IMPERFECT INFORMATION LEADS TO COMPLETE MARKETS IF DIVIDENDS ARE DIFFUSIONS

BY FRANK RIEDEL [1]

A pure exchange economy with a financial market is studied where aggregate dividends are modeled as a diffusion. The dynamics of the diffusion are allowed to depend on factors which are unobservable to the agents and have to be estimated. With perfect information, the asset market would be incomplete because there are more factors than traded assets. Imperfect information reduces the number of observable risks, but also the number of admissible portfolio strategies. It is shown that, as long as the observable dividend stream is a diffusion, the asset market is complete. It is therefore possible to establish the existence of an equilibrium with dynamically complete markets that leads to the same allocation as the Arrow-Debreu equilibrium.

1. INTRODUCTION

It is known since the seminal work of Arrow (1953) that an asset market reduces substantially the number of markets needed to reach efficiency. In a continuous-time setting, an infinite number of contingent forward markets is needed to establish an Arrow-Debreu-equilibrium whereas a finite number of assets is sufficient to establish the same

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Pareto-optimal allocation. In Brownian frameworks, the number of assets needed to span a complete market is equal to the number of independent Brownian motions plus one. If there are less assets, however, some risks cannot be hedged, and markets are incomplete. Our point is that imperfect information may help to overcome this deficiency of the market.

The model starts with a market that contains some unhedgeable risks. Essentially, this is displayed by the fact that there are more independent Brownian motions than non-numéraire assets. I assume that the information available to the agents is given by dividends and stock prices only. All other risks are unobserved.

Imperfect information has two effects. First, it reduces the dimension of the consumption space, because a consumption plan must be formulated in observable terms. In this sense, it is easier to have a complete asset market, since the number of observable risks to hedge is smaller. On the other hand, the space of admissible portfolios is also smaller. This makes it more difficult to have a complete market.

The question naturally arises which one of the effects dominates and whether complete markets might result. We show that one has almost always complete markets if trading occurs continuously and dividends and asset prices are continuous semimartingales.

Continuous-time asset markets with imperfect information have been previously studied in a number of papers. The contributions of Detemple (1986), Dothan and Feldman (1986) and Feldman (1989) study a production economy where an unobservable factor is explicitly modeled as an autoregressive process. They describe the filtering procedure used by the agents and derive the corresponding equilibria. They assume, however, that a complete asset market is exogenously given whether there is perfect information or not.

Karatzas and Xue (1991) have a model where the number of independent Brownian motions also outnumbers the number of assets. They take asset prices as exogenously given and show that the utility maximization problem of a single agent, who observes asset prices only, is equivalent to the corresponding problem with complete asset markets and a smaller consumption space. Their paper inspired my approach. The difference is that I develop the full equilibrium with heterogeneous agents where asset prices are derived endogenously.

The paper is organized as follows. The next section describes the model. In Section 3, the estimating procedure of rational Bayesian agents is developed. An important martingale representation result is imported from Filtering Theory which is the key fact needed to show market completeness later on. The well known Negishi method is used in Section 4 to find an Arrow-Debreu equilibrium for the economy. Section 5 is devoted to the proof of market completeness. The well-known argument of Duffie and Huang (1985) is used to show how the Pareto efficient allocation of the Arrow-Debreu equilibrium can be supported by a Arrow-Radner equilibrium.

2. A Competitive Financial Market with Imperfect Information

Some people come together on a market place. All of them own a share of a stock paying aggregate output of the economy as a dividend. Agents know the dividends paid by the index in the past. There may be other factors determining the course of the economy, but these are not observable to the agents in my model.

There is one perishable good for consumption and investment. The horizon of the economy is $T < \infty$. Uncertainty is modeled by a filtered probability space $(\Omega, \mathcal{A}, P, (\mathcal{F}_t)_{t\geq 0})$ endowed with a k-dimensional Brownian motion W adapted to $\mathcal{F}$. In addition, there is a $\mathcal{F}_0$-measurable random variable η taking values in $\mathbb{R}^l$. Being $\mathcal{F}_0$-measurable, η is independent of the Brownian motion W.

Aggregate output K of the economy is exogenously given and grows at a rate X, that is

$$\frac{dK_t}{K_t} = dX_t\,, \quad K_0 = 1\,. \tag{1}$$

The evolution of X is described by the stochastic differential equation

$$\begin{aligned} dX_t &= \phi(X,Y)_t dt + \sum_{\iota=1}^{k} \xi^{\iota}(X)_t dW_t^{\iota} \\ X_0 &= 0\,. \end{aligned} \tag{2}$$

X depends in general also on other factors, Y, described as an $\mathbb{R}^l$-valued stochastic process. Y is modeled as the solution to

$$\begin{aligned} dY^{\iota} &= \mu^{\iota}(X,Y)_t dt + \sum_{\kappa=1}^{k} \zeta^{\iota\kappa}(X,Y)_t dW_t^{\kappa} \qquad (\iota = 1 \ldots l) \\ Y_0 &= \eta\,. \end{aligned} \tag{3}$$

Here, ϕ and μ are $\mathbb{R}$ and $\mathbb{R}^l$-valued, respectively. The diffusion coefficients ξ and ζ take their values in $\mathbb{R}^k$ and $\mathbb{R}^{l\times k}$, resp. It is assumed that the functionals ϕ, μ, ξ,and ζ appearing above are nonanticipative, and satisfy a global Lipschitz condition in the sense of the following definition(compare (Protter 1995, p. 195)):

Definition 2.1 Let $\mathbb{L}$ be the space of continuous, $\mathcal{F}$-adapted processes. A functional $\upsilon : \mathbb{L}^m \to \mathbb{L}^n$ is called

- *nonanticipative,* iff for all $A, B \in \mathbb{L}^m$ and all stopping times τ,

$$A^\tau = B^\tau \Rightarrow \upsilon(A)^\tau = \upsilon(B)^\tau\,,$$

- *Lipschitz,* iff there exists an increasing process K such that for all $A, B \in \mathbb{L}^m$,

$$\|\upsilon(A)_t - \upsilon(B)_t\|_{\mathbb{R}^n} \le K_t \sup_{s\le t} \|A_s - B_s\|_{\mathbb{R}^m} \;\; a.e.$$

The system (2) (3) has a pathwise unique, strong solution, compare Protter, Theorem 7, p. 197.

Remark 2.1 *It is crucial for our result, that the diffusion coefficients $\xi^\iota(X)_t$ do not depend on the unobservable process Y, see (Kunita 1979). The author is grateful to Marcel Rindisbacher for pointing this out.*

The random variable η may be interpreted as containing the values of some unknown parameters of the diffusion model.

Next, we assume that the diffusion coefficient is bounded away from zero:

Assumption 2.1 *The diffusion coefficient ξ is strongly nondegenerate:*

$$\text{(4)} \qquad \textit{there is an } \epsilon > 0 \textit{ with } \quad ||\xi(X)_t|| \quad \ge \quad \epsilon \quad a.e.$$

Finally we impose an integrability condition on the solution X of the stochastic differential equation (2):

Assumption 2.2

$$E\int_0^T |\,\phi(X,Y)_t\,|\,dt \quad < \quad \infty$$
$$E\int_0^T ||\xi(X)_t||^2\,dt \quad < \quad \infty\,.$$

This assumption is satisfied, if the coefficients of the stochastic differential equation satisfy a linear growth condition.

All publicly available information is contained in the growth rate of aggregate output X, the factors Y are unobservable. Denote by $\mathcal{G}_t = \mathcal{F}^X_{t+}$ the right-continuous filtration generated by X. Let $\mathcal{O}(\mathcal{G})$ denote the σ-field of all events $A \in \mathcal{A} \otimes \mathcal{B}([0,T])$ which are progressively measurable with respect to the observation filtration $\mathcal{G}$. Henceforth, every event $A \in \mathcal{O}(\mathcal{G})$ and every $\mathcal{O}(\mathcal{G})$-measurable process are called observable.

I agents live in the economy characterized by a state-independent felicity function $u^i(t,c)$. Agents are risk averse and felicity functions are smooth:

Assumption 2.3 *The felicity functions $u^i : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ are in $C^{2,3}$; for all t, $u^i(t,\cdot)$ is strictly increasing and strictly concave.*

In addition, we impose the Inada-condition[2] on the felicity function to simplify the analysis:

Assumption 2.4 *Marginal felicity is infinite at zero:*

$$\lim_{c\to 0} \frac{\partial}{\partial c} u^i(t,c) = \infty \,.$$

According to their preferences, agents choose a consumption rate $(c_t(\omega))$ which has to be an observable, square integrable process. The consumption space is

$$L = \left\{ c \in \mathcal{L}^2(\Omega \times [0,T], \mathcal{O}(\mathcal{G}), P \otimes dt) : c \geq 0 \quad P \otimes dt - a.e. \right\} .$$

[2]The assumption 2.4 leads to interior solutions for the maximization problems and allows the use of differential methods. In principle, it is possible to treat the case of finite marginal utility at zero, too, cf. (Karatzas, Lehoczky, and Shreve 1990). However, we did not want to burden the argument with the technical difficulty of a mixture of both types of felicity functions because it is not the essential point in the argument.

This space is smaller than the space with perfect information since consumption plans are not allowed to depend on the unobservable factors Y. The felicity function u^i induces a preference ordering over L represented by a von Neumann-Morgenstern type utility function U^i:

$$U^i(c) \quad = \quad E\int_0^T u^i(t, c_t)dt\,. \tag{5}$$

In the literature on existence of equilibria it is typically assumed that the aggregate output process K is bounded away from zero, cf. (Duffie 1986), (Duffie and Zame 1989) and (Karatzas, Lehoczky, and Shreve 1990). Here, K is strictly positive because it is equal to

$$\begin{aligned} K_t &= \exp\left(X_t - \frac{1}{2}\,[X]_t\right) \\ &= \exp\left(\int_0^t \xi(X)_u dW_u + \int_0^t \left(\phi(X,Y)_u - \frac{1}{2}\left|\left|\xi(X)_u\right|\right|^2\, du\right)\right)\,, \end{aligned} \tag{6}$$

but it need not be bounded away from zero. We replace the boundedness assumption with the weaker

Assumption 2.5 *Marginal felicities of initial endowments are square integrable:*

$$\frac{\partial}{\partial c}u^i(t, e^i_t) \in L\,.$$

The assumption ensures that marginal felicities give rise to positive and linear price functionals on L. This is needed to ensure the existence of an Arrow-Debreu equilibrium for the economy, compare (Aliprantis 1997) and (Mas-Colell and Zame 1991, especially Example 6.5).

Agents have two investment possibilities. On the one hand, there is the risky stock yielding the dividend K. It is traded at some price S. Moreover, there is a market for lending and borrowing at a short rate r, an observable process. $\beta_t = \exp(\int_0^t r_u du)$ is called the money market account. The space of admissible asset prices is the space of observable Itô-processes. This class is large enough to contain possible equilibrium prices S and β, which are determined endogenously.

Agent i initially owns a share s^i of the index, $0 < s^i \leq 1$. If she does not trade at all, she receives dividends at a rate $e^i_t = s^i K_t$. The shares add up to 1, $\sum_{i=1}^I s^i = 1$.

A spot market for the consumption good opens at every point in time t where the consumption good is traded at a price ψ, taken here to be an Itô-process like S. Using ψ as a state-price deflator, the deflated gain process for the index is

$$G_t = S_t \psi_t + \int_0^t K_u \psi_u du$$

and the deflated value of the money market account is

$$H_t = \beta_t \psi_t \, .$$

Definition 2.2 The triple of observable processes (θ^1, θ^2, c) is called a *portfolio/consumption strategy at initial cost* $x \geq 0$ if

$$c \in L \tag{7}$$

$$\int_0^T \left| \theta^1_u \right|^2 d[G]_u < \infty \quad a.e. \tag{8}$$

$$\int_0^T \left| \theta^2_u \right|^2 d[H]_u < \infty \quad a.e. \tag{9}$$

and the deflated value of the portfolio $V_t = (\theta^1_t S_t + \theta^2_t \beta_t)\psi_t$ satisfies

$$V_t = x + \int_0^t \theta^1_u dG_u + \int_0^t \theta^2_u dH_u - \int_0^t c_u \psi_u du \tag{10}$$

$$\text{for all } t \quad V_t \geq 0 \quad a.e. \tag{11}$$

If (θ^1, θ^2, c) is a portfolio/consumption strategy at cost $\tilde{x} \leq x$, then c is said to be *affordable at cost* x.The set of all consumption streams affordable at cost x if prices are (S, β, ψ) is denoted by $C(x, S, \beta, \psi)$.

Remark 2.2 *To avoid the usual change of numéraire argument, the deflated model is used right from the definition of portfolio processes.*

Given the initial value S_0 of the index, agents are endowed with a capital $x^i = s^i S_0$. They choose a consumption strategy that maximizes their utility over all strategies which are affordable at cost x^i. In equilibrium, markets clear.

Definition 2.3 An *Arrow-Radner equilibrium* consists of asset prices (S, β), a spot price for the consumption good ψ and portfolio/consumption strategies $(\theta^{1i}, \theta^{2i}, c^i)_{i=1,\dots,I}$ such that agents are rational,

$$c^i \quad \in \quad \arg\max_{d \in C(x^i, S, \beta, \psi)} U^i(d)\,,$$

and all markets clear:

$$\begin{aligned}
\sum_{i=1}^{I} \theta^{1i} &= 1 \quad P \otimes dt - a.e. \\
\sum_{i=1}^{I} \theta^{2i} &= 0 \quad P \otimes dt - a.e. \\
\sum_{i=1}^{I} c^i &= K \quad P \otimes dt - a.e.
\end{aligned}$$

Since our focus is on market completeness, we give the definition of completeness we need here:

Definition 2.4 Let the prices (S, β, ψ) be given. The asset market is called *complete* if for all $c \in L$ there exists a portfolio (θ^1, θ^2) that finances c at some initial cost x, formally

$$L = \cup_{x \geq 0} C(x, S, \beta, \psi)\,.$$

3. Updating Your Beliefs - The Innovation Process

Faced with imperfect information, the agents engage in Bayesian inference. They do not observe the drift coefficient ϕ, but rather estimate it on the basis of the observed realizations $(X_s)_{s \le t}$ of the rate of aggregate output growth. The difference between the observed growth rate and the estimated drift, $dX_t - \pi_t dt$, is white noise in the sense that it is a martingale differential. This martingale is a stochastic integral with respect to an *observable* Brownian motion $\hat{W}$, called the innovation process in Filtering Theory. As we shall show, it has an important spanning property, also called the martingale representation property; every observable local martingale is a stochastic integral with respect to $\hat{W}$. The spanning property is the key to complete markets, as the following sections will demonstrate.

We start by simplifying a bit the representation of the observed process X. Set $\sigma(X)_t = ||\xi(X)_t||$.

LEMMA 3.1 *The process*

$$B_t := \sum_{\iota=1}^{k} \int_0^t \frac{\xi^\iota(X)_u}{\sigma(X)_u} dW_u^\iota$$

is a Brownian motion with respect to $(\mathcal{F}_t)$.

PROOF : By assumption 2.1, $\sigma(X) > 0$ *a.e.* Moreover,

$$E \int_0^T \frac{\xi^\iota(X)_u^2}{\sigma(X)_u^2} du \le T < \infty .$$

B is therefore well defined and a continuous martingale with quadratic variation $[B]_t = t$. By Lévy's theorem (Theorem 3.16 of (Karatzas and Shreve 1991)), B is a Brownian motion. □

The observed process has a representation

$$X_t = \int_0^t \phi(X, Y)_u du + \int_0^t \sigma(X)_u dB_u$$

according to the preceding Lemma. Let

$$\pi_t = E\left[\,\phi(X,Y)_t \,|\, \mathcal{G}_t\right]$$

be the agents' estimate of the unobservable drift ϕ. The remainder which is not "explained" by the estimate π is $M_t = X_t - \int_0^t \pi_u du$.

LEMMA 3.2 *M is an observable martingale with quadratic variation*

$$[M]_t = \int_0^t \sigma(X)_u^2 du\,.$$

PROOF : M is observable, since so are X and π. By Assumption 2.2, M is integrable. For $t, s > 0$, we have

$$E\left[M_{t+s} - M_t |\, \mathcal{G}_t\right] = E\left[\int_t^{t+s} \sigma(X)_u dB_u + \int_t^{t+s} (\phi_u - \pi_u) du \,\middle|\, \mathcal{G}_t\right] =$$

$$= E\left[E\left[\int_t^{t+s} \sigma(X)_u dB_u \,\middle|\, \mathcal{F}_t\right] \middle|\, \mathcal{G}_t\right] + \int_t^{t+s} E\left[E\left[\phi_u | \mathcal{G}_u\right] - \pi_u \,|\, \mathcal{G}_t\right] du = 0\,.$$

Hence, M is a martingale. Its quadratic variation is equal to the quadratic variation of X, since the difference $M - X$ is of bounded variation. □

COROLLARY 1 *The process*

$$\hat{W}_t = \int_0^t \frac{1}{\sigma(X)_u} dM_u \tag{12}$$

is an observable Brownian motion.

PROOF : The stochastic integral is well defined because of

$$E \int_0^T \frac{1}{\sigma(X)_u^2} d\,[M]_u = T < \infty.$$

Moreover, the process $\hat{W}_t$ defined by (12) is a continuous martingale with quadratic variation

$$\left[\hat{W}\right]_t = t\,.$$

We conclude, again by Lévy's theorem, that $\hat{W}$ is a Brownian motion. □

According to the preceding Corollary, the canonical decomposition of X in observable terms is

$$X_t = \int_0^t \pi_u du + \int_0^t \sigma(X)_u d\hat{W}_u.$$

In principle, we are now back in the standard model with perfect information. The only thing which remains to be shown, is the martingale representation property. It is, a priori, not clear, whether the innovation process $\hat{W}$ is a basis for the space of observable martingales. However, this is, as long as the diffusion coefficient depends only on the observable process X, the case:

THEOREM 3.1 *Every observable local martingale N admits a representation*

$$N_t = N_0 + \int_0^t \theta_u d\hat{W}_u \tag{13}$$

for some observable process θ with

$$\int_0^T \theta_t^2 dt < \infty \quad P - a.e.$$

The proof is given in the Appendix.

4. Existence of an Arrow-Debreu Equilibrium

In the last section, it was demonstrated how the agents deal with the arrival of new information. A second important step is to establish the existence of an Arrow-Debreu equilibrium. If the asset market is complete, a point which shall be demonstrated in the next section, it is possible to implement the allocation of this equilibrium in an Arrow-Radner equilibrium.[3]

Imperfect information poses no problem to general equilibrium theory. The agents know that they can only demand consumption streams of the type $c_t = c_t((X_s)_{s\le t})$ which depend on the information carried by X only. Thus, the filtration $\mathcal{F}$ one uses for defining the concept of equilibrium given perfect information is replaced with the smaller filtration $\mathcal{G}$. The same argument as in the case of perfect information yields the existence of an equilibrium.

In the general equilibrium framework, $e^i = s^i K$ is the endowment of agent i. Equilibrium is defined thus

Definition 4.5 An *Arrow-Debreu equilibrium* consists of a positive linear functional $\Psi : L \to \mathbb{R}$ and an allocation $(c^i)_{1\le i\le I}$ such that the consumption market clears, $\sum_i c^i = K$, and agents maximize utility over all consumption bundles d, which are budget feasible:

$$c^i \in \arg \max_{\Psi(d)\le\Psi(e^i)} U^i(d)$$

Remark 4.3 *The classical existence proof of (Arrow and Debreu 1954) is not valid here because the consumption space is infinite dimensional. Yet, the existence theory for infinite dimensional commodity spaces with a finite number of agents is by now well understood, even though infinite marginal felicity at zero was not covered by the first proofs. For a survey of general equilibrium theory with infinite dimensional commodity spaces, refer to*

[3]This argument has been introduced into the literature on continuous-time markets by (Duffie and Huang 1985). The idea goes back to (Radner 1972) and (Arrow 1953), who proved in a discrete-time setting how financial assets economize on the number of markets needed to reach efficiency.

the article by (Mas-Colell and Zame 1991). (Dana 1993) treats the case of additive smooth felicity functions with infinite derivative at zero.

The method of proof we use to establish the existence of an Arrow-Debreu equilibrium was invented by Negishi (1960). It uses the social welfare properties of equilibria to establish the existence of an equilibrium.

DEFINITION 4.6 An allocation $(c^i)_{i=1,\dots,I} \in L^I$ is *Pareto efficient* if it is market clearing, $\sum_i c^i = K$, and it solves the social welfare problem

$$\sum_i \lambda^i U^i(d^i) \quad = \quad \max_{\sum d^i = K} \tag{14}$$

for some vector $\lambda \in \Delta^{I-1} = \{z \in \mathbb{R}^I; z^i \geq 0, \sum z^i = 1\}$.

REMARK 4.4 *The set of efficient allocations is described by vectors λ in a compact, convex subset of the finite dimensional space $\mathbb{R}^I$. The typical fixed-point argument will be carried out on this space. By use of this trick, the problems caused by the infinite dimensional commodity space are avoided.*

Let us solve the social welfare problem (14) for arbitrary aggregate endowments e.

LEMMA 4.3 *Let $\lambda \gg 0$. The value function*

$$\bar{U}(e;\lambda) = \max_{\sum_i d^i = e, d^i \in L} \sum \lambda^i U^i(d^i)$$

has a von Neumann-Morgenstern representation

$$\bar{U}(e;\lambda) = E \int_0^T v(t, e_t, \lambda) dt \,.$$

The felicity function $v(t,x,\alpha)$ is the solution of

$$v(t,x,\alpha) = \max_{\sum y^i = x} \sum_i \alpha^i u^i(t,y^i)\,. \tag{15}$$

v is strictly increasing and strictly concave in x. It is three times continuously differentiable in x and α and twice continuously differentiable in t.

The solutions $y^i(t,x,\alpha)$ of (15) are determined by the conditions

$$\alpha^i \frac{\partial}{\partial y} u^i(t,y^i) = \nu \tag{16}$$

$$\sum y^i = x \tag{17}$$

for a suitable Lagrange multiplier ν. The marginal felicity is

$$\frac{\partial}{\partial x} v(t,x,\alpha) = \nu\,. \tag{18}$$

The proof is standard.

The function $\bar{U}$ has the properties of a utility function as the preceding lemma shows. Therefore, it is usually interpreted as the utility function of a representative agent.

The Negishi method is inspired by the second welfare theorem: Every Pareto efficient allocation can be sustained as an equilibrium with transfer payments. It is possible to redistribute wealth among the agents and to define a price Ψ in a manner that Ψ is an equilibrium price for the allocation. If for some allocation, no transfer payments are needed, an equilibrium is found.

Denote by $c^i(\lambda)_t = y^i(t,K_t,\lambda)$ the Pareto efficient allocation associated with λ. The candidate for the price process $\psi(\lambda)$ is the marginal felicity of the representative agent at aggregate endowment. Set

$$\psi(\lambda)_t = \frac{\partial}{\partial x} v(t,K_t,\lambda)\,. \tag{19}$$

The money agent i needs in addition to his endowment e^i to buy the consumption stream $c^i(\lambda)$ is given by $b^i(\lambda) = E\int_0^T \psi(\lambda)_t(c^i(\lambda)_t - e^i_t)\,dt$. If $b^i(\lambda)$ units of money are given to or taken from agent i, then she is able to afford $c^i(\lambda)$. Note that the sum of payments is $\sum_i b^i(\lambda) = 0$. The final step consists in looking for a zero of the transfer map b.

THEOREM 4.2 *There is a strictly positive vector* $\lambda^\star \gg 0$ *with* $b(\lambda^\star) = 0$*. In particular,* $(\Psi(\lambda^\star), c(\lambda^\star))$ *is an Arrow-Debreu equilibrium.*

The proof follows from the standard fixed-point argument applied to the (finite-dimensional) function ε defined by $\varepsilon(\lambda)^i = \frac{1}{\lambda^i}b^i(\lambda)$, which has the properties of an excess demand function, compare (Dana 1993, Section 2, especially Theorem 2.5).

5. Completeness of the Market and Existence of a Financial Equilibrium

The Arrow-Debreu equilibrium derived in the last section yields an efficient allocation $c(\lambda)$ which is achieved if a complete set of forward markets exists at time 0. In the following, the Arrow-Debreu price process $\psi = \psi(\lambda)$ is used as a state-price deflator. Note that by (19)

$$\psi(\lambda)_t = \frac{\partial}{\partial x}v(t, K_t, \lambda) \tag{20}$$

and according to Lemma 4.3 ψ is a smooth function of Itô-processes, hence itself an Itô-process with some representation $d\psi = \alpha^1 dt + \alpha^2 d\hat{W}$.

If ψ is used as a deflator, $\beta\psi$ is a local martingale. Having this in mind, define the short rate r and the market price of risk p as

$$r = -\frac{\alpha^1}{\psi} \tag{21}$$

and

$$p = -\frac{\alpha^2}{\psi},$$

hence

$$\frac{d\psi}{\psi} = -r\,dt - p\,d\hat{W}. \tag{22}$$

Note that ψ is continuous and strictly positive. For almost all ω, the path (ψ_t) is bounded away from zero. Therefore, the integrals $\int_0^T |r_u|\,du$ and $\int_0^T p_u^2 du$ exist almost surely.

Thus,

$$H_t = \exp\left(-\int_0^t p_u d\hat{W}_u - \frac{1}{2}\int_0^t p_u^2\,du\right)$$

and $\beta_t = \exp\left(\int_0^t r_u du\right)$ are well defined. Note that H is the deflated value of the money market account β because of

$$\psi\beta = H.$$

Define the ex-dividend price of the index as

$$S_t\psi_t := E\left[\int_t^T K_u\psi_u\,du \,\middle|\, \mathcal{G}_t\right]. \tag{23}$$

The associated deflated gain process G is the observable martingale

$$G_t = E\left[\int_0^T K_u\psi_u\,du \,\middle|\, \mathcal{G}_t\right].$$

Being a positive martingale, G has a representation

$$G_t = G_0 - \int_0^t \gamma_u G_u d\hat{W}_u \tag{24}$$

by Theorem 3.1.

The two assets give rise to two local martingales and there is one Brownian motion $\hat{W}$ which generates the space of martingales. The usual argument along the lines of Black and Scholes (1973) allows to establish the completeness of the market as long as the diffusion coefficients satisfy a certain condition.

THEOREM 5.3 *If*

$$pS\psi \neq \gamma G \quad P \otimes dt - a.e. \tag{25}$$

holds, then the asset market is complete. The initial capital needed to finance a consumption stream $c \in L$ is given by the Arrow-Debreu price $\Psi(c) = E\int_0^T \psi_t c_t\, dt$.

PROOF : Let $c \in L$ be given. The martingale

$$M_t = E\left[\int_0^T c_t \psi_t dt \,\middle|\, \mathcal{G}_t\right] \tag{26}$$

is a stochastic integral by Theorem 3.1:

$$M_t = \Psi(c) + \int_0^t m_u d\hat{W}_u\,. \tag{27}$$

Set

$$V_t = E\left[\int_t^T c_u \psi_u\, du \,\middle|\, \mathcal{G}_t\right] \tag{28}$$

For (t, ω) fixed, let θ^1, θ^2 be the solution to the linear system

$$-\theta^1 \gamma G - \theta^2 pH = m \tag{29}$$

$$\theta^1 S\psi + \theta^2 H = V\,. \tag{30}$$

The system has almost always a solution because the determinant is

$$H(-\gamma G + pS\psi) \neq 0$$

by assumption (25). V is the deflated value of the portfolio formed by (θ^1, θ^2) because of (30). By (28), V is nonnegative and equal to

$$V_t \quad = \quad M_t - \int_0^t c_u \psi_u du \,,$$

by (26). (27) yields

$$V_t \quad = \quad \Psi(c) + \int_0^t m_u d\hat{W}_u - \int_0^t c_u \psi_u du \,.$$

Use (29) to obtain

$$\begin{aligned} V_t &= \Psi(c) - \int_0^t (\theta_u^1 \gamma_u G_u + \theta_u^2 p_u H_u) d\hat{W}_u - \int_0^t c_u \psi_u du \\ &= \Psi(c) + \int_0^t \theta_u^1 dG_u + \int_0^t \theta_u^2 dH_u - \int_0^t c_u \psi_u du \,. \end{aligned}$$

Hence, (θ^1, θ^2) finances c at initial cost $\Psi(c)$. □

With complete asset markets, it is possible to finance all consumption streams by trading in the asset market. It is no wonder then, that the Pareto efficient allocation of the Arrow-Debreu equilibrium is also the allocation of a financial equilibrium. Denote by

$$\tilde{C}(e, \Psi) = \{c \in L; \Psi(c) \leq \Psi(e)\}$$

the budget set of an agent with endowment e in the Arrow-Debreu framework where Ψ is the price functional associated with ψ.

LEMMA 5.4 *The budget sets in the Arrow-Debreu- and the Arrow-Radner framework coincide:*

$$C(x, S, \beta, \psi) = \tilde{C}(e^i, \Psi) \,.$$

PROOF : Remember that the initial capital is the same in both cases:

$$x^i = s^i S_0 = s^i E \int_0^T K_t \psi_t \, dt = \Psi(e^i) \,.$$

If c is budget-feasible in the Arrow-Debreu sense, $c \in \tilde{C}(e^i, \Psi)$, then, by Theorem 5.3, there exists a portfolio (θ^1, θ^2) that finances c at initial cost $\Psi(c) \leq \Psi(e^i) = x^i$. Hence, $c \in C(x^i, S, \beta, \psi)$.

If, on the other hand, (θ^1, θ^2, c) is a portfolio at initial cost $x \leq x^i$, then the sum of the deflated value V of the portfolio and cumulated consumption is

$$V_t + \int_0^t c_u \psi_u du = x + \int_0^t \theta^1_u dG_u + \int_0^t \theta^2 dH_u \,.$$

Because of $V \geq 0$, the right-hand side is a nonnegative local martingale, hence a supermartingale. It follows

$$\Psi(c) = E \int_0^T c_t \psi_t \, dt \leq E \left(V_T + \int_0^T c_u \psi_u du \right) \leq x$$

and $c \in \tilde{C}(e^i, \Psi)$. □

The Arrow-Radner equilibrium cannot be far away - and here it is:

THEOREM 5.4 *If the condition (25) holds, there is an Arrow-Radner equilibrium with the allocation $c(\lambda)$ of the Arrow-Debreu equilibrium of Theorem 4.2 and asset prices S and β as defined in (23) and (21).*

PROOF: Choose portfolio strategies $(\theta^{1i}, \theta^{2i})$ that finance c^i for agent i at cost $x^i = s^i S_0$ as in the proof of Theorem 5.3. In particular, the deflated value of i's portfolio is

$$V^i_t = E\left[\int_t^T c^i_u \psi_u du \,\middle|\, \mathcal{G}_t \right] . \tag{31}$$

Since the budget sets coincide by Lemma 5.4, c^i maximizes utility over all affordable consumption streams for agent i. The consumption market clears because c is the allocation of an Arrow-Debreu equilibrium. It remains to show that the asset market clears. Set $\theta^1 = \sum_i \theta^{1i}$, $\theta^2 = \sum_i \theta^{2i}$ and $V = \sum_i V^i$. Then

$$\begin{aligned}
V_t &= \sum_i \left(s^i S_0 + \int_0^t \theta_u^{1i} dG_u + \int_0^t \theta_u^{2i} dH_u - \int_0^t c_u^i \psi_u du \right) \\
&= S_0 + \int_0^t \theta_u^1 dG_u + \int_0^t \theta_u^2 dH_u - \int_0^t K_u \psi_u du \\
&= S_0 + G_t - G_0 + \int_0^t (\theta_u^1 - 1) dG_u + \int_0^t \theta_u^2 dH_u - \int_0^t K_u \psi_u du \\
&= S_t \psi_t + \int_0^t (\theta_u^1 - 1) dG_u + \int_0^t \theta_u^2 dH_u \,.
\end{aligned}$$

On the other hand, (31) and the clearing of the consumption market imply

$$V_t = E\left[\int_t^T K_u \psi_u du \,\middle|\, \mathcal{G}_t \right] = S_t \psi_t \,. \tag{32}$$

Hence, for all t

$$0 = \int_0^t (\theta_u^1 - 1) dG_u + \int_0^t \theta_u^2 dH_u \,.$$

Therefore, the quadratic variation is zero, too:

$$(\theta^1 - 1)^2 \gamma^2 G^2 + (\theta^2)^2 p^2 H^2 = 0 \, a.e.$$

$p_t = \gamma_t = 0$ is precluded almost everywhere by condition (25); Therefore, $\theta_t^1 = 1$ or $\theta_t^2 = 0$ and what is needed is that both equations hold. But if $\theta_t^1 = 1$ then by the definition of the portfolio value $V_t = S_t \psi_t + \theta_t^2 H_t$ and by (32), $\theta_t^2 = 0$. If $\theta_t^2 = 0$, then $V_t = \theta_t^1 S_t \psi_t$ and again by (32) $\theta_t^1 = 1$. The asset market clears. □

6. Conclusion

The main result of the paper is that imperfect information can lead to dynamically complete asset markets if the only information available to the agents is the dividend stream paid by the traded assets and if this dividend stream is a diffusion, that is a continuous stochastic process whose dynamics are driven by a Brownian motion. Faced with imperfect information, rational agents engage in Bayesian inference about the unobservable factors. In particular, they are able to obtain an observable white noise process - called the innovation process in filtering theory. It turns out that this process is sufficient to describe all the observable noise - technically, this process is a martingale generator. As long as a genericity condition holds, it follows that the equilibrium stock price is also a martingale generator and by the familiar argument one can construct an Arrow-Radner type stochastic equilibrium that leads to the same allocation as the Arrow-Debreu equilibrium.

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APPENDIX

1. Proof of the Martingale Representation Property

Let $\mathcal{G}_\infty = \sigma\left(\bigcup_{t=0}^{\infty} \mathcal{G}_t\right)$ be the information generated by the observed process from zero up to infinity. Consider the measurable space $(\Omega, \mathcal{G}_\infty)$ in conjunction with the filtration $\mathcal{G}$ and the observed process X.

Definition A.7 Let A, B be two continuous, observable processes and $x \in \mathbb{R}$. A probability measure Q on $(\Omega, \mathcal{G}_\infty)$ is called a *solution to the martingale problem* (x, X, A, B),if

1. $X_0 = x \quad Q - a.e.$,
2. $M_t := X_t - x - A_t$ is an observable local martingale with respect to Q,
3. $[M]_t = B_t \quad Q - a.e.$

It is clear from (2) and Lemma 3.2, that P is a solution to the martingale problem $\left(0, X, \int_0^t \pi_s ds, \int_0^t \sigma(X)_s^2 ds\right)$.

Lemma A.5 *P is the unique solution (on $\mathcal{G}_\infty$) of the martingale problem* $\left(0, X, \int_0^t \pi_s ds, \int_0^t \sigma(X)_s^2 ds\right)$.

Proof : Let Q be a solution of the martingale problem $\left(0, X, \int_0^t \pi_s ds, \int_0^t \sigma(X)_s^2 ds\right)$. Then, the process $M_t = X_t - \int_0^t \pi_s ds$ is a local martingale under Q with quadratic variation $[M]_t = \int_0^t \sigma(X)_u^2 du$. Proceeding as in Corollary 1, one shows that $\tilde{W}_t = \int_0^t \sigma(X)_u^{-1} dM_u$ is a Brownian motion under Q. The stochastic differential equation

$$Z_t = \int_0^t \pi_s ds + \int_0^t \sigma(Z)_s d\hat{W}_s \tag{33}$$

has a pathwise unique, strong solution, compare Protter, Theorem 7, p.197. By a result of Yamada and Watanabe (1971), pathwise uniqueness yields uniqueness in the sense of probability law. Since $\left((X,\hat{W}),(\Omega,\mathcal{G}_\infty,P),(\mathcal{G}_t)\right)$ and $\left((X,\tilde{W}),(\Omega,\mathcal{G}_\infty,Q),(\mathcal{G}_t)\right)$ are two weak solutions of the stochastic differential equation (33), we have $\mathcal{L}(X|P) = \mathcal{L}(X|Q)$, and $P = Q$ (on $\mathcal{G}_\infty$) follows. □

COROLLARY 2 *For every observable local martingale N there exists an observable process θ with*

$$\int_0^T \theta_u^2 du \quad < \quad \infty \quad P-a.e.$$

$$N_t \quad = \quad N_0 + \int_0^t \theta_u d\hat{W}_u \,.$$

PROOF : By Theorem 2 of (Jacod 1977) and the preceding Lemma A.5, every locally square integrable martingale N can be written as a stochastic integral with respect to the martingale $M_t = X_t - \int_0^t \pi_s ds$:

$$N_t \quad = \quad N_0 + \int_0^t \hat{\theta}_u dM_u \tag{34}$$

for some M-integrable process $\hat{\theta}$. Since we have $dM_t = \sigma(X)_t d\hat{W}_t$,

$$N_t = N_0 + \int_0^t \theta_u d\hat{W}_u$$

follows with $\theta_t = \hat{\theta}_t \sigma(X)_t$. It remains to extend this property to all local martingales. This is done by showing that the martingale representation of locally square integrable martingales implies the continuity, hence local square-integrability, of *all* local martingales(compare the methodology in Protter, Section IV.,3). Let N be a local martingale and (τ_n) a fundamental sequence. For fixed n, set $Z := N_\infty^{\tau_n} 1_{\{\tau_n<\infty\}}$. Define

$$\tilde{N}_t^J := E\left[\, Z 1_{\{|Z|\le J\}} \,\middle|\, \mathcal{G}_t\right] \,.$$

Then $\tilde{N}^J$ is a bounded martingale, hence continuous by the martingale representation (34). By Doob's submartingale inequality,

$$P\left(\sup_{t\leq T}\left|\tilde{N}_t^J - N_t^{\tau_n}\right| > \epsilon\right) \leq \epsilon^{-1}E\left[\left|\tilde{N}_\infty^J - N_{\tau_n}\right|\right] \longrightarrow_{J\to\infty} 0\,.$$

Hence, there is a subsequence $(\tilde{N}^{J_k})$ which converges uniformly to N^{τ_n} a.e., and N^{τ_n} is continuous. This property extends to N and we are done.

□