of casual empiricism. We are mainly interested in the economic aspects of advertising and advertising wars and less in the details of the craft which is the subject of marketing research. The economics literature on advertising has focussed on two separate yet related issues: First, the competitive or anti-competitive effects of advertising. Second, the question whether advertising is too much or too little from the perspective of social welfare.

On the issue of competitive effects of advertising, Kaldor in his seminal 1950 contribution suggests a "concentration-effect" which, depending on the circumstances may operate at the manufacturing, wholesale or retail level. Advertising may facilitate or simply accelerate industry concentration through the creation of brands, differentiated products and "goodwill". Much of the later debate centers around the question whether and why incumbent firms have an advantage in advertising and can use it to put barriers to entry. See Bain (1956), Schmalensee (1974), Comanor and Wilson (1979) among others.

On purely theoretical grounds, it may be difficult to separate causes and effects of advertising, if the degree and effectiveness of advertising depend on market conditions which in turn are modified by advertising.[1] In the sequel, we shall assume an oligopolistic industry structure. This assumption permits comparative statics of the equilibrium levels of advertising with respect to the number of oligopolists. Thus to some extent, the effect of the market structure on the intensity of advertising can be studied in our model. The converse question of the causes of the prevailing market structure — and of advertising as a potential cause — is beyond the scope of our investigation.

[1] According to Dorfman and Steiner (1954), there will be no advertising under perfect competition and heavy advertising under imperfect competition. The specific conclusion of zero advertising under perfect competition need not obtain in other models; see e.g. Stigler and Becker (1977), Stegeman (1991).

Regarding the question of whether firms buy the socially optimal amount of advertising, the direct costs and benefits to the individual firm are evident. As a rule, it pays for its own advertisements. Its benefits derive from the fact that *ceteris paribus* its advertising effort affects the demand for its product positively. This may occur through a gain of market share at the expense of other firms or an increase of demand for the entire industry. [2] In the first case, in the absence of greater general demand, it is possible that individual demand shifts — which each are profitable for the respective firm — neutralize each other. Then the aggregate effect of advertising may be zero or insignificant, whereas firms incur substantial advertising costs. Such a Prisoner-Dilemma-like situation can arise in our model.

As for consumer welfare, the literature distinguishes between "informative advertising" and "persuasive" advertising. As Kaldor (1950) noted, this distinction is a matter of degree; whereas all advertising is persuasive in intention and informative in character, the motive of persuasion can be very strong in some cases and relatively weak in others. Informative advertising can be beneficial to consumers to the extent that it reduces search costs. It may inform consumers about existence of a product, its characteristics, its price or price distribution, the location of its vendors, etc.

The implications of persuasive advertising for consumer welfare are much more controversial. For instance, consider the partial equilibrium model of oligopoliostic competition that we are going to analyze. Suppose one ignores the usual concerns whether consumer surplus is an adequate measure of welfare and compares consumer surpluses for different levels of advertising. Are these valid welfare comparisons, if the shift of the demand curves is brought about by a shift in consumer tastes?

For their model with explicit utility functions which have advertising as an argument, Dixit and Norman (1978) argue that different market outcomes should be compared on the basis of constant preferences, say pre-advertising or post-advertising tastes. They find excessive advertising in terms of pre-advertising and post-advertising tastes. Fisher and McGowan (1979) argue that each outcome should be evaluated on the basis of the tastes that brought it about. See also the reply by Dixit and Norman (1978).

Fisher and McGowan's argument against Dixit and Norman's "shifting tastes"

[2] Advertising may also help deter entry as noted earlier.

approach hints at the main alternative, the "stable tastes" approach pioneered by Stigler and Becker (1977) and others, applied by Nichols (1985) and further developed by Becker and Murphy (1993).[3] Apart from some striking conclusions, the appeal of the stable tastes approach lies in the fact that it can rely on standard methods of economic analysis without resorting to explanations from other social sciences.

Although consumer preferences are not explicitly modelled in our partial equilibrium setting of oligopolistic competition, the distinction between shifting and stable tastes remains relevant. Namely, one can follow either Dixit and Norman or Fisher and McGowan when comparing total (consumer plus producer) surpluses. For some symmetric versions of our model, such a choice is unnecessary. We side with Fisher and McGowan, if we have to choose. Let us add that the prevailing taxonomy, shifting versus stable tastes, is neither exclusive nor exhaustive. As a conceivable third alternative, consider the case of two chemically and physically identical laundry detergents. Some consumers are well aware of this fact and indifferent between the two. If prices are significantly different, a consumer will choose the cheaper alternative. Otherwise, the consumer chooses at random or, as a repeat buyer, sticks with the previously chosen brand with occasional experimentation. All that advertising does is to raise consumer awareness of a brand so that consumers choose it or experiment with it with higher probability. But advertising does not make the brand more valuable to consumers. If the advertised brand happens to be unavailable in the store while the other brand is available, consumers are not disappointed, since they still assess the two brands as equally good. Last but not the least, there are indirect welfare effects of advertising beyond the scope of this paper and most of the literature. For example, if for whatever reasons, advertising could boost general consumer demand, then it might be helpful in preventing, mitigating or shortening economic recessions. As another example, if advertising happens to have a "concentration-effect", then one has to deal with pros and cons of industry concentration as well.

Our subject is advertising wars where explicitly or by implication one brand takes on a particular competing brand and vice versa. Such targeted ad campaigns are comparative in nature and suggest the superiority of one's own brand in some di-

[3] Stigler and Becker assume that consumers do not care about goods per se, but their attributes -or charecteristics in the tradition of Lancaster. Becker and Murphy postulate that advertisements and goods advertised are complements in stable metautility functions.

mension(s). But they may turn negative and stress the inferiority of the competing brand. Because of its surge in recent political campaigns and its rise in commercial campaigns, comparative advertising has received increasing attention in both popular and specialized media. However, the tactical details of comparative advertising such as optimal framing of ads are of secondary concern to us here - despite some potentially exciting economic, ethical, legal and marketing issues. What exactly constitutes "competitive bashing" or comparative advertising is a matter of degree and perception. For attempts to define negative advertising and assess its effectiveness, see James and Hensel (1991) and Sorescu and Gelb (2000). For our purposes, we need not and do not specify whether the content of a firm's ads is positive or negative, though we implicitly assume that the firm chooses whatever format works best for it. Thus in our reduced form model of advertising wars, a firm simply determines the amount of advertising against each of the other firms.

We develop a simple game theoretic model wherein targeted comparative advertising will have a positive impact on the demand of the advertised brand and a negative impact on the demand of the targeted competing brand. The player set consists of a finite number of firms. Our static game has two stages. In the first stage, the players decide whom to advertise against. In the second stage, given the advertising configuration determined in the first stage, they compete as Cournot oligopolists. In section 2, we introduce the second-stage Cournot model. In section 3, we develop the static two-stage model of comparative advertising and provide sufficient conditions for the existence of a subgame perfect Nash equilibrium. Under these conditions, the equilibrium advertising efforts are unique and positive. Later in the section, we revisit some of the welfare issues raised above.

In section 4, we consider the perfectly symmetric case where one can explicitly solve for subgame perfect Nash equilibrium. In this particular case, equilibrium prices and quantities are the same as in the Cournot model without advertising. Hence, each firm would gain if they refrained from advertising. We go on to study the sustainability of collusion with respect to advertising in the infinitely repeated two-stage game. Obviously, the firms could make further gains by colluding in advertising and output decisions. It turns out that under certain conditions, collusion in advertising and output can be supported as a subgame perfect equilibrium outcome of the infinitely repeated game whenever collusion in advertising alone can be supported. In the symmetric static model, we further find, among other things, that individual advertising levels decrease with the number of firms whereas aggregate advertising expenditures increase in the number of firms.

In section 5, we consider deviations from symmetry in a duopoly, focusing on one asymmetry at a time: asymmetry in intercepts, in advertising cost parameters and in advertising effectiveness parameters. For certain parameter values, the "smaller" firm, the one with the smaller demand intercept without advertising, not only advertises more but also produces more than the other firm. The firm with higher advertising cost parameter may advertise more than the other firm. For certain parameter constellations, advertising can serve as an entry deterrence device. Differences in advertising effectiveness can have similar effects.

In section 6, we explore possible links between the degree of product differentiation and effectiveness of comparative advertising. Section 7 contains concluding comments. Section 8 contains proofs and derivations.

2 The Cournot Model

In this section, we present the Cournot model, that will constitute the second stage of our two-stage advertising model. There are a finite number $n \geqslant 2$ of firms belonging to the set $N = \{1, 2, \ldots, n\}$. Generic firms are denoted i, j or k. p_i denotes firm i's price and q_i denotes its quantity. Firms produce imperfect substitutes. So the inverse demand function for firm i assumes the form

$$p_i = \alpha_i - q_i - \varepsilon \cdot \sum_{k \neq i} q_k \tag{1}$$

where $\alpha_i > 0$ and $0 < \varepsilon < 1$.

We note that the above demand function emerges from a quality augmented version of a standard utility function introduced by Vives (1999) and further employed by Billand and Bravard (2006). The utility function is given by

$$U(q_1, q_2, \ldots, q_n) = \sum_{i=1}^{n} \alpha_i \cdot q_i - \frac{1}{2}\left(\sum_{i=1}^{n} q_i^2 + 2\varepsilon \sum_{i=2}^{n} \sum_{k<i} q_i q_k\right) + I$$

where I stands for the expenditure of the consumer on other goods. If we maximize this utility function subject to a budget constraint

$$\sum_{i=1}^{n} p_i \cdot q_i + I \leq R$$

where R is the total income of the consumer, the first order condition determining the optimal consumption of the product sold by firm i on the market is:

$$\frac{\partial U}{\partial q_i} = \alpha_i - q_i - \varepsilon \cdot \sum_{k \neq i} q_k - p_i = 0$$

which gives rise to the inverse demand function (1).

For each firm, we assume a constant marginal cost c_i and a fixed cost C_i^F, resulting from the first-stage advertising decision so that its total costs are

$$C_i = c_i \cdot q_i + C_i^F. \tag{2}$$

With profits given by: $\pi_i = p_i \cdot q_i - C_i$, we solve the Cournot equilibrium in subsection 8.1 where we also develop the necessary and sufficient conditions (21) for positive equilibrium quantities. In the latter case, Cournot equilibrium quantities and profits are given by:

$$q_i = \frac{1}{2-\varepsilon}\left[\alpha_i - c_i\right] - \frac{\varepsilon}{2-\varepsilon} \cdot \frac{1}{2+(n-1)\varepsilon} \sum_{k \in N} \left[\alpha_k - c_k\right] \tag{3}$$

$$\pi_i = q_i^2 - C_i^F. \tag{4}$$

From (3), we obtain $\partial q_i / \partial \alpha_i = \frac{1}{2-\varepsilon} \cdot \left(1 - \frac{\varepsilon}{2+(n-1)\varepsilon}\right)$ and $\partial q_i / \partial \alpha_j = -\frac{1}{2-\varepsilon} \cdot \frac{\varepsilon}{2+(n-1)\varepsilon}$ for $i \neq j$. Since $\varepsilon < 2$, $\partial q_i / \partial \alpha_i > 0$ and $\partial q_i / \partial \alpha_j < 0$ for $i \neq j$.

For later reference, let us also report the collusive outcome. If firms collude to maximize joint profits, the quantity chosen and the profits earned are given by

$$q_i = \frac{1}{2(1-\varepsilon)}\left[\alpha_i - c_i\right] - \frac{\varepsilon}{2(1-\varepsilon)} \cdot \frac{1}{1-(n-1)\varepsilon} \sum_{k \in N} \left[\alpha_k - c_k\right]; \tag{5}$$

$$\pi_i = \frac{1}{2}\left[\alpha_i - c_i\right] q_i - C_i^F. \tag{6}$$

3 The Two-Stage Model of Advertising

In this section, we consider a two-stage game played by n firms where in the second stage, the firms engage in Cournot competition described by the model of the previous section.

In the first stage of the game, a (pure) strategy for firm i is a vector $s_i = (s_{i1}; .....; s_{i,i-1}; s_{i,i+1}; ...; s_{in})$ where $s_{ij} \in \mathbb{R}_+$ for each $j \in N \backslash \{i\}$. Throughout the paper, we restrict our attention to pure strategies. The set of first-period (pure) strategies of firm i is denoted by S_i. Since firm i has the option of advertising against each of the other $n-1$ competitors, $S_i = \mathbb{R}_+^{n-1}$. The set $S = S_1 \times S_2 \times S_3 \times \ldots \times S_n$ is the joint strategy space of all firms. A strategy profile $s = (s_{1,} s_{2,} \cdots, s_n) \in S$ defines an advertising outcome. Also denote $S_{-i} = \prod_{j \neq i} S_j$, the set of strategy profiles of all firms but i. For analytical reasons, we take s_{ij} to be the advertising levels which cause advertising expenditures $e_{ij} = s_{ij}^2 \cdot \phi_{ij}$, giving rise to the cost functions (9) below. Obviously, it would be possible to cast the model in terms of advertising expenditures instead.

To describe the effects of advertising, we introduce parameters $\theta_{ij}^i > 0$ and $\theta_{ij}^j > 0$ for all i and $j \neq i$. If firm i chooses the advertising level of $s_{ij} \geqslant 0$ against firm j, then there is a positive gain for firm i in the sense that its demand increases by $s_{ij} \cdot \theta_{ij}^i$. If conversely firm j chooses an advertising level s_{ji} against firm i, the latter's demand falls by $s_{ji} \cdot \theta_{ji}^i$. To be precise, advertising affects the intercepts α_i of the second-stage inverse demand functions (1) as follows:

$$\alpha_i = \beta_i + \sum_{j \neq i} s_{ij} \cdot \theta_{ij}^i - \sum_{j \neq i} s_{ji} \cdot \theta_{ji}^i \tag{7}$$

where $\beta_i > 0$ is exogenously given. Notice that (1) can be viewed as a demand relation where i's demand depends on its own price and the quantities produced by others.

$$q_i = \alpha_i - p_i - \varepsilon \cdot \sum_{k \neq i} q_k \tag{8}$$

We have stated that the second-stage fixed costs in (2) are advertising expenses determined by first-stage decisions. Specifically, we assume numbers $\phi_{ij} > 0$ for all i

and $j \neq i$. Then for firm i the cost of advertising is given by

$$C_i^F = \sum_{j \neq i} s_{ij}^2 \cdot \phi_{ij}. \tag{9}$$

In the second stage, the firms maximize profits, given their cost functions, the (inverse) demand functions and the strategy profile of the first stage. The Cournot outcome is unique and is given by (3) and (4) in case it is positive. It depends on the first-stage profile $s \in S$ via (7) and (9). Hence, for every strategy profile s of the first stage, there will be a unique Cournot-Nash equilibrium in the second stage. Let $q_i(s)$ denote the equilibrium quantity chosen by firm i given the strategy profile s of the first stage and $\pi_i(s)$ be the resultant equilibrium profits. In the remainder of this section, we utilize the explicit expressions for $q_i(s)$ and $\pi_i(s)$ to determine gains and losses from advertising (subsection 3.1), present sufficient conditions for the existence of the subgame perfect equilibrium (subsection 3.2), and assess equilibrium welfare (subsection 3.3).

3.1 Gains and Losses from Advertising

If one disregards advertising costs, then as a rule, an advertiser gains from comparative advertising and the targeted firm loses. Moreover, the profits of third parties tend to be affected as well. Each effect has two components. By (3), (4), (7), and (9), firm i's advertising impacts its own profits as follows:

$$\begin{aligned} \frac{\partial \pi_i(s)}{\partial s_{ij}} &= 2q_i(s) \cdot \left[\frac{\partial q_i(s)}{\partial \alpha_i} \cdot \frac{\partial \alpha_i}{\partial s_{ij}} + \frac{\partial q_i(s)}{\partial \alpha_j} \cdot \frac{\partial \alpha_j}{\partial s_{ij}} \right] - \frac{\partial C_i^F}{\partial s_{ij}} \\ &= 2q_i(s) \cdot \frac{1}{(2-\varepsilon)} \cdot \left[\left(1 - \frac{\varepsilon}{2+(n-1)\varepsilon} \right) \theta_{ij}^i + \frac{\varepsilon}{2+(n-1)\varepsilon} \theta_{ij}^j \right] \\ &\quad -2s_{ij} \cdot \phi_{ij} \end{aligned}$$

Hence there are two sources of gain from strategic advertising, namely, the fact that the advertising firm's demand is increasing and the fact that the demand of the firm advertised against is decreasing. The loss of course stems from the cost of advertising. Hence as long as the cost of advertising is sufficiently small, a firm always gains by advertising.

If firm i advertises against firm j, then the profits of the latter are affected as follows:

$$\begin{aligned}\frac{\partial \pi_j(s)}{\partial s_{ij}} &= 2q_j(s) \cdot \left[\frac{\partial q_j(s)}{\partial \alpha_j} \cdot \frac{\partial \alpha_j}{\partial s_{ij}} + \frac{\partial q_j(s)}{\partial \alpha_i} \cdot \frac{\partial \alpha_i}{\partial s_{ij}}\right] \\ &= -2q_j(s) \cdot \frac{1}{(2-\varepsilon)} \cdot \left[\left(1 - \frac{\varepsilon}{2+(n-1)\varepsilon}\right)\theta_{ij}^j + \frac{\varepsilon}{2+(n-1)\varepsilon}\theta_{ij}^i\right]\end{aligned}$$

There are also two sources of loss for the firm being advertised against, namely a decline of its own demand and a rise of its rival's demand.

Finally consider the third parties, that is firms k different from i and j. Such a firm's profit is affected by i's advertising against j as follows:

$$\begin{aligned}\frac{\partial \pi_k(s)}{\partial s_{ij}} &= 2q_k(s) \cdot \left[\frac{\partial q_k(s)}{\partial \alpha_j} \cdot \frac{\partial \alpha_j}{\partial s_{ij}} + \frac{\partial q_k(s)}{\partial \alpha_i} \cdot \frac{\partial \alpha_i}{\partial s_{ij}}\right] \\ &= 2q_k(s) \cdot \frac{1}{2-\varepsilon} \cdot \frac{\varepsilon}{2-(n-1)\varepsilon}\left[\theta_{ij}^j - \theta_{ij}^i\right]\end{aligned}$$

Thus third parties stand to gain if the loss of the firm incurred by the firm advertised against is greater than the gain of the advertising firm (where advertising costs are ignored).

3.2 Subgame Perfect Nash Equilibrium

We have solved for the Cournot Nash equilibrium in the second stage, obtaining quantities $q_i(s)$ and profits $\pi_i(s)$ as functions of the first-period strategy profile. To obtain the Subgame Perfect Nash Equilibrium (SPNE), we can use backward induction and solve for a Nash equilibrium $s^* \in S$ of the one-stage game based on continuation payoffs $\pi_i(s)$. To this end, put

$$V_{ij} = \frac{1}{2-\varepsilon} \cdot \left[\left(1 - \frac{\varepsilon}{2+(n-1)\varepsilon}\right)\theta_{ij}^i + \frac{\varepsilon}{2+(n-1)\varepsilon}\theta_{ij}^j\right]$$

for $i \neq j$. Then the previous expression $\partial \pi_i(s)/\partial s_{ij}$ reduces to

$$\frac{\partial \pi_i(s)}{\partial s_{ij}} = 2q_i(s)V_{ij} - 2s_{ij} \cdot \phi_{ij}.$$

Since $V_{ij} > 0$, the first order condition for maximization with respect to s_{ij} becomes

$$s_{ij} = q_i(s)V_{ij}/\phi_{ij} \tag{10}$$

where $q_i(s)$ is given by (3) and (7). If the resulting system of linear equations in the $n(n-1)$ variables s_{ij}, $i \neq j$ has a strictly positive solution s^*, then $(s^*, q(.)) = (s_i^*, q_i(.))_{i \in N}$ is a candidate for a SPNE of the two-stage game. For a wide range of parameter values, such an s^* exists and is unique and $(s^*, q(.))$ is a SPNE indeed.

Proposition 1 *Suppose (22) holds for all $i \in N$. Then for sufficiently large ϕ_{ij}, $i \neq j$, there exists a subgame perfect equilibrium $(s^*, q(.))$ of the two-stage game. Moreover, s^* is unique and satisfies $0 < s_{ij}^* < 1$ for all $i \neq j$.*

The proof and conditions (22) can be found in subsection 8.2. In essence, the proposition says that if the cost of advertising is sufficiently large, then there will be a positive but limited amount of advertising.

3.3 Equilibrium Welfare

Under certain conditions, like in the special case considered in the next section, the equilibrium advertising efforts of firm i against firm j and vice versa will offset each other. Hence, the advertising expenses constitute a net loss from a social welfare perspective. Neither producers nor consumers are ultimately affected by the possibility of comparative advertising, except for advertising costs. Still, each individual firm has an incentive to advertise up to a certain point. Such a scenario is reminiscent of the Prisoner's Dilemma and has been described verbally already by Pigou (1924, p. 176): "It may happen that the expenditures on advertising made by competing monopolists simply neutralize one another, and leave the industrial position exactly as it would have been if neither had expended anything. For clearly, if each of two rivals makes equal efforts to attract the favour of the public away from the other, the total result is the same if neither had made any effort at all." Consumers may have gained, of course, if the advertising efforts have created the perception of higher product quality. With comparative advertising, however, the utility effects may also neutralize each other and Pigou's verdict of wasteful advertising may be well be justified even if consumer welfare is taken into account.

In general, straightforward welfare conclusions may prove impossible. First, our existence result relies on the quadratic form of the cost terms $s_{ij}^2 \cdot \phi_{ij}$ in (9). This gives rise to second order terms

$$\partial^2 \pi_i(s)/\partial s_{ij}^2 = 2V_{ij}^2 - 2\phi_{ij}$$

which becomes negative for sufficiently large ϕ_{ij}. On the other hand, linear cost terms $s_{ij} \cdot \phi_{ij}$ would yield second order terms

$$\partial^2 \pi_i(s)/\partial s_{ij}^2 = 2V_{ij}^2 > 0.$$

Consequently best responses and SPNE would fail to exist. Secondly, advertising efforts need not be offsetting across firms and, therefore equilibrium quantities and prices may be affected by advertising as we show in Section 5. In that case, the welfare of firms can still be evaluated. But as explained in the introduction, the assessment of consumer welfare tends to be more problematic if consumers respond to comparative advertising. To the extent that comparative advertising is uninformative and merely persuasive, in other words is an attempt to manipulate consumer preferences, the question arises how to account for the part of the change in consumer surplus that is attributable to a shift of the consumers' willingness to pay. Thirdly, as a rule, the SPNE is only implicitly given which makes it difficult to draw firm conclusions. For these reasons, we specialize in the next section.

Before we turn to this special case, let us report (without the straightforward but lengthy analytical details) some general facts. In the spirit of Stigler and Becker (1977), Fisher and McGowan (1979), and Becker and Murphy (1993), let us take preferences as stable.[4] In concrete terms, let us measure welfare by means of total surplus that is the sum of producer surplus (industry profits) and consumer surplus. Obviously one source of inefficiency is the very fact that the firms behave as Cournot oligopolists and hence do not choose quantities at levels where price equals marginal cost. But that is a well-known fact and we are primarily interested in inefficiencies related to strategic advertising. So let us suppose that firms continue to behave as Cournot oligopolists but the social planner can now select levels of strategic advertising in a bid to maximize welfare. Then as a rule, the first order conditions for welfare

[4] For lack of an explicit description of preferences, this is an implicit assumption, though.

maximizing advertising levels differ from the system (10). In the special perfectly symmetric case of the next section, the welfare maximizing advertising levels turn out to be zero so that Pigou's verdict holds even if consumer welfare is taken into account. However, the welfare maximizing advertising levels are not always zero. The latter occurs for an almost symmetric model with differential impact of advertising as follows. There are constants $\alpha > 0$, $c > 0$, $\theta > 0$ and $\phi > 0$ such that for all i and $j \neq i : \beta_i = \alpha$, $c_i = c$, $\theta^i_{ij} = \theta^j_{ji} = 2\theta$, $\theta^j_{ij} = \theta^i_{ji} = \theta$, $\phi_{ij} = \phi$. Provided that the existence and uniqueness result of Proposition 1 holds, we find that for sufficiently large ϕ, the welfare maximizing advertising levels are positive but less than the SPNE levels. We conjecture that for large, but not too large ϕ, the welfare maximizing advertising levels may exceed the SPNE levels.

4 The Symmetric Case

In the perfectly symmetric case, one can explicitly solve for the SPNE. This permits detailed welfare analysis and a study of the effects and sustainability of collusion. The perfectly symmetric case is given by constants $\alpha > 0, c > 0, \theta > 0$, and $\phi > 0$ such that for all i and $j \neq i : \beta_i = \alpha$, $c_i = c$, $\phi_{ij} = \phi$, $\theta^j_{ij} = \theta^i_{ji} = \theta^i_{ij} = \theta^j_{ji} = \theta$. Because of the last identities, advertising has an off-setting impact, if firms choose equal advertising levels.

4.1 Subgame Perfect Nash Equilibrium

As a immediate consequence of Proposition 1, we obtain

Corollary 1 *Suppose $\alpha > c$. Then for sufficiently large ϕ, there exists a unique subgame perfect equilibrium $(s^*, q(.))$ of the two stage game with the property that $0 < s^*_{ij} < 1$ for all $i \neq j$.*

Under the symmetry assumptions, the SPNE can be easily found. Namely, we obtain $V_{ij} = \theta/(2-\varepsilon)$ for all $i \neq j$. Hence (10) reduces to $s_{ij} = q_i(s) \cdot \theta/((2-\varepsilon)\phi)$. Now set

$$s^*_{ij} = \frac{1}{2-\varepsilon} \cdot \frac{\alpha - c}{2+(n-1)\varepsilon} \cdot \frac{\theta}{\phi} \tag{11}$$

for $i \neq j$. With that choice, (7) and (3) become $\alpha_i = \alpha$ for all i and

$$q_i(s^*) = \frac{\alpha - c}{2 + (n-1)\varepsilon} \tag{12}$$

for all i, respectively, and the first order conditions (10) are satisfied. Therefore:

Corollary 2 *Suppose $\alpha > c$. Then for sufficiently large ϕ, the SPNE advertising levels are given by (11) and the equilibrium quantities are given by (12).*

This SPNE possesses several interesting properties. First and foremost, as expected the equilibrium quantities are unaffected by the benefit and cost parameter θ and ϕ and the corresponding equilibrium levels of advertising. The pairwise levels of advertising in (11) are linear in the benefit-cost ratio θ/ϕ. They are decreasing in n indicating that the smaller the number of firms, the more intensive is the advertising. Individual advertising expenditures are given by $(n-1)e$ where $e = \frac{1}{\phi}\left(\frac{1}{2-\varepsilon}\frac{\alpha-c}{2+(n-1)\varepsilon}\theta\right)^2$. Therefore, a rise in the cost parameter ϕ leads to a reduction of advertising levels and a decline in individual advertising expenditures. Individual advertising expenditures increase in the number of firms if n is less than $1 + 2/\varepsilon$ and decrease otherwise. While levels of advertising decline in the number of firms and individual advertising expenditures may decrease, total advertising expenditure $E = n(n-1)e$ always increases with the number of firms.

4.2 Beneficial Collusion

Firms have an incentive to collude with respect to advertising or output decisions, or both. Let us focus on advertising first. The equilibrium advertising levels (11) amount to a total advertising expenditure $E = n(n-1)e$. Since the equilibrium quantities (12) are unaffected by advertising, consumer welfare is unaffected by advertising. Hence the advertising expenditure E incurred by firms is a clear welfare loss. This would be a justification for a ban on comparative advertising. Germany, for instance, has traditionally banned comparative advertising, because it was considered a form of “improper competition”. Another way to avoid the welfare loss would be collusion among firms - which might be tolerated by regulators as long as consumer protection is not an issue. Compared to the SPNE outcome each firm can gain

$$g = (n-1)e$$

if they refrain from advertising, assuming that the second-stage Cournot competition persists. But given that no body else advertises, the firm has an incentive to deviate from the collusive outcome. If it does not advertise, its payoff is given by (4) and (12) with $C_i^F = 0$. Let h be the additional payoff the firm receives, if it chooses optimal levels of advertising. Since, $\partial\pi_i(s)/\partial s_{ij} = 2q_i(s)V_{ij}$ at $s = (0, \ldots, 0)$ which is again equal to $2q_i(s^*)V_{ij} > 0$ where $q_i(s^*)$ is given by (12), $h > 0$ holds. If ϕ is sufficiently high, one can calculate h precisely namely,

$$h = \frac{(\alpha - c)^2 \cdot \theta^2(n-1)}{(2 + (n-1)\varepsilon)^2((2-\varepsilon)^2\phi - (n-1)\theta^2)}$$

and in fact, $h > g$ is possible.

4.3 Sustainable Collusion

While there exist incentives to deviate from the collusive outcome in the two-stage game, collusion may be sustainable if the two-stage game is infinitely repeated. Suppose there are periods $t = 0, 1, 2, \ldots$. In each period the two-stage game is played. Suppose firm i's time preferences are given by a discount factor $\delta_i \in (0, 1)$. We can assume that in each period, firms choose advertising levels in the first stage and play a Cournot equilibrium in the second stage, given these advertising levels. By a standard argument there exists a subgame perfect Nash equilibrium in trigger strategies for the repeated game, without advertising along the equilibrium path, provided $g \cdot \frac{\delta_i}{1-\delta_i} > h$ or $\delta_i > \frac{h/g}{1+h/g}$ for all i. This means that the collusive outcome is sustainable for sufficiently large discount factors.

The firms have an incentive to collude with respect to advertising and output levels, since without advertising, the collusive payoffs given by (5) and (6) exceed Cournot payoffs given by (3) and (4), say by $g' > 0$. One can calculate g' precisely. Namely,

$$g' = \frac{(\alpha - c)^2(n-1)^2\varepsilon^2}{4(1 + (n-1)\varepsilon)(2 + (n-1)\varepsilon)}.$$

Then, $G = g + g'$ is a firm's payoff gain from two-fold collusion relative to its SPNE payoff in the two-stage game. Clearly, there exist stronger incentives for two-fold collusion than collusion in advertising only. But, one might suspect that the incentives for deviation are also stronger. It turns out that in the infinitely repeated two-stage game, the incentives to deviate from two-fold collusion can be less than the incentives to deviate from collusion in advertising only.

Let h' denote the additional payoff the firm receives, if it deviates optimally from collusion in the Cournot model without advertising. The key observation is that in each period, the two-stage game is played. Therefore, if in any period, a player deviates in the first stage, the other players have the opportunity for instantaneous retaliation in the second stage of the same period. To be concise, let us state

Proposition 2 *Suppose $h' \leqslant h$. If for some discount factor $\delta_i, i \in N$, collusion in advertising can be supported by a subgame perfect equilibrium in trigger strategies, then there exist discount factors $\delta_i' < \delta_i, i \in N$, such that collusion in advertising and output can be supported by a subgame perfect equilibrium in trigger strategies.*

The essence of the argument is as follows. Suppose that a firm considers a first deviation in the first stage of some period. Then the trigger strategies can be such that play reverts from collusion in output to Cournot equilibrium play in the second stage of that period and from collusion in advertising and output to the SPNE of the two-stage game in all subsequent periods. Hence, the one-time undiscounted gain from such deviation is equal to $h - g'$ which is less than h, the one-time undiscounted gain from such a deviation when only collusion in advertising was implemented, whereas the subsequent forgone benefits from collusion is equal to G in each period which is more than g.

Suppose next that in some period, a firm considers a first deviation in the second stage of that period. Then h', the one-time undiscounted gain from such a deviation is at most h. But the potential deviator forgoes the (undiscounted) collusive benefits G in every subsequent period when play reverts from collusion in advertising and output to the SPNE of the two-stage game. Since G is higher than g, the forfeited benefits from collusion in advertising only, it follows that smaller discount factors suffice to sustain collusion in advertising and output.

It can be shown that

$$h' = \frac{(\alpha - c)^2 (n-1)^2 \varepsilon^2}{16(1 + (n-1)\varepsilon)^2}.$$

Hence, $h' \leqslant h$ if and only if

$$\frac{(n-1) \cdot \varepsilon^2 \cdot (2 + (n-1)\varepsilon)^2 \cdot ((2-\varepsilon)^2 \phi - (n-1)\theta^2)}{16(1 + (n-1)\varepsilon)^2} \leqslant 1 \tag{13}$$

For instance, if $n = 2, \varepsilon = 0.5, \theta = 1$ and $\phi = 2$, the left hand side of inequality (13) is equal to 175/1052 and hence the condition is obviously satisfied. One can replace the condition $h' \leq h$ by the weaker condition $h' < h + g'$, if $\delta_i > 1/2$ is assumed for all i.

5 The Asymmetric Case

Next, we will analyze the case where we relax the assumption of complete symmetry. However, a comprehensive analysis is impossible since in order to get explicit solutions, we need to impose certain restrictions. We will consider a duopoly primarily because it is easy to check the second order conditions, and also because it yields sharp results. Hence, let $n = 2$. We will further assume throughout that $\theta_{12}^1 = \theta_{12}^2 = \theta_1$ as well as $\theta_{21}^1 = \theta_{21}^2 = \theta_2$ and $c_1 = c_2 = c$.

Then, $V_{12} = \dfrac{\theta_1}{2-\varepsilon}$ and $V_{21} = \dfrac{\theta_2}{2-\varepsilon}$. The second order conditions are given by

$$\frac{\partial^2 \pi_i(s)}{\partial s_{ij}^2} = 2 \cdot V_{ij}^2 - 2 \cdot \phi_{ij} < 0.$$

In order to ensure that second order conditions are satisfied, assume $V_{ij}^2 < \phi_{ij}$. Then, equilibrium levels of quantity and advertising are given by

$$\begin{aligned}
q_1 &= \left(\frac{\alpha_1 - c}{2-\varepsilon} - \frac{\varepsilon}{2-\varepsilon} \cdot \frac{\beta_1 + \beta_2 - 2 \cdot c}{2+\varepsilon} \right); \\
q_2 &= \left(\frac{\alpha_2 - c}{2-\varepsilon} - \frac{\varepsilon}{2-\varepsilon} \cdot \frac{\beta_1 + \beta_2 - 2 \cdot c}{2+\varepsilon} \right); \\
s_{12} &= \frac{q_1 \cdot V_{12}}{\phi_{12}}; \\
s_{21} &= \frac{q_2 \cdot V_{21}}{\phi_{21}}.
\end{aligned}$$

Further, at the equilibrium, the following condition will hold:

$$\frac{s_{12} \cdot \phi_{12}}{s_{21} \cdot \phi_{21}} = \frac{q_1 \cdot V_{12}}{q_2 \cdot V_{21}} \tag{14}$$

Given that it is practically impossible to have any meaningful analysis by considering all types of asymmetries simultaneously, we shall focus on one asymmetry at a time.

We shall consider an asymmetry in intercepts, namely, $\beta_1 > \beta_2$ with symmetry with regard to all other parameters. Then, (14) becomes

$$\frac{s_{12}}{s_{21}} = \frac{q_1}{q_2}.$$

In this case, it will always be true that advertising moves in the same direction as quantity if, starting from a position of symmetry we introduce any asymmetry. Below we show that depending on V_{ij}, either firm may end up with the higher levels of both advertising and quantity.

We shall consider asymmetry in advertising cost parameters, namely, $\phi_{12} > \phi_{21}$ with symmetry in other parameters. Then, (14) becomes

$$\frac{s_{12} \cdot \phi_{12}}{s_{21} \cdot \phi_{21}} = \frac{q_1}{q_2}.$$

Hence, it follows that

$$\frac{q_1 \cdot s_{21}}{q_2 \cdot s_{12}} > 1$$

and so we can rule out the possibility that $q_1 \leqslant q_2$ and $s_{12} \geqslant s_{21}$. But many other possibilities are open. We shall show below that starting from symmetry if we introduce an asymmetry, then quantity and advertising move in the same direction. But either the high cost firm or the low cost firm may produce and advertise more.

Finally, we shall consider an asymmetry in advertising effectiveness parameters, namely, $\theta_1 > \theta_2$ with symmetry in other parameters. Then, (14) becomes

$$\frac{s_{12}}{s_{21}} = \frac{q_1 \cdot \theta_1}{q_2 \cdot \theta_2}.$$

Hence it follows that

$$\frac{q_1 \cdot s_{21}}{q_2 \cdot s_{12}} < 1$$

and so we can rule out the possibility that $q_1 \geqslant q_2$ and $s_{12} \leqslant s_{21}$. But again many other possibilities are open almost all of which may occur as we show below. In fact, it is possible that advertising and quantity move in opposite directions if we introduce an asymmetry in advertising effectiveness starting from a position of symmetry.

In the case of asymmetries in cost and advertising effectiveness, note that in equilibrium, $q_1 + q_2 = \dfrac{2(\alpha - c)}{2 + \varepsilon}$ which is a constant. Hence, starting from a position of symmetry if we introduce any asymmetry, either the quantities will remain unchanged (which is never the case as we show below) or they will move in opposite directions. It will be never the case that they will move in the same direction.

5.1 Asymmetry in Intercepts

Let us consider an asymmetric duopoly where the firms differ with regard to the initial intercept of the demand function. Specifically, let $\beta_1 > \beta_2$. Assume symmetry in all other dimensions, namely, for all i and $j \neq i : c_i = c$, $\phi_{ij} = \phi$, $\theta_{ij}^j = \theta_{ji}^i = \theta_{ij}^i = \theta_{ji}^j = \theta$. We refer to firm 1 as the large firm and firm 2 as the small firm. In complete absence of advertising, the large firm will produce more. Namely,

$$\begin{aligned} q_1 &= \left(\frac{\beta_1 - c}{2 - \varepsilon} - \frac{\varepsilon}{2 - \varepsilon} \cdot \frac{\beta_1 + \beta_2 - 2 \cdot c}{2 + \varepsilon} \right); \\ q_2 &= \left(\frac{\beta_2 - c}{2 - \varepsilon} - \frac{\varepsilon}{2 - \varepsilon} \cdot \frac{\beta_1 + \beta_2 - 2 \cdot c}{2 + \varepsilon} \right); \\ s_{12} &= 0; \\ s_{21} &= 0. \end{aligned}$$

Let us now introduce the possibility of advertising. Note that

$$V_{12} = V_{21} = \left(\frac{\theta}{2 - \varepsilon} \right) V.$$

Hence sufficient conditions for maximization require

$$\frac{\partial^2 \pi_i(s)}{\partial s_{ij}^2} = 2 \cdot V^2 - 2 \cdot \phi < 0$$

where $i = 1, 2$ and $j \neq i$. In order to ensure that the second order conditions of maximization hold in the second stage, we assume $\phi > V^2$. Now, which firm would produce and advertise more depends entirely on the cost structure. We can demarcate two ranges of costs:

Case 1: $V^2 < \phi < 2 \cdot V^2$

In this "low cost" range, actually the smaller firm both produces and advertises more compared to the bigger firm. The optimal amounts chosen are given by

$$\begin{aligned} q_1 &= \frac{1}{2} \left[\frac{\beta_1 + \beta_2 - 2 \cdot c}{2 + \varepsilon} - \frac{(\beta_1 - \beta_2) \cdot \phi}{(2 - \varepsilon)(2 \cdot V^2 - \phi)} \right]; \\ q_2 &= \frac{1}{2} \left[\frac{\beta_1 + \beta_2 - 2 \cdot c}{2 + \varepsilon} + \frac{(\beta_1 - \beta_2) \cdot \phi}{(2 - \varepsilon)(2 \cdot V^2 - \phi)} \right]; \\ s_{12} &= \frac{q_1 \cdot V}{\phi}; \\ s_{21} &= \frac{q_2 \cdot V}{\phi}. \end{aligned}$$

The possibility of advertising results in reversion of sizes.

Case 2: $\phi > 2 \cdot V^2$

In this "high cost" range, the bigger firm both produces and advertises more compared to the smaller firm. The optimal amounts chosen are given by

$$\begin{aligned} q_1 &= \frac{1}{2}\left[\frac{\beta_1+\beta_2-2\cdot c}{2+\varepsilon}+\frac{(\beta_1-\beta_2)\cdot\phi}{(2-\varepsilon)(\phi-2\cdot V^2)}\right]; \\ q_2 &= \frac{1}{2}\left[\frac{\beta_1+\beta_2-2\cdot c}{2+\varepsilon}-\frac{(\beta_1-\beta_2)\cdot\phi}{(2-\varepsilon)(\phi-2\cdot V^2)}\right]; \\ s_{12} &= \frac{q_1\cdot V}{\phi}; \\ s_{21} &= \frac{q_2\cdot V}{\phi}. \end{aligned}$$

In other words, unless advertising costs are sufficiently high, the smaller firm would advertise more aggressively than the bigger firm.

We illustrate using a specific example and reaction curves in Figure 1. Let us start with a situation of complete symmetry. Let $\beta_1 = \beta_2 = \alpha = 6$, $c_1 = c_2 = c = 1$, $\varepsilon = 0.5$, $\theta_{12}^1 = \theta_{21}^1 = \theta_{12}^2 = \theta_{21}^2 = \theta = 3$. Let $\phi_{12} = \phi_{21} = \phi = 6$. The reaction curve of firm 1 is indicated by a blue line and the reaction curve of firm 2 is indicated by a red line. Then the Cournot equilibrium is given by the intersection of the two reaction curves at point A. Each firm produces an equilibrium amount of 2 and advertises an amount 0.67. Now, let us introduce a small asymmetry by assuming $\beta_1 = 6.1$, $\beta_2 = 6$. Note that the parameter are consistent with the range given by Case 1. If the levels of advertising did not change, then this would involve an rightward shift of the reaction curve and a shift of the equilibrium point from A to B. However in response to the above change in intercepts, advertising levels change. Firm 1 advertises an amount of 0.64 and firm 2 advertises an amount 0.71. Hence the reaction curve of firm 1 shifts leftward (to an extent that the shift completely counteracts the original rightward shift and hence the net shift is also leftward) and that of firm 2 shifts rightward. So, the equilibrium moves to point C and firm 1 produces an amount 1.92 and firm 2 produces an amount 2.12.

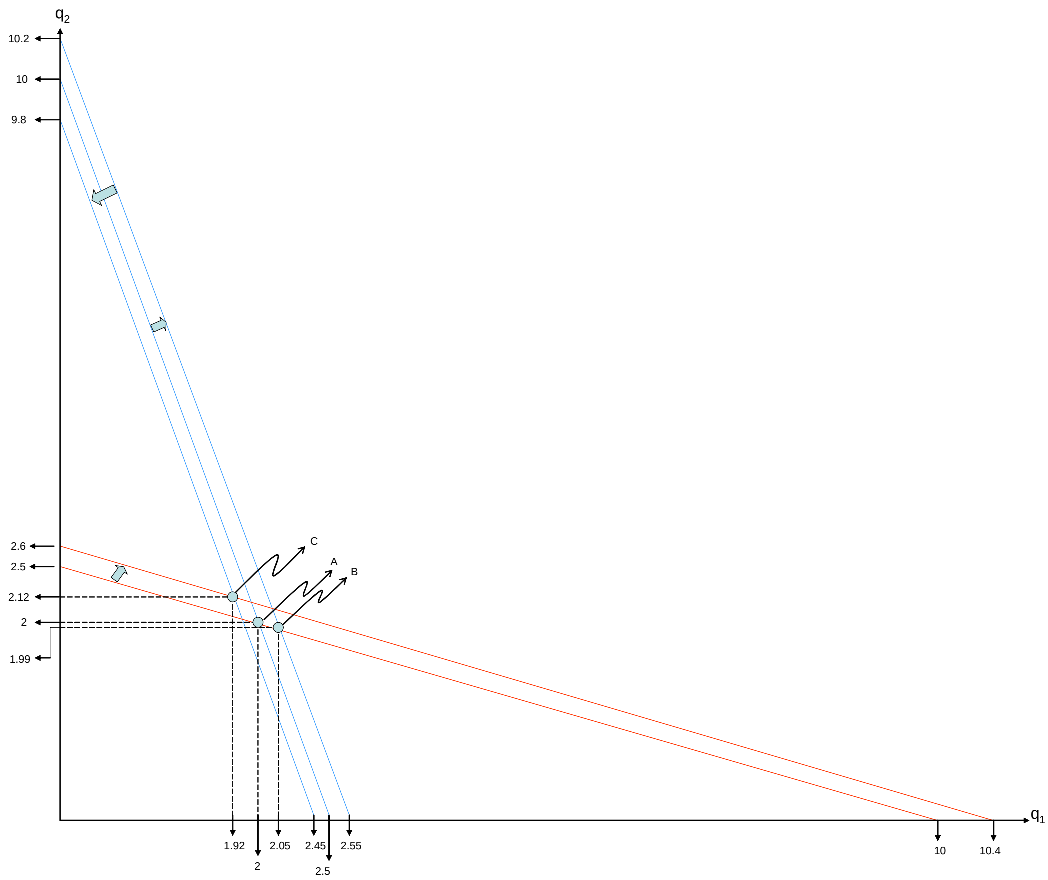

Figure 1

Next consider the case when $\phi_{12} = \phi_{21} = \phi = 9$ and other parameter values remain unchanged. Consider a similar change in intercepts. Now the parameters are consistent with Case 2. We illustrate this in Figure 2 and we use a similar labelling. The points A and B are identical to analogous points in Figure 1. But the point C is different. Now, in the case with perfect symmetry both firms advertise an amount 0.44. Once we introduce change in the intercepts, firm 1 advertises at 0.52 and firm 2 at 0.38. Consequently, the reaction curve of firm 1 moves even further rightward and that of firm 2 moves leftward. The equilibrium moves to point C and firm 1 produces an amount 2.32 and firm 2 produces an amount 1.72.

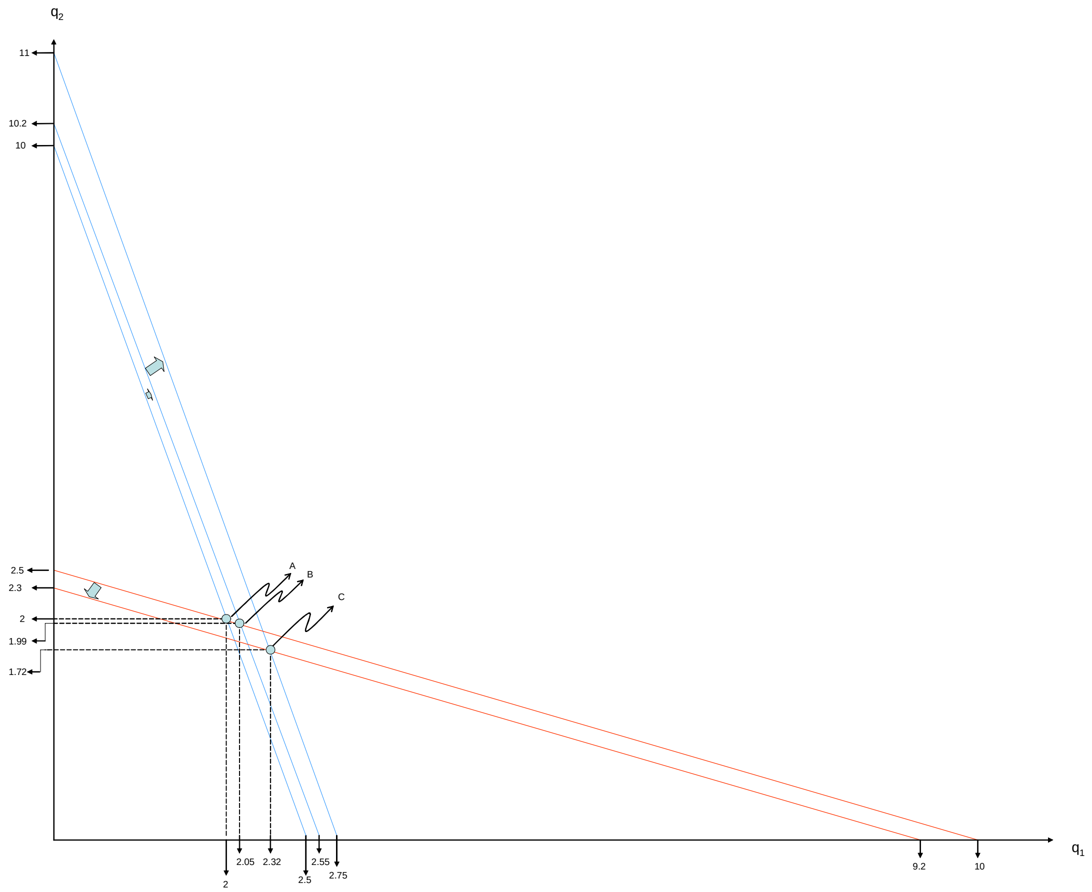

Figure 2

5.2 Asymmetry in Advertising Cost Parameters

Let us consider a "low cost" firm and a "high cost" firm. Specifically, let $\phi_{12} > \phi_{21}$. Firm 1 is the "high cost" firm and firm 2 is the "low cost" firm. Assume symmetry in all other dimensions, namely, for all i and $j \neq i : \beta_i = \alpha$, $c_i = c$, $\theta^j_{ij} = \theta^i_{ji} = \theta^i_{ij} = \theta^j_{ji} = \theta$. In complete absence of advertising both firms will produce exactly the same amount, namely,

$$
\begin{aligned}
q_1 &= \frac{\alpha - c}{2 + \varepsilon}; \\
q_2 &= \frac{\alpha - c}{2 + \varepsilon}; \\
s_{12} &= 0; \\
s_{21} &= 0.
\end{aligned}
$$

Now, let us introduce the possibility of advertising. We have

$$V_{12} = V_{21} = \left(\frac{\theta}{2-\varepsilon}\right) = V \text{ (say)}.$$

Also, in the second stage, sufficient conditions for maximization require

$$\frac{\partial^2 \pi_i(s)}{\partial s_{ij}^2} = 2 \cdot V^2 - 2 \cdot \phi_{ij} < 0$$

where $i = 1, 2$ and $j \neq i$. In order to ensure that the second order conditions of maximization hold in the second stage, we assume

$$\phi_{12} > \phi_{21} > V^2.$$

Once we solve for SPNE, the expressions are rather complicated. The SPNE are given by

$$\begin{aligned}
q_1 &= \frac{2(\alpha - c)}{(2+\varepsilon)} \cdot \left[\frac{\phi_{12}(\phi_{21} - 2 \cdot V^2)}{\phi_{12}(\phi_{21} - 2 \cdot V^2) + \phi_{21}(\phi_{12} - 2 \cdot V^2)}\right]; \\
q_2 &= \frac{2(\alpha - c)}{(2+\varepsilon)} \cdot \left[\frac{\phi_{21}(\phi_{12} - 2 \cdot V^2)}{\phi_{12}(\phi_{21} - 2 \cdot V^2) + \phi_{21}(\phi_{12} - 2 \cdot V^2)}\right]; \\
s_{12} &= \frac{q_1 \cdot V}{\phi_{12}}; \\
s_{21} &= \frac{q_2 \cdot V}{\phi_{21}}.
\end{aligned}$$

However, once we take ratios, the expressions simplify quite a bit. Namely,

$$\begin{aligned}
\frac{q_1}{q_2} &= \frac{\phi_{12}}{\phi_{21}} \cdot \left[\frac{\phi_{21} - 2 \cdot V^2}{\phi_{12} - 2 \cdot V^2}\right]; \\
\frac{s_{12}}{s_{21}} &= \left[\frac{\phi_{21} - 2 \cdot V^2}{\phi_{12} - 2 \cdot V^2}\right].
\end{aligned}$$

We will consider the following cost ranges.

Case 1: $\phi_{12} > \phi_{21} > 2 \cdot V^2 > V^2$

Both cost parameters lie in the "high cost" range. We find $s_{21} > s_{12}$. Hence indeed we get the expected result, namely the high cost firm advertises less and the low cost firm advertises more. Also, $q_1 < q_2$, namely, the high cost firm produces less than the low cost firm.

Case 2: $2 \cdot V^2 > \phi_{12} > \phi_{21} > V^2$

Both cost parameters lie in the "low cost" range. We find $s_{21} < s_{12}$. Now the high cost firm advertises more and the low cost firm advertises less. Also, $q_1 > q_2$, namely, the high cost firm produces more than the low cost firm.

Case 3: $\phi_{12} > 2 \cdot V^2 > \phi_{21} > V^2$

Finally, consider the case where one of the cost parameters is high and the other one low. It immediately follows that

$$\frac{1}{\phi_{12}} + \frac{1}{\phi_{21}} < \frac{3}{2 \cdot V^2}.$$

This time the interior solution yields a negative result for the quantity and advertising level of one of the firms and hence we obtain a boundary solution. One of the firms drops out of the market and the other becomes a monopoly but still has to advertise to keep the other firm out. Advertising takes the form of entry deterrence. The high cost firm will drop out if

$$\frac{1}{\phi_{12}} + \frac{1}{\phi_{21}} < \frac{1}{V^2} < \frac{3}{2 \cdot V^2}.$$

The low cost firm will drop out if

$$\frac{1}{V^2} < \frac{1}{\phi_{12}} + \frac{1}{\phi_{21}} < \frac{3}{2 \cdot V^2}.$$

Again, if cost parameters are not sufficiently high, we get seemingly counter-intuitive results.

We illustrate the results with an example using reaction curves in Figure 3. Let us start with a situation of complete symmetry. Let $\beta_1 = \beta_2 = \alpha = 6$, $c_1 = c_2 = c = 1$, $\varepsilon = 0.5$, $\theta_{12}^1 = \theta_{21}^1 = \theta_{12}^2 = \theta_{21}^2 = \theta = 3$. Let $\phi_{12} = \phi_{21} = \phi = 6$. The reaction curve of firm 1 is indicated by a blue line and the reaction curve of firm 2 is indicated by a red line. Then the Cournot equilibrium is given by the intersection of the two reaction curves at point A. Each firm produces an equilibrium amount of 2 and advertises an amount 0.67. Now, let us introduce a small asymmetry by assuming $\phi_{12} = 6.1$, $\phi_{21} = 6$. The parameter are consistent with the range given by Case 2. If the levels of advertising did not change, the reaction curves do not change and the equilibrium point remains at A. However in response to the above change in intercepts, advertising levels change. Firm 1 advertises an amount of 0.68 and firm 2 advertises an amount 0.64. Hence the reaction curve of firm 1 shifts rightward and that of firm 2 shifts leftward. So, the equilibrium moves to point B and firm 1 produces an amount 2.07 and firm 2 produces an amount 1.93.

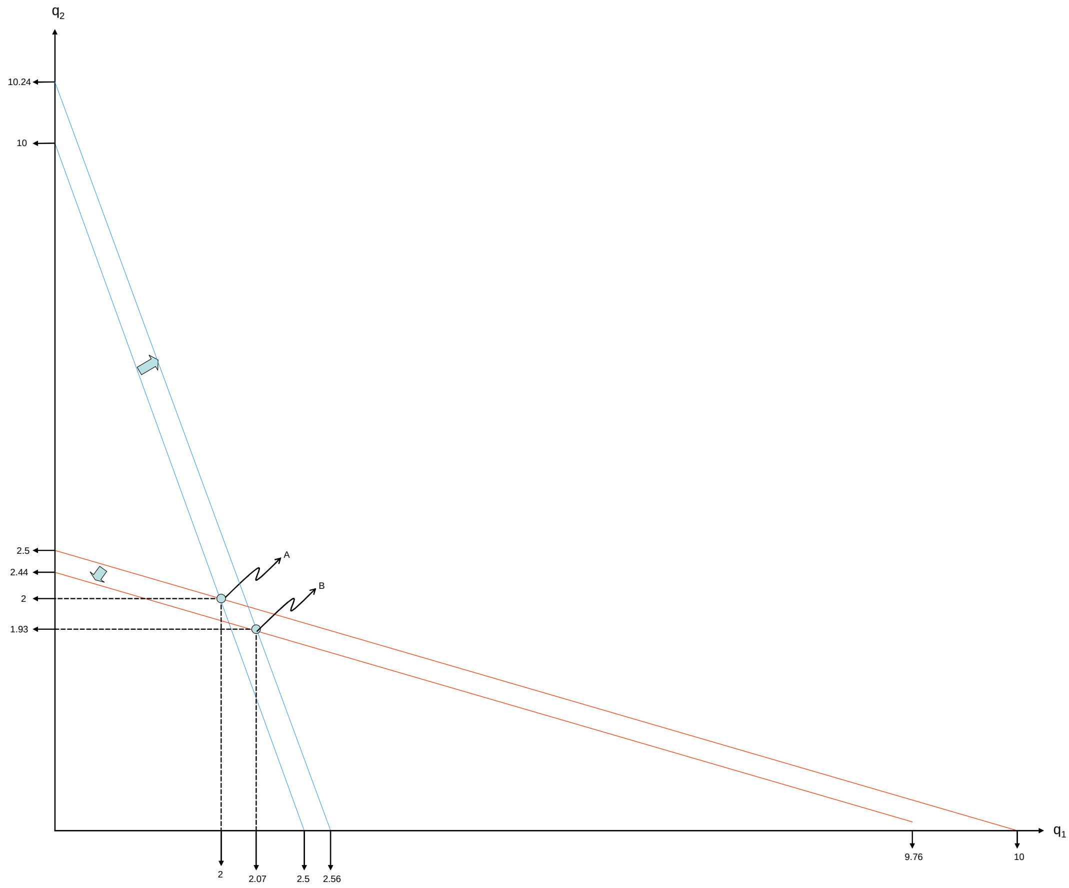

Figure 3

Next consider the case when initially $\phi_{12} = \phi_{21} = 9$ and thereafter we increase ϕ_{12} to 9.1. All other parameter values remain unchanged. Now the parameters are consistent with Case 1. We illustrate this in Figure 4 and we use a similar labelling. The points A is identical to analogous points in Figure 3. But the point B is different.

Now, in the case with perfect symmetry both firms advertise an amount 0.44. Once we introduce change in the intercepts, firm 1 advertises at 0.42 and firm 2 at 0.46. Consequently, the reaction curve of firm 1 moves leftward and that of firm 2 moves rightward. The equilibrium moves to point B and firm 1 produces an amount 1.92 and firm 2 produces an amount 2.08.

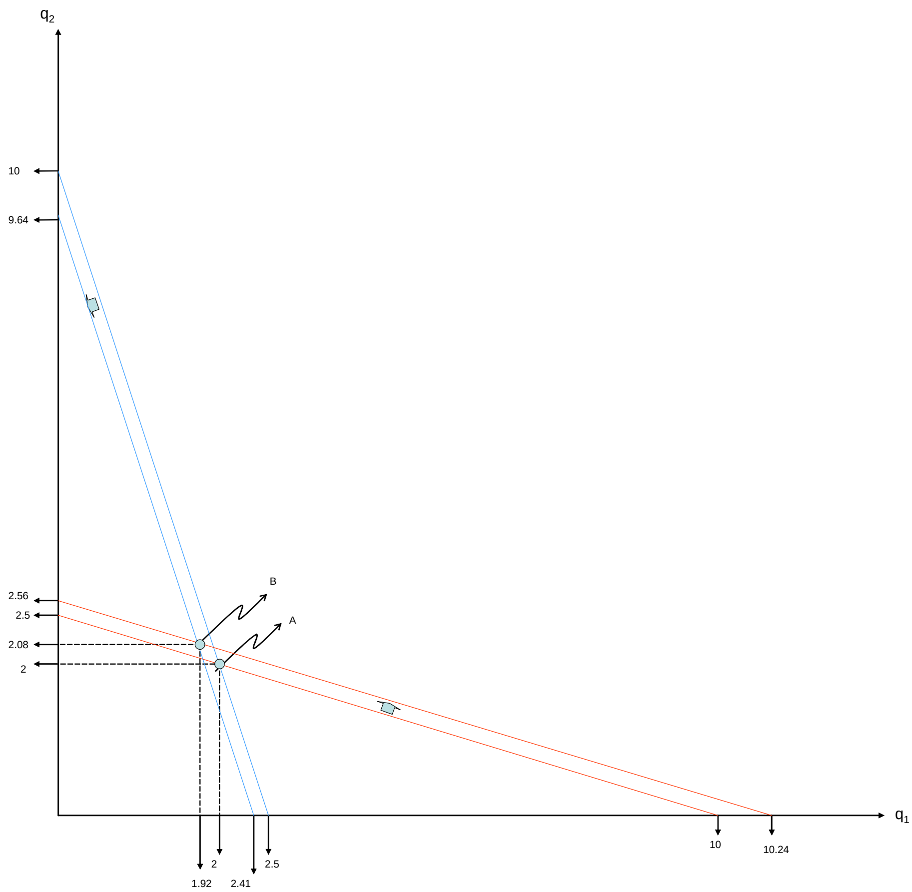

Figure 4

5.3 Asymmetry in Advertising Effectiveness Parameters

Finally, let us consider the case where one firm's advertising is exogenously more effective than the other firm. This can happen if one firm is marketing a relatively well known brand and the other firm has very little or no brand presence, or is a new entrant into the market. Then, comparative advertising gives the latter more brand recognition and would drive experimenting buyers to buy the good in question.

Specifically, assume $\theta^1_{12} = \theta^2_{12} = \theta_1 > \theta_2 = \theta^1_{21} = \theta^2_{21}$. Assume symmetry in all other dimensions, namely, for all i and $j \neq i : \beta_i = \alpha$, $c_i = c$, $\phi_{ij} = \phi$. In complete absence of advertising both firms will produce exactly the same amount, namely,

$$
\begin{aligned}
q_1 &= \frac{\alpha - c}{2+\varepsilon}; \\
q_2 &= \frac{\alpha - c}{2+\varepsilon}; \\
s_{12} &= 0; \\
s_{21} &= 0.
\end{aligned}
$$

Let us now introduce the possibility of advertising. We have

$$
\begin{aligned}
V_{12} &= \left(\frac{\theta_1}{2-\varepsilon}\right); \\
V_{21} &= \left(\frac{\theta_2}{2-\varepsilon}\right).
\end{aligned}
$$

Obviously, $V_{12} > V_{21}$. In the second stage, sufficient conditions of maximization require

$$
\frac{\partial^2 \pi_i(s)}{\partial s_{ij}^2} = 2 \cdot V_{ij}^2 - 2 \cdot \phi < 0
$$

where $i = 1, 2$ and $j \neq i$. In order to ensure that the second order conditions of maximization hold in the second stage, we assume

$$
\phi > V_{12}^2 > V_{21}^2.
$$

With advertising, the SPNE are given by

$$
\begin{aligned}
q_1 &= 2\left(\frac{\alpha - c}{2+\varepsilon}\right) \cdot \left(\frac{\phi - 2 \cdot V_{21}^2}{2 \cdot \phi - 2 \cdot V_{21}^2 - 2 \cdot V_{12}^2}\right); \\
q_2 &= 2\left(\frac{\alpha - c}{2+\varepsilon}\right) \cdot \left(\frac{\phi - 2 \cdot V_{12}^2}{2 \cdot \phi - 2 \cdot V_{21}^2 - 2 \cdot V_{12}^2}\right); \\
s_{12} &= \frac{q_1 \cdot V_{12}}{\phi}; \\
s_{21} &= \frac{q_2 \cdot V_{21}}{\phi}.
\end{aligned}
$$

Taking ratios, we get

$$
\begin{aligned}
\frac{q_1}{q_2} &= \left(\frac{\phi - 2 \cdot V_{21}^2}{\phi - 2 \cdot V_{12}^2}\right); \\
\frac{s_{12}}{s_{21}} &= \left(\frac{\theta_1}{\theta_2}\right) \cdot \left(\frac{\phi - 2 \cdot V_{21}^2}{\phi - 2 \cdot V_{12}^2}\right).
\end{aligned}
$$

We will consider the following cost ranges.

Case 1: $\phi > 2 \cdot V_{12}^2 > 2 \cdot V_{21}^2$

The cost parameters are in the "high cost" range. Then, $q_1 > q_2$ and $s_{12} > s_{21}$. Hence, we get the expected result, namely, the firm with greater effectiveness of advertising both advertises more and produces more.

Case 2: $2 \cdot V_{12}^2 > 2 \cdot V_{21}^2 > \phi > V_{12}^2 > V_{21}^2$

The cost parameter are in the "low cost" range. Also, this assumes $2 \cdot V_{21}^2 > V_{12}^2$ or $\sqrt{2} \cdot \theta_2 > \theta_1 > \theta_2$. First, $q_1 < q_2$, hence the firm with less effectiveness of advertising produces more.

With regard to the levels of advertising, we can consider three sub-cases.

Case 2A: $\dfrac{2 \cdot V_{21}^2 - \phi}{2 \cdot V_{12}^2 - \phi} < \dfrac{\theta_2}{\theta_1} < 1$

In this case, the firm (firm 2) with lower effectiveness of advertising actually advertises more than the other firm (firm 1).

Case 2B: $\dfrac{2 \cdot V_{21}^2 - \phi}{2 \cdot V_{12}^2 - \phi} = \dfrac{\theta_2}{\theta_1} < 1$

In this case, the levels of advertising are equal.

Case 2C: $\dfrac{\theta_2}{\theta_1} < \dfrac{2 \cdot V_{21}^2 - \phi}{2 \cdot V_{12}^2 - \phi} < 1$

In this case, firm 1 advertises more than firm 2.

Case 3: $2 \cdot V_{12}^2 > \phi > 2 \cdot V_{21}^2$

In this case, the interior solution yields a negative result for the quantity and advertising levels of one of the firms. So we obtain a boundary solution. One of the firms drop out of the market. The other firm becomes a monopoly but has to keep advertising to keep its rival out. Advertising takes the form of entry-deterrence.

The firm with more effectiveness in terms of advertising will drop out if

$$V_{12}^2 + V_{21}^2 > \phi > 2 \cdot V_{21}^2.$$

The firm with less effectiveness in terms of advertising will drop out if

$$2 \cdot V_{12}^2 > \phi > V_{12}^2 + V_{21}^2.$$

We illustrate the results in Figure 5, depicting the reaction curves of the following example. Let us start with a situation of complete symmetry. Let $\beta_1 = \beta_2 = \alpha = 6$,

$c_1 = c_2 = c = 1$, $\varepsilon = 0.5$, $\theta_{12}^1 = \theta_{21}^1 = \theta_{12}^2 = \theta_{21}^2 = \theta = 3$. Let $\phi_{12} = \phi_{21} = \phi = 6$. The reaction curve of firm 1 is indicated by a blue line and the reaction curve of firm 2 is indicated by a red line. Then the Cournot equilibrium is given by the intersection of the two reaction curves at point A. Each firm produces an equilibrium amount of 2 and advertises an amount 0.67. Now, let us introduce a small asymmetry by assuming $\theta_{12}^1 = \theta_{12}^2 = \theta_1 = 3.1$ and $\theta_{21}^1 = \theta_{21}^2 = \theta_2 = 3$. The parameter are consistent with the range given by Case 2A. If the levels of advertising did not change, the reaction curves of firm 1 would move rightward and that of firm 2 would move leftward. The equilibrium point moves from A to B and firm 1 produces more. However in response to the above change, advertising levels change. Firm 1 advertises an amount of 0.61 and firm 2 advertises an amount 0.75. Hence the reaction curve of firm 1 shifts leftward and that of firm 2 shifts rightward. So, the equilibrium moves to point C and firm 1 produces an amount 1.76 and firm 2 produces an amount 2.24.

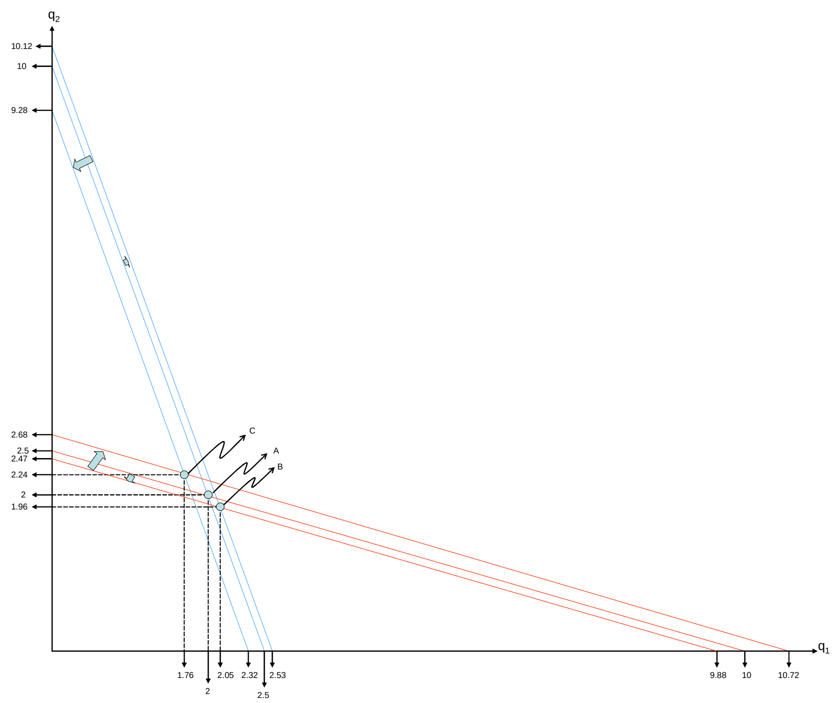

Figure 5

Now, consider $\beta_1 = \beta_2 = \alpha = 6$, $c_1 = c_2 = c = 1$, $\varepsilon = 0.5$, $\theta_{12}^1 = \theta_{21}^1 = \theta_{12}^2 = \theta_{21}^2 = \theta = 3$. Let $\phi_{12} = \phi_{21} = \phi = 9$. We illustrate the reaction curves in Figure 6. The reaction curve of firm 1 is indicated by a blue line and the reaction curve of firm 2 is indicated by a red line. Then the Cournot equilibrium is given by the intersection of the two reaction curves at point A. Each firm produces an equilibrium amount of 2 and advertises an amount 0.44. Now, let us introduce a small asymmetry by assuming $\theta_{12}^1 = \theta_{12}^2 = \theta_1 = 3.1$ and $\theta_{21}^1 = \theta_{21}^2 = \theta_2 = 3$. The parameter are consistent with the range given by Case 1. If the advertising levels did not change, the reaction curves of firm 1 would move rightward and that of firm 2 would move leftward. The equilibrium point moves from A to B and firm 1 produces more. However, in response to the above change advertising levels change. Firm 1 advertises an amount 0.63 and firm 2 advertises an amount 0.28. Consequently the reaction curve of firm 1 moves even further rightward and that of firm 2 moves even further leftward. The equilibrium point moves to C and firm 1 produces 2.74 and firm 2 produces 1.26.

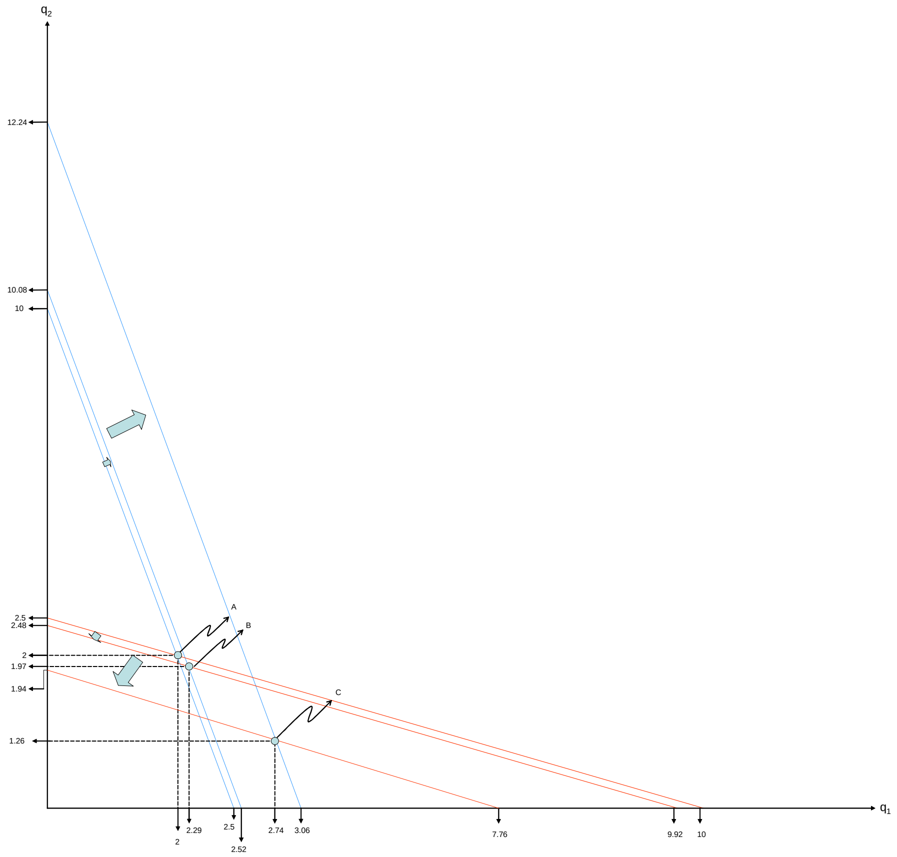

Figure 6

The asymmetric case serves to illustrate the richness of possibilities based on parameter values as to what sort of firm would come out on top in case there are advertising wars. Often what would happen in terms of advertising in equilibrium may run contrary to what a casual observer may expect. Advertising may take the form of entry deterrence in many situations. Monopoly firms may advertise against potential rivals who have not entered that particular market segment.[5]

6 Product Differentiation

In this section we explore two plausible premises about the link between the degree of product differentiation and the effectiveness of comparative advertising. One assumption is that less horizontal product differentiation makes comparative advertising more effective. The other assumption is that comparative advertising is an attempt to create or alter perception of vertical product differentiation.

6.1 Horizontal Product Differentiation and Advertising

To convey the general idea, let us assume that each firm places itself (its product) in an abstract product space represented by a metric space. If two firms i and k are close in the product space, then their products are close substitutes and comparative advertising between them is highly effective.

Formally, the inverse demand functions (1) are replaced by the slightly more general form

$$p_i = \alpha_i - q_i - \sum_{k \neq i} \varepsilon_{ik} \cdot q_k \tag{15}$$

Let d_{ik} denote the distance between firms i and k. Then, the assumption that less product differentiation make products closer substitutes and comparative advertising more effective amounts to the following two conditions:

(a) $\varepsilon_{ik} = \varepsilon_{ki}$ is strictly decreasing in d_{ik}.

(b) θ_{ik}^i and θ_{ik}^k are strictly decreasing in d_{ik}.

Notice that $d_{ki} = d_{ik}$ and by symmetry the effect of k's advertising against i decreases as well if they locate further apart. One would expect that *ceteris paribus* a firm will advertise more against close than against distant competitors. Ford's Lincoln may be pitted primarily against General Motor's Cadillac and not against Fiat's Uno. This

[5] Of course, firms cannot advertise against nonexistent rivals, but they may advertise against multi-product firms who are prospective rivals for the market segment in question.

could even happen only when (a) holds while the $\theta' s$ are independent of distance so that the impact of horizontal product differentiation is transmitted through only one channel. Consider, as a numerical example, the situation of three firms where firms 1 and 2 sell (almost) identical products whereas firm 3 sells a product quite different from the other two.

Example: There are three firms. Let us assume that there exists a number $\alpha > 0$ such that $\beta_i - c_i = \alpha$ for each firm i. We postulate that $d_{12} = 0$ or $d_{12} \approx 0$ whereas d_{13} and d_{23} are very large. Let us further assume that this implies $\varepsilon_{12} \approx 1$, $\varepsilon_{13} \approx 0$ and $\varepsilon_{23} \approx 0$. For simplicity we set $\varepsilon_{12} = 1$ and $\varepsilon_{23} = \varepsilon_{13} = 0$. Hence, at the second stage the market is divided into a segment served by firms 1 and 2 and a segment served by firm 3. However, we allow for advertising spillovers from one segment of the market to the other, assuming $\theta^i_{ij} = \theta^j_{ij} = 1$ for each pair of firms i and j. Let the advertising cost parameters be given as $\phi_{ij} = 8$ for each pair of firms i and j. Then they are sufficiently large to satisfy second order conditions with respect to advertising.

In the second-stage Cournot equilibrium of the duopoly formed by firms 1 and 2, the equilibrium profits are

$$\pi_1 = \frac{1}{9} \cdot (2\alpha_1 - \alpha_2)^2 - C^F_1, \; \pi_2 = \frac{1}{9} \cdot (2\alpha_2 - \alpha_1)^2 - C^F_2 \tag{16}$$

where $\alpha_1 = \alpha + s_{12} - s_{21} + s_{13} - s_{31}$ and $\alpha_2 = \alpha + s_{21} - s_{12} + s_{23} - s_{32}$. The second-stage monopoly profit for firm 3 is

$$\pi_3 = \frac{1}{4} \cdot \alpha_3^2 - C^F_3 \tag{17}$$

where $\alpha_3 = \alpha + s_{31} - s_{13} + s_{32} - s_{23}$. To determine equilibrium advertising levels, we exploit symmetry and set $s_{12} = s_{21}, s_{13} = s_{23}, s_{31} = s_{32}$. This reduces the first order conditions (10) to the following three equations immediately derived from (16) and (17):

$$\begin{aligned} 8s_{12} &= \frac{3}{9}(\alpha + s_{13} - s_{31}); \\ 8s_{13} &= \frac{2}{9}(\alpha + s_{13} - s_{31}); \\ 8s_{31} &= \frac{1}{4}(\alpha + 2s_{31} - 2s_{13}). \end{aligned}$$

The solution is

$$s_{12}^* = \frac{43.5}{1048}\alpha,\ s_{13}^* = \frac{29}{1048}\alpha,\ s_{31}^* = \frac{33}{1048}\alpha.$$

Thus, indeed, advertising between the two duopolists is fiercer than across market boundaries. Interestingly enough, the monopolist advertises somewhat more against a duopolist than vice versa. The reason is that for equal advertising levels, the α_i are equal whereas the monopolist's profits and marginal benefits from advertising are clearly higher as a comparison of (16) and (17) shows.

Now, suppose in the example, the impact of horizontal product differentiation gets transmitted through both channels, i.e. both (a) and (b) hold. We can achieve this by setting $\theta_{12}^1 = \theta_{12}^2 = \theta_{21}^1 = \theta_{21}^2 = \widehat{\theta} > 1$ while keeping all other parameters fixed at their previous levels. Then, only the first of the first order conditions changes and becomes

$$8s_{12} = \frac{3}{9}\widehat{\theta}(\alpha + s_{13} - s_{31}).$$

Therefore, the new equilibrium advertising levels are $\widehat{s}_{12} = \widehat{\theta}s_{12}^*$, $\widehat{s}_{13} = \widehat{\theta}s_{13}^*$, $\widehat{s}_{31} = s_{31}^*$. The advertising war among the duopolists has intensified as to be expected while advertising across market boundaries has not changed.

Symmetric Case: For arbitrary fixed locations, the analysis becomes very complicated. In contrast, the situation becomes fully transparent in the symmetric case with two firms to which we turn next. Let us assume $n = 2$ and make the symmetry assumptions of the previous section. Moreover, we assume (a) and (b). Since there are only two firms, we can drop the subscript from ε and θ. In the absence of advertising, (3) and (4) become $q_i = (\alpha - c)/(2 + \varepsilon)$ and $\pi_i = (q_i)^2$, respectively. Consequently a firm's Cournot equilibrium profits is decreasing in ε and by (a), increasing with its distance from the other firm. In a two stage game, where firms first choose locations on the unit interval and then choose quantities, these comparative statics yield maximum product differentiation:

Proposition 3 *Suppose the product space is the unit interval. In a subgame perfect equilibrium of the two-stage game, where firms first choose locations and then choose quantities, one firm locates at one end-point and the other firm locates at the opposite end-point.*

Next we introduce advertising and consider a three stage game. In the first stage, firms choose locations in the product space. In the second stage, they choose advertising levels. In the final stage, they choose quantities. Notice that when locations are fixed, ε and θ are determined and the two firms face a symmetric situation as in the previous section. Hence the subgame perfect equilibrium outcomes of stages two and three are given (11) and (12). For $n = 2$ these expressions simplify to $s_i = ((\alpha - c)/(4 - \varepsilon^2)) \cdot (\theta/\phi)$ and, as before, $q_i = (\alpha - c) / (2 + \varepsilon)$. We have already seen that a firm favors maximum product differentiation, if we ignore advertising. The comparative statics of the equilibrium advertising levels s_i reinforce this conclusion: s_i is increasing in ε and θ. Hence a firm spends less on advertising if it is located further away from the other firm. Therefore, q_i increases and $C_i^F = \phi \cdot s_i^2$ decreases, if firm i moves further away from the other firm. Consequently, i's profit $\pi_i = {q_i}^2 - C_i^F$ is increasing with its distance from the other firm. In the three-stage game, these comparative statics yield again maximum product differentiation:

Proposition 4 *Suppose the product space is the unit interval. In a subgame perfect equilibrium of the three-stage game where firms choose first locations, next advertising levels, and finally quantities, one firm locates at one endpoint and the other firm locates at the opposite endpoint.*

In the classical Hotelling model of a linear city, firms first choose locations on the interval and then compete as Bertrand duopolists for consumers uniformly distributed on the interval. If a firm moves closer to its competitor, it experiences two opposing effects on its profits. The strategic effect refers to the loss the firm incurs because of fiercer price competition. The market share effect refers to the increased demand for the firm's product. D'Aspremont, Gabsewicz and Thisse (1979) have shown that the strategic effect dominates the market share effect so that maximum product differentiation results in the subgame perfect equilibrium. Here we have obtained a similar result in a very different context: the losses from moving closer to one's competitor outweigh the gains. In particular, if a firm reduces its distance from its competitor, then its advertising becomes more effective. But the competitor's advertising becomes also more effective. Consequently, the ensuing advertising war intensifies to the detriment of both firms' profits.

6.2 Advertising as Perceived Product Differentiation

In this section, we briefly elaborate on the possibility that functionally equivalent products may be perceived as different as a consequence of persuasive advertising. If advertising is merely aimed at brand recognition, otherwise identical products may be perceived as horizontally differentiated. If advertising is comparative, otherwise identical products may be perceived as vertically differentiated. This kind of virtual product differentiation can coexist with actual product differentiation and amplify and alleviate the effects of the latter. To be more specific, let us reconsider a modified inverse demand function of the form (15). In our previous interpretation, we had assumed that the degree of actual horizontal product differentiation determines the magnitude of substitution coefficients ε_{ik} and of the marginal advertising effect parameters θ_{ik}^{i} and θ_{ik}^{k} in (7). Now let us make the opposite assumption that actual product differentiation is non-existent or ineffective. By postulating (7) and (1) or (15), we have always presumed that comparative advertising makes the own product more desirable and the product of the firm advertised against less desirable. In that sense, the effect of comparative advertising is similar to raising the quality of one's own product or lowering the quality of the other product. In addition to affecting the intercepts (constant terms) α_i of the inverse demand functions, advertising could influence the substitutability of products in a more direct way, for example as follows:

$$\varepsilon_{ij} = \varepsilon - s_{ij} \cdot \theta_{ij}^{i} + s_{ji} \cdot \theta_{ji}^{i} \tag{18}$$

This formulation reflects the intuition that if a firm advertises against another firm, then its own product becomes more of a substitute for the other firm's product whereas the other firm's product becomes less of a substitute for one's own product. This specification allows for the possibility of $\varepsilon_{ij} \neq \varepsilon_{ji}$. In other words, substitutability is no longer a symmetric property — as should be the case with vertical product differentiation. One can reanalyze the symmetric two stage game of section 4 under new assumptions. In particular, let us assume $n = 2, c = 0, \phi_{ij} = \phi$, and $\alpha_i = \beta_i = \alpha$ so that α_i does not respond to advertising. On the other hand, let us assume (18) with $\theta_{ij}^{i} = \theta_{ij}^{j} = \theta_{ji}^{j} = \theta_{ji}^{j} = \theta$. Then once more (12) obtains for the SPNE quantities.

In contrast, the SPNE advertising levels are now given by

$$s_{ij}^* = \frac{1}{2-\varepsilon}\left(\frac{\alpha}{2+\varepsilon}\right)^2 \frac{\theta}{\phi}. \tag{19}$$

Although (11) and (19) are not directly comparable because of different roles of θ in (7) and (18), they share several qualitative features. In particular, in both instances, equilibrium advertising is positive and linear in the benefit-cost ratio θ/ϕ.

6.3 Two-dimensional Product Differentiation: a Reinterpretation of the Model

So far we have argued that comparative advertising creates the perception of vertical product differentiation. It takes only a minor conceptional step to say that advertising "is" vertical product differentiation, at least in the case of a symmetric duopoly. In that case, our formal model can be converted into a model with vertical product differentiation by means of the following reinterpretation: $s_1 = s_{12}$ is the quality choice of firm 1 and $s_2 = s_{21}$ is the quality choice of firm 2. As a consequence of Corollary 2, one obtains

Corollary 3 *For $\alpha - c > \theta$ and ϕ sufficiently large, in the SPNE of the two stage game where firms make first quality and then quantity choices, the duopolists choose identical positive qualities, that is minimum vertical product differentiation.*

By (19), a similar conclusion holds under the alternative specification (18). Next suppose that the spectrum of product variety (horizontal product differentiation) is given by the unit interval and the range of product quality (vertical product differentiation) is given by the nonnegative real numbers. Then the analogue of Proposition 4 holds.

Corollary 4 *In a subgame perfect equilibrium of the three-stage game where the two firms decide first on variety, next on quality and finally on quantities, maximum horizontal product differentiation and minimum vertical product differentiation (with a positive quality level) result.*

Thus as a by-product, our analysis yields an interesting result on two-dimensional product differentiation. Our model is a special reduced form in that individual consumer choice is not explicitly specified. Fully specified models of multidimensional product differentiation prove hard to analyze. Economides (1989) considers quality variations in a duopoly of locationally differentiated products with linear transportation costs. In his two-stage game, variety is chosen in the first stage and quality and prices are chosen in the second stage. In his three-stage game, variety choice occurs in the first stage, quality choice in the second stage, and price choice in the last stage. Equilibria in pure strategies need not exist. They do exist for certain parameter ranges in which case the SPNE outcome for both models is maximum horizontal product differentiation combined with minimum quality differentiation. Neven and Thisse (1990) consider quality variations in a duopoly of locationally differentiated products with quadratic transportation costs. Potential qualities are restricted to an interval $[\underline{q}, \overline{q}]$. In their two-stage game, variety and quality are chosen in the first stage and prices are chosen in the second stage. They identify numbers $K_h > K_v > 0$ such that for $\overline{q} - \underline{q} \geqslant K_v$, there exists an equilibrium with maximum horizontal product differentiation and minimum vertical product differentiation (at quality $\overline{q}$) and for $\overline{q} - \underline{q} \leqslant K_h$, there exists an equilibrium with minimum horizontal product differentiation (at location 1/2) and maximum vertical product differentiation. In particular, for $\overline{q} - \underline{q} \in [K_v, K_h]$, both types of equilibria exist. As Neven and Thisse (1990) have noted their findings as well as those of Economides and ours suggest that "firms have a tendency to select similar strategies with respect to some characteristics, if at the same time they are sufficiently differentiated along the remaining dimensions." Neven and Thisse further assert that a similar result can be established with two vertical characteristics. Indeed, Vandenbosch and Weinberg (1995) study a two-stage game of product competition in two vertical dimensions followed by price competition and show the existence of a subgame perfect equilibrium with MaxMin differentiation, that is maximum differentiation in one dimension and minimum differentiation in the other dimension. However sometimes an equilibrium with maximum differentiation in both dimensions (or with maximum differentiation in one dimension and intermediate differentiation in the other) may also exist. Irmen and Thisse (1998) find that in a location game with $n \geqslant 2$ characteristics, firms tend to maximize differentiation in the dominant characteristic and to minimize in the others when the salience coefficient of the former is sufficiently large, corroborating findings by Tabuchi (1994) for $n = 2$ and Ansari, Economides and Steckel (1998) for $n = 2, 3$.

7 Concluding Comments

In this paper, we perform a theoretical analysis of advertising wars where firms engage in comparative advertising against each other. This practice has received a fair amount of media attention in recent years. Any rendition of comparative advertising episodes would be incomplete without mentioning the cola wars; see, e.g. Prince (2000). In 1975, Pepsi launched a widely publicized taste test called the "Pepsi Challenge" in which customers were asked to sample both Pepsi and Coke side by side without being aware of the labels. The alleged superior performance of Pepsi in this test was widely advertised and led to impressive increase in sales. Over the years, both soft drink giants have launched numerous spot ads against each other.

The basic premise of our analysis is that disregarding costs, a firm's advertising against another firm benefits the advertiser and harms the target. James and Hensel (1991) summarize a number of studies in the marketing literature on the impact of negative advertising on brand perception and induced brand demand based on consumer surveys. Several authors find that comparative advertising is particularly beneficial to new brands. Comparative advertising can also be very effective if it makes undisputed claims like ads by Visa against American Express or for Aleve against Tylenol. Comparative advertisements invite retaliation with potentially devastating effects for both sides and likely benefits to third parties. Examples are AT&T versus MCI, Pizza Hut versus Papa John's, Tylenol versus Advil. In our static model, we obtain advertising wars, namely positive levels of advertising, provided that advertising is not too expensive. The general version of the model allows for (positive and negative) side-effects on third parties as shown in subsection 3.1, a feature notably absent from most of the literature.

In the perfectly symmetric version of the model, we obtain that advertising levels are positive in equilibrium, but second-stage quantities and prices are the same as with zero advertising. Indeed, zero advertising would be socially optimal. This result supports the time-honored contention — dating back to Pigou (1924) at least — that advertising efforts by competitors might just neutralize each other and prove wasteful. Netter (1982) reports empirical evidence for the mutual cancelling of advertising efforts. Indeed, some firms "have chosen disarmament after years of ad warfare proved fruitless — such as Unilever's Ragu and Campbell Soup Co.'s Prego" as Neff (1999) reports. But other long-lasting feuds keep going. Take for instance, television ads for Advil against Tylenol, for Aleve against Tylenol and for Pine-Sole against Lysol.

If in fact advertising efforts by close competitors neutralize each other, then there are potential gains from (explicit or tacit) collusive agreements to refrain from comparative advertising. This possibility motivated us to study the sustainability of collusion in the dynamic version of our model. Empirically, collusion may be hard to identify and to our knowledge has not been systematically investigated at the brand level.[6] However, the research of Alston et al. (2001) hints at potentially huge gains from collusion at the more aggregate level. They start from the premise that profits from generic advertising by a producer group often come partly at the expense of producers of closely related commodities. They compare a scenario where different producer groups cooperate and choose their advertising expenditures jointly to maximize the sum of profits across groups, and a scenario where they optimize independently. They calibrate an example using 1998 data for U.S. beef and pork and find that the non-cooperatively chosen expenditure on beef and pork is more than three times the cooperative optimum.

Collusion is one possible explanation why comparative advertising is not predominant. Other reasons are regulatory restrictions and their legal repercussions. Moreover, comparative advertising may just not be effective in some markets. While it may not be predominant, comparative advertising seems far from negligible. According to Neff (1999), among ads in the ARS database used in a 1997 study, 12 % had direct comparisons while 16 % used indirect comparisons.

In the asymmetric case, general conclusions are hard to reach. We show using an asymmetric duopoly in section 5 that modest parameter changes may lead to dramatically different outcomes.

[6] Though Scherer and Ross (1990, pp. 596f) mention a few overt collusive attempts by cigarette and automobile manufacturers which failed with one exception.

In section 6, we relate product differentiation and comparative advertising. We first explore the assumption that less horizontal product differentiation makes comparative advertising more effective. We illustrate the implication of this assumption in an asymmetric example with three firms. We then consider the symmetric case with two firms and the unit interval as the product space. In a subgame perfect equilibrium of the three-stage game where firms first choose locations, next advertising levels, and finally quantities, one firm locates at one end-point and the other firm locates at the opposite end-point.[7] Next, we discuss the assumption that comparative advertising effort is an attempt to create or alter consumer perception of vertical product differentiation. Finally, we identify comparative advertising effort with quality choice and obtain a result on two-dimensional product differentiation. Prior to us, Economides (1989) also considers two-dimensional product differentiation and mentions advertising effort as an example of a quality feature whose cost is independent of output. Research effort falls into this category as well. But other quality features like use of better material, say stainless steel instead of ordinary steel often contribute to the unit cost of output. There is also a crucial difference between true and perceived quality attributes of consumer durables — which brings us to the earlier debate on the effects of advertising on consumer welfare. True high quality attributes like reliability, size and speed are embodied in the product and tend to last whereas perceived quality is subject to change, in particular if the ads creating the perception are discontinued or countered by the competition.

8 Proofs and Derivations

8.1 Cournot Equilibrium

Let $Q = \sum_i q_i$. Firm i chooses a positive best response if and only if

$$\alpha_i - c_i > \varepsilon \cdot \sum_{k \neq i} q_k. \tag{20}$$

[7] Some of the literature on persuasive advertising, e.g., Von der Fehr and Stevik (1998), Bloch and Manceau (1999) adopts Hotelling's linear city model with exogenously given firm locations, because it provides a convenient way to model individual consumer responses to advertising.

In that case, the best response of firm i is given by the first order condition

$$2q_i = \alpha_i - c_i - \varepsilon \cdot \sum_{k \neq i} q_k.$$

Therefore,

$$q_i = \frac{1}{2-\varepsilon}\left[\alpha_i - c_i\right] - \frac{\varepsilon}{2-\varepsilon} Q.$$

Summation over all firms yields:

$$\begin{aligned} Q &= \frac{1}{2-\varepsilon} \sum_{k \in N} (\alpha_k - c_k) - \frac{n\varepsilon}{2-\varepsilon} Q; \\ Q &= \frac{1}{2-\varepsilon+n\varepsilon} \sum_{k \in N} (\alpha_k - c_k). \end{aligned}$$

(3) and (4) follow. Now replace the summation index k in (20) by j. Next replace each q_j by the corresponding right hand expression in (3). This results in the following necessary and sufficient conditions for positive equilibrium quantities

$$\left[(2-\varepsilon) + \frac{(n-1)\varepsilon^2}{2+(n-1)\varepsilon}\right](\alpha_i - c_i) > \left[\varepsilon - \frac{(n-1)\varepsilon^2}{2+(n-1)\varepsilon}\right] \sum_{k \neq i} (\alpha_k - c_k) \qquad (21)$$

for all i.

The conditions (21) are satisfied if the terms $\alpha_i - c_i$ are identical or not too different across firms. When the conditions are not met, one can distinguish a set F of firms who choose a positive output in equilibrium and the set $N \backslash F$ of firms who choose a zero output in equilibrium. Then the counterparts of (3) and (21) hold for forms i and k restricted to F.

8.2 Existence

We are going to prove the existence claim of Proposition 1 by means of a fixed point argument. We assume $V_{ij} > 0$ for all $i \neq j$ and (1) (see below) for all $i \in N$.

An SPNE has the form $(s^*, q(\cdot))$. Among other things, we have to associate a Cournot equilibrium $q(s) = (q_1(s), \ldots, q_n(s))$ to each strategy profile s of the first stage. It is straightforward to show that given any strategy profile s of the first stage, a Cournot equilibrium of the subsequent subgame exists. In general, Cournot equilibrium is not unique. But it is unique in the present setting. Namely, it follows from the analysis of the previous subsection that equilibrium outputs are uniquely determined, given the set M of firms that choose positive quantities in equilibrium. It remains to be shown that M is unique. Now, take any Cournot equilibrium given s. By keeping the equilibrium quantities of all but two firms fixed and analyzing the reduced game between the remaining two firms, say i and j, one finds that $\alpha_i - c_i \geq \alpha_j - c_j$, then the corresponding equilibrium outputs satisfy $q_i \geq q_j$. Let us label firms so that $\alpha_1 - c_1 \geq \alpha_2 - c_2 \geq \ldots \geq \alpha_n - c_n$. Next, suppose there are two Cournot equilibria, one where the set of firms with positive output is$K = \{1, \ldots, k\}$, and one where the set of firms with positive output is $M = \{1, \ldots, m\}$, with $0 \leq k < m \leq n$. $k = 0$ means $K = \emptyset$. Let Q_K and Q_M denote the corresponding aggregate equilibrium outputs. Since m chooses zero output in the first equilibrium, $\alpha_m - c_m \leq \varepsilon \cdot Q_K$. Using the corresponding formula for aggregate output, we obtain

$$(2 + (k-1)\varepsilon)(\alpha_m - c_m) \leq \varepsilon \sum_{i \in K} (\alpha_i - c_i) \, .$$

Further $\varepsilon(\alpha_m - c_m) \leq \varepsilon(\alpha_i - c_i)$ for $i = k+1, \ldots, m$. Adding these $m - k$ inequalities yields

$$(2 + (m-1)\varepsilon)(\alpha_m - c_m) \leq \varepsilon \sum_{i \in M} (\alpha_i - c_i) \, ,$$

which in turn implies
$\alpha_m - c_m \leq \varepsilon Q_M$ and $q_m = ([\alpha_m - c_m] - \varepsilon Q_M) / (2 - \varepsilon) \leq 0$ for m's output in the second equilibrium, a contradiction. Hence, to the contrary, the set of firms which choose positive output in equilibrium is unique. This completes the proof that there is a unique Cournot equilibrium $q(s)$ for each s.

The next step is to establish existence of a strategy profile $s^* \in S$ that satisfies the first order conditions (10). In two further steps, we show that the pair $(s^*, q(.))$ is an SPNE and that s^* is unique. The arguments of all three steps require that the coefficients $\phi_{ij}, i \neq j$, be sufficiently large. Given the assumption that (22) holds for all i, we shall show the existence of a number $\varphi > 0$ such that if $\phi_{ij} > \varphi$ for all $i \neq j$, then the conclusion of Proposition 1 holds.

Without advertising, the conditions (21) become

$$\left[(2-\varepsilon)+\frac{(n-1)\varepsilon^2}{2+(n-1)\varepsilon}\right](\beta_i-c_i) > \left[\varepsilon-\frac{(n-1)\varepsilon^2}{2+(n-1)\varepsilon}\right]\sum_{k\neq i}(\beta_k-c_k) \qquad (22)$$

for all i. The conditions (22) are satisfied if the terms $\beta_i - c_i$ are identical or not too different across firms. Now suppose they are satisfied. Then, because of (7) there exists a number $b \in (0,1)$ such that if $0 \leqslant s_{ij} \leqslant b$ for all $i \neq j$, then the conditions (21) hold. Choose $b > 0$ with this property and and let $B = [0,b]^{n(n-1)} \subset S$. Then for $s \in B$ and $i \in N$, $q_i(s)$ is given by (3) and is positive. Since each $q_i(s)$ depends continuously on $s \in B$ and B is compact, there exists $K > 0$ such that $q_i(s) \leqslant K$ for all $i \in N$ and $s \in B$. Therefore there exists $\varphi_1 > 0$ such that if $\phi_{ij} \geqslant \varphi_1$ for all $i \neq j$, then $0 < q_i(s).V_{ij}/\phi_{ij} \leqslant b$ for all $s \in B$ and $i \neq j$. Let us choose such a $\varphi_1 > 0$ and suppose $\phi_{ij} \geqslant \varphi_1$ for all $i \neq j$. Then we can define a continuous mapping $\sigma : B \to B$ by setting $\sigma_{ij}(s) = q_i(s)V_{ij}/\phi_{ij}$, for all $s \in B$ and $i \neq j$. Since B is non-empty, compact and convex, σ has a fixed point, by Brouwer's fixed point theorem. A fixed point s^* of σ satisfies $s^* = \sigma(s^*)$. Hence for any $i \neq j : s^*_{ij} = \sigma_{ij}(s^*)$ $= q_i(s^*)V_{ij}/\phi_{ij}$. That is, s^* satisfies the first order conditions (10). Hence $(s^*, q(.))$ is an SPNE, provided that for each firm i, $\pi_i(s_i, s^*_{-i})$ is maximized at s^*_i. We distinguish two cases.

Case 1: $s_{ij} \leqslant b$ for all $j \neq i$. Recall that (3) and (4) hold for any choice of $s = (s_i, s^*_{-i})$ such that $s_{ij} \leqslant b$ for all $j \neq i$. Because of (7) and (9), there exists $\psi_i > 0$ such that if $\phi_{ij} > \psi_i$ for all $j \neq i$, then firm i's profit function $\pi_i(s_i, s^*_{-i})$ has a negative definite Hessian matrix and, therefore, is a strictly concave function of $s_i \in [0,b]^{n-1}$. Choose such a $\psi_i > 0$ for each i and set $\varphi_2 = \varphi_1 + \sum_i \psi_i$. Suppose $\phi_{ij} \geqslant \varphi_2$ for all $i \neq j$. Then because of the first order conditions (10), s^*_i is the unique best response against s^*_{-i} in $[0,b]^{(n-1)}$.

Case 2: $s_{ij} > b$ for some $i \neq j$. Let $\|\cdot\|$ denote the Euclidean norm on $\mathbb{R}^{n-1}$. The best that can happen to firm i is that all other firms choose zero advertising and zero output. In that case, its monopoly profit is

$$\Pi_i(s_i) = \left(\beta_i - c_i + \sum_{j\neq i} s_{ij}\theta^i_{ij}\right)^2 \Big/ 4 - \sum_{j\neq i} s^2_{ij}\phi_{ij}$$

if firm i chooses $s_i \in S_i$ in the first stage and its profit maximizing quantity given s_i in the second stage. There exists $\vartheta_i > 0$ such that if $\phi_{ij} > \vartheta_i$ for all $j \neq i$, then

$\Pi_i(s_i) < 0$ for $\|s_i\| > b$. Choose such a $\vartheta_i > 0$ for all i. Set $\varphi_3 = \varphi_2 + \sum_i \vartheta_i$. Suppose $\phi_{ij} > \varphi_3$ for all $i \neq j$. Then $s_{ij} > b$ for some $i \neq j$ implies $\|s_i\| > b$ and $\pi_i(s_i, s^*_{-i}) \leqslant \Pi_i(s_i) < 0 < \pi_i(s^*)$. Combined with Case 1 this means s^*_i is the unique best response to s^*_{-i} in S_i. Since this holds true for any i, $(s^*, q(.))$ is an SPNE indeed. Moreover, $0 < s^*_{ij} < 1$ for all $i \neq j$ as asserted.

It remains to be shown that s^* is unique. To this end, consider the system of linear equations given by (10), (3) and (7). It can be summarized in the form $s = y + As$ or $(I - A)s = y$ where I is the $n(n-1) \times n(n-1)$ identity matrix and $y + As$ is the right-hand side of the system (10). The constant vector y and the square matrix A are determined by the model parameters. Ceteris paribus, the entries in A can be made arbitrarily small in absolute value by setting the advertising cost parameters $\phi_{ij}, i \neq j$ sufficiently large. Choose $\varphi_4 \geqslant \varphi_3$ so that this holds true if $\phi_{ij} \geqslant \varphi_4$ for all $i \neq j$. Suppose that in fact $\phi_{ij} \geqslant \varphi_4$ for all $i \neq j$. In that case, the matrix $I - A$ has a dominant diagonal and consequently is non-singular [by Theorem 1 of McKenzie (1960)]. Therefore, the system $(I - A)s = y$ has a unique solution s^*. Hence, there exists only one SPNE where all firms choose positive quantities and advertising levels.

Under the hypothesis that (22) holds for all i, we have shown existence of a number $\varphi_4 > 0$ such that if $\phi_{ij} > \varphi_4$ for all $i \neq j$, then the conclusion of Proposition 1 holds. This completes the proof.

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