A Martingale Result for Convexity Adjustment in the Black Pricing Model

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Abstract

This paper explains how to calculate convexity adjustment for interest rates derivatives when assuming a deterministic time dependent volatility, using martingale theory. The motivation of this paper lies in two directions. First, we set up a proper no-arbitrage framework illustrated by a relationship between yield rate drift and bond price. Second, making approximation, we come to a closed formula with specification of the error term. Earlier works (Brotherton et al. (1993) and Hull (1997)) assumed constant volatility and could not specify the approximation error. As an application, we examine the convexity bias between CMS and forward swap rates.

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1 Introduction

The motivation of this paper is to provide a proper framework for the convexity adjustment formula, using martingale theory and no-arbitrage relationship. The use of the martingale theory initiated by Harrison, Kreps (1979) and Harrison, Pliska (1981) enables us to define an exact but non explicit formula for the convexity. We show that making approximation, we can derive previous results, first discovered by Brotherton-Ratcliffe and Iben (1993) and later by Hull (1997) and Hart (1997). The approach hereby considered has the great advantage to enables us to specify the error of the approximation. We extend results derived in the Black Scholes framework to time dependent volatility, often referred as the model of Black (1976). This is more in agreement with the consideration of practitioners who commonly use time dependent volatility to best fit the market prices.

The convexity adjustment hereby derived is of considerable interest to measure the convexity adjustment required by a security paying only once a swap rate. The rate of this kind of security is named in the fixed income market as the CMS rate.

The formula, first discovered by Brotherton-Ratcliffe and Iben (1993) and later by Hull (1997), is an analytic approximation of the difference between the expected yield and the forward yield, collectively referred to as the convexity adjustment. It assumes a constant yield volatility σ. Brotherton-Ratcliffe and Iben (1993) show that the convexity adjustment for yield bond was given by:

$$-\frac{1}{2}\left(y_0^f\right)^2 \sigma^2 T \frac{h''\left(y_0^f\right)}{h'\left(y_0^f\right)} \tag{1}$$

where y_0^f denotes the value today of the forward bond yield, $h(y)$ the price of the bond that provides coupons equal to the forward bond yield and that is assumed to be a function of its yield y, $h'(y)$ and $h''(y)$ the first and second partial derivatives of the bond price $h(y)$ with respect to its yield. Hull (1997) shows that this convexity adjustment can be extended to derivatives with payoff depending on swap rates. Hart (1997) sharpened the approximation with a Taylor expansion up to the four terms. However, all proofs, based on Taylor expansion, never introduced any error of the approximation. This was precisely the motivation of this paper. It shows that, when a proper no-arbitrage framework is assumed, formulae similar to (1) can be derived with an exact definition of the error term.

The remainder of this paper is organized as follows. In section 2, we give some insight about convexity. In section 3, we derive convexity adjustment from a no-arbitrage proposition implied by martingale condition. We show how to derive an approached formula, with a control on the error term. Monte Carlo simulations confirm the efficiency of the approached closed formula. In section 4, we explicit the convexity adjustment required for a CMS rate. We conclude briefly giving some further developments.

2 Insights about convexity

Convexity is a puzzling notion, which has been gained many meanings. In this section, we give a more specific definition and explain on a rough model how to lock in the convexity adjustment using a static hedge.

2.1 The definition of the convexity

For fixed income markets, convexity has emerged as an intriguing and challenging notion. Taking correctly this effect into account could provide competitive advantage for financial institutions. This paper tries to give insights and intuition about convexity.

One main difficulty is to give a unified framework for all the different meanings of convexity. Indeed, it is true that the notion of convexity refers to different situations, which can be sometimes seen as having almost nothing in common. Sometimes used as the gamma ratio for interest rate options, as an indicator of risk for bonds portfolios, as a measurement of the curvature of some financial instruments or as a small adjustment quantity for a wide variety of interest rate derivatives, convexity has become a synonym for small adjustment in fixed income markets, related somehow to the notion of mathematical convexity and more generally related to a second order differentiation term. A more restrictive definition would lead to abandon some particular case of the notion of convexity. Furthermore, the notion of convexity is quite disturbing since concavity is sometimes seen as a negative convexity, leading to quid pro quo and misunderstandings. The situations which are of particular interest for practitioners can be classified into two types with different causes of adjustment:

- the bias due to correlation between the interest rate underlying the financial contract and the financing rate. An example is the bias between forward and futures contracts. This correlation, capitalized by the margin calls of the futures contract, leads to a more expensive (respectively cheaper) futures contract in the case of positive (respectively negative) correlation.

- the modified schedule adjustment. Even if the analysis is the same for the two sub-cases above, it is traditionally divided into two categories depending on the type of rates:

  - One period interest rate (money market rates, zero coupon rate) and bond yield. An example is the difference between plain vanilla products and in-arrear ones, or in advance ones. Another one is the differentiation between forward yield rate and expected bond yield. Furthermore, a modified formula for every type of path dependent interest rate option, like Asian options, multi European options is required.

– Swap rates. These products are called by the market CMS products for constant maturity swap. A convexity adjustment is required between forward swap rate and expected swap rate, often called in the markets the CMS rate. Indeed, this analysis is very similar to the previous case. It comes as well from a modified schedule.

For practitioners, the two sub-cases have long been separated because they were concerning different products. As a result, they were seen as two types of adjustment. Indeed, the two required convexity adjustments are coming from a modified schedule of the rate.

In this paper, we concentrate on the distinction between forward and expected bond yield as well as swap rate.

2.2 A rough model

As pointed out in our definition section, one should make a distinction between the convexity adjustment required between futures and forward contract (correlation convexity) and the other adjustment (modified schedule adjustment). As a general rule for the second type of situation, it is necessary to make a convexity adjustment when an interest rate derivative is structured so that it does not incorporate the natural time lag implied by the interest rate. This is the case obviously of in-arrears and in-advance products where the rate is observed and paid at the same time. This is as well the case of the CMS rate where the swap rate instead of being paid during the whole life of the swap is only paid once.

Let us now explain intuitively the convexity defined as the difference between forward rate and expected rate. We examine the case of bond but it is exactly the same analysis for swap rate. Since the relationship between bond price and the bond yield Y is non-linear, it is not correct to say that the expected yield is equal to the yield of a forward bond, and called the forward yield. Similarly, it is not correct to say that the expected swap rate should be equal to the forward swap rate.

This can be well understood by taking a two states world. The bond price can be either P_1, P_2 with equal probability $\frac{1}{2}$. The corresponding yields are Y_1, Y_2. In this binomial world, the expected price P_e is given by the different possible price with their corresponding probabilities $P_e = \frac{1}{2}P_1 + \frac{1}{2}P_2$. The forward yield Y^f is the yield corresponding to the expected price P_e. The expected yield Y_e is the one given by the expected value of the yield $Y_e = \frac{1}{2}Y_1 + \frac{1}{2}Y_2$.

However, since the relationship between price and yield is decreasing and convex, the two given yields, forward and expected one, are not equal and the expected yield Y_e is above the forward one as figure 1 shows it.

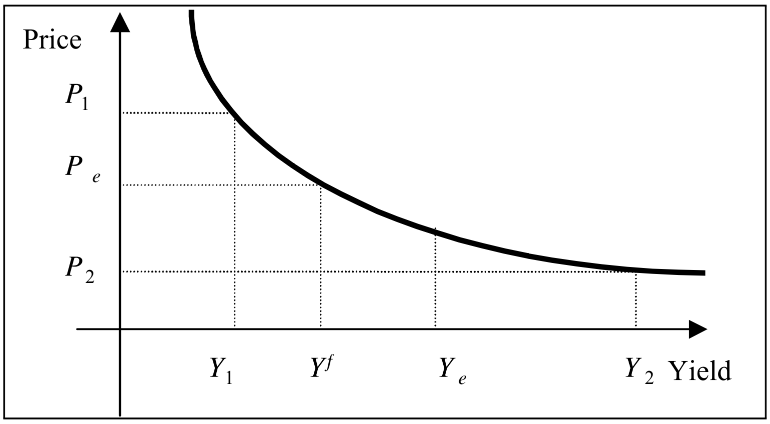

Figure 1: Convexity of the bond price with respect to its yield. This graphic shows that the expected yield denoted by Y_e is higher than the corresponding forward yield Y^f

These results can be derived in a more general stochastic framework. The Jensen inequality on convex functions tells us that the forward price defined as the expected value under the risk neutral probability of the price $\mathbb{E}(P(Y))$ should be higher than the bond price with a yield equal to the expected rate $P(\mathbb{E}(Y))$

$$\mathbb{E}(P(Y)) > P(\mathbb{E}(Y))$$

Using the fact that the bond price is a decreasing function, we get that the expected bond rate defined as the expected value of the yield $\mathbb{E}(Y)$ is higher than the forward bond rate corresponding to the forward price $\mathbb{E}(P(Y))$ ($Y^f = P^{-1}(\mathbb{E}(P(Y)))$). The difference between the expected yield and the forward yield $Y^e - Y^f$ is called the convexity adjustment and defined by

$$Y^e - Y^f = \mathbb{E}(Y) - P^{-1}(\mathbb{E}(P(Y))) \tag{2}$$

With these rough modelling framework, we can already get interesting results. When a bond or a security price is a convex function of the interest rate, the expected bond yield $\mathbb{E}(Y)$ is always above the forward bond yield $P^{-1}(\mathbb{E}(P(Y)))$.

This can as well applied to swap rates. Indeed, a receiver swap, swap where one receives the fixed rate and pays the floating one, is also a convex decreasing function of the swap rate. The only difference comes from the fact that the swap price contrary to the bond price can be negative. This is illustrated by figure 2. Since only hypotheses on the monotony and convexity of the function are required for deriving our result above (2), we conclude that the expected swap rate is above the forward swap rate.

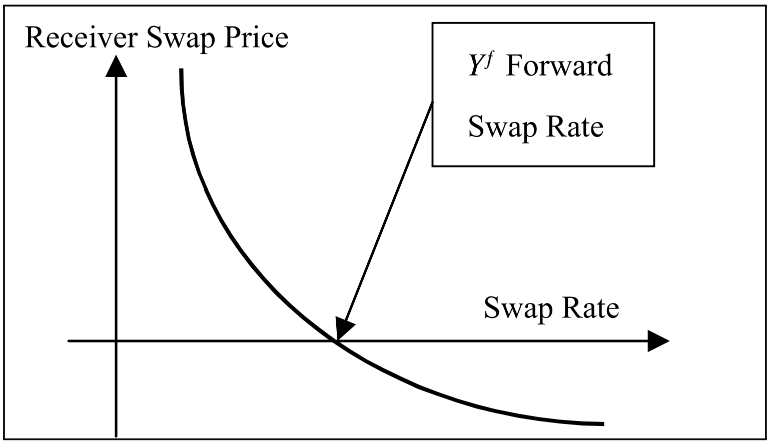

Figure 2: Convexity of the swap price with respect to its swap rate: The relationship between the receiver swap price and the swap rate is convex and decreasing. The only difference between swap and bond contract lies in the possible negative values of the receiver swap

As a general conclusion of this subsection, expected bond yield or swap rate should be higher than the corresponding forward for convex contract and lower for concave one.

2.3 Static hedge: locking the convexity

Intuitively, the difference between the forward yield and the expected yield is due to the fact that the underlying bond price is a decreasing convex function of the yield. We can take advantage of this by a static hedge. Let us consider a continuous trading economy. The uncertainty in this economy is characterized by the probability space (Ω, F, Q) where Ω is the state space, F is the $\sigma-$algebra representing measurable events, and Q is the risk neutral probability measure. We denote by y_t^f the value of the forward yield at time t. We denote by $h\left(y_t^f\right)$ the pay-off of a security depending on the forward yield. We denote by y_0^f the value today of the forward yield. We define by σ the constant volatility of the forward yield at time T when compared with the today forward yield. This means that the square difference between the forward yield at time T and the today value is proportional to the volatility times the time elapsed times the square of the today value of the forward yield:

$$\frac{\mathbb{E}_{Q_T}\left(\left(y_T^f - y_0^f\right)^2\right)}{\left(y_0^f\right)^2} = \sigma^2 T \tag{3}$$

All this analysis is made for yield bond for clarity reason. However, this can be adapted easily to swap rate. We consider the following portfolio:

- a forward contract on the forward yield with a strike at the today value of the forward yield. The payoff at time T is simply the difference between the forward yield at time $T : y_T^f$ and the strike: today value of the forward yield y_0^f.

- a hedging portfolio composed of n forward contract(s) on the bond set at at-the-money strike. The payoff of the forward contract is therefore the difference between the non-linear security pay-off at time T $h\left(y_T^f\right)$ and the price if the yield were the value of the today forward yield $h\left(y_0^f\right)$. This is an hedging portfolio since the variation of the forward contract on the forward yield y_T^f are offset by the variation of the forward contract on the bond. Since the forward contract is set at at-the-money strike, this contract is of zero value.

Since the value of the total portfolio is equal to the sum of its two components, with the second one of zero value, the total value of the portfolio is equal to the value of the first portfolio, given at maturity time T by the expected difference between the forward yield and the today value of the forward yield, which is exactly the definition of the convexity adjustment. The value today of the total portfolio is therefore the convexity adjustment times the zero coupon maturing at time T. The determination of the convexity adjustment is consequently equivalent the one of the global portfolio. Its expression is given by the following proposition:

Proposition 1 *Convexity adjustment*
The value of the portfolio denoted by P *is given by*

$$\frac{P}{B(0,T)} = -\frac{1}{2}\frac{h''\left(y_0^f\right)}{h'\left(y_0^f\right)}\left(y_0^f\right)^2\sigma^2 T \tag{4}$$

Proof: By means of a change of probability measure, from risk neutral to forward neutral probability measure, the price P of the all portfolio can be written as the expected value of the payoff under the forward neutral probability measure Q_T times the zero-coupon bond maturing at the payment time T:

$$P = B(0,T)\,\mathbb{E}_{Q_T}\left(\left(y_T^f - y_0^f\right) + n * \left(h\left(y_T^f\right) - h\left(y_0^f\right)\right)\right)$$

Using a Taylor expansion up to the second order around the today forward yield, we get that the pay-off of the hedging portfolio at time T can be expressed as a simple function of the difference between the forward yield at time $T : y_T^f$ and the today value of the forward yield y_0^f.

$$h\left(y_T^f\right) - h\left(y_0^f\right) = \left(y_T^f - y_0^f\right)h'\left(y_0^f\right) + \frac{1}{2}\left(y_T^f - y_0^f\right)^2 h''\left(y_0^f\right) + o\left(\left(y_T^f - y_0^f\right)^2\right)$$

we can assume that the difference between the value at time T of the forward yield y_T^f and its today value y_0^f is small since the forward yield at time T should be close to its initial value. This is a not very rigorous assumptions but it is an assumption often used by practitioners. The total value of the portfolio can therefore be expressed as a quadratic function of the difference between the value at time T of the forward yield y_T^f and its today value y_0^f

$$P = B(0,T)\,\mathbb{E}_{Q_T}\left[\left(y_T^f - y_0^f\right)\left(1 + nh'\left(y_0^f\right)\right) + \frac{1}{2}n\left(y_T^f - y_0^f\right)^2 h''\left(y_0^f\right)\right]$$

To eliminate the first order risk (role of our hedging strategy), the quantity of the hedging portfolio should exactly offset the variation of the forward contract(up to the first order):

$$n = -\frac{1}{h'\left(y_0^f\right)}$$

The quantity n is positive and confirms that the second component of the global portfolio is a hedge against the variation of the first one. The value of the global portfolio is therefore coming only from the second order risk or gamma risk. Getting all the deterministic term out of the expectation leads to the following expression:

$$P = -\frac{1}{2}B(0,T)\frac{h''\left(y_0^f\right)}{h'\left(y_0^f\right)}\mathbb{E}_{Q_T}\left[\left(y_T^f - y_0^f\right)^2\right]$$

Using the strong assumption (3) about the pseudo "volatility" σ, we get that the price of the total portfolio can be expressed as a function of the today value of the forward yield y_0^f and the parameter of "volatility" σ

$$P = -\frac{1}{2}B(0,T)\frac{h''\left(y_0^f\right)}{h'\left(y_0^f\right)}\left(y_0^f\right)^2\sigma T$$

which is exactly the result (4).□

3 Calculating the convexity adjustment

In this section, we show how to derive the convexity adjustment required when assuming a time dependent volatility, hypotheses similar to the Black model. The difference between our model and the Black model lies in the fact that in our model, the drift term is supposed to be stochastic. However, when we take a deterministic approximation of our drift term, our model becomes a standard Black model.

Our proof is based on the martingale theory. We obtain that the martingale condition implies a strong condition on the drift term of the forward yield. Making approximations, we obtain as a particular case (when the volatility is constant) the traditional formula for the convexity adjustment (1), obtained by Brotherton-Ratcliffe and Iben (1993) and later by Hull (1997). However, the motivation of this approach is to specify the error of the approximation. Monte Carlo simulations prove that the error is relatively small.

3.1 Pricing framework

We consider a continuous trading economy with a limited trading horizon $[0, \tau]$ for a fixed $\tau > 0$. The uncertainty in the economy is characterized by the probability space (Ω, F, Q) where Ω is the state space, F is the $\sigma-$algebra representing measurable events, and Q is the risk neutral probability measure uniquely defined in complete markets (Harrison, Kreps (1979) and Harrison, Pliska (1981)). We assume that information evolves according to the augmented right continuous complete filtration $\{F_t, t \in [0, \tau]\}$ generated by a standard one-dimensional Wiener Process $(W_t)_{t\in[0,\tau]}$.

We assume as well that the price at time t of the bond can be defined as a function $h(.)$ of the bond yield at time t : $\left(y_t^f\right)_{t<\tau}$: $h\left(y_t^f\right)$. The two stochastic variables $\left(y_t^f\right)_{t<\tau}$, $\left(h\left(y_t^f\right)\right)_{t<\tau}$ are supposed to be adapted to the information structure $(F_t)_{t\in[0,\tau]}$. We examine a bond security which payoff is paid at time T. Following the work of El Karoui et al. (1995), the no-arbitrage condition and the markets' completeness assumption enable us to define a unique forward neutral probability measure Q_T, under which the price $h\left(y_t^f\right)$ is a martingale. Under this probability measure Q_T, the volatility of the forward yield rate y_t^f is supposed to have a deterministic volatility function depending only on the time, leading to the following diffusion:

$$\frac{dy_t^f}{y_t^f} = \mu_t dt + \sigma_t dW_t$$

where the drift term is stochastic. Since the volatility is supposed to be a deterministic function of time, this is sometimes referred as a "Black" model. However, the drift is stochastic. It is therefore different from the standard Black model where a deterministic drift. We denote zero coupon bond price at time t, maturing at time $T > t$ by $B(t, T)$. The following theorem gives us the necessary condition for the drift term so that the price $h\left(y_t^f\right)$ is a martingale.

3.2 Convexity adjustment formula

Theorem 2 *Convexity Adjustment formula*
Under the hypotheses above, the drift term should satisfy the following non-

arbitrage condition

$$\mu_t = -\frac{\frac{1}{2}h''\left(y_t^f\right)\sigma_t^2 y_t^f}{h'\left(y_t^f\right)} \tag{5}$$

Proof: the Ito lemma gives

$$dh\left(y_t^f\right) = h'\left(y_t^f\right) y_t^f \sigma_t dW_t + \left[y_t^f h'\left(y_t^f\right)\mu_t + \frac{1}{2}h''\left(y_t^f\right)\left(\sigma_t y_t^f\right)^2\right] dt$$

Under the forward neutral probability Q_T, the price of the bond $h\left(y_t^f\right)$ is a martingale. This implies that the drift term $y_t^f h'\left(y_t^f\right)\mu_t + \frac{1}{2}h''\left(y_t^f\right)\left(\sigma_t y_t^f\right)^2$ should be equal to zero, which leads to the necessary condition (5).□

We take the following definition of the convexity adjustment:

Definition 3 *The convexity adjustment is defined as the difference between the expected yield under the forward neutral probability measure and the forward yield, leading to the exact but non explicit formula:*

$$\mathbb{E}_{Q_T}\left(y_T^f / F_0\right) - y_0^f \tag{6}$$

The above definition provides us with an exact but not tractable formula of the convexity adjustment. Assuming that the drift term can be approximated by its value with the forward yield equal to the today forward yield y_0^f, we get a closed and tractable formula given the following theorem:

Theorem 4 *Under the assumptions above, we can prove that the convexity adjustment for the expected bond yield can be approached by*

$$y_0^f\left(e^{-\frac{1}{2}\frac{h''\left(y_0^f\right)}{h'\left(y_0^f\right)} y_0^f \int_0^T \sigma_t^2 dt} - 1\right) \tag{7}$$

where $h'\left(y_0^f\right)$ *and* $h''\left(y_0^f\right)$ *denote the first and second derivatives of the bond price with respect to its yield* y *taken at the point* y_0^f.

Proof: Calculating the expected yield under the forward neutral probability gives:

$$\mathbb{E}_{Q_T}\left(y_T^f / F_0\right) = \mathbb{E}_{Q_T}\left(y_0^f e^{\int_0^T\left(\mu_t - \frac{1}{2}\sigma_t^2\right)dt + \int_0^T \sigma_t dW_t}\right)$$

Using the fact that we assume that the drift term can be approximated by a purely deterministic formula given when approximating the forward yield rate y_t^f

by its initial value y_0^f (very rough approximation) $\mu_t - \frac{1}{2}\sigma_t^2 \simeq -\frac{1}{2}\frac{h''\left(y_0^f\right)}{h'\left(y_0^f\right)} y_0^f \sigma_t^2 - \frac{1}{2}\sigma_t^2$ leads to

$$\mathbb{E}_{Q_T}\left(y_T^f / F_0\right) \simeq y_0^f \exp\left(-\frac{1}{2}\frac{h''\left(y_0^f\right)}{h'\left(y_0^f\right)} y_0^f \int_0^T \sigma_t^2 dt\right)$$

Consequently, using its definition (3), the convexity adjustment is given by the final result (7).□

An approximation of the theorem formula is then given by a Taylor expansion of the exponential up to the first order, leading to an extension, to time dependent volatility, of the formula Iben (1993)

$$-\frac{1}{2}\frac{h''\left(y_0^f\right)}{h'\left(y_0^f\right)} \left(y_0^f\right)^2 \int_0^T \sigma_t^2 \tag{8}$$

Corollary 5 *Black Scholes formula*
When the volatility is constant, the convexity adjustment derived here leads exactly to the one obtained by Brotherton-Ratcliffe and Iben (1993) and later by Hull (1997)

Proof: Using the approximation formula (8) with a constant volatility leads to the result.□

The calculation in the proof is not very rigorous in the sense that we assumed that the drift term μ_t could be approximated by the initial deterministic value equal to $\frac{-\frac{1}{2}h''\left(y_0^f\right) y_0^f \sigma_t^2}{h'\left(y_0^f\right)}$. A more complex framework should take into account this approximation. This implies two interesting remarks. First, it means that the Black Scholes convexity adjustment used by markets is a very rough approximation formula when assuming a deterministic volatility. One assumes that the stochastic drift term is indeed deterministic. Second, this approximation is highly depending on the initial value of the forward yield rate y_0^f. If this forward yield rate is unstable, one should think about using an average of the past observations.

We can now specify the error term as the difference between our closed formula (7) and the intractable one (6). We can see that in the difference the two term y_0^f simplify each other, leading to an error term given by:

$$\mathbb{E}_{Q_T}\left[y_0^f \left(e^{\int_0^T \left(\mu_t - \frac{1}{2}\sigma_t^2\right) dt + \int_0^T \sigma_t dW_t} - e^{-\frac{1}{2}\frac{h''\left(y_0^f\right)}{h'\left(y_0^f\right)} y_0^f \int_0^T \sigma_t^2 dt} \right) \right]$$

Using a change of probability measure (Girsanov theorem), we can see that this expression is under a probability measure denoted by $\widetilde{Q}$ a difference between two terms, where the Radon Nykodim derivative of $\widetilde{Q}$ with respect to Q_T is given by $e^{\int_0^T \sigma_t dW_t - \frac{1}{2}\int_0^T \sigma_t^2 dt}$. Using the Taylor Lagrange theorem, we get that there exits a parameter θ_t between 0 and t so that this difference of terms can be expressed as the difference between the two rates y_t and y_0^f times the derivatives of the exponential:

$$\mathbb{E}_{\widetilde{Q}}\left[y_0^f\left(e^{-\int_0^T \frac{\frac{1}{2}h''\left(y_{\theta_t}^f\right)y_{\theta_t}^f\sigma_t^2}{h'\left(y_{\theta_t}^f\right)}dt}\int_0^T g\left(y_{\theta_t}^f\right)dt\left(y_t^f - y_0^f\right)\right)\right]$$

where the function $g()$ denotes the derivatives of the function $\frac{\frac{1}{2}h''(y)\,y}{h'(y)}\sigma_t^2$ with respect to y. This is not very satisfactory but it is the only result we get for an estimate of the error term. Indeed, results could be derived with more knowledge about the function h. This implies of course to specify more the diffusion equation of y. Without any further information, nothing very specific can be derived on the error term. Another way to measure the error term is by means of Monte Carlo simulations.

3.3 Monte Carlo simulations of the error

In the previous section, we have assumed that the forward yield rate y_t^f can be approximated by the today value of the forward yield y_0^f. In this subsection, we analyze, by means of Monte-Carlo simulations how big the error is. We consider a derivative that provides the payoff equal to the one-year zero coupon rate in T years multiplied by a principal of 100. For the simplicity of the simulation, we take a constant volatility $(\sigma_t = \sigma)$ equal to 20 % and a forward rate of 10%. Since our bond is a one year zero coupon, its payoff is equal to the discounted value of the unique unity coupon.

$$h(y) = \frac{1}{1+y}$$

The no-arbitrage condition (5) implies that the yield should have the following diffusion

$$\frac{dy_t^f}{y_t^f} = \frac{\sigma^2 y_t^f}{1+y_t^f}dt + \sigma dW_t$$

with the initial value $y_t^f = y_0^f$. The aim of the Monte Carlo simulation is to examine the quality of the approximation done for the convexity adjustment. We compute the expected yield $\mathbb{E}^Q[y_T]$ which is called in table 1 by theoretical yield (calculated with a Sobol sequence Quasi-Monte Carlo with 20.000 draws) and compare it to the approached formula for the convexity adjustment

$$y_0^f\ e^{-\frac{1}{2}\frac{h''\left(y_0^f\right)}{h'\left(y_0^f\right)}y_0^f\sigma^2 T} = y_0^f\ e^{\frac{y_0^f}{1+y_0^f}\sigma^2 T}$$

The results are given in the table 1. These are simulations for different value of the expiry time T : 3, 5 and 10 years. It means that our derivatives is paying the one-year zero-coupon rate determined at time T and paid at time T. The price of this derivative is therefore the forward rate with a convexity adjustment times the principal 100 discounted by the zero coupon bond maturing at time T. The results show that the approximation is quite efficient and can therefore been used as a good estimator of the convexity adjustment required for the derivatives concerned.

Time T	Approached yield	Theoretical yield (MC)	Approached Price	Theoretical Price (MC)
3	10.1097	10.1099	7.59556	7.59572
5	10.1835	10.1844	6.32314	6.32373
10	10.3703	10.3888	4.83783	4.84646

Table 1: Results of the simulation for the expected rate. The simulations show that the rough approximation is quite valid.

4 CMS rate

Interest rate derivatives are often structured as to include a period of time between the date of the fixing of the interest rate and the date of the payment. In general, when the payoff on a derivative depends on a t-period rate, it is common to include a time lag of exactly the maturity of the rate between the fixing of the rate and the corresponding payment. This is traditionally called as the natural time lag. The interesting point is that each derivatives which includes a natural time lag does not need a convexity adjustment (Hull (1997) page 407). However, for some derivatives, this natural time lag is not respected. These derivatives require a convexity adjustment. This is the case of in arrear products and CMS products. This section targets at the problem of the CMS and shows how to apply our results of the section 3 to this particular question.

4.1 Introduction

The CMS rate is the rate of a contract that pays only once the swap rate. Because a regular swap rate should be paid during the whole period, this product includes a modified schedule. Swap price are a convex function of the rates. Therefore, as explained in the first section, the expected swap rate should not be equal to the forward swap rate. The difference should be positive because of the convexity of the function.

This result can be proved in a very basic way. We want to calculate the expected value of an annual swap rate assumed to have n payments at date $T+i$ with $i = 1...n$. Let us denote by y_0^f the forward swap rate, and by y_t^f the swap rate at time t. A useful relationship between a receiver swap price with a fixed rate equal to the forward swap rate and the swap rate is the following. The receiver swap price P_{Swap} is equal to difference between the forward swap rate and the swap rate times the swap sensitivity.

$$P_{Swap}(t) = \sum_{i=0}^{n} B(t, T+i) \left(y_0^f - y_t^f \right) \tag{9}$$

Therefore, introducing this quantity, we get that the expected swap rate can be calculated as:

$$\mathbb{E}_{Q_T} \left[y_T^f \right] = \mathbb{E}_{Q_T} \left[-\frac{\sum_{i=0}^{n} B(T, T+i) \left(y_0^f - y_T^f \right)}{\sum_{i=0}^{n} B(T, T+i)} + y_0^f \right]$$

Knowing that the swap sensitivity $\sum_{i=0}^{n} B(t, T+i)$ is positively correlated with the receiver swap price $\sum_{i=0}^{n} B(t, T+i) \left(y_0^f - y_t^f \right)$ for every time t, we get that the two variables, the opposite of the inverse of the sensitivity of the swap $-\frac{1}{\sum_{i=0}^{n} B(T,T+i)}$ and the receiver swap $\sum_{i=0}^{n} B(T, T+i) \left(y_0^f - y_T^f \right)$ are positively correlated. A simple result is that when two stochastic variables X_1 and X_2 are positively correlated, the expectation of their product is bigger than the product of their expectation

$$\mathbb{E}[X_1 X_2] \geq \mathbb{E}[X_1] \mathbb{E}[X_2]$$

In the case of a strictly positive correlation, the inequality is strict. Since the forward swap is exactly at the money (fixed rate equal to the forward rate), its expected value should be equal to zero. This leads to the final result that the expected swap rate should be higher than the corresponding forward swap:

$$\mathbb{E}_{Q_T}[y_T] > y_0^f$$

4.2 Hedging strategy

The hedging point of view is interesting as well. If an investor who is long a CMS rate hedges it like a forward swap rate, he will make almost surely profit. Let us show how to make an arbitrage in this situation. The hedging strategy should cost today exactly the discounted swap rate $y_0^f B(0, T)$.

Take the following strategy. An investor is:

- long a CMS rate which maturity is denoted by T, with an underlying swap rate of an n years maturity.

He hedges it as if the CMS contract was giving him the forward swap rate. A hedging strategy is to replicate synthetically the forward swap rate:

- long the corresponding forward receiver swap with an amount equal to the inverse of the forward swap sensitivity $\frac{1}{\sum_{i=1}^{n}\mathbb{E}_{Q_T}[B(T,T+i)]}$ with a fixed rate equal to the forward swap rate. Since the receiver swap is with a fixed rate equal to the forward rate, the value today of this swap is 0.

- short at the same time a risk free bond maturing at time T with an investment amount equal to the forward swap rate y_0^f. The value of this zero coupon bond is today $y_0^f B(0,T)$.

We verify that the hedge cost, today, is the discounted forward swap rate $y_0^f B(0,T)$. Let us now examine our total portfolio. It is long a CMS rate, long a forward receiver swap, short a zero coupon. The total value Π_T of the portfolio at time T is:

$$\Pi_T = \left[\left(y_T^f - y_0^f\right) + \frac{P_{Swap}(T)}{\sum_{i=1}^{n}\mathbb{E}_{Q_T}\left[B(T,T+i)\right]}\right]$$

using again the useful relationship between swap price and swap rate (9), we get

$$\Pi_T = \left[\frac{-P_{Swap}(T)}{\sum_{i=1}^{n}B(T,T+i)}\right] + \left[\frac{P_{Swap}(T)}{\sum_{i=1}^{n}\mathbb{E}_{Q_T}\left[B(T,T+i)\right]}\right]$$

Using the fact that to be short the receiver swap is equivalent to be long the corresponding payer swap, the first position is exactly long a payer swap with a stochastic amount $\frac{1}{\sum_{i=1}^{n}B(T,T+i)}$. Denoting by $P_{P\_Swap}(T)$ the price of the payer swap, we get that our total portfolio can be decomposed into two sub-portfolios:

- portfolio 1: the sum of the CMS rate and the zero coupon bond times the forward swap rate. Its value at time T is equal to a payer swap $P_{P\_Swap}(T)$ with a stochastic amount $\frac{1}{\sum_{i=1}^{n}B(T,T+i)}$

- portfolio2: the forward receiver swap with an amount equal to the inverse of the forward swap sensitivity $\frac{1}{\sum_{i=1}^{n}\mathbb{E}_{Q_T}[B(T,T+i)]}$.

Let us examine different scenari for the interest rates.

- If the swap rate realized at time T is exactly the forward swap rate, the two portfolios have zero value.

- If the swap rate y_T^f is above the forward swap rate $-y_0^f$, the portfolio 1 increases because of two things: first, because the payer swap ends in the money and second , because the sensitivity of the swap has decreased, which is equivalent to an increase of the inverse of the swap sensitivity. In contrast, the portfolio 2 decreases only because the receiver swap ends out of the money, and which offsets only the profit realized on the payer swap. Therefore, in this case, the total portfolio will increase.

- If the swap rate is below the forward swap rate, the payer swap ends out of the money whereas the receiver swap ends in the money by the same amount. However, the loss on the payer swap of the portfolio 1 is offset by the decrease of the inverse of the swap sensitivity, leading again, to a positive value for the total portfolio.

As a conclusion, we can see that whenever the swap rate are above or below the forward swap rate, our total portfolio ends in the money. This positive value is due to the convexity effect. We see on this example that the static hedge does not hedge against the convexity term. Since this effect is depending obviously on the importance of the move between the swap rate and the forward one, in either directions, this should be related somehow to the volatility. A hedging strategy that hedges against the convexity term should therefore have a volatility component by including some options like swaptions. However, since swaptions are not perfect substitute for the convexity term, the hedge needs to be rebalance dynamically. Many questions remain unsolved. Which option should I take and more specifically which option maturity and strike should I choose? These questions are depending mainly on the market data for the example examined. The answer is outside from the scope of this paper.

4.3 Pricing CMS rate

The price of a bond that gives the forward swap rate at each different date, and with no principal exchanged at the end of the swap is given by

$$h(y) = \frac{F}{(1+y)^{T_1}} + ... + \frac{F}{(1+y)^{T_n}}$$

This leads to the following calculation for the convexity adjustment denoted by CA (equation (8))

$$CA = \frac{1}{2} \frac{(T_1+1)\,T_1 \frac{F}{\left(1+y_0^f\right)^{T_1+2}} + ... + (T_n+1)\,T_n \frac{F}{\left(1+y_0^f\right)^{T_n+2}}}{T_1 \frac{F}{\left(1+y_0^f\right)^{T_1+1}} + ... + T_n \frac{F}{\left(1+y_0^f\right)^{T_n+1}}} \left(y_0^f\right)^2 \int_0^T \sigma_t^2 dt$$

This shows us that it is only because of some volatility on the swap rate that the CMS rate is different from the forward swap rate. Our result shows that the influence of the volatility is linear in the volatility of the whole process $\int_0^T \sigma_t^2 dt$.

5 Conclusion

In this paper, we have seen that using martingale theory enables us to give a more robust proof of the convexity adjustment formula in the Black framework.

Looking for a definition of convexity, we classify the convexity adjustments into two categories: a correlation convexity, futures versus forward contracts and a modified schedule convexity, mainly the rest of the convexity adjustments. We explain on a static hedge the origin of the convexity. We derive convexity adjustment from a no-arbitrage proposition implied by martingale condition. This enables us to give a definition of the convexity adjustment, with no approximation. Then making approximation, we show how to get a tractable closed formula, which encompassed previous results. We specify the error term between the approached closed formula and the exact but non explicit formula. We show that under certain conditions, this error term can be bounded by a "modified" Laplace Transform of the yield variable. Monte Carlo simulations prove us that the error is relatively small. One can consider the approached formula a good estimate of the convexity adjustment.

There are many possible extensions to this paper. The first one is to relax the hypothesis of a Black diffusion. This is more in agreement with the use of term structure models by financial institutions. However, the problem turns to be non-linear and complex. Its solving requires sophisticated approximation techniques like Wiener chaos, Cramers-Moyal expansion or the theory of stochastic perturbation (see Benhamou (2000) for a discussion and a solution by means of Wiener Chaos). A second development concerns the pricing of in-arrear derivatives. These derivatives are well known for their convexity component. An approximate pricing can be obtained by using forward rates modified by the correct convexity adjustment, as explained in this article. Last but not least, the same methodology could be applied to the convexity adjustment of futures against forward contracts, fact that has been studied empirically by French (1983), Park and Chen (1985) and Viswanath (1989) and that is still little explored.

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