Monopolistic Pricing in the Banking Industry: a Dynamic Portfolio Model

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Abstract

This work develops a portfolio model of the banking firm where both the size and composition of the portfolio are jointly determined. The model provides a quite simple micro-foundation of the credit channel of the transmission of monetary policy. It allows analysing the pricing policies of the banking firm, and shows how interest rate shocks and credit quality shocks (the real shocks that change expected default costs) affect the equilibrium level of loans and deposits.

1 Introduction

The aim of this work is to analyse the behaviour of banking intermediaries as interest rate shocks and credit quality shocks affect the economy. In particular we want to study how the equilibrium level of loans and deposits changes, and how the pricing policies of banks are affected, by different types of shocks.

To address these problems we have chosen to build up a general portfolio model, based on the assumption that the banking firm is risk-neutral. Standard portfolio models, explicitly or implicitly, superimpose a concave utility function on the profit function of the bank, so that the second moment of the probability distribution of the underlying independent variables matter. The problem with this approach is that the concavity of the function is assumed rather than properly justified, and this assumption contrasts with standard microeconomic practice. This kind of assumption has been justified on the basis of the existence of non-linear bankruptcy costs, which influence the decision process of managers, delegated agents of the shareholders. This approach is problematic, though. Firstly, it is not general, since it could not describe the behaviour of private firms. Secondly, bankruptcy costs cannot be strictly convex, as the approach would require: when managers' private information makes them believe that the value of assets is lower than value of liabilities, they have an incentive to bear any risk, since they have nothing to loose. Besides this analysis does not take into account another very important factor that works in the opposite direction. When the value of the firm is divided between equity and debt, it can be shown that debt is a concave function, while equity is a convex function, of the value of the underlying investment projects.[1] The risk attitude of banks, which have peculiar liabilities and assets, and of industrial firms, must be completely different. For this reason it is necessary to rely on properly micro-founded models, where the nature of all relevant revenue and cost factors is properly specified. In other words, the risk attitude of the firm cannot be assumed *a priori*, it must be an endogenous outcome of the model.

Banks provide jointly two different sets of services: payment services and financial intermediation. In principle the banking firm should be treated as a multi-product firm, separating the interest rate margins that constitute revenues from the intermediation services from the fees generated by the payment services. Despite that, the almost entire analysis of the banking industry has focused almost exclusively on financial intermediation. The study of the payment industry is in fact quite straightforward as long as transaction costs are kept out of the picture, since, as Fama has shown,[2] general equilibrium analysis can be applied in a quite standard fashion. Most of the peculiarities of the industry seem accordingly to depend on the peculiar structure of assets and liabilities of banks, which depend on the intermediation services that banks provide.[3] As compared to other industries, the standard treatment of banking firms results in a more detailed treatment of some aspects of the industry, including notably the relationship between deposits and other assets, loans in particular. As a counterpart there is a great deal of simplification in other regards. The cost functions are simple, and there is no technical progress. This has permitted to study the effect of transaction costs modifying the basic framework.[4]

In this work we emphasize the role of payment services, introducing industrial cost functions which reflect some of the main features of the industry. We show that in presence of imperfect information, the focus on the joint provision of financial intermediation and payment services produces a richer, dynamic, framework. Transaction costs (search costs in particular, which are empirically significant in both the market for loans and the market for deposits[5]), produce in fact two important consequences: they cause path dependence in the demand functions and they generate market power.[6] We show that in this

[1]See Merton [24] and Flannery [15].

[2]See Fama [13].

[3]Sealey and Lindley have reconciled the literature on banking intermediation with the traditional theory of the firm, showing that deposits have to be considered an intermediate product that enters as an input in the production function of the final product: loans facilities. See Sealey and Lindley [23]

[4]For example, because of the relevance of search costs, deposits have been modelled as a quasi-fixed input.

[5]See for the market of the US the empirical analysis of Flannery [14] and Hess [19] and [20].

[6]See Salop [29] and Salop and Stiglitz [30] and [31]. Beside, a vast literature has shown that banks

situation, any non-linear cost function generates implicit adjustment costs. Thus we have a dynamic problem even without explicitly postulating *ad hoc* adjustment costs for the stock of deposits or loans.

We specify just one, very simple, non-linearity: a stochastic default cost function. Such a cost function captures a fundamental aspect of the banking activity, the ability of banks to finance opaque investment projects whose risk the market cannot price. We still rely on the standard simplifications, such as constant technology and constant returns to scale in the payment services provision.

A crucial assumption of the model is that the quantity of loans issued affects the behaviour over time of the demand for deposits. This kind of relationship is implicitly assumed in macroeconomic monetary models, whenever concepts such as *inside money* or *endogenous money creation* are used.[7] We provide a simple micro-foundation of such a process, based on the assumption that banks compel borrowers to deposits a fraction of every loan issued.[8] This process allows the banker to choose optimally the amount of loans issued in order to obtain the optimal size of the portfolio. Loans are an "investment" that generates deposits. We assume that the asset portfolio of the bank is composed of loans and bonds. The equilibrium composition of the portfolio and the size of the portfolio are thus jointly determined, in contrast to the traditional assumption of portfolio separation.[9] We obtain a very simple solution for the value of the roots, and it is possible to analyse both the dynamic properties of the system and the equilibrium composition of the portfolio.

Another important peculiarity of this work is that all variables are defined in real terms, which contrasts with the standard practice of the literature on banking, even if it is more in line with the standard assumptions of the theory of the firm. The need to define the variables in real terms comes from the dynamic properties of the model. Defining the portfolio in real terms makes it possible to have real variables that have stable growth ratios even when the nominal variables diverge. This approach is particularly valuable since the evolution of the financial systems over the last decades has seen a continuous growth of the size of banking intermediaries while the banking industry's share of the financial intermediation has declined. In general, the treatment of the financial sector in real terms suggests the neutrality of money, and it poses the question of price level determinacy. However we finesse both issues here by introducing the price level

benefit from monopoly power in both the market for deposits and the market for loans.

[7] The peculiar institutional framework of contemporary banks, based on the joint provision of depository and lending services, can be explained viewing the bank as an institution specialized in the provision of liquidity on demand to both households and firms. See Diamond and Rajan [10].

[8] As we will show the available empirical evidence on firms' demand for money supports this assumption.

[9] Elyasiani, Kopecky and Van Hoose [11] have shown that the hypothesis of portfolio separation in the case of banking intermediaries is not empirically supported.

as an exogenous process. This permits moving directly from real to nominal variables. Therefore this analysis of the banking firm does not imply money neutrality (since the exogenous price level process can affect real bank behaviour) and does not imply price level indeterminacy.

Finally, monetary policy choices are not explicitly introduced in the analysis, even if discount window borrowing would be possible to study readily. Besides, the construction of a general equilibrium model of the market for payment services is beyond the scope of this work and we simply assume the existence of substitution between currency and deposits. We do not explicitly introduce the market for currency, relying on a reduced form equation that describes the demand for deposits as a function of the main relevant variables. An example of a simple micro-foundation of our assumptions in a general equilibrium framework can be found in Freeman and Kydland [16].

Three important limitations of the model must be spelled out. We assume price and cost flexibility and neutrality, so that inflation has no direct effect on costs and revenues. The only market imperfections we consider are linked to limited information. This limited information is the underlying source of both market power and the peculiar structure of the cost functions in the model. Secondly, we choose not to deal with liquidity problems, on the assumption that they are adequately managed through compulsory reserve requirement and deposit insurance. Liquidity costs could easily be introduced in the model, but they would complicate the results without increasing the understanding of the problems that we want to study. Finally, we largely disregard the influence of net worth, and we introduce no markets in bank equity in the analysis. We discuss this limitation of the model to some extent, although in quite general terms. This limitation is almost standard in microeconomic theory of banking. It is so even though the role of banking intermediaries usually rests on limited availability of information, as here, and when information is not perfect the Modigliani-Miller theorem does not hold. Consequently, the composition of the liabilities of the firm matters and equity markets can have a role. This reinforces the fact that, while standard, our disregard of equity markets is a limitation of our work.

2 The environment

2.1 The banking firm

The model is in discrete time, and has the following time structure. At the beginning of every period, households and firms dispose of a certain amount of funds from previous periods. Households take decisions regarding their portfolio allocation and their consumption plans for the period. Firms plan their investments for the period and determine their finance needs. Deposits serve in order to carry out transactions and as a financial investment. At the end of every period households and firms dispose of an amount of funds that reflects the evolution of the value of their assets, the income of the period and their consumption choices. Firms assure themselves the liquidity they require to carry out their transactions by resorting to loans. The feedback process of loans on deposits that we describe can be understood as resulting from the provision of liquidity to firms: at the end of the period households receive part of the liquidity generated trough loans. This assumption fits well with Ramey's [28] findings of cointegration between M1 and business M1.

The bank can invest its deposits in two types of assets: loans and bonds. Besides the bank must hold a fraction of its deposits as reserves (that could possibly provide a return).

2.1.1 Cost functions

The analysis of the problem of the banking firm in its most general form is impossible without specifying a simplified cost structure. A frequent solution is to suppose separability of the cost structure in terms of the three major components of the portfolio: bonds, loans and deposits. Formally:

$$C(K,L) = C\big(D(K,L)\big) + C\big(L(K,L)\big) + C\big(B(K,L)\big).$$

This is a simplification we will follow.[10]

We choose to describe the cost of servicing deposits and loans as a linear function of the quantity. The cost of check clearing and other desk operations is in fact linked to the number of transactions made by the customers, but for simplicity we can assume that the cost is proportional to the amount of deposits and loans,[11] since there are no obvious reasons for it to be convex. In fact, the cost might be concave, because of an element of fixed costs. But within the context of an infinite horizon problem without entry or exit, fixed costs can be neglected. Besides, the large empirical literature regarding the existence of scale economies in the banking system has not led to undisputed conclusions.[12] The

[10] This formulation supports the existence of a separate production function for each class of assets and for deposits. The simplification is not a big problem as long as the eventual economies of scope between assets and liabilities or among assets are not crucial for the problem studied. The available empirical evidence on the relevance of economies of scope among different components of the portfolio has not produced any conclusive result, and is quite controversial. This is not surprising, though, because complementarities and economies of scope do not arise between the provision of deposit services and loans, as normally assumed. They arise between the two separate economic functions that banks fulfil: the provision of payment services and financial intermediation. Consequently, the empirical analysis is complex, because revenues and cost of one service are often confused with revenues or costs of the other and vice versa.

[11] A detailed study of the industrial costs of deposit is provided by Osborne [27], and our assumptions are compatible with it.

[12] The most recent empirical evidence regarding the return to scale of banks is in Weelock and Wilson

fact that banks of widely different size survive in almost every country would indicate that economies of scale certainly are not overwhelming. We can assume that:

$$\frac{\partial C(D_t)}{\partial D_t} > 0 \qquad \frac{\partial^2 C(D_t)}{\partial {D_t}^2} = 0; \qquad \frac{\partial C(L_t)}{\partial L_t} > 0 \qquad \frac{\partial^2 C(L_t)}{\partial {L_t}^2} = 0. \tag{1}$$

We can express the relevant cost functions simply as:

$$C(D_t) = uD_t \qquad C(L_t) = zL_t, \tag{2}$$

where u and z are positive real numbers.

The assumptions regarding the structure of costs linked with the provision of financial intermediation services are crucial. Banks normally face two principal kinds of costs: default costs and liquidity costs both stochastic. They are in fact essentially due to the uncertainty resulting form shocks that may hit borrowers or depositors. The former shocks might lead to defaults by borrowers, the latter may cause a bank run. We will focus on default costs.

One of the most relevant functions of banks is to evaluate uncertain investments, whose risk the market cannot price because it lacks the information necessary to attribute a probability distribution to the outcome of the investment. To undertake this activity banks must invest resources to obtain and process the relevant information. We assume that the returns of the investment in information are decreasing, since the available stock of knowledge represents a binding constraint.[13] Besides we assume that the outcome of this investment is uncertain, since it may depend on factors whose uncertainty is radical.[14] We formalize this idea introducing a stochastic quadratic default cost function on loans.

Formally:

$$D(L_t) = \frac{1}{2} v L_t^2, \tag{3}$$

where

$$\frac{\partial D(L_t)}{\partial L_t} > 0 \qquad \frac{\partial^2 D(L_t)}{\partial {L_t}^2} < 0, \tag{4}$$

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[40]. They showed that after 1985 there is evidence of increasing returns to scale for small and medium size banks, while the restriction of constant returns to scale could not be rejected for large banks. The finding of relevant return to scale is probably due to the progressive deregulation of the banking sector.

[13]This implies that banks cannot increase direct lending at will without reducing the efficiency of their monitoring and screening processes. Increasing direct lending indefinitely would mean that sooner or later they would finance investment projects of decreasing quality.

[14]The concept of radical uncertainty has been largely used by Keynes and Shackle to describe the outcome of processes that the existing knowledge does not allow to forecast with a reasonable degree of accuracy. One of Keynes' examples is the price of copper in fifty years time.

and with

$$v = v_d + \varepsilon^d \qquad \text{with} \qquad E[\varepsilon_d] = 0 \quad E[\varepsilon_d^2] = \sigma_d^2. \tag{5}$$

We assume the cost functions to be constant over time, in order to obtain a closed-form solution. Under the assumption of rational expectations, cost functions are homogeneous of degree one with respect to inflation; both marginal costs coefficients and interest rates are proportionally shifted by variations of the price level. The same cost functions can be used to describe the problem of the bank when the relevant variables are defined in nominal terms, or when they are defined in real terms, as a ratio over the price level.

Finally, we assume that the market for bonds is efficient and competitive. This implies that expected default costs on bonds are a linear function of the quantity purchased. Consequently we do not need to introduce explicitly an expected default cost function for bonds, assuming that the relevant interest rates on bonds are net of default costs.[15]

2.1.2 Revenues

The main stream of profits of the bank stems from the difference between the interest rate r_{Lt}, that the bank charges on loans, and the interest rate r_{Dt}, that it pays to depositors. As mentioned, for simplicity we assume that banks do not buy shares, and that the only available alternative to the issue of loans is the purchase of bonds. The alternative source of revenues is given by the spread between the interest rate on bonds r_{Bt} and the rate on deposits.

We assume that the rate on bonds is set exogenously, and that the bank is price taker in the market for bonds. Since banks normally hold reserves in the form of cash, or non-interest-bearing deposits at the central bank, reserves do not provide a return, and reserve requirements are for the bank equivalent to a tax on deposits.

We assume that expectations are rational and financial markets are efficient. Consequently, expected inflation is always fully incorporated in all market interest rates, and we can consider all interest rates to be real interest rates.

2.1.3 The budget constraint

The budget constraint is the following:

$$L_t^N + F_t^N + R_t^N = D_t^N + NW_t^N, \tag{6}$$

[15] The introduction of non-linear default cost for bonds would complicate the analysis without changing the results in a relevant way.

where the upper index indicates that the variables are in nominal terms. The value of F_t^N represents the amount of assets that are invested on assets, such as bonds. The value of L_t^N represents the amount of loans issued by the bank. NW_t^N is the net worth of the bank, and we assume that it remains constant over time: $NW_{t+1}^N = NW_t^N = NW^N$. Because of a monopolistic framework, profits are not pushed down to the normal rate. In addition, we assume that there is a one hundred per cent dividend payout, so that all profits are distributed to shareholders in every period. We then define the variables in real terms, as ratios with respect to the price level, and real deposits become $D_t = \frac{D_t^N}{P_t}$, loans become $L_t = \frac{L_t^N}{P_t}$, and analogously for the other variables. Thus we obtain:

$$L_t + F_t + R_t = D_t + NW, \tag{7}$$

The bank can buy securities or invest in loans only the part of deposits that it does not keep as reserve. Defining with q the legal reserve coefficient, so that $R_t = qD_t$, the equation becomes:

$$L_t + F_t = (1-q)D_t + NW_t. \tag{8}$$

2.2 The market for banking services

2.2.1 The demand for deposits

Households and firms demand deposits not just as a financial asset for portfolio allocation, but mainly because banks provide them with transaction services.[16] The provision of payment services implies the establishment of mutual trust between bank and depositor. This generates substantial search costs for the depositors and transaction costs for the bank, which implicitly furnishes a guarantee to the counterparts of the transactions undertaken by its customers. Banks consequently charge depositors a fixed cost for the provision of deposit services, which makes even more expensive to hold multiple bank

[16] The market for payment services has always been highly competitive. Historically, commercial banks needed to compete with note-issuing banks (prior to the arrival of state-owned central banks). In order to get remunerated for the payment services that they provide (by means of checks, bookkeeping entries and credit cards) banks charge fees on the transactions undertaken. On the contrary, transactions by means of banknotes, whose technology is much simpler and cheaper, do not require the payment of fees. As a consequence, commercial banks have to attract depositors offering an interest rate that banknotes do not pay. The technological developments of the 20th century have reduced the competitive pressure from banknotes, whose role has become smaller. But new competitors have come out. At the beginning of the twentieth century savings institutions, which were developed initially exclusively to provide financial intermediation services, have been allowed to provide payment services by means of the gyro. Only later they have been allowed to issue loans, becoming in all respect analogous to commercial banks. More recent technological developments have allowed money market mutual funds and other financial intermediaries to provide many of the payment services that banks provide at a low cost. As a consequence the need to pay interest rates has increased.

accounts.

Firms may find it profitable to hold multiple accounts, but their choice of the optimal allocation of their deposits among banks is a function of their need to pay for finance. Since firms are normally net debtors, their main financial problem is the minimization of the cost of debt, and they normally hold deposits exclusively to the extent that it is necessary in order to manage their commercial transactions. In practice though, banks compel firms to deposit a fraction of the loans they issue. In the logic of simple intermediation services this type of behaviour is apparently not very logical, since banks apparently give with one hand and take back with the other. But it is perfectly consistent, taking into account the provision of payment services. Banks compel borrowers to deposit the liquidity that firms hold in order to manage their payments and to face any type of shocks in order to manage the transactions of the borrowers, earning fees on the payment services provided. An explanation of this kind was suggested by Sprenkle to explain the actual amount of firms' deposits, since it is impossible to justify the observed amount on the basis of inventory theoretic models.[17]. This behaviour plays a fundamental role in banking intermediation, and generates the economies of scope between the two different types of services, financial intermediations and payment services. By means of this kind of implicit contractual agreements, bankers can monitor the liquidity of the borrowers in real time, obtaining the fundamental stream of information that allows them to evaluate and price the risk of the firms' investment projects. This simple link between the amount of loans issued and the amount of firms' deposits allows a simple formalization of the process of liquidity creation, due to the convertibility of deposits on demand.

We can conclude that because of the relevance of search costs in the provision of transaction services, depositors do not easily switch from one bank to another when fees and interest rates are marginally changed. Flannery [14] and Hess [20] have conclusively shown the empirical relevance of transaction costs (search costs in particular) in the market for deposits. Deposits are, in fact, increasingly described as quasi-fixed inputs. Since search costs allow the banking firm to charge non-competitive prices,[18] we assume that monopolistic competition is the normal market structure.[19]

Banks need to pay an interest rate on deposits because of the competition of intermediaries other from normal commercial banks, such as money market mutual funds. These inter-industry competitors can in fact offer interest rates not too far from those on bonds. As a consequence we will assume ultimately that each bank has some monopoly power on the price of deposits, while market interest rate on bonds affect negatively the demand

[17] See Sprenkle [35] and [36].

[18] See Salop [29] and Salop and Stiglitz [30] and [31].

[19] The existence of intra-industry monopoly power in the banking industry of the US has been empirically confirmed by Cosimano and Mc Donald [8].

for deposits.

In order to obtain a demand schedule for deposits services, we decompose the demand of households and firms. The demand of both classes of agents is assumed to depend on two different interest rates, the own rate on deposits and the rate on bonds, that is an opportunity cost. For simplicity, we assume that transaction fees do not affect the demand for deposits. This can be justified considering interest rates on deposits to be net of transaction fees, which is acceptable as long as the average amount of transactions conducted for a given sum deposited is constant. Having ruled out the influence of technology shocks, this simplification should be acceptable.

Following the usual assumption, we model agents' transaction demand for deposit services as a function of income. We assume that real output is an AR(1) process:

$$Y_{t+1} = \gamma_Y Y_t + \varepsilon^Y_{t+1} = Y_t + g_Y Y_t + \varepsilon^Y_{t+1}. \tag{9}$$

We also assume that the general price level is another AR(1) process, defined as

$$P_{t+1} = \gamma_P P_t + \varepsilon^P_{t+1} = P_t + \Pi P_t + \varepsilon^P_{t+1}. \tag{10}$$

Nominal household's deposits depend (positively) on the level of nominal income,[20] the own interest rate on deposits and (negatively) on the interest rate on bonds. But the household income demand for nominal deposits is less than unitarily elastic:

$$ID^N_t = (Y_t P_t)^{1/\eta}, \qquad \eta > 1 \qquad . \tag{11}$$

As a result of the behaviour over time of income and prices, in the next period,

$$E_t[ID^N_{t+1}] = E_t[(\gamma_Y Y_t + \varepsilon^Y_{t+1})(\gamma_P P_t + \varepsilon^P_{t+1})]^{1/\eta}. \tag{12}$$

For simplicity, we assume that the correlation between the two error terms is zero.

$$E_t[ID^N_{t+1}] = (\gamma_Y \gamma_P)^{1/\eta}[Y_t P_t]^{1/\eta}. \tag{13}$$

Thus:

$$E_t[ID^N_{t+1}] = (\gamma_Y \gamma_P)^{1/\eta} ID^N_t. \tag{14}$$

Expressed in real terms, the demand for deposits is:

$$E_t ID_{t+1} = \frac{ID^N_{t+1}}{P_t + 1} = E_t\left[\frac{(\gamma_Y \gamma_P)^{1/\eta} ID^N_t}{P_{t+1}}\right] = \frac{(\gamma_Y \gamma_P)^{1/\eta} ID^N_t}{\gamma_P P_t} = \delta \frac{ID^N_t}{P_t} = \delta ID_t, \tag{15}$$

[20] Since their transaction demand is assumed to be a function of income.

$$\text{with} \qquad \delta = \frac{(\gamma_Y \gamma_P)^{1/\eta}}{\gamma_P}. \tag{16}$$

To the income component of the demand for deposits, we must add the interest rate component:

$$D_t^h = ID_t + f_2 r_t^D - f_3 r_t^B. \tag{17}$$

Interest rates on bonds are assumed to follow a pure random walk process:

$$r_{t+1}^B = r_t^B + \varepsilon_{t+1}^B \tag{18}$$

$$\text{with} \qquad E[\varepsilon_t^B] = 0 \qquad E[\varepsilon_{t+i}^B \varepsilon_{t+j}^B] = \sigma_B^2 \quad i = j, \quad E[\varepsilon_{t+i}^B \varepsilon_{t+j}^B] = 0 \quad i \neq j.$$

Banks ultimately set interest rates on deposits as a function of the rate on bonds. Consequently the rate on deposits becomes a linear combinations of the rate on bonds and some non stochastic parameters. So they can be assumed to follow a different random walk process, which is correlated with the one on bonds:

$$r_{t+1}^D = r_t^D + \varepsilon_{t+1}^D \tag{19}$$

$$\begin{aligned} \text{with} \qquad & E[\varepsilon_t^D] = 0 \quad E[\varepsilon_{t+i}^D \varepsilon_{t+j}^D] = \sigma_D^2 \quad i = j, \quad E[\varepsilon_{t+i}^D \varepsilon_{t+j}^D] = 0 \quad i \neq j \\ \text{and} \qquad & E[\varepsilon_{t+i}^D \varepsilon_{t+j}^B] = Cov(DB) \quad i = j, \quad E[\varepsilon_{t+i}^D \varepsilon_{t+j}^B] = 0 \quad i \neq j. \end{aligned}$$

Consequently,

$$E[D_{t+1}^h] = \delta ID_t + f_2 r_t^D - f_3 r_t^B. \tag{20}$$

$$E[D_{t+1}^h] = \delta D_t^h + (1-\delta) f_2 E[r_{t+1}^D] - (1-\delta) f_3 E[r_{t+1}^B]. \tag{21}$$

I assume that firm's real deposits depend on both rates as before, and on the quantity of real loans issued by the bank. We assume that deposits depend on the amount of loans of the current period.[21]

$$D_t^f = \kappa L_t + f_4 r_t^D - f_5 r_t^B. \tag{22}$$

The coefficient κ captures the effect of the feedback of loans on deposits. For simplicity loans are assumed to affect deposits for the following period only.[22] We make the further

[21] Alternatively the dependence can be assumed to be lagged, and deposits of the current period depend on loans of the previous one. It can be shown that the results do not change in a relevant way.

[22] This assumption is necessary in order to make the model tractable. But it can be justified considering that the lag in the operation of the feedback should not be too long: firms keep part of their loans as deposits, and in general most of the portfolio of retail banks is made up of short-term loans.

assumption that:

$$(1-q)\kappa > 1-\delta. \tag{23}$$

This assumption guarantees that issuing loans will raise deposits because of sufficient inertia in the exogenous component of the demand for deposits. The assumption is not very restrictive since $1-\delta$ is always very small, as can be easily verified, and the value of q is also small, unless the reserve coefficient is enormous.[23] [24]

Summing deposits of firms and households we can obtain the expected level of deposits of the bank as:

$$E[D_{t+1}] = \delta D_t + g_3 E[r^D_{t+1}] - g_4 E[r^B_{t+1}] + \kappa E[L_{t+1}] \tag{24}$$

$$\text{where} \qquad g_3 = [f_4 + (1-\delta) f_2] \qquad \text{and} \qquad g_4 = [f_5 + (1-\delta) f_3].$$

With our formulation, interest rates on bonds and deposits affect equilibrium levels, rather than the dynamic behaviour of deposits. In the long run in fact the former should depend on the productivity of capital and the average time-preference coefficient, while the latter should be different only for the marginal costs of banking services. These factors are unlikely to follow a trend, either deterministic or stochastic. This is reflected in our assumption of a unit root in the stochastic process. Indeed, the interest rate is increasingly modelled as a mean-reverting stochastic process, such as the Uhlenberg-Ulbeck one in continuous time.

In conclusion, the demand for deposits services has three components. One component is completely exogenous and cannot be influenced by the bank in any way. It depends on the behaviour of income and prices. The second depends on the portfolio choices of the bank, and results from the fact that a fixed proportion of loans feeds back into deposits. The third component is the interest rate on deposits.

2.2.2 The demand for loans

The costliness of information generates monopoly power in the market for loans. Relationship lending in fact allows the bank to price monopolistically, and the higher return due to the market power makes the higher risks of the project worthwhile.[25] All the avail-

[23] In countries like the UK its value is zero.

[24] When this condition is not satisfied, monetary authorities never need to worry about the influence of the issue of loans on part of banks. With narrow banking this would be the case, the condition would in fact become $\delta > 1$, and it would never hold.

[25] Sharpe [34] has shown that establishing long-term relationships with its customers, a bank learns more than others about the business and the capability of the borrower. This information asymmetry generates a rent that allows banks to finance risky projects whose information is very opaque, which cannot be financed

able empirical evidence confirms that the direct lending activity of the banking industry is scarcely competitive.[26] Consequently we introduce in the problem of the bank a demand curve for loans, which the banks estimates.

We assume that the demand for loans can be summarised by the following linear equation:

$$E_t[L_{t+1}] = a - bE_t[r^L_{t+1}] + dE_t[r^B_{t+1}] + \eta_{t+1}, \tag{25}$$

where η_{t+1} is a white noise error term, and we do not explicitly specify the factors that affect the intercept.[27]

3 Solution

3.1 Intertemporal maximization

The firm maximizes its expected profits over an infinite horizon period. The problem of the banking firm, for every pair of positive real numbers (v, u), can be expressed as:

$$\underset{\{F_t, D_t\}_0^\infty}{Max}$$

$$\Pi = \sum_{t=0}^{\infty} \beta^t \left[r^L_t L_t + r^B_t F_t - r^D_t D_t - \frac{1}{2} v L_t^2 - u_t D_t - z L_t \right], \tag{26}$$

s.t.

$$L_t + F_t + R_t = D_t + NW, \tag{27}$$

$$R_t = qD_t, \tag{28}$$

$$L_t = a - br^L_t + dr^B_t, \tag{29}$$

$$D_t = \delta D_{t-1} + \kappa L_t + g_3 r^D_t - g_4 r^B_t. \tag{30}$$

in the market. Establishing the relationship and developing their knowledge, banks provide a valuable service, they create the knowledge necessary to price the risk. The price that firms pay for this service is the monopolistic rent that they pay on loans.

[26]The empirical tests for the presence of market power were traditionally performed studying the behaviour of the rate on loans, which has been found to be stickier than the rate on bonds, in different estimates conducted in different periods of time and different countries. This evidence though was not conclusive, since the stickiness of the rate can be explained as well as the outcome of credit rationing, or the result of implicit contracts for the smoothing of interest rate shocks. An important recent result has been provided by Cosimano and Mc Donald [8], who, studying the effect of a change in reserve requirements on bank profits, have conclusively proved that banks in the US exploit significant market power in the market for loans.

[27]For an analysis of the factors that affect the intercept term of a linear demand curve for loans, see Bertoni Mazzoleni and Szëgo [4].

The logical structure of the profit function is very simple: revenues come from the interest rate spreads, the costs that must be deducted are the cost functions, as previously defined. The discount factor is $\beta^t = \frac{1}{(1+r)^t}$, where r, is the banker's discount rate. I assume that the correlation among all error terms are zero, except for the positive correlations between respectively, the default cost and the interest rate on bonds, and between the demand for loans and the interest rate on bond. Deposits are the state variable of the problem, while F_t, the amount of bonds held in the portfolio, is the control variable of the bank.

3.1.1 Monopolistic pricing and the dynamic constraint

Some features of the model are standard: the demand function for loans, in particular, solves for the interest rate on loans, and its value is substituted in the profit function. The main peculiarity of the model lies in the deposit demand schedule, because its presence makes stocks relevant, and the model becomes dynamic. To understand why, the equation can be solved for the quantity of loans. From:

$$D_t = \delta D_{t-1} + g_3 r_t^D - g_4 r_t^B + \kappa L_t, \tag{31}$$

the following obtains:

$$L_t = \frac{1}{\kappa}\Big\{D_t - \delta D_{t-1} - g_3 r_t^D + g_4 r_t^B\Big\}. \tag{32}$$

Substituting this function for L_t in the profit function, we can observe that the quadratic cost on loans works as a quadratic adjustment cost on deposits. The model thus becomes formally identical to a standard dynamic one.[28] But its structure is much simper than that of other dynamic models of banking since we did not need to introduce *ad hoc* other adjustment cost functions.[29]

In its dynamic properties the model is very close to an investment model. Deposits are the state variable, and play the role of capital. The bank can increase the stock of deposits issuing loans, so that the quantity of loans is akin to the level of investment. The only difference is that we have chosen the quantity of bonds held in the portfolio, F_t, as a control variable, so that loans are obtained residually. The model would be simpler adopting the quantity of loans as a control variable, since the solution would then be reduced to a first order difference equation. But the choice of the quantity of bonds makes it possible to obtain a simultaneous solution for both the optimal size and the optimal composition of the portfolio. Thus, this alternative solution allows a much richer framework. Given

[28] It can be shown that a different specification of the feedback process would produce identical results.

[29] Most dynamic models of banking, such as Elyasiani, Kopecky and Van Hoose [11] and Cosimano [6] and [7], simply assume the presence of quadratic adjustment cost for deposits, loans or both.

this preferable choice, the solution is composed of a system of two first order difference equations.[30]

3.1.2 Euler equation

After some manipulations (shown in the appendix), the following difference equation can be obtained from the Euler equations of the problem:

$$E[F_{t+1}] = \frac{1-(1-q)\kappa}{\delta\beta}F_t + \frac{(1-q)\left\{\delta^2\beta - [1-(1-q)\kappa]\right\}}{\delta\beta}D_t + \\ + \frac{\beta\delta + (1-q)\kappa - 1}{\delta\beta}NW + (1-q)X_t + \frac{1-(1-q)\kappa - \delta H}{\delta\beta}E\left[\frac{1}{\alpha}Z_{t+1}\right], \tag{34}$$

where:

$$(bv+2)/b = \alpha \qquad \varepsilon_t^L = \frac{1}{b}\eta_t, \tag{35}$$

and

$$Z_{t+1} = \left[\left(1-\frac{d}{b}\right)(\delta\beta - L) + (1-q)\kappa L\right]r_{t+1}^B - (\delta\beta - L)\varepsilon_{t+1}^L \\ +\kappa\left[r_t^R q - r_t^D - u\right] + (\beta\delta - 1)\left(z - \frac{a}{b}\right), \tag{36}$$

$$X_t = g_3 r_t^D - g_4 r_t^B. \tag{37}$$

Equation (34) together with the original dynamic constraint, given by the demand condition, (which we rewrite after substituting the budget constraint) form a system of differ-

[30] The model could easily (at lest in abstract) be modified to a Cournot model, without altering the main results. The problem of every individual bank would in this case include the market share as an unknown of the problem, and it would take into account the result of the same optimisation problem performed by the others banks. We would now have n firms facing the respective n maximization problems, that include the problems of the competitors in the price setting equation. And each individual firm's problem would now include as an unknown the value of the market shares $\psi = \frac{L}{L_j}$ and $\chi = \frac{D}{D_j}$. The n equations would provide the optimal supply functions. The condition of aggregation of the loan and deposits supply schedules provides the two extra equations that allow closing the system:

$$L = \sum_{j=1}^{n} L_j, \quad D = \sum_{j=1}^{n} D_j, \tag{33}$$

where n is the number of firms.

ence equations:

$$E[D_{t+1}] = \frac{\delta}{1-\kappa(1-q)}D_t - \frac{\kappa}{1-\kappa(1-q)}E[F_{t+1}] + \\ + \frac{\kappa}{1-\kappa(1-q)}NW + \frac{1}{1-\kappa(1-q)}E[X_{t+1}]. \tag{38}$$

3.2 Stability conditions

The solution of this class of dynamic models is normally obtained as a function of the roots of the system, which are in general quite complex. But because of the simplicity of the structure of the model, there is a closed form solution for the eigenvalues of the system that allows studying its stability. The eigenvalues are:

$$\frac{1}{\delta} \qquad \text{and} \qquad \beta\delta. \tag{39}$$

If one of the two roots is larger, while one is smaller than one, it is possible to solve the model partially forward and partially backward, and obtain a saddle-path equilibrium. This implies the existence of a unique convergent trajectory, on which the rational expectations equilibrium lies. Necessary and sufficient condition are:

$$\begin{cases} \delta > 1 & \text{and} & \beta\delta > 1 \\ \delta < 1 & \text{and} & \beta\delta < 1 \end{cases}$$

In order to understand these conditions we must recall the expression for δ, from Equation (16):

$$\delta = \frac{(\gamma_Y\gamma_P)^{1/\eta}}{\gamma_P} = \gamma_Y^{\frac{1}{\eta}}\gamma_P^{\frac{1-\eta}{\eta}}. \tag{40}$$

Thus $\delta < 1$ implies $\gamma_Y < \gamma_P^{\eta-1}$. The income demand for deposits can grow at a faster rate than prices, but it must not be of exponential order higher than $\eta - 1$. This insures that the demand converges to a finite value as the time horizon tends to infinity. In the remainder of the work we will assume that the condition $\delta < 1$ holds.

When this condition is satisfied, the condition regarding the other eigenvalue is satisfied *a fortiori*, since β is a discount factor. Interestingly, it then follows that the dynamic of system is not influenced either by the cost coefficients, or by the feedback process (neither would it be by reserve requirements). This is due the particularly simple structure of the model, wherein costs on deposits are linear. But even in more complex models, with more non-linear aspects, the stability of the system would depend fundamentally on the same two variables as here: the discount factor and the coefficient of the lagged term in the deposit demand condition (which, in turn, depends on the income demand for deposits).

In order to guarantee the stability of the system, a typical transversality condition must also be satisfied. This additional transversality condition is in the appendix.

3.3 Rational Expectations Equilibrium

The Rational Expectations Equilibrium of the system follows through substitution. Substituting Equation (34) in Equation (38), we obtain a second order difference equation relating to the stock of bonds (as shown in the appendix):

$$E[F_{t+1}] - \left[\frac{1}{\delta\beta} + \delta\right]F_t + \frac{1}{\beta}F_{t-1} = \frac{[1-\delta]\left[\beta\delta - 1\right]}{\delta\beta}NW +$$
$$-E\left[\frac{1}{\beta\alpha}\right]Z_t + E\left[\frac{1-(1-q)\kappa}{\delta\beta\alpha}Z_{t+1}\right] + \frac{(1-q)(\delta\beta-1)}{\delta\beta}X_t + \frac{(1-q)\delta}{1-\kappa(1-q)}\Delta X_t. \tag{41}$$

Following the same procedure (also in the appendix), we can write the value of D_t as:

$$E[D_{t+1}] - \left[\frac{1}{\delta\beta} + \delta\right]D_t + \frac{1}{\beta}D_{t-1} =$$
$$= -\frac{\kappa}{\delta\beta}E\left[\frac{1}{\alpha}Z_{t+1}\right] - \frac{1-(1-\delta\beta)\kappa(1-q)}{\delta\beta[1-\kappa(1-q)]}X_t + \frac{1}{1-\kappa(1-q)}E[X_{t+1}]. \tag{42}$$

Using the expectation lag operator H, such that $H^{-j}E_{s-1}x_s = E_{s-1}x_{s+j}$, the left hand side of the equation can be expressed as:

$$E[F_{t+1}] - [\frac{1}{\delta\beta} + \delta]F_t + \frac{1}{\beta}F_{t-1} = (1-\lambda_1 H)(1-\lambda_2 H)E[F_{t+1}]. \tag{43}$$

Where λ_1 and λ_2 are the reciprocal of the roots of the system. The right hand side can be rewritten as:

$$1-(\lambda_1+\lambda_2)H + \lambda_1\lambda_2 H^2, \qquad \text{so that:}$$
$$-(\lambda_1+\lambda_2) = \frac{1}{\delta\beta} + \delta \qquad \text{and} \qquad \lambda_1\lambda_2 = \frac{1}{\beta}. \tag{44}$$

Thus, as stated earlier, the eigenvalues are:

$$\lambda_1 = \delta \qquad \lambda_2 = \frac{1}{\beta\delta}. \tag{45}$$

Next, Equation (41) can be rewritten as:

$$(1-\lambda_1 H)F_{t+1} = \frac{1}{(1-\lambda_2 H)}E_t\left\{\frac{[1-\delta]\left[\beta\delta-1\right]}{\delta\beta}NW + \right.$$

$$\left. + \frac{1-(1-q)\kappa-\delta H}{\delta\beta\alpha}Z_{t+1} + \frac{(1-q)(\delta\beta-1)}{\delta\beta}X_t + \frac{(1-q)\delta}{1-\kappa(1-q)}\Delta X_t\right\}. \tag{46}$$

From Equation (36):

$$Z_{t+1} = \left[\left(1-\frac{d}{b}\right)(\delta\beta-L)+(1-q)\kappa H\right]r^B_{t+1} - (\delta\beta-H)\varepsilon^L_{t+1}$$
$$-\kappa\left[r^D_t+u\right]+(\beta\delta-1)\left(z-\frac{a}{b}\right), \tag{47}$$

the constant terms can be removed, obtaining:

$$Z'_{t+1} = \left[\left(1-\frac{d}{b}\right)(\delta\beta-H)+(1-q)\kappa H\right]r^B_{t+1} - (\delta\beta-L)\varepsilon^L_{t+1} - \kappa r^D_t, \tag{48}$$

$$C = (1-\beta\delta)\left[\frac{a}{b}-z\right]-\kappa u. \tag{49}$$

We are assuming that $\lambda_1 < 1$ and $\lambda_2 > 1$. The right-hand side can be solved forward, applying the algorithm developed by Sargent.[31] Applying the transversality condition of the problem (discussed in the Appendix), Equation (41) can be solved as:

$$F_{t+j+1} = \delta F_{t+j} + (1-\delta)NW - \frac{1-(1-q)\kappa-\delta H}{\beta\delta}\sum_{i=1}^{\infty}\left(\beta\delta\right)^i E_{t+i}\left[\frac{Z'_{t+j+i+2}}{\alpha}\right] +$$
$$-\left[\frac{(1-q)(\delta\beta-1)}{\delta\beta}+\frac{(1-q)(1-H)}{1-\kappa(1-q)}\right]\sum_{i=1}^{\infty}\left(\beta\delta\right)^i X_{t+j+i+1} - \frac{1-(1-q)\kappa-\delta}{(1-\beta\delta)\alpha}C + c\lambda_2^t, \tag{50}$$

where c is an arbitrary constant. The application of the transversality condition implies the imposition of a value of zero on the constant term c. Zero is in fact the only possible value for the constant which makes the solution finite as $t \to \infty$, since $\lambda_2 = 1/\beta\delta > 1$.[32]

3.4 Composition of the portfolio

Loans

The rational expectation equilibrium quantity of loans can be easily obtained from the budget constraint $L = (1-q)D - F + NW$, after obtaining the equilibrium values of

[31] See Sargent [33] p. 176.
[32] See Sargent [33], p. 174 e p. 198.

deposits and bonds.[33] The value is:

$$L_{t+j+1} = \delta L_{t+j} + \delta NW + \frac{1-\delta}{(1-\delta\beta)\alpha} C + \frac{1-\delta H}{\delta\beta} \sum_{i=0}^{i+1} \left(\delta\beta\right)^i E_{t+i}\left[\frac{Z'_{t+j+i+1}}{\alpha}\right] + \\ - \frac{1-q}{1-\kappa(1-q)}\left[(1+\delta)X_t - \delta^2\beta X_t - \delta X_{t-1}\right], \qquad (51)$$

Since we have assumed that interest rates follow a random walk process, the correlation between interest rates and default costs is time-invariant, and we can rewrite the expression in terms of the current and lagged values and the covariance as follows:

$$L_{t+j+1} = \delta L_{t+j} + \delta NW + \frac{1-\delta}{\alpha}\left\{\frac{(1-q)\kappa}{(1-\delta\beta)} - \left(1 - \frac{d}{b}\right)\right\} \times \\ \left[r^B_{t+j+1} + COV\left(r^B, \frac{1}{\alpha}\right)\right] + (\bar{A}_2 + \bar{A}_3 L) r^B_{t+j} + \\ + (\bar{C}_1 + \bar{C}_2 L + \bar{C}_3 L^2) r^D_{t+j+1} - \frac{z}{\alpha} - \frac{\kappa u}{(1-\delta\beta)\alpha} + \frac{a}{b\alpha} + \frac{1-\delta}{(1-\delta\beta)} COV\left(L^D, \frac{1}{\alpha}\right). \qquad (52)$$

The former expression has a simple interpretation. The equilibrium quantity of loans is a function of its lagged value and of the expected future values of a set of variables. These variables are: the quantity of net worth; the current and lagged values of interest rate on bonds, interest rate return on required reserves, and interest rate on deposits; the coefficients of the industrial costs; the intercept of the demand for loans; and two terms that describe the covariance between default costs and, respectively, interest rates on bonds and the demand for loans. All of these factors have the expected sign, and their interpretation is the same in most regards as it would be in a static monopolistic model. So the equilibrium quantity is an increasing function of aggregate demand, as indicated by the intercept of the demand curve, and a decreasing function of all cost terms. In the following sections we will focus the discussion on the most relevant results, in particular those concerning the impact of interest rates on bonds, the inertia in deposits demand and the respective correlations between default costs and interest rates and default costs and demand. We will say something about the neglected influence of the influence of net worth in a separate section.

Consider first the importance of market power in our dynamic framework. As noted in Equation (35) before:

$$\alpha = \frac{bv+2}{b}, \qquad (53)$$

It is then easy to verify that the solution depends in a fundamental way on the coefficient b, which measures the interest rate sensitivity of the demand for loans and gives the slope

[33] We change the logical order for ease of exposition.

of the demand curve. Examination of Equation (52) shows that most terms of the solution, interest rates and costs in particular, are multiplied by $1/\alpha$, while the intercept of the demand curve is multiplied by $1/b\alpha$. This makes a big difference since $1/\alpha$ is an increasing function of b, while $1/b\alpha$ is a decreasing function of b. In fact, when the value of b is not large, the main positive influence on the equilibrium is the intercept of the demand curve. Otherwise (if the elasticity of the demand for loans is large), the main positive influence is the interest rate on bonds. This is easily understood since:

$$\frac{1}{\alpha}=\frac{b}{bv+2}\begin{cases}\lim_{b\to 0}=0\\ \lim_{b\to\infty}=\frac{1}{v}\end{cases}\qquad\qquad \frac{1}{b\alpha}=\frac{1}{bv+2}\begin{cases}\lim_{b\to 0}=\frac{1}{2}\\ \lim_{b\to\infty}=0\end{cases}$$

It follows that as competitive pressures increase, the relevance of the intercept of the demand curve proportionally declines, and the issue of loans becomes dependent exclusively on the margin between the interest rate spread and industrial costs.

The main result of this section thus far is that the equilibrium quantity of loans is an increasing and concave function of the interest rate on bonds.

The effect of interest rates on bonds on the issue of loans depends on two different factors: the interest rate sensitivity of the demand for loans, which is positive,[34] and the direct effect on banks, on the supply side of the market. The supply of loans is affected by the rate on bonds in two different, contrasting, ways. The first is a standard negative portfolio composition effect (analogous to that in static analysis): the rate on bonds is opportunity cost on loans. The second is a positive effect, which is due to the feedback process linking the size of the portfolio to the issue of loans. Higher rates in fact increase the return of both components of the assets portfolio, loans and bonds. This represents a positive incentive for the issue of loans, which always dominates the negative opportunity-cost direct effect, based on Equation (23).[35] Thus, according to our model, the equilibrium quantity in the market for loans is an increasing function of the interest rates on bonds, independently of the expected level of the default cost,[36] and independently of the impact of the interest rate on bonds on the demand for loans.

It must be emphasized, though, that this result does depend on the assumption that the bank is able to issue bonds to finance the issue of loans. Quite specifically, we have not imposed a positive value on the equilibrium level of bonds in the portfolio, F. In case

[34]this effect is amplified by the market power of the bank.

[35]It can in fact be seen that the first of the three terms in the curled bracket that multiplies r^B_{t+j+1} is always larger than one (which is the value of the second), thanks to our basic assumption of Equation (23).

[36]We do not need to consider the book value of the existing stock of bonds, which would be reduced by the higher rate, because we are not considering the liquidity risk. In our framework the bank always takes bonds to maturity, and since we have an infinite horizon, and deposits are quasi-fixed, the duration of the liabilities is always longer than the duration of the assets. As a consequence, as Samuelson [32] had originally shown, higher interest rates increase the profits of the bank.

of a negative equilibrium value, the bank issues rather than purchases bonds. If the bank could not issue bonds, as it may be the case of small banks, the issue of loans might be constrained by the availability of deposits. This could alter the model.

The covariance between the rate of interest and the reciprocal of the cost function, $COV(r^B, \frac{1}{\alpha})$, in turn, has a positive sign. Thus a positive correlation between the rate of interest and the default cost has a *negative* effect on the issue of loans. As a result, if higher rates on bonds are correlated with higher default costs on loans, the bank issues less loans. Accordingly, the correlation reduces the impact of interest rate shocks. The same logic holds in case of a positive correlation between the demand for loans and default costs, so that $COV(L^D, \frac{1}{\alpha}) > 0$. Not surprisingly, if a buoyant demand raises future default cost, the bank issues a proportionate lower quantity of loans, and buys more bonds instead. The correlation between default cost and the interest rate makes the equilibrium quantity of loans a concave function of the interest rate.

The level of the contemporaneous and the twice lagged coefficient of the rate on deposits ($\bar{C}_1$ and $\bar{C}_3$ of Equation (52)) affects the issue of loans negatively, while the effect of the lagged value is ambiguous (shown by the coefficient $\bar{C}_2$), but normally positive. The rate on deposits is in fact a cost that the bank has to face in order to issue loans and it reduces proportionally the profitability of loans. As we would expect, industrial costs have a negative impact too, and the influence of marginal cost of loans exceeds the influence of the marginal cost of deposits.

An increase in the inertia of the demand for deposits (a larger δ) increases the value of the backward looking part of the equation and the positive influence of net worth. Besides it increases the negative influence of the costs of deposits (the industrial cost and the interest rate), since $\frac{\partial L_{t+j+1}}{\partial\delta\,\partial u} < 0$. The impact on the interest rate coefficient is more complex, and it depends on the particular value of δ. But for reasonable values of the income elasticity of demand for deposits, whenever trend inflation is not very high the sign of $\frac{\partial L_{t+j+1}}{\partial\delta\,\partial r^B_{t+j+1}}$ is negative.[37] Besides we have shown that the influence of the value of the interest rate coefficient becomes dominant only when the market is competitive.

[37] When the value of δ is below the following threshold:

$$\delta < \left[1 - \sqrt{\frac{(1-q)\kappa(1-\beta)}{1-\frac{d}{b}}}\;\right] \Big/ \beta, \tag{54}$$

the sign of the cross derivative is positive, as shown in the Appendix. From Equation (16),

$$\delta = \frac{(\gamma_Y\gamma_P)^{1/\eta}}{\gamma_P} = \gamma_Y^{\frac{1}{\eta}}\gamma_P^{\frac{1-\eta}{\eta}}. \tag{55}$$

Therefore, making some numerical examples, it can be shown that when the feedback coefficient is not very small (not much smaller than 0.1) inequality (54) holds exclusively when expected inflation is high (10% or more), or the coefficient η is large (i.e. when the income elasticity of deposits demand is low).

Therefore we can conclude that a higher degree of inertia produces a reduced issue of loans, at least when inflation is moderate. Since δ is an increasing function of real income expected growth, a higher expected trend of income growth apparently reduces the issue of loans (in real terms). It must be added though that the demand for loans is strongly correlated with income. Formally, the coefficient a becomes a function of real income $a(Y)$, and since it normally plays a very relevant role, this correlation is likely to revert the previous results. A moderate increase of expected inflation raises the issue of loans. When trend inflation becomes very high though, the issue of real loans may actually decline.

Finally, we can observe that loans are a decreasing and concave function of default cost. It is important to observe that the impact of default costs, measured by the coefficient v, grows with the own interest rate sensitivity of the demand for loans, so that the stronger the competitive pressure, the higher the influence of default costs.

3.4.1 Bonds

We have already obtained the general solution for the equilibrium quantity of bonds, in Equation (50). Following the assumption that interest rates obey a random walk process, the solution can be expressed as:

$$F_{t+j+1} = \delta F_{t+j} + (1-\delta)NW - \frac{1-(1-q)\kappa-\delta}{(1-\beta\delta)\alpha}\left\{(1-\beta\delta)\left[\frac{a}{b}-z\right]-\kappa u\right\}+$$
$$+\frac{1-(1-q)\kappa-\delta}{(1-\beta\delta)\alpha}\left[\left(1-\frac{d}{b}\right)(1-\delta\beta)-(1-q)\kappa\right]COV\left(r^B,\frac{1}{\alpha}\right)+(A_1+A_2L+$$
$$+A_3L^2)r^B_{t+j+1}+(C_1+C_2L+C_3L^2)r^D_{t+j+1}-\left[1-(1-q)\kappa-\delta\right]COV\left(L^D,\frac{1}{\alpha}\right), \quad (56)$$

where

$$A_1 = \frac{[1-(1-q)\kappa-\delta]\left[\left(1-\frac{d}{b}\right)(1-\delta\beta)-(1-q)\kappa\right]}{(1-\beta\delta)\alpha}-(1-q)g_4. \quad (57)$$

Since the structure of the solution is analogous to that of loans, the same kind of consideration regarding the structure of the solution hold in this case, as for example for the importance of the coefficient b. Assumption (23) guarantees that

$$\frac{1-(1-q)\kappa-\delta}{(1-\beta\delta)\alpha} < 0. \quad (58)$$

The sign of A_1 is positive when the value of the interest rate sensitivity of the demand for deposits, g_4 is small, since the first term of the expression is always positive.[38] The

[38]The exact formulation of the other coefficients is shown in the appendix.

first impact of the rate on bonds on the amount of bonds held in the portfolio is the result of a positive scale effect due to the fact that higher rates increase the return of the portfolio. This implies that loans and bonds in the portfolio are complements, driving the bank to purchase bonds when the issue of loans increases the amount of deposits. A negative, contrasting effect is due to the negative dependence of the demand for deposits on the rate on bonds, which represents the opportunity cost of holding deposits. Higher rates, reducing the demand for deposits, tend to reduce the equilibrium level of deposits, reducing accordingly the equilibrium level of the portfolio of assets different from loans. Consequently the final effect of an increase of the interest rate on the portfolio of bonds cannot be established *a priori*.

The covariance between the rate of interest and the reciprocal of the cost function, $COV(r^B, \frac{1}{\alpha})$, has a positive sign. Thus any positive correlation between the rate of interest and the default cost would have a *negative* effect on the purchase of bonds, reducing the positive influence of the interest rate. The covariance between the demand for loans and default costs $COV(L^D, \frac{1}{\alpha})$ is positive. Therefore, any positive correlation between the demand for loans and default costs implies a lower equilibrium holding of bonds.

The sign of the coefficient A_2 depends on the specific values of the coefficients, so that the lagged value of the rate on bonds may be either positive or negative. The twice-lagged value of the rate is normally negative.

Industrial costs on both loans and deposits have a negative sign, since the profit margin becomes tighter as either increases, and the optimal size of the portfolio becomes smaller. The impact of the interest rates on deposits is ambiguous. The rise in the cost to the bank tends to reduce the optimal size of the portfolio of assets, but the increase in returns to depositors augments the demand for deposits with a positive effect on the size of the portfolio. The final impact depends on the specific values of the coefficients.

Default costs shrink the size of the forward-looking part of the equation, reducing the portfolio of bonds. Default costs reduce the optimal quantity of loans and, as a consequence, the size of the whole portfolio. We did not specify whether the bonds that the bank buys were risk-free or high yield risky bonds. In the second case, assuming that default costs are a linear function of the quantity purchased, these costs would proportionally shrink the net returns of bonds.

3.4.2 Deposits

The general solution for deposits is the following:

$$D_{t+j+1} = \frac{1}{\delta\beta}D_{t+j} + \frac{\kappa}{\beta\delta}\sum_{i=1}^{\infty}\left(\frac{1}{\delta}\right)^{i} E_{t+i}\left[\frac{Z'_{t+j+i+2}}{\alpha}\right] - \frac{\kappa}{\beta\delta\alpha(1-\delta)}C + \\ + \sum_{i=1}^{\infty}\left(\frac{1}{\delta}\right)^{i} E_{t+i}[X_{t+j+i+1}] - \frac{1-(1-\delta\beta)\kappa(1-q)}{\delta\beta[1-\kappa(1-q)]}X_t. \quad (59)$$

Since interest rates obeys a random walk process, it follows that:

$$D_{t+j+1} = \delta D_{t+j} + \left\{\frac{\kappa\left[(1-q)\kappa - \left(1-\frac{d}{b}\right)(1-\delta\beta)\right]}{(1-\beta\delta)\alpha} - g_4\right\} r^B_{t+j+1} + \\ + \hat{C}_1 r^D_{t+j+1} + \hat{A}_2 r^B_{t+j} + \hat{C}_2 r^D_{t+j} - \frac{\kappa}{\alpha}\left\{\left[z - \frac{a}{b}\right] + \frac{\kappa u}{1-\beta\delta}\right\} + \\ + \frac{\kappa\left[(1-q)\kappa - \left(1-\frac{d}{b}\right)(1-\delta\beta)\right]}{(1-\beta\delta)\alpha} COV\left(r^B, \frac{1}{\alpha}\right) + \kappa COV\left(L^D, \frac{1}{\alpha}\right). \quad (60)$$

The intertemporal equilibrium level of deposits depends negatively on the costs of both deposits and loans. The level of deposits grows with the own interest rate, as a function of the sensitivity of the demand for deposits. The lagged value of the own rate has a negative impact.

Based on Equation (23), the sign of the equilibrium quantity of deposits as a function of contemporaneous interest rate on bonds, can be positive or negative, depending on two contrasting effects. The standard demand side effect is negative, since the rate on bonds is the opportunity cost of deposits. But in this model, an opposite influence occurs from the supply side of the market. Since the size of the portfolio depends on the issue on loans, and profits are an increasing function of interest rates on bonds, the interest rate has a positive influence on the equilibrium level of deposits. The available evidence regarding the demand for money would support the assumption that the demand effect dominates in normal conditions. But in periods of high inflation the supply side effect may dominate.[39] Besides, even in the case of deposits the market power plays an important role. The relevance of the supply side effect is a function of the competitive structure of the market. When the bank has a relevant market power it seems unlikely that the supply effect may dominate the standard money-demand effect. But in a highly competitive environment the influence of the supply-side is relevant, and the equilibrium quantity of deposits (and of money aggregates where deposits are predominant, such as M1) is likely not to be very sensitive to the interest rate on bonds. The sign of the lagged value of the rate depends on

[39] See Hoffman, Rasche and Tieslau [21].

the specific values of the coefficients, and it cannot be decided *a priori*, but its importance is minor.

These results provide a theoretical rationale for the empirical evidence in Chari, Christiano and Eichenbaum [5], which shows that M1 has a positive correlation with future values of the interest rate, while the correlation with contemporaneous and past values is negative. Because of the assumption that future interest rates follow a random walk (and accordingly the deterministic component is expected to remain constant), in our formulation, the contemporaneous interest rate synthesises the effect of both future and contemporaneous rates. But it can easily be seen from the general solution [40] that the negative demand side effect exclusively affects the contemporaneous value. On the contrary, because of the supply side, the equilibrium level of deposits is an increasing function of all future expected values of the interest rate on bonds. [41]

The covariance between the interest rate on bonds and the reciprocal of the default costs reduces the effect of the interest rate, with a negative effect on the level of deposits. Thus, the stronger the positive correlation between interest rates and default costs, the smaller the equilibrium level of deposits. And the same happens for the covariance between the demand for loans and the reciprocal of the default cost.

Default costs shrink the size of the forward-looking part of the equation, reducing the size of the portfolio. These costs are the true constraint on the size of the portfolio, putting a limit to the liquidity creation. But the positive correlation between shocks in the demand for loans and default costs reduces the equilibrium level of deposits still further.

The sign of: $\frac{\partial D_{t+j+1}}{\partial \delta \, \partial r^B_{t+j+1}}$ is positive, but, on the other hand, an increase of the inertia of deposits demand increases the impact of the costs of deposits. As long as the feedback coefficient is not very small though, the positive effect always dominates.

3.5 The interest rate solutions

3.5.1 The interest rate on loans

The interest rate on loans can easily be obtained by substituting the solution (51) for the quantity of loans in the demand condition. For simplicity, we use the solution of (52),

[40] Equation (60).

[41] Besides, it must be observed that contemporaneous and past values of the variables influence the lagged value of the dependent variable too. In order to evaluate the influence of the lagged values of the rate it would be necessary to solve the equation even in the backward direction. This can be seen from the general solution in Equation (60).

but the result is general. We obtain:

$$r^L_{t+j+1} = \frac{1}{b}\Big\{(d-\bar{A}_1)r^B_{t+j+1} + G\Big\} =$$
$$= \frac{1}{b}\Big\{\Big[d - \frac{(1-\delta)\Big[(1-q)\kappa - \Big(1-\frac{d}{b}\Big)(1-\delta\beta)\Big]}{(1-\delta\beta)\alpha}\Big]r^B_{t+j+1} + G\Big\}. \tag{61}$$

where

$$G = -\delta L_{t+j+1} + \Big(1-\frac{1}{b\alpha}\Big)a + (1-\delta)NW + \frac{z}{\alpha} + \frac{\kappa u}{(1-\delta\beta)\alpha} +$$
$$-(\bar{A}_2 L + \bar{A}_3 L^2)r^B_{t+j+1} - (\bar{C}_1 + \bar{C}_2 L + \bar{C}_3 L^2)r^D_{t+j+1} +$$
$$+\frac{1-\delta}{\alpha}\Big[\Big(1-\frac{d}{b}\Big) - \frac{(1-q)\kappa}{1-\delta\beta}\Big]COV\Big(r^B, \frac{1}{\alpha}\Big) - \frac{1-\delta}{(1-\delta\beta)}COV\Big(L^D, \frac{1}{\alpha}\Big). \tag{62}$$

The value of the interest rate on loans depends fundamentally on the difference between the value of the coefficients of the intercept of the demand curve and the value of the cost factors (industrial costs and interest rates on deposits). Any other factor is of a much smaller order, including the interest rate on bonds in the case when the demand for loans is elastic. As for the equilibrium quantity of loans, it can be seen that the main other factor which affects the value of the rate on loans is the slope of the demand curve. As we would expect, the rate is a decreasing function of b, the interest rate sensitivity of the demand for loans.

Defining $A = 1 + \frac{\partial COV\left(r^B, \frac{1}{\alpha}\right)}{\partial r^B}$, we can obtain:

$$\frac{\partial r^L_{t+j+1}}{\partial r^B_{t+j+1}} = \frac{d}{b} - \Big[\frac{(1-\delta)\Big[(1-q)\kappa - \Big(1-\frac{d}{b}\Big)(1-\delta\beta)\Big]}{(1-\delta\beta)b\alpha}\Big]A =$$
$$\frac{d}{b} - \Big[\frac{(1-\delta)\Big[(1-q)\kappa - \Big(1-\frac{d}{b}\Big)(1-\delta\beta)\Big]}{(1-\delta\beta)(bv+2)}\Big]A. \tag{63}$$

We assume that A is non-negative. When the demand for loans is not affected by the market for bonds ($d = 0$), the sign of the derivative is always negative, because of the fundamental assumption of the model, shown in Equation (23). In this situation though the assumption that the bank can increase the liabilities issuing bonds is not acceptable, because banks then would be the only firms with access to the bond market. As a consequence, in this case we should introduce the non-negativity constraint, which would be binding. In this situation the issue of loans would be severely constrained by the availability of deposits, and since higher rates on bonds normally reduce the equilibrium level of deposits, the equilibrium level of loans could not be increased and the interest rate on loans could not decline.

In the general case, when the bank can freely issue bonds, the sign of the derivative is positive if there is some substitution between bonds and loans in the demand of firms, so that d is not irrelevant. The second term of the expression is in fact of a small magnitude, and it is a decreasing function of the slope of the demand curve, b. As we would expect, when the demand curve is flat, shocks on the rate on bonds, which affect in particular the supply of loans, produce a large impact on the quantity of loans and a small effect on the interest rate. Since the supply curve is quite steep, shocks to the rate on bonds, which shift the demand curve up produce an increase of the interest rate on loans. We can conclude that in general the rate on loans is an increasing function of the rate on bonds. This relationship is the stronger the higher the substitution between bonds and loans.

It seems reasonable to assume that the correlation between interest rates and default costs becomes larger for higher values of the interest rates.[42] This implies a negative value for the derivative of the covariance, and, as a consequence, $A < 1$. It can be easily seen that the correlation reduces the value of the right-hand side of the first inequality. Consequently, the correlation between interest rates and default costs drives the interest rate on loans to move in the same direction as the interest rate on bonds, reducing the influence of the supply-side of the market. Higher rates on bonds in fact imply a larger issue of loans, and eventually lower rates on loans; but the positive correlation means that higher interest rates are associated with higher default costs, and as a consequence this negative effect on the interest rate is proportionally reduced. We can conclude that the interest rate on loans is a concave function of the interest rate on bonds.

Since $\frac{\partial r^L_{t+j+1}}{\partial v} > 0$, while the second derivative is negative, the interest rate on loans is an increasing, concave function of expected default costs. The equilibrium quantity of loans is a decreasing, concave function of default costs: higher expected default costs reduce the equilibrium quantity of loans, increasing the equilibrium interest rate.

The other variables of the model have an impact on the interest rate on loans which is opposite than the effect on the quantity of loans. Because of the path dependence, caused by the dynamics of the model, the lagged quantity of loans has a negative influence on the rate. The once lagged coefficient of the rate on bonds is normally negative, while the second lag has a positive influence. The contemporaneous and the twice lagged coefficient of the rate on deposits affect positively the rate on loans, while the effect of the once lagged

[42] The correlation may be caused by two kinds of factors. It may be due to asymmetric information problems that cause the pooling of borrowers (which in the extreme case cause a more than proportional increase in defaults when interest rates rise, as the literature on credit rationing has emphasised), or it may be due to the reduced value of the net worth of borrowers, which declines with the interest rate, increasing the probability of default. In both cases the correlation is likely to be an increasing function of the interest rate itself. This implies that the sensitivity of the rate on loans to variation of the interest rate on bonds must be non-linear, it must be a concave function.

value is ambiguous, but normally negative. Finally we can observe that the correlation between the demand and the default costs tends to raise the rate on loans. The higher covariance implies lower expected profits, a reduced issue of loans, and higher interest rates on loans.

3.5.2 The spread

The spread between the interest rate on loans and the interest rate on bonds can be obtained as:

$$s_{t+j+1} = r^L_{t+j+1} - r^B_{t+j+1} =$$
$$= \left\{ \frac{1}{b} \left[d - \frac{(1-\delta)\left[(1-q)\kappa - \left(1-\frac{d}{b}\right)(1-\delta\beta)\right]}{(1-\delta\beta)\alpha} \right] - 1 \right\} r^B_{t+j+1} + \frac{G}{b}. \quad (64)$$

The sign of

$$\frac{\partial s_{t+j+1}}{\partial r^B_{t+j+1}} = \frac{d}{b} - 1 - \left[\frac{(1-\delta)\left[(1-q)\kappa - \left(1-\frac{d}{b}\right)(1-\delta\beta)\right]}{(1-\delta\beta)b\alpha} \right] A, \quad (65)$$

depends on the factors that we have discussed in the previous section, and it is always negative, since the own coefficient b is always larger than the cross coefficient d of the demand curve, and the term in the final bracket is negative. [43] The spread is always a decreasing function of the interest rate on bonds, because the rate on bonds has a limited impact on the rate on loans, even when loans and bonds are close substitutes for firms. Finally, the spread is an increasing function of all the factors that enter in the term G with a positive sign, which include the industrial costs of both loans and deposits and the interest rates on deposits.

[43] Unless for the case of an extreme interest rate sensitivity of the correlation between the interest rate and default costs, which would change the sign of A. We will discuss this possibility in a separate section.

4 Implications

4.1 Trends

4.1.1 Output growth

A permanent increase in the factor of growth of the economy raises the value of the coefficient δ, which measures the degree of inertia in deposits demand:

$$\frac{\partial\delta}{\partial\gamma_Y} = \frac{1}{\eta}(\gamma_Y\gamma_P)^{\frac{1-\eta}{\eta}} > 0 \qquad (66)$$

We assume that the increase is not large enough for the value of δ to be larger than one, as in the rest of the analysis.

If income would not affect the demand for loans, the impact of variations of output growth on the equilibrium quantity of loans would normally be negative, at least when trend inflation is moderate. A lower inertia of deposits demand, in fact, allows the bank to generate more deposits trough the issue of loans, increasing profits. This would imply that variations in the stock of loans are counter-cyclical. The demand for loans though is strongly correlated with income. Since the expected value of future demand plays a key role in the model, variations of the expected trend of income are fundamental, since they permanently shift the demand curve. Persistent shocks, as those generated by productivity shocks, generate a pro-cyclical response from the banking system. When the influence of the demand for loans is fundamental for the problem of the bank, so whenever the bank benefits from relevant market power in the market for loans, the intermediation of the banking system is strongly pro-cyclical, producing a financial accelerator. This effect is amplified by the cyclical nature of expected defaults.

4.1.2 Expected inflation

We have not explicitly described the market for means payment different form deposits, currency and other interest bearing means of payment, such as money market mutual funds. Nevertheless our formulation of the demand for deposits implies some strong assumptions regarding the substitution between different means of payment. In particular the assumption of a positive, but less than unitary, nominal income elasticity of the demand of deposits, implies that the demand for deposits in nominal terms is an increasing but concave function of the price level. The substitution between deposits and currency, whose demand is a declining function of the price level, explains the assumption that the demand is increasing. Assuming that the demand is a concave function implies that the demand for other interest bearing means of payments grows more with the price level than the demand of deposits.

Because of our long-run approach, we assume perfect flexibility of all prices, so that nominal costs and interest rates vary in the same proportion and real values are not affected. Consequently, an increase of expected inflation reduces the value of the coefficient δ:

$$\frac{\partial \delta}{\partial \gamma_P} = \frac{1-\eta}{\eta} \gamma_Y \frac{1}{\eta} \gamma_P^{\frac{1-2\eta}{\eta}} < 0. \tag{67}$$

In general the increase of expected inflation produces an increase of the issue of loans, because higher expected inflation reduces the degree of inertia. The effect of an increase of trend inflation on the issue of loans may become uncertain only when trend inflation is very high. In the last case the impact of the variation of the price level may become relatively larger than the increase of the level of the nominal variables, so that in real terms the equilibrium value becomes lower.

4.2 Deviations from trend

4.2.1 A positive interest rate shock

The results obtained so far allow to understand the reaction of the banking industry to market interest rate shocks. The fundamental conclusion that we have drawn in the former section, with this regard, is that the issue of loans is an increasing concave function of the interest rate on bonds, since bonds and loans are complementary assets in the portfolio of the bank. Besides interest rates on loans are normally a positive but concave function of the rate on bonds, while the spread between the rate on loans and the rate on bonds is a decreasing function of the rate on bonds. We have obtained these results under the assumption of risk neutrality for the bank; the non-linearity are due to the presence of a convex default cost function in the profit function. Since default costs are correlated with the rate on bonds, profits vary non-linearly with interest rates.

Since both the equilibrium quantity of loans and the interest rate on loans are increasing but concave functions, the bank tends to smooth any transitory shock. The reason is that changes in the stock of deposits, and consequently of loans, are costly. So any shock which is expected to be reverted over a not too long horizon does not produce a variation of the equilibrium quantity of loans, and consequently of the interest rate on loans. This prediction of the model is line with the empirical evidence,[44] and we reach the same conclusion of Fried and Howitt [17], namely that the bank has a strong incentive to provide insurance against these shocks. According to Fried and Howitt though, banks smooth interest rate shocks because insuring their customers they obtain higher average profits,

[44] See Berlin and Mester [2].

reducing the average default probability of borrowers. In our model, on the contrary, the bank smoothes shocks even if average default costs are unchanged, because profits are a concave function of the interest rate, and a higher variance of the rate reduce profits. And, contrary to Fried and Howitt's, these results would hold even in the case of a perfectly competitive market for loans. They are particularly strong when the bank benefits of relevant market power: profit margins, in fact, depend in this case largely on the level of aggregate demand and to a minor degree only from the interest rate on bonds.

The model allows to study how the size and composition of the portfolio changes in reaction to variations the market interest rates which are expected to be permanent. A permanent interest rate shock pushes the bank to increase the size of the portfolio issuing more loans, since loans are an increasing function of the interest rate on bonds. The larger issue of loans produces a higher supply of deposits services, so the issue of loans is partially self-financed. The impact of the higher rate on the demand of deposits is negative, so the reduction of the demand for deposits may often be larger than the increase of the supply. In this case, the bank finances the increased issue of loans by switching away from bonds. Bonds and loans are complementary assets, but the equilibrium quantity of bonds is more volatile than the quantity of loans. The bank faces decreasing returns to scale on loans, while returns to scale on bonds are constant; this implies that bonds holdings largely represent a buffer, which is increased or reduced in function of the needs of the issue of loans.

The final effect on the interest rate on loans of an increase of the interest rate on bonds depends on the market power of the bank. If the demand for loans is sensitive to the rate on bonds and the market power of the bank relevant the rate on loans is an increasing function of the rate on bonds. When competitive pressure is sufficiently strong though, the relationship could even become negative, unless the coefficient is a function of interest rate too, $d(r_B)$, so that the demand for loans increases non-linearly with the interest rate on bonds.

When the correlation between interest rates and default costs is relevant, higher interest rates on bonds may imply higher rates even when the substitution between bonds and loans is poor. Variations of interest rates produce a reduction of the borrower's cash flow that is proportional to the initial level of the rate. So we can expect the effect of an interest rate shock on default costs to be dependent on the initial level of the rates. In the case of high initial levels of the interest rate, influence of the interest rate is likely to be largely offset by the covariance of the default costs. Following the same line of reasoning, in the case of heavy shocks we might expect that the effects are not symmetric in the case of a positive or negative shock, and in particular that they depend on the initial level of the interest rate.

Introducing the distinction between small and large firms (defining as small those firms which do not have access to financial markets and have to rely on banks for external finance), it is possible to appreciate the benefit that the bank's risk insurance provides. Banks in fact provide insurance not only to small firms which rely heavily on bank's lending as a source of finance. Banks provide insurance as well to all those borrowers, such as large corporations, who can easily substitute bonds for loans. The reason is that the spread between the rates is always reduced by a positive the shock, because the rate on loans is sticky. And the viscosity of the rate on loans is a decreasing function of the elasticity of the demand for loans. This explains why large firms, whose activities are largely financed issuing bonds, are always willing to pay fees to get access to the more expensive, but more reliable, credit provided by banks.

4.2.2 Credit quality shocks

Different types of shocks of real origin affect borrowers, producing variations of current and expected default costs.

To keep the analysis simple, we have assumed that expected default costs are the same as the cost of the current period, and that expectations are revised when the current cost changes. As a consequence, variations of the default cost coefficient are caused by shocks that are regarded as permanent ones by the banker.

Higher default costs reduce the size of the whole portfolio and increase the interest rate on loans. In the case of a negative shock affecting permanently the average credit quality of borrowers, our model predicts that banks reduce the issue of loans and raise interest rates on loans.[45] Accordingly, the bank would not provide insurance against this kind of shocks.

The behaviour of the bank is likely to be different in the case of shocks which are expected to be temporary. These types of shocks are likely to be smoothed, because both the equilibrium quantity of loans and the equilibrium interest rate on loans are decreasing but concave functions of default costs. Since any adjustment of the stock of deposits and loans is costly, the impact of these shocks must be very small. Even in this case, though, implicit contracts between the bank and its customers would never imply a reduction of interest rates charged in reaction to the shock.

Even in the case of real shocks that produce a deterioration of credit quality, banks may provide some sort of insurance against the shock. But in order to understand the meaning of insurance in this context, we must extend the discussion beyond the strict limits of the model and consider the impact of shocks on the available alternative external sources of finance.

[45] This prediction is in line with the empirical findings of Berlin and Mester [2].

In general, when heavy shocks hit the economy, it may become very difficult for the less informed lenders of the market to properly price certain risks, and the bond market reacts abruptly to the shocks. In some cases the market may eventually altogether dry out for some borrowers, because of the insurgence of a lemon problem. On the contrary, in the case of banks, as long as their information allows the formulation of expectations regarding future default costs, banks reduce the amounts involved in the provision of direct lending facilities, but to a much a smaller degree than the bond market. In this sense banks provide insurance even against this type of shocks. Besides, since some borrowers are pushed to rely exclusively on banks, the demand for loans may surge as bonds are not available any more to finance certain risky projects. A strong enough increase of the demand could in principle push the bank to lend more, since the demand would in this case grow more than proportionally as default costs rise. But in this case the correlation between the demand for loans and default costs would probably be strong and the negative impact of the covariance would be relevant. Consequently it seems unlikely that banks may be willing to substitute the bond market to a large extent when large firms are hit by credit quality shocks.

The extent to which banks may be willing to provide insurance when sharp variations of relative prices produce idiosyncratic shocks, depends critically on the diversification of the portfolio. But even a bank with a perfectly diversified portfolio would not smooth entirely these shocks. The reason is that average expected default costs would always change after the shock. There is no way to allocate the quantity of lending to different classes of lenders in order to obtain a perfect hedge, since not just the direction, but even the size of the shocks is unpredictable. Besides many prices, as in the case of commodities or many financial products, are set in global markets, while on the contrary banking markets are highly segmented and heavily regulated, limiting the possible diversification of the portfolio of loans. Since most banks operate in close national markets, it may useful in this regard to distinguish between small and large firms lending. Banks specialised in small firms lending or consumer credit can in fact achieve a larger diversification of the portfolio. Consequently, they can provide more insurance against credit quality shocks because they can achieve both a lower average default costs, and a lower variance of defaults on the portfolio of loans. [46]

4.3 Credit rationing

According to the models of Stiglitz Weiss [37] and [38], credit rationing may occur because asymmetric information causes adverse selection and moral hazard problems,

[46] Our model supports the empirical results of Berlin and Mester [2] and [3].

producing a positive correlation between interest rates on loans and default costs. When the correlation is so strong that higher interest rates generate a more than proportional increase in default costs, reducing profits, credit becomes rationed, because the supply curve becomes backward-bending.

In our model the interest rate on loans is in normal conditions a monotonically increasing function of the rate on bonds. So the correlation between interest rates on bonds and default costs implies a similar correlation between interest rates on loans and default costs. We have not specified the correlation coefficient for the rate on loans, but because of the structure of the solution, it must be lower than the coefficient between bonds and default costs. This is acceptable, since the asymmetry of information is less severe for the bank than for the market. The situation described by Stiglitz and Weiss, of a reduction of the equilibrium quantity of loans following a positive interest rate shock, would occur in our model if, and only if, the coefficient $A = 1 + \frac{\partial COV\left(r^B, \frac{1}{\alpha}\right)}{\partial r^B}$ becomes negative, as from Equation (52). When this is the case the interest rate on loans grows more with the rate on bonds, and consequently the spread declines less as the rate on bonds goes up, as it is evident from Equation (65). This implies that the stickiness of the rate on loans is not a good test for the existence of credit rationing. The factors which produce credit rationing make the rate on loans more sensitive to movements of the rate on bonds. Consequently tests of credit rationing based on the behaviour of interest rates are inconclusive. This result is in line with the evidence produced by Berger and Udell [1] suggesting that, in the case of the market of the US, credit rationing is not a significant phenomenon. They have shown that commitment loans (insured against rationing) and normal loans do not behave differently, and they have concluded that loan interest rates stickiness is caused by interest rate smoothing.

5 Extension and limitations

The main limitations of the model is the absence of a proper analysis of the capital structure of the bank. This limitation can only partially be addressed without making the model much more complex.

5.1 Capital ratios

It can be useful to study the importance of the level of initial capital under the very restrictive assumption of a strong equity rationing. In this case net worth has a very modest influence on the problem, since the bank can generate liabilities issuing loans. Consequently, it is mainly employed in the purchase of bonds, and only a minor proportion

δ of the amount is used for the issue of loans.[47] Besides, the bank does not need a buffer because there is no penalty if profits become negative during a period.

The relevance of net worth in this case is very limited. Yet this result is not trivial, because since a higher capitalisation allows the bank to lend more creating liquidity, it could be supposed that deposits should be positively correlated with net worth. On the contrary, the model shows that a higher net worth increases only marginally issue of loans, *without* any influence on the process of liquidity creation.

The model can easily be extended to analyse the impact of the legal requirement of a minimum ratio between capital and loans. If the net worth of the bank has to cover at least a fixed proportion of the loans issued, since in our model the bank would never have an incentive to keep a higher than necessary share of capital, we could assume that:

$$L_t + F_t + R_t = D_t + NW_t, \tag{68}$$

$$R_t = qD_t, \tag{69}$$

$$NW_t = \theta L_t. \tag{70}$$

so that the budget constraint becomes the following:

$$(1-\theta)L_t = (1-q)D_t - F_t. \tag{71}$$

It can be easily seen that in this case the results of the model would change in a simple way. The term in net worth obviously disappears, and in the final result both intercept terms are multiplied for $1-\theta$. The effect of this legal requirement is to reduce the forward looking part of the solution. As a consequence in this case the size of the portfolio is reduced in proportion to the legal requirement coefficient, while the composition of the assets portfolio is not affected. Clearly the amount of deposits is reduced as well, because deposits positively depend on the main spread Z_t. The impact of the imposition of capitalisation coefficients is similar to the effect of reserve coefficients on deposits, but it is much stronger. This result explains why in order to control the growth of monetary aggregates in periods of high inflation, the introduction of constraints on the issue of loans is effective, while the increase of reserve coefficient may not.

5.2 Remaining limitations

This model does not allow a correct assessment of the importance of the own capital and of capital requirements, for two reasons. In first instance the extreme volatility of

[47] This result does not depend on the lag structure adopted, because it can be shown that it remains unchanged adopting a different structure for the lags of the feedback.

deposits induced by bank runs is relevant for this kind of problems, and the introduction of a stochastic error in the demand for deposits schedule could not solve the problem.[48] Second, and even more important, the stock market has not been introduced in the analysis, and it is a relevant omission, because we are explicitly analysing conditions under which the Modigliani-Miller theorem does *not* hold. The explicit introduction of the stock market would be quite complex, since we should model agency problems in condition of opaque information, and it is far beyond the scope of this work.

6 Conclusion

The equilibrium in the market for deposits has never been the object of dedicated studies, since it seems safe to assume that banks are always willing to supply any amount of deposit services that the market requires. Banks have in fact mainly been studied in their role of financial intermediaries. But it has normally been neglected that banks can pay deposits a low interest rate because they provide a whole range of payment services, and the demand for deposits is largely a demand for the payment services that deposits allow. The provision of payment services implies heavy costs, and the historical development of different banking systems has been shaped by the evolution of the technology of the transaction services industry. In this respect the banking industry is not different from any other industry, as was pointed out by Fama, and industrial costs cannot be disregarded. As a consequence, the equilibrium of the market for deposits must be properly micro-founded on the basis of the profit function of the banking firm.

Static portfolio models generate portfolio separability theorems, which imply that the size of the bank does not depend on the composition of the portfolio of assets or liabilities. Traditional portfolio models, though, do not take into account the crucial role that the limited and costly availability of information plays in banking. Once we take into account the implications of the imperfect availability of information, a dynamic analysis is required. Consequently portfolio separation theorems do not hold, and the composition of both assets and liabilities has to be determined jointly with the optimal size of the banking firm.

In this work we have provided a model of banking intermediation that explicitly accounts for the links between assets and liabilities and where size and composition of the portfolio are jointly determined. This allows a proper micro-foundation of the analysis of the impact of banking intermediation on the transmission of different types of shock. Our model shows that banks always smoothes the impact of shocks, both interest rates shocks

[48]Bank runs are in fact subject to radical uncertainty, since it is not possible to define a meaningful probability distribution on these type of events.

and credit quality shocks. Since banks need to maximise profits over a long interval of time because of the path dependence of their problem, and since variations of the quantity of loans imply adjustment costs, the smoothing of shocks, minimising the variance of the equilibrium quantities of the assets in the portfolio, produces higher profits. Besides, the model shows that banks provide further insurance against interest rate shocks to those borrowers that have access to the bond market, reducing the spread between the rate on loans and the rate on bonds when positive interest rate shocks occur. This because the interest rate on loans is sticky, and holds as long as the bank has any market power.

Default costs reduce the optimal amount of any asset and liability of the portfolio and are the main constraint on the size of the bank. This implies that regulatory requirements regarding the write-off of bad loans are crucial not just for the efficiency of the banking industry, but for the health of the economic system as a whole. When banks are allowed to roll over loans that should be written off, the constraint on the size of the portfolio is virtually removed, since default costs can be indefinitely postponed. Any investment could in this case be financed, without any selection, producing permanent distortions in the productive structure.

One of the peculiar results of the model is that banks that have a wide access to financial markets are normally induced to increase the issue of loans after a positive interest rates shock. This result puts in question the relevance of the direct lending effect of the credit channel. In accordance to the prediction of the model, the empirical support for the direct lending effect comes almost exclusively from small banks, which have a limited access to financial markets. Another result relevant for monetary theory is that the supply of deposits is an increasing function of interest rates on bonds. When portfolio separation does not hold, in fact, the optimal supply of deposits depends on the return of the assets. This implies that higher interest rates, raising the returns of assets, produce a larger supply of deposits. This supply side effect is normally neglected in monetary theory, but it can explain the limited interest rate elasticity that empirical estimations of the demand for money have found. Even the instability of monetary aggregates may largely be dependent on the links between bank's assets and liabilities, as central banks need in many circumstances to control the issue of loans in order to keep under control the growth of monetary aggregates.

Finally we can observe that when banking institutions maximise profits, their intermediation is normally pro-cyclical since the size of the portfolio is an increasing function of income. This seems to imply that banking intermediaries may generate a financial accelerator, whenever bank loans positively affect income.

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Appendix

Appendix: solution of the model with contemporaneous feedback

The Lagrangian of the problem is the following:

$$\ell = \sum_{t=0}^{\infty} \beta^t \Big\{ \Big\{ -\frac{1}{b}\big[(1-q)D_t - F_t + NW\big] + \frac{a(Y_t)}{b} + \frac{d}{b} r_t^B + \varepsilon_t^L \Big\} \big[(1-q)D_t + \\ -F_t + NW\big] - \frac{1}{2} v \Big[(1-q)D_t - F_t + NW\Big]^2 - r_t^D D_t - uD_t - z\big[(1-q)D_t - F_t + NW\big] + \\ + r_t^B F_t - \mu_t \big\{ D_t - \delta D_{t-1} - \kappa\big[(1-q)D_t - F_t + NW\big] - g_3 r_t^D + g_4 r_t^B \big\} \Big\}. \quad (72)$$

The first order conditions are:

$$\frac{\partial \ell}{\partial F_{t+j}} = \beta^{t+j} \Big\{ \Big(v + \frac{2}{b} \Big) \Big[(1-q)D_{t+j} - F_{t+j} + NW\Big] + \\ - \frac{a(Y_{t+j})}{b} - \frac{d}{b} r_{t+j}^B + z - \varepsilon_{t+j}^L + r_{t+j}^B - \kappa \mu_{t+j} \Big\} = 0, \quad (73)$$

$$\frac{\partial \ell}{\partial D_{t+j}} = \beta^{t+j} E_{t+j} \Big[(1-q)\Big[\frac{a(Y_{t+j})}{b} + \frac{d}{b} r_{t+j}^B + \varepsilon_{t+j}^L\Big] + \\ -\Big(v + \frac{2}{b} \Big) \Big[(1-q)^2 D_{t+j} - (1-q)F_{t+j} + (1-q)NW\Big] - r_{t+j}^D - u - (1-q)z + \\ -\Big[1 - \kappa(1-q)\Big]\mu_{t+j} + \beta\delta E\left[\mu_{t+j+1}\right]\Big] = 0, \quad (74)$$

$$\frac{\partial \ell}{\partial \mu_{t+j}} = -D_{t+j} + \delta D_{t+j-1} + \kappa E\Big[(1-q)D_{t+j} - F_{t+j} + NW\Big] + g_3 r_{t+j}^D - g_4 r_{t+j}^B = 0. \quad (75)$$

Defining $\left(\frac{bv+2}{b}\right) = \alpha$ and $\frac{a(Y_t)}{b} + \frac{d}{b} r_t^B + \varepsilon_t^L = \omega_t$. For $j = T$, Condition (73) implies:

$$\beta^{t+T} E \Big\{ \alpha \Big[(1-q)D_{t+T} - F_{t+j} + NW\Big] - \omega_{t+T} + z + r_{t+T}^B - \mu_{t+T} \Big\} = 0. \quad (76)$$

The transversality condition of the problem is:

$$\lim_{T\to\infty} \beta^{t+T} E_t\Big\{\alpha\Big[(1-q)D_{t+T} - F_{t+j} + NW\Big] - \omega_{t+T} + z + r^B_{t+T} - \mu_{t+T}\Big\} = 0. \tag{77}$$

To keep the notation simpler, we omit from now on the index j, to show it only when it will be necessary, in the final solutions. From Equation (73) we can obtain an equation that can be solved in order to get a solution for the multiplier μ_t.

$$\mu_t = \frac{1}{\kappa}\Big\{\alpha\Big[(1-q)D_t - F_t + NW\Big] - \omega_t + z + r^B_t\Big\}. \tag{78}$$

This value can be substituted in the other first order condition, shown in equation (74), that provides the Euler equation of the system:

$$\begin{aligned}\beta^t\Big\{(1-q)(\omega_t - z) - r^D_t - u - \alpha\Big[(1-q)^2 D_t - (1-q)F_t + (1-q)NW\Big] + \\ -\frac{1-\kappa(1-q)}{\kappa}\Big\{\alpha\Big[(1-q)D_t - F_t + NW_t\Big] - (\omega_t - z) + r^B_t\Big\} + \\ +\frac{\delta\beta}{\kappa}E_t\Big\{\alpha\Big[(1-q)D_{t+1} - F_{t+1} + NW\Big] - (\omega_{t+1} - z) + r^B_{t+1}\Big\}\Big\} = 0.\end{aligned} \tag{79}$$

Rearranging and dividing everything by β^t we obtain a difference equation for F_t and D_t. Since $X_t = E_t[X_t]$ we can include everything under the expectation, from which we will from now on omit the time index, since all expectations are at time t.

$$\begin{aligned}E\Big\{\delta\beta\alpha F_{t+1} = \alpha F_t + \delta\beta(1-q)\alpha D_{t+1} - \alpha(1-q)D_t + \big[\beta\delta\alpha - \alpha\big]NW + \\ +\kappa\big[(1-q)(\omega_t - z) - r^D_t - u\big] + \big[1-\kappa(1-q)\big]\big[\omega_t - r^B_t - z\big] - \delta\beta\big[\omega_{t+1} - r^B_{t+1} - z\big]\Big\}.\end{aligned} \tag{80}$$

The next step is to simplify the resulting expression and to substitute for the value of D_t in Equation (80), the expression of the dynamic constraint that is obtained from Equation (75).

$$\begin{aligned}E\Big\{\delta\beta\alpha F_{t+1} = \alpha F_t + \delta\beta(1-q)\alpha\Big[\frac{\delta}{1-(1-q)\kappa}D_t - \frac{\kappa}{1-(1-q)\kappa}F_{t+1} + \\ +\frac{\kappa}{1-(1-q)\kappa}NW + \frac{g_3 r^D_{t+1} - g_4 r^B_{t+1}}{1-\kappa(1-q)}\Big] - \alpha(1-q)D_t + \big(\beta\delta\alpha - \alpha\big)NW + \\ +\kappa\big[(1-q)(\omega_t - z) - r^D_t - u\big] + \big[1-\kappa(1-q)\big]\big[\omega_t - r^B_t - z\big] - \delta\beta\big[\omega_{t+1} - r^B_{t+1} - z\big]\Big\},\end{aligned} \tag{81}$$

and

$$E\Big\{\delta\beta\frac{\alpha}{1-(1-q)\kappa}F_{t+1}=\alpha F_t+\delta\beta(1-q)\alpha\Big\{\frac{\delta}{1-(1-q)\kappa}D_t+$$
$$+\frac{\kappa}{1-(1-q)\kappa}NW+\frac{g_3r^D_{t+1}-g_4r^B_{t+1}}{1-\kappa(1-q)}\Big\}-\alpha(1-q)D_t+\big(\beta\delta\alpha-\alpha\big)NW+$$
$$+\kappa\big[(1-q)r^B_t-r^D_t-u\big]+(\delta\beta-L)\left[r^B_{t+1}-\omega_{t+1}\right]+(\beta\delta-1)z\Big\}. \tag{82}$$

Dividing both sides of the former equation for α, and introducing the lag operator L, we obtain:

$$E\left[F_{t+1}\right]=\frac{1-(1-q)\kappa}{\delta\beta}F_t+\frac{(1-q)\left[\delta^2\beta-\left[1-(1-q)\kappa\right]\right]}{\delta\beta}D_t+$$
$$+\frac{\beta\delta+(1-q)\kappa-1}{\delta\beta}NW+(1-q)[g_3r^D_t-g_4r^B_t]+$$
$$+E\Big\{\frac{1-(1-q)\kappa}{\delta\beta\alpha}\Big\{\kappa\big[(1-q)r^B_t-r^D_t-u\big]+(\delta\beta-L)\left[r^B_{t+1}-\omega_{t+1}\right]+(\beta\delta-1)z\Big\}. \tag{83}$$

As before we end up with a system of two equations, the other is obtained from the dynamic constraint:

$$D_t=\frac{\delta}{1-\kappa(1-q)}D_{t-1}-\frac{\kappa}{1-\kappa(1-q)}F_t+\frac{\kappa}{1-\kappa(1-q)}NW+\frac{g_3r^D_t-g_4r^B_t}{1-\kappa(1-q)}=0. \tag{84}$$

We can simplify the value of the second intercept term as:

$$Z_{t+1}=\kappa\big[(1-q)r^B_t-r^D_t-u\big]+(\delta\beta-L)\left[r^B_{t+1}-\omega_{t+1}\right]+(\beta\delta-1)z=$$
$$(\delta\beta-L)\left[r^B_{t+1}-r^D_{t+1}-\frac{a}{b}-\frac{d}{b}r^B_{t+1}+r^D_{t+1}-\varepsilon^L_{t+1}\right]+\kappa\big[(r^R_t-r^D_t)q+$$
$$+(1-q)(r^B_t-r^D_t)-u\big]+(\beta\delta-1)z=\Big\{\delta\beta\Big(1-\frac{d}{b}\Big)+\Big[(1-q)\kappa+$$
$$-\Big(1-\frac{d}{b}\Big)\Big]L\Big\}r^B_{t+1}-(\delta\beta-L)\varepsilon^L_{t+1}-\kappa\big[r^D_t+u\big]+(\beta\delta-1)\Big(z-\frac{a}{b}\Big). \tag{85}$$

Solution of the system

Defining $X_t=g_3r^D_t-g_4r^B_t$, the first equation can be rewritten as:

$$D_t=\frac{\delta\beta}{(1-q)\left[\delta^2\beta-\left[1-(1-q)\kappa\right]\right]}E\left[F_{t+1}\right]-\frac{\left[1-(1-q)\kappa\right]}{(1-q)\left[\delta^2\beta-\left[1-(1-q)\kappa\right]\right]}F_t+$$
$$-\frac{\beta\delta+\left[(1-q)\kappa-1\right]}{(1-q)\left[\delta^2\beta-\left[1-(1-q)\kappa\right]\right]}]NW-\frac{\delta\beta}{\left[\delta^2\beta-\left[1-(1-q)\kappa\right]\right]}X_t+$$
$$-\frac{1-(1-q)\kappa}{(1-q)\left[\delta^2\beta-\left[1-(1-q)\kappa\right]\right]}E\left[\frac{1}{\alpha}Z_{t+1}\right]. \tag{86}$$

Substituting the former in the other equation, we obtain a second order difference equation in $E[F_t]$.

$$
\begin{aligned}
E[F_{t+1}] - \frac{[1-(1-q)\kappa]}{\delta\beta}F_t + \frac{\kappa}{1-\kappa(1-q)}\frac{(1-q)\left[\delta^2\beta-[1-(1-q)\kappa]\right]}{\delta\beta}F_t + \\
-\frac{\delta}{1-\kappa(1-q)}F_t + \frac{\delta}{1-\kappa(1-q)}\frac{[1-(1-q)\kappa]}{\delta\beta}F_{t-1} = \\
= \frac{(1-q)\left[\delta^2\beta-[1-(1-q)\kappa]\right]}{\delta\beta}\left[\left[-\frac{\delta}{1-\kappa(1-q)}+1\right]\times\right. \\
\frac{\beta\delta+[(1-q)\kappa-1]}{(1-q)\left[\delta^2\beta-[1-(1-q)\kappa]\right]}NW + \frac{\kappa}{1-\kappa(1-q)}NW + \\
-\frac{\delta}{1-\kappa(1-q)}\frac{\delta\beta}{\left[\delta^2\beta-[1-(1-q)\kappa]\right]}X_{t-1} + \frac{\delta\beta}{\left[\delta^2\beta-[1-(1-q)\kappa]\right]}X_t + \\
+\frac{1}{1-\kappa(1-q)}X_t - \frac{\delta}{1-\kappa(1-q)}\frac{1-(1-q)\kappa}{(1-q)\left[\delta^2\beta-[1-(1-q)\kappa]\right]}E\left[\frac{1}{\alpha}\right]Z_t + \\
\left.+\frac{1-(1-q)\kappa}{(1-q)\left[\delta^2\beta-[1-(1-q)\kappa]\right]}E\left[\frac{1}{\alpha}Z_{t+1}\right]\right]. \qquad (87)
\end{aligned}
$$

We now study separately the left-hand side.

$$
\begin{aligned}
E[F_{t+1}] - \frac{[1-(1-q)\kappa]}{\delta\beta}F_t + \frac{\kappa}{1-\kappa(1-q)}\frac{(1-q)\left[\delta^2\beta-[1-(1-q)\kappa]\right]}{\delta\beta}F_t + \\
-\frac{\delta}{1-\kappa(1-q)}F_t + \frac{\delta}{1-\kappa(1-q)}\frac{[1-(1-q)\kappa]}{\delta\beta}F_{t-1} = E[F_{t+1}] - \left[\frac{1}{\delta\beta}+\delta\right]F_t + \frac{1}{\beta}F_{t-1}. \qquad (88)
\end{aligned}
$$

We split the right-hand side in different pieces, to begin with net worth.

$$
\begin{aligned}
\frac{(1-q)\left[\delta^2\beta-[1-(1-q)\kappa]\right]}{\delta\beta}\times\left[\left[-\frac{\delta}{1-\kappa(1-q)}+1\right]\times\right. \\
\left.\frac{\beta\delta+[(1-q)\kappa-1]}{(1-q)\left[\delta^2\beta-[1-(1-q)\kappa]\right]} + \frac{\kappa}{1-\kappa(1-q)}\right]NW = \frac{[1-\delta]\left[\beta\delta-1\right]}{\delta\beta}NW. \qquad (89)
\end{aligned}
$$

The coefficient of Z_t is:

$$
\begin{aligned}
\frac{(1-q)\left\{\delta^2\beta-[1-(1-q)\kappa]\right\}}{\delta\beta}\left[-\frac{\delta}{1-\kappa(1-q)}\times\right. \\
\frac{1-(1-q)\kappa}{(1-q)\left[\delta^2\beta-[1-(1-q)\kappa]\right]}E\left[\frac{1}{\alpha}\right]Z_t + \frac{1-(1-q)\kappa}{(1-q)\left[\delta^2\beta-[1-(1-q)\kappa]\right]}\times \\
\left.E\left[\frac{1}{\alpha}Z_{t+1}\right]\right] = -\frac{1}{\beta}E\left[\frac{1}{\alpha}\right]Z_t + \frac{1-(1-q)\kappa}{\delta\beta}E\left[\frac{1}{\alpha}Z_{t+1}\right]. \qquad (90)
\end{aligned}
$$

The value of the coefficient of X_t is:

$$\frac{(1-q)\left[\delta^2\beta-[1-(1-q)\kappa]\right]}{\delta\beta}\left[-\frac{\delta}{1-\kappa(1-q)}\frac{\delta\beta}{\left[\delta^2\beta-[1-(1-q)\kappa]\right]}X_{t-1}+\right.$$
$$\left.+\frac{\delta\beta}{\left[\delta^2\beta-[1-(1-q)\kappa]\right]}X_t+\frac{1}{1-\kappa(1-q)}X_t\right]=$$
$$=\frac{(1-q)(\delta\beta-1)}{\delta\beta}X_t+\frac{(1-q)\delta}{1-\kappa(1-q)}\Delta X_t. \qquad (91)$$

The final result is the following:

$$E[F_{t+1}]-\left[\frac{1}{\delta\beta}+\delta\right]F_t+\frac{1}{\beta}F_{t-1}=\frac{[1-\delta]\left[\beta\delta-1\right]}{\delta\beta}NW+$$
$$-\frac{1}{\beta}E\left[\frac{1}{\alpha}\right]Z_t+\frac{1-(1-q)\kappa}{\delta\beta}E\left[\frac{1}{\alpha}Z_{t+1}\right]+\frac{(1-q)(\delta\beta-1)}{\delta\beta}X_t+\frac{(1-q)\delta}{1-\kappa(1-q)}\Delta X_t. \qquad (92)$$

The difference equation for deposits can be obtained with the same procedure.

Bonds

Isolating the constant terms, and using the properties that $\frac{1}{(1-\lambda_2 L)}a=\frac{-(\lambda_2 L)^{-1}}{1-(\lambda_2 L)^{-1}}a=\frac{a}{1-\lambda_2}$, and $\frac{1}{(1-\lambda_2 L)}bX_t=b\frac{-(\lambda_2 L)^{-1}}{1-(\lambda_2 L)^{-1}}X_t=-b\sum_{i=1}^{\infty}\left(\frac{1}{\lambda_2}\right)^i X_{t+i+1}$ (where a and b are arbitrary constant terms), applying the transversality condition of the problem, Equation (41) can be solved as:

$$F_{t+j+1}=\delta F_{t+j}+(1-\delta)NW-\frac{1-(1-q)\kappa-\delta H}{\beta\delta}\sum_{i=1}^{\infty}\left(\beta\delta\right)^i E_{t+i}\left[\frac{Z'_{t+j+i+2}}{\alpha}\right]+$$
$$-\left[\frac{(1-q)(\delta\beta-1)}{\delta\beta}+\frac{(1-q)(1-H)}{1-\kappa(1-q)}\right]\sum_{i=1}^{\infty}\left(\beta\delta\right)^i X_{t+j+i+1}-\frac{1-(1-q)\kappa-\delta}{(1-\beta\delta)\alpha}C. \qquad (93)$$

Since we have assumed that interest rates follow a random walk process, the correlation between interest rates and default costs is time-invariant, and we can rewrite the expression treating all future terms as constants:

$$\begin{aligned}
F_{t+j+1} = \delta F_{t+j} + (1-\delta)NW - \frac{1-(1-q)\kappa-\delta}{(1-\beta\delta)\alpha}\Big\{(1-\beta\delta)\Big[z-\frac{a}{b}\Big]+\kappa u\Big\} + \\
+\Big\{\frac{[1-(1-q)\kappa-\delta]\Big[\Big(1-\frac{d}{b}\Big)(\delta\beta-1)+(1-q)\kappa\Big]}{(\beta\delta-1)\alpha} - (1-q)g_4\Big\} r^B_{t+j+1} + \\
-\Big\{\frac{[1-(1-q)\kappa-\delta]\Big[\Big(1-\frac{d}{b}\Big)(\delta\beta-1)+(1-q)\kappa\Big]\kappa}{(\beta\delta-1)\alpha} - (1-q)g_3\Big\} r^D_{t+j+1} + \\
+\Big\{\frac{[\delta^2\beta+1-(1-q)\kappa]\Big(1-\frac{d}{b}\Big)-[1-(1-q)\kappa](1-q)\kappa}{\beta\delta\alpha} + \\
-\Big[\frac{(1-q)(1-\delta\beta)}{\delta\beta}+\frac{(1-q)\delta^2\beta}{1-\kappa(1-q)}\Big]g_4\Big\} r^B_{t+j} + \\
+\Big\{\frac{1-(1-q)\kappa}{\beta\delta\alpha}\kappa+\Big[\frac{(1-q)(1-\delta\beta)}{\delta\beta}+\frac{(1-q)\delta^2\beta}{1-\kappa(1-q)}\Big]g_3\Big\} r^D_{t+j} + \\
+\Big\{\frac{\delta\Big[(1-q)\kappa-\Big(1-\frac{d}{b}\Big)\Big]}{\beta\delta\alpha}+\frac{(1-q)\delta}{1-\kappa(1-q)}g_4\Big\} r^B_{t+j-1} + \\
-\Big[\frac{\delta\kappa}{\beta\delta\alpha}+\frac{(1-q)\delta}{1-\kappa(1-q)}g_3\Big] r^D_{t+j-1} + \\
+\frac{1-(1-q)\kappa-\delta}{(\beta\delta-1)\alpha}\Big[\Big(1-\frac{d}{b}\Big)(\delta\beta-1)+(1-q)\kappa\Big]COV\Big(r^B,\frac{1}{\alpha}\Big) + \\
-\Big[1-(1-q)\kappa-\delta\Big]COV\Big(L^D,\frac{1}{\alpha}\Big).
\end{aligned} \tag{94}$$

The former expression can be exposed as:

$$\begin{aligned}
F_{t+j+1} = \delta F_{t+j} + (1-\delta)NW - \frac{1-(1-q)\kappa-\delta}{(1-\beta\delta)\alpha}\Big\{(1-\beta\delta)\Big[z-\frac{a}{b}\Big]+\kappa u\Big\} + \\
+(A_1+A_2L+A_3L^2)r^B_{t+j+1}+(C_1+C_2L+C_3L^2)r^D_{t+j+1} + \\
+\frac{1-(1-q)\kappa-\delta}{(\beta\delta-1)\alpha}\Big[\Big(1-\frac{d}{b}\Big)(\delta\beta-1)+(1-q)\kappa\Big]COV\Big(r^B,\frac{1}{\alpha}\Big) + \\
-\Big[1-(1-q)\kappa-\delta\Big]COV\Big(L^D,\frac{1}{\alpha}\Big),
\end{aligned} \tag{95}$$

where

$$A_1 = \frac{[1-(1-q)\kappa-\delta]\Big[\Big(1-\frac{d}{b}\Big)(\delta\beta-1)+(1-q)\kappa\Big]}{(\beta\delta-1)\alpha} - (1-q)g_4$$
$$C_1 = -\frac{[1-(1-q)\kappa-\delta]\Big[\Big(1-\frac{d}{b}\Big)(\delta\beta-1)+(1-q)\kappa\Big]\kappa}{(\beta\delta-1)\alpha} + (1-q)g_3$$
$$A_2 = \frac{[\delta^2\beta+1-(1-q)\kappa]\Big(1-\frac{d}{b}\Big)-[1-(1-q)\kappa](1-q)\kappa}{\beta\delta\alpha} +$$
$$-\Big[\frac{(1-q)(1-\delta\beta)}{\delta\beta}+\frac{(1-q)\delta^2\beta}{1-\kappa(1-q)}\Big]g_4$$
$$C_2 = \frac{1-(1-q)\kappa}{\beta\delta\alpha}\kappa + \Big[\frac{(1-q)(1-\delta\beta)}{\delta\beta}+\frac{(1-q)\delta^2\beta}{1-\kappa(1-q)}\Big]g_3$$
$$A_3 = \frac{\delta\Big[(1-q)\kappa-\Big(1-\frac{d}{b}\Big)\Big]}{\beta\delta\alpha} + \frac{(1-q)\delta}{1-\kappa(1-q)}g_4$$
$$C_3 = -\Big[\frac{\delta\kappa}{\beta\delta\alpha}+\frac{(1-q)\delta}{1-\kappa(1-q)}g_3\Big]. \tag{96}$$

Deposits

Following the same steps, but substituting this time the value of F_t obtained form the demand condition in the Euler equation, we obtain:

$$E[D_{t+1}] - \Big[\frac{1}{\delta\beta}+\delta\Big]D_t + \frac{1}{\beta}D_{t-1} = -\frac{\kappa}{\delta\beta}E\Big[\frac{1}{\alpha}Z_{t+1}\Big] +$$
$$-\frac{1-(1-\delta\beta)\kappa(1-q)}{\delta\beta[1-\kappa(1-q)]}X_t + \frac{1}{1-\kappa(1-q)}E[X_{t+1}]. \tag{97}$$

We can obtain the value of D_t solving the previous equation. The solution is given by:

$$D_{t+j+1} = \frac{1}{\delta\beta}D_{t+j} + \frac{\kappa}{\beta\delta}\sum_{i=1}^{\infty}\Big(\frac{1}{\delta}\Big)^i E_{t+i}\Big[\frac{Z'_{t+j+i+2}}{\alpha}\Big] - \frac{\kappa}{\beta\delta\alpha(1-\delta)}C +$$
$$+\sum_{i=1}^{\infty}\Big(\frac{1}{\delta}\Big)^i E_{t+i}[X_{t+j+i+1}] - \frac{1-(1-\delta\beta)\kappa(1-q)}{\delta\beta[1-\kappa(1-q)]}X_t. \tag{98}$$

Under the assumption that interest rates follow a random walk process, it can finally be simplified as:

$$
\begin{aligned}
D_{t+j+1} = \delta D_{t+j} + \Bigg\{ \frac{\kappa\left[\left(1-\frac{d}{b}\right)(\delta\beta-1)+(1-q)\kappa\right]}{(1-\beta\delta)\alpha} - g_4 \Bigg\} r^B_{t+j+1} + \Bigg[g_3 + \\
-\frac{\kappa^2}{(1-\beta\delta)\alpha}\Bigg] r^D_{t+j+1} + \Bigg\{ \left[\frac{1}{\delta\beta}+\frac{\kappa(1-q)}{1-\kappa(1-q)}\right] g_4 + \frac{\kappa\left[\left(1-\frac{d}{b}\right)-(1-q)\kappa\right]}{\delta\beta\alpha} \Bigg\} r^B_{t+j} + \\
-\Bigg\{ \left[\frac{1}{\delta\beta}+\frac{\kappa(1-q)}{1-\kappa(1-q)}\right] g_3 - \frac{\kappa^2}{\delta\beta\alpha} \Bigg\} r^D_{t+j} - \frac{\kappa\left\{(1-\beta\delta)\left[z-\frac{a}{b}\right]+\kappa u\right\}}{(1-\beta\delta)\alpha} + \\
+\frac{\kappa\left[\left(1-\frac{d}{b}\right)(\delta\beta-1)+(1-q)\kappa\right]}{(1-\beta\delta)\alpha} COV\left(r^B,\frac{1}{\alpha}\right) + \kappa COV\left(L^D,\frac{1}{\alpha}\right),
\end{aligned}
\tag{99}
$$

and

$$
\begin{aligned}
D_{t+j+1} = \delta D_{t+j} + \hat{A}_1 r^B_{t+j+1} + \hat{C}_1 r^D_{t+j+1} + \\
+\hat{A}_2 r^B_{t+j} + \hat{C}_2 r^D_{t+j} - \frac{\kappa\left\{(1-\beta\delta)\left[z-\frac{a}{b}\right]+\kappa u\right\}}{(1-\beta\delta)\alpha} + \\
+\frac{\kappa\left[\left(1-\frac{d}{b}\right)(\delta\beta-1)+(1-q)\kappa\right]}{(1-\beta\delta)\alpha} COV\left(r^B,\frac{1}{\alpha}\right) + \kappa COV\left(L^D,\frac{1}{\alpha}\right),
\end{aligned}
\tag{100}
$$

where,

$$
\begin{aligned}
\hat{A}_1 = -\frac{\kappa\left[\left(1-\frac{d}{b}\right)(1-\delta\beta)-(1-q)\kappa\right]}{(1-\beta\delta)\alpha} - g_4 \qquad \hat{C}_1 = \left[g_3 - \frac{\kappa^2}{(1-\beta\delta)\alpha}\right] \\
\hat{A}_2 = \left[\frac{1}{\delta\beta}+\frac{\kappa(1-q)}{1-\kappa(1-q)}\right] g_4 + \frac{\kappa\left[\left(1-\frac{d}{b}\right)-(1-q)\kappa\right]}{\delta\beta\alpha} \\
\hat{C}_2 = -\left[\frac{1}{\delta\beta}+\frac{\kappa(1-q)}{1-\kappa(1-q)}\right] g_3 + \frac{\kappa^2}{\delta\beta\alpha}.
\end{aligned}
\tag{101}
$$

Loans

The rational expectation equilibrium quantity of loans can be easily obtained from the budget constraint $L = (1-q)D - F + NW$:

$$
\begin{aligned}
L_{t+j+1} = \delta L_{t+j} - (1-\delta)NW + \frac{1-\delta}{(1-\delta\beta)\alpha}C + \frac{1-\delta H}{\delta\beta}\sum_{i=0}^{i+1}\left(\delta\beta\right)^i E_{t+i}\left[\frac{Z'_{t+j+i+1}}{\alpha}\right] + \\
-\frac{1-q}{1-\kappa(1-q)}\left[(1+\delta)X_t - \delta^2\beta X_t - \delta X_{t-1}\right]
\end{aligned}
\tag{102}
$$

and:

$$L_{t+j+1} = \delta L_{t+j} - (1-\delta)NW + \frac{(1-\delta)\left[\left(1-\frac{d}{b}\right)(\delta\beta-1)+(1-q)\kappa\right]}{(1-\delta\beta)\alpha} r^B_{t+j+1} +$$
$$-\frac{(1-\delta)\kappa}{(1-\delta\beta)\alpha} r^D_{t+j+1} - \left\{\frac{[\delta^2\beta+1]\left(1-\frac{d}{b}\right)-(1-q)\kappa}{\beta\delta\alpha} +\right.$$
$$\left.-\frac{(1-q)[1+\delta(1-\delta\beta)]g_4}{1-\kappa(1-q)}\right\} r^B_{t+j} - \left\{\frac{1}{\beta\delta\alpha}\kappa - \frac{(1-q)(1+\delta(1-\delta\beta)g_3}{1-\kappa(1-q)}\right\} r^D_{t+j} +$$
$$-\left\{\frac{\delta\left[(1-q)\kappa-\left(1-\frac{d}{b}\right)\right]}{\beta\delta\alpha} + \frac{(1-q)\delta g_4}{1-\kappa(1-q)}\right\} r^B_{t+j-1} +$$
$$+\left[\frac{\delta\kappa}{\beta\delta\alpha} + \frac{(1-q)\delta g_3}{1-\kappa(1-q)}\right] r^D_{t+j-1} - \frac{z}{\alpha} - \frac{\kappa u}{(1-\delta\beta)\alpha} + \frac{a}{b\alpha} +$$
$$+\frac{(1-\delta)\left[\left(1-\frac{d}{b}\right)(\delta\beta-1)+(1-q)\kappa\right]}{(1-\delta\beta)\alpha} COV\left(r^B, \frac{1}{\alpha}\right) + \frac{1-\delta}{(1-\delta\beta)} COV\left(L^D, \frac{1}{\alpha}\right). \quad (103)$$

$$L_{t+j+1} = \delta L_{t+j} - (1-\delta)NW +$$
$$-\frac{z}{\alpha} - \frac{\kappa u}{(1-\delta\beta)\alpha} + \frac{a}{b\alpha} + (\bar{A}_1 + \bar{A}_2 L + \bar{A}_3 L^2) r^B_{t+j+1} + (\bar{C}_1 + \bar{C}_2 L + \bar{C}_3 L^2) r^D_{t+j+1} +$$
$$+\frac{(1-\delta)\left[\left(1-\frac{d}{b}\right)(\delta\beta-1)+(1-q)\kappa\right]}{(1-\delta\beta)\alpha} COV\left(r^B, \frac{1}{\alpha}\right) + \frac{1-\delta}{(1-\delta\beta)} COV\left(L^D, \frac{1}{\alpha}\right), \quad (104)$$

where

$$\bar{A}_1 = \frac{(1-\delta)\left[\left(1-\frac{d}{b}\right)(\delta\beta-1)+(1-q)\kappa\right]}{(1-\delta\beta)\alpha} \qquad \bar{C}_1 = -\frac{(1-\delta)\kappa}{(1-\delta\beta)\alpha}$$
$$\bar{A}_2 = -\frac{[\delta^2\beta+1]\left(1-\frac{d}{b}\right)-(1-q)\kappa}{\beta\delta\alpha} + \frac{(1-q)[1+\delta(1-\delta\beta)]g_4}{1-\kappa(1-q)}$$
$$\bar{C}_2 = -\frac{1}{\beta\delta\alpha}\kappa + \frac{(1-q)(1+\delta(1-\delta\beta)g_3}{1-\kappa(1-q)}$$
$$\bar{A}_3 = -\frac{\delta\left[(1-q)\kappa-\left(1-\frac{d}{b}\right)\right]}{\beta\delta\alpha} - \frac{(1-q)\delta g_4}{1-\kappa(1-q)} \qquad \bar{C}_3 = \frac{\delta\kappa}{\beta\delta\alpha} + \frac{(1-q)\delta g_3}{1-\kappa(1-q)}. \quad (105)$$

The interest rate on loans

The interest rate on loans can easily be obtained substituting the solution (51) for the quantity of loans in the demand condition:

$$L_{t+j+1} = a - br^L_{t+j+1} + dr^B_{t+j+1} + \eta_{t+j+1}, \quad (106)$$

For simplicity we use the solution of (52), but the result is general.

$$L_{t+j+1} = \delta L_{t+j} - (1-\delta)NW - \frac{z}{\alpha} - \frac{\kappa u}{(1-\delta\beta)\alpha} + \frac{a}{b\alpha} +$$
$$+(\bar{A}_1 + \bar{A}_2 L + \bar{A}_3 L^2) r^B_{t+j+1} + (\bar{C}_1 + \bar{C}_2 L + \bar{C}_3 L^2) r^D_{t+j+1} + \frac{1-\delta}{(1-\delta\beta)} COV\left(L^D, \frac{1}{\alpha}\right) +$$
$$+\frac{(1-\delta)\left[(1-q)\kappa - \left(1-\frac{d}{b}\right)(1-\delta\beta)+\right]}{(1-\delta\beta)\alpha} COV\left(r^B, \frac{1}{\alpha}\right), \tag{107}$$

and

$$r^L_{t+j+1} = \frac{1}{b}\Big\{a + d r^B_{t+j+1} - \delta L_{t+j+1} + (1-\delta)NW + \frac{z}{\alpha} + \frac{\kappa u}{(1-\delta\beta)\alpha} - \frac{a}{b\alpha} - (\bar{A}_1 +$$
$$+\bar{A}_2 L + \bar{A}_3 L^2) r^B_{t+j+1} - (\bar{C}_1 + \bar{C}_2 L + \bar{C}_3 L^2) r^D_{t+j+1} - \frac{1-\delta}{(1-\delta\beta)} COV\left(L^D, \frac{1}{\alpha}\right)\Big\} +$$
$$-\frac{(1-\delta)\left[(1-q)\kappa - \left(1-\frac{d}{b}\right)(1-\delta\beta)+\right]}{(1-\delta\beta)\alpha} COV\left(r^B, \frac{1}{\alpha}\right); \tag{108}$$

focusing on the relationship with the contemporaneous rate on bonds, we obtain:

$$r^L_{t+j+1} = \frac{1}{b}\Big\{(d - \bar{A}_1) r^B_{t+j+1} + G\Big\}, \tag{109}$$

where

$$G = -\delta L_{t+j+1} + \left(1 - \frac{1}{b\alpha}\right)a + (1-\delta)NW + \frac{z}{\alpha} + \frac{\kappa u}{(1-\delta\beta)\alpha} +$$
$$-(\bar{A}_2 L + \bar{A}_3 L^2) r^B_{t+j+1} - (\bar{C}_1 + \bar{C}_2 L + \bar{C}_3 L^2) r^D_{t+j+1} +$$
$$-\frac{(1-\delta)\left[(1-q)\kappa - \left(1-\frac{d}{b}\right)(1-\delta\beta)\right]}{(1-\delta\beta)\alpha} COV\left(r^B, \frac{1}{\alpha}\right) - \frac{1-\delta}{(1-\delta\beta)} COV\left(L^D, \frac{1}{\alpha}\right). \tag{110}$$

Finally,

$$r^L_{t+j+1} = \frac{1}{b}\Big\{\Big[d - \frac{(1-\delta)\left[(1-q)\kappa - \left(1-\frac{d}{b}\right)(1-\delta\beta)\right]}{(1-\delta\beta)\alpha}\Big] r^B_{t+j+1} + G\Big\}. \tag{111}$$

The cross-derivative

Given

$$L_{t+j+1} = \delta L_{t+j} - (1-\delta)NW + \frac{1-\delta}{\alpha}\Big\{\frac{(1-q)\kappa}{(1-\delta\beta)} - \left(1 - \frac{d}{b}\right)\Big\} \times$$
$$\left[r^B_{t+j+1} + COV\left(r^B, \frac{1}{\alpha}\right)\right] + \ldots \quad , \tag{112}$$

and defining $A = 1 + \partial COV\left(r^B, \frac{1}{\alpha}\right)/\partial r^B$, we obtain:

$$\frac{\partial L_{t+j+1}}{\partial\delta\ \partial r^B_{t+j+1}} = \frac{A}{\alpha}\left\{(1-\delta)\frac{(1-q)\kappa\beta}{(1-\delta\beta)^2} - \left[\frac{(1-q)\kappa}{(1-\delta\beta)} - \left(1-\frac{d}{b}\right)\right]\right\}. \tag{113}$$

$$\frac{\partial L_{t+j+1}}{\partial\delta\ \partial r^B_{t+j+1}} > 0 \qquad \text{when} \qquad (1-\delta)\frac{(1-q)\kappa\beta}{(1-\delta\beta)^2} > \frac{(1-q)\kappa}{1-\delta\beta} - \left(1-\frac{d}{b}\right), \tag{114}$$

$$(1-\delta)(1-q)\kappa\beta + \left(1-\frac{d}{b}\right)(1-\delta\beta)^2 > (1-q)\kappa(1-\delta\beta), \tag{115}$$

$$(1-q)\kappa(\beta-1) + \left(1-\frac{d}{b}\right)(1-\delta\beta)^2 > 0, \tag{116}$$

$$\frac{\left(1-\frac{d}{b}\right)(1-\delta\beta)^2}{1-\beta} > (1-q)\kappa, \qquad \delta < \left[1-\sqrt{\frac{(1-q)\kappa(1-\beta)}{1-\frac{d}{b}}}\right]\Big/\beta. \tag{117}$$