

Dynamic State Tameness*

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Abstract

An extension of the idea of state tameness is presented in a dynamic framework. The proposed model for financial markets is rich enough to provide analytical tools that are mostly obtained in models that arise as the solution of SDEs with deterministic coefficients. In the presented model the augmentation by a shadow stock of the price evolution has a Markovian character. As in a previous paper, the results obtained on valuation of European contingent claims and American contingent claims do not require the full range of the volatility matrix. Under some additional continuity conditions, the conceptual framework provided by the model makes it possible to regard the valuation of financial instruments of the European type as a particular case of valuation of instruments of American type. This provides a unifying framework for the problem of valuation of financial instruments.

1 Introduction

State Tameness was introduced in Londoño [5] in a setting of a general semimartingale process driven by Brownian motions, in order to give a full characterization of non existence of arbitrage that has an algebraic appealing character with an economic justification. It also provided theorems for valuation of financial instruments of European and American type. When analytical tools are needed for the study of financial markets, as it is the case for the problem of optimal consumption and investment, the general semimartingale framework is too weak, and the standard approach is usually to impose very strong conditions (e.g. deterministic coefficients) in order to obtain a rich theory (Karatzas and Shreve [3]). In this paper we propose a model for financial markets that

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captures the characteristic properties that a general formulation of the problem of optimal consumption and investment, we believe should have. We think that the main contribution of this work is that it provides tools and a framework to solve dynamic problems as the mentioned above. (See for instance Londoño [6].)

The model is inspired by heuristic considerations when the author tried to formulate the problem of optimal investment and consumption as explained in Londoño [6]. In particular, in order to model a typical consumer that changes preferences for consumption partners, due for instance to aging, it is needed that the model should allow for the computation of optimal strategies of investment and consumption after any given time. As it is explained in Londoño [6], the latter problem reduces to computing the portfolio that finances a given optimal wealth. Typically, the key tool to obtain those optimal portfolios, is the representation theorem for Brownian martingales as stochastic integrals (see for instance Londoño [5]). In the framework proposed Kunita [4, Exercise 3.2.10] can be used instead. However, in order to apply the cited theorem, it is necessary that the randomness underlying the process after any given time be generated by a Brownian motion that does not carry any information from the past. It is therefore unavoidable the use of two-parameter filtrations to model the cited problem. Finally, we use the concept of non-arbitrage of state tame portfolios to restrict the class of models we propose to study.

In few words, the model of Market that we propose is a model where the underlying source of randomness is a finite dimensional Brownian motion, where the evolution of the process between any two times depends on the *evolution* of the Brownian motion and the state of the process at the initial time of the interval. In this model the state is characterized by the price of the stocks and the value of a “shadow stock” that captures the evolution of the economy (see Section 3). If the model is obtained as the result of solving some stochastic differential equations of coefficients satisfying some Lipschitz condition, the proposed model becomes a Brownian flow of homeomorphisms (see Section 3). We point out that there are important classes of processes that do not fit in the latter framework. Processes that do not satisfy the Lipschitz continuity property, and processes driven by Levy processes (not necessary continuous), are important examples (see Kager and Scheutzow [2]).

Using Arnold’s terminology (Arnold [1]), the model of prices proposed is a crude cocycle over the metric dynamical system defined by the Brownian motion (see remarks at the end of Section 2). Arnold’s formulation does not require the evolution of the process to be adapted to the evolution of the underlying Brownian motion. In fact when the random dynamical system is the solution of an SDE driven by a continuous spatial $(\mathcal{F}_{s,t})$ forward semimartingale helix, satisfying some smoothness conditions, both definitions are equivalent (See Section 2).

Next we briefly describe the contents of the paper. For the sake of completeness, in Section 2 we recall some spaces of functions, following Kunita [4]. Next, we define the notion of consistent process and we give some examples. The author is not aware of a similar definition in the current literature. Sec-

tion 3 defines the model of prices. In order to provide a consistent framework the model for the price process of n stocks and a hidden variable proposed is a $(n + 1)$ dimensional process. Its k^{th} -point motion, for any positive integer k , satisfies the Markov property (see the proof of Theorem 4.2.1 in Kunita [4]). In other words the price of the stocks does not necessarily satisfy a Markov structure, in our model, but the augmentation by a “hidden” variable does. We also define within the model proposed basic structures in finance as wealth, income and portfolio structures; an example is provided. It should be emphasized that although we try to be as close as possible in definitions and notation for the above concepts to Karatzas and Shreve [3], all of the above structures need a precise definition that goes along with our framework, since this is the first time this model is proposed. After the conceptual framework is established the math is rather straightforward. In Section 4, a characterization of non existence of arbitrage opportunities is given, mainly following the proof given in Londoño [5]. Sections 5 and 6 provide the corresponding theory for valuation of financial instruments of European and American type. Although some ideas of the proofs given in Londoño [5] can be easily changed to be adapted to the current model, special care must be taken with the smoothness in the price variable that the model implies. As in the mentioned paper full range of the volatility matrix is not required. However the most interesting feature of the framework proposed within the valuation of derivatives is that under some additional condition on continuity of the expected values of some random variables (see Condition 1), the problem of valuation of financial instruments of the European type is a particular case of the problem of valuation of financial instruments of the American type (see Theorem 2 and Theorem 3).

2 Some Definitions

First we introduce some notation which will be frequently used in this paper. Let $\mathbb{D} \subset \mathbb{R}^k$ be a open connected set. Let m be a non-negative integer. We denote by $C^{m,\delta}(\mathbb{D}; \mathbb{R}^n)$ the the Fréchet space of m -times continuous differentiable functions whose m -order derivatives are δ -Hölder continuous with semi-norms $\|f\|_{m,\delta;K}$ defined in Kunita [4, Section 3.1] where $K \subset \mathbb{D}$ is a compact set and $0 \leq \delta \leq 1$. In case $m = 0$ (or $\delta = 0$) we denote $C^{m,\delta}(\mathbb{D}; \mathbb{R}^n)$ simply by $C^\delta(\mathbb{D}; \mathbb{R}^n)$ ($C^m(\mathbb{D}; \mathbb{R}^n)$). We also denote by $\tilde{C}^{m,\delta}(\mathbb{D}; \mathbb{R}^n)$ the Fréchet space of continuous functions $g: \mathbb{D} \times \mathbb{D} \rightarrow \mathbb{R}^n$ which are m -times continuously differentiable with respect to each variable and whose m -order derivatives with respect to both variables are δ -Hölder continuous with semi-norms $\|g\|_{m,\delta;K}$, where K is a compact set, as described in Kunita [4, Section 3.1]. In case $m = 0$ we denote $\tilde{C}^m(\mathbb{D}; \mathbb{R}^n)$ by $\tilde{C}(\mathbb{D}; \mathbb{R}^n)$ and $\tilde{C}^{m,\delta}(\mathbb{D}; \mathbb{R}^n)$ by $\tilde{C}^\delta(\mathbb{D}; \mathbb{R}^n)$. In case $\delta = 0$ we use the notations $\tilde{C}^{m,\delta}(\mathbb{D}; \mathbb{R}^n)$ and $\tilde{C}^m(\mathbb{D}; \mathbb{R}^n)$ interchangeably.

We assume a d -dimensional Brownian Motion $\{W(t), \mathcal{F}_t; 0 \leq t \leq T\}$ starting at 0 defined on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ where $\mathcal{F} = \mathcal{F}_T$ and $\{\mathcal{F}_t, 0 \leq t \leq T\}$ is the \mathbf{P} augmentation by the null sets of the natural filtration $\mathcal{F}_t^W = \sigma(W(s), 0 \leq s \leq t)$. Let $(\mathcal{F}_{s,t})_{s_0} = \{\mathcal{F}_{s,t}, s_0 \leq s \leq t \leq T\}$ be the two

parameter filtration where $\mathcal{F}_{s,t}$ is the least sub σ -field containing all null sets and $\sigma(W_s(u) \mid s \leq u \leq t)$, where $W_s(u) \equiv W(u) - W(s)$. In the case that $s_0 = 0$, we just write $(\mathcal{F}_{s,t})$ as an abbreviation for $(\mathcal{F}_{s,t})_0$. See Arnold [1] and Kunita [4] for detailed accounts of two parameter filtrations.

Next, we give some definitions frequently used in this paper. Let $0 \leq s_0 \leq T$ be a fixed number. We shall say that a family of processes $\{\varphi(s,t), s_0 \leq s \leq t \leq T\}$ with values in some euclidean space is a $(\mathcal{F}_{s,t})_{s_0}$ *progressive measurable process with two parameters after time s_0* if for each s with $s_0 \leq s \leq T$, $\{\varphi(s,t), s \leq t \leq T\}$ is a $\mathcal{F}_{s,t}$ progressive measurable processes. In addition, if for each $s_0 \leq s \leq T$, the process $\{\varphi(s,t), s \leq t \leq T\}$ is a continuous $\mathcal{F}_{s,t}$ -semimartingale, then we say that the process $\{\varphi(s,t), s_0 \leq s \leq t \leq T\}$ is a continuous $(\mathcal{F}_{s,t})_{s_0}$ -semimartingale with two parameters.

Let $\mathbb{D} \subset \mathbb{R}^k$, be an open set. Let $\varphi(s,t,x,\omega)$, $s_0 \leq s \leq t \leq T$, $x \in \mathbb{D}$ be a \mathbb{R}^n -valued random field on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$. We call it a *measurable process with two parameters after time s_0 with values in \mathbb{R}^n* if for each $x \in \mathbb{D}$, $\varphi(\cdot, \cdot, x)$ is a progressive measurable process with two parameters after time s_0 . We say that the φ is a $C^{m,\delta}(\mathbb{D}; \mathbb{R}^n)$ -*process with two-parameters after time s_0* if for each $s_0 \leq s \leq T$ the process $\varphi_s: t \rightarrow \varphi(s,t,\cdot)$, is a measurable random field with values in $C^{m,\delta}(\mathbb{D}; \mathbb{R}^n)$. In addition, if $\varphi(s,t,x)$ is a continuous $(\mathcal{F}_{s,\cdot})$ process for each $x \in \mathbb{D}$ and $s_0 \leq s \leq T$, then we shall say that φ is a *continuous $C^{m,\delta}(\mathbb{D}; \mathbb{R}^n)$ -process with two parameters after time s_0* . For a definition of measurable random fields see Kunita [4].

Let φ be a $C^{m,\delta}(\mathbb{D}; \mathbb{R}^n)$ -process with two parameters after time s_0 . Assume that $\varphi(s,t,x)$, $x \in \mathbb{D}$ is a family of continuous semimartingales decomposed as $\varphi(s,t,x) = \varphi_{loc}(s,t,x) + \varphi_{fv}(s,t,x)$, where φ_{loc} is a continuous $C^{m,\delta}(\mathbb{D}; \mathbb{R}^n)$ -local-martingale with two parameters, and φ_{fv} is a continuous $C^{m,\delta}(\mathbb{D}; \mathbb{R}^n)$ -process of bounded variation with two parameters after time s_0 . We say that φ is a *continuous $C^{m,\delta}(\mathbb{D}; \mathbb{R}^n)$ -semimartingale with two parameters after time s_0* .

For each $0 \leq s \leq T$ we define a σ -field \mathcal{P}_s of progressive measurable sets after time s as the σ -field of sets $P \in \mathcal{B}([s,T]) \otimes \mathcal{F}_{s,T}$, the product σ -field, such that $\chi_P(t,\omega)$, $t \geq s$, is a $\mathcal{F}_{s,t}$ progressive measurable (in t) process, where χ is the indicator function. Define the measure μ_s on \mathcal{P}_s by $\mu_s(P) = \mathbf{E} \int_s^T \chi_P(s,\omega) dt$. Assume that φ is a $C(\mathbb{D}; \mathbb{R}^n)$ -semimartingale with two parameters after time s_0 , with decomposition $\varphi = \varphi_{loc} + \varphi_{fv}$ as above. A pair (a,b) where $a(s,t,x,y)$ and $b(s,t,x)$ are measurable random fields $\mathcal{F}_{s,t}$ -progressive measurable in t , for all $x,y \in \mathbb{D}$, $s_0 \leq s \leq T$, is said to be the *local characteristics of φ* , if $(a(s,\cdot,x,y), b(s,\cdot,x))$ is the local characteristic of $\varphi_s \equiv \varphi(s,\cdot,\cdot)$ (see Kunita [4]) for any $s \leq t \leq T$. In addition, a pair (σ,b) where $\sigma(s,t,x)$ is a measurable random field with values in $L(\mathbb{R}^d; \mathbb{R}^n)$, where $L(\mathbb{R}^d; \mathbb{R}^n)$ denotes the set of matrices with size $n \times d$, $(\mathcal{F}_{s,t})$ -progressive measurable in t , for all $x \in \mathbb{D}$, $s_0 \leq s \leq T$, and b is as above is said to be the *volatility and drift processes of φ* if

$$\varphi_{loc}(s,t,x)(\omega) = \int_s^t \sigma(s,u,x) dW_s(u),$$

for all x, s, t and ω . Assuming a Brownian filtration, if φ is a continuous $C(\mathbb{D}: \mathbb{R}^n)$ -semimartingale with two parameters after time s_0 , then there exists a pair (σ, b) of volatility and drift processes of φ , as a consequence of Kunita [4, Exercise 3.2.10]. It follows that the pair of volatility and drift processes of a continuous $C(\mathbb{D}: \mathbb{R}^n)$ -semimartingale with two parameters after time s_0 , is unique in the sense that for each $x \in \mathbb{D}$, the processes $\sigma(s, \cdot, x)$, and $b(s, \cdot, x)$ are determined uniquely up to μ_s -measure 0. Moreover, if we define $a = \sigma\sigma' = \{\sigma(s, t, x)\sigma(s, t, y); x, y \in \mathbb{D}, s \leq t \leq T\}$, then φ is a process with local characteristic (a, b) , and a similar remark to the one made for the uniqueness of the volatility and drift processes applies to the uniqueness of the local characteristic.

We say that φ has local characteristic and drift of class $C^{m,\delta}(\mathbb{D}: \mathbb{R}^n)$ if φ_s has local characteristic and drift of class $\tilde{C}^{m,\delta}(\mathbb{D}: \mathbb{R}^n)$ for any time $s \geq s_0$. Similarly we say that φ has volatility and drift processes of class $C^{m,\delta}(\mathbb{D}: \mathbb{R}^n)$. If φ is a continuous semimartingale with volatility and drift of class $C^{m,\delta}$, for $\delta > 0$, then its local characteristic belongs to the class $\tilde{C}^{m,\delta}$, and it follows by a well known result that φ has a modification to a continuous semimartingale of class $C^{m,\epsilon}$ for any $\epsilon < \delta$. (This follows as a consequence of Kunita [4, Theorem 3.1.1]). Reciprocally, if φ is a continuous semimartingale of class $C^{m,\epsilon}(\mathbb{D}: \mathbb{R}^n)$ it follows (as a consequence of Kunita [4, Exercise 3.2.10 (iii)]) that the volatility and drift can be chosen to be progressive measurable processes of class $C^{m,\delta}$ for any $\delta < \epsilon$.

Let $\varphi(s, t, x), x \in \mathbb{D}$ and $\psi(s, t, x), x \in \mathbb{D}$ be measurable processes with two parameters after time s_0 with values in \mathbb{R}^n and \mathbb{D} , respectively; in addition, it is assumed that $\psi(s, s, x) = x$ for all $x \in \mathbb{D}$, and $0 \leq s \leq T$. We say that the process φ is a ψ -consistent process if for each $s_0 \leq s \leq s' \leq T$ there exists a set $N_{s,s'} \in \mathcal{P}_{s'}$ with $\mu_{s'}(N_{s,s'}) = 0$, such that $\varphi(s, t, x) = \varphi(s', t, \psi(s, s', x))$ for all $(t, \omega) \notin N_{s,s'}$ and all $x \in \mathbb{D}$. We say that the process φ is a consistent process if φ is a φ -consistent process.

Let $\tau = \{\tau(s, x), x \in \mathbb{D}, s_0 \leq s \leq T\}$ be a family of stopping times with values in $[s_0, T]$. It is assumed that for each $s_0 \leq s \leq T, x \in \mathbb{D}$, $\tau(s, x)$ is a stopping time relative to the filtration $\{\mathcal{F}_{s,t}; s \leq t \leq T\}$, and that $\tau(s, x)(\omega)$ is a measurable random field that is lower semi-continuous with respect to (s, x) . We say that a family τ as above is a measurable family of stopping times after time s_0 ; we say that the random field $\psi(s, t, x), s_0 \leq s \leq t \leq \tau(s, x), x \in \mathbb{D}$ is a measurable process of two parameters after time s_0 with random time τ , if $\psi_\tau = \{\psi(s, \tau(s, x) \wedge t, x), s_0 \leq t \leq T; x, s_0 \leq s \leq T\}$ where $s \wedge t = \min\{s, t\}$ is a measurable process with two parameters after time s_0 ; we say that the family τ is a ψ -consistent family of stopping times after time s_0 , if for each $s_0 \leq s \leq T$ there exist $N_s \in \mathcal{P}_s, \mu_s(N_s) = 0$ with $\tau_s(x) = \tau_{t \wedge \tau}(\psi(s, t \wedge \tau, x))$ for all $(t, \omega) \notin N_s$ and all x , and in this case we say that (ψ, τ) is a consistent stopping structure. We say that the consistent stopping structure (ψ, τ) is of class $C^{m,\delta}(\mathbb{D}: \mathbb{R}^k)$ if ψ_τ is a process of class $C^{m,\delta}(\mathbb{D}: \mathbb{R}^k)$. Given a consistent stopping structure (ψ, τ) , we say that a family of \mathbb{R}^n -valued processes $\varphi = \{\varphi(s, t, x), s \leq t \leq \tau_s(x); x \in \mathbb{D}, s_0 \leq s \leq T\}$ is a ψ -consistent process with

random time τ , if φ_τ is a ψ_τ -consistent measurable process with two parameters after time s_0 . Similarly, we say that φ is a process of class $C^{m',\delta'}(\mathbb{D}; \mathbb{R}^n)$ if φ_τ is a process of the same class.

Before we start discussing a model for financial markets using the above terminology, we do a digression on how this fits in the framework of the theory of Random Dynamical Systems introduced by L. Arnold and his school (see Arnold [1]). The notation on the following paragraph is local to it. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be the probability space defined above. Let $\theta: \mathbb{R} \times \Omega \rightarrow \Omega$ be the P -preserving flow on Ω defined by $\theta(t, \omega) \equiv W. - W_t$; without loss of generality we assume that θ is defined on $\mathbb{R} \times \Omega$. Assume that $F: \mathbb{R}^n \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}^d$ is a continuous spatial helix forward $(\mathcal{F}_{s,t})$ semimartingale with forward local characteristic of class $\tilde{C}^{m,\delta}$, for $m \geq 1$ and $\delta > 0$. Assume that $\varphi(s, t, \cdot)(\omega)$ is the trajectory random field of the differential equation

$$d\varphi(s, t, x) = F(\varphi(s, t, x), dt), \quad \varphi(s, s, x) = x.$$

It is known that φ has a jointly measurable modification (also denoted by φ), such that (φ, θ) is a perfect cocycle, with the property that

$$\varphi(s, s+t)(\omega) = \varphi(0, t)(\theta_s \omega).$$

It follows that the above structure is both a perfect cocycle and a φ consistent process. More details can be found in Mohammed and Scheutzow [7, Theorem 2.1].

We believe that the proposed class of processes has desirable properties that arise in many areas. We are motivated by the finding of an appropriate framework for the problem of optimal consumption and investment in finance. An elaborated discussion on the heuristics behind our approach can be found in Londoño [6].

3 A mathematical formulation of the model.

Fix $0 \leq s_0 \leq T$; we assume $n+1$ stocks whose *evolution price process* P is a consistent $C(\mathbb{R}_+^{n+1}; \mathbb{R}_+^{n+1})$ -semimartingale with volatility and drift processes of class $C^{0,\epsilon}(\mathbb{R}_+^{n+1}; \mathbb{R}^{n+1})$ for some $\epsilon > 0$, where \mathbb{R}_+ denotes the set of real positive numbers. For $0 \leq i \leq n$ we define *the price per-share process for the i -stock*, P_i , to be the P -consistent $C(\mathbb{R}_+^{n+1}; \mathbb{R}_+)$ -semimartingale process $P_i = \{P_i(s, t, p) = \pi_i \circ P(s, t, p), p \in \mathbb{R}^{n+1}, s_0 \leq s \leq t \leq T\}$ where π_i denotes the projection on the i -component; it follows (see e.g., Karatzas and Shreve [3], and Kunita [4, Exercise 3.2.10]) that the evolution price-per-share process for each stock obeys the differential equations

$$dP_i(s, t, p) = P_i(s, t, p) \left[b_i(s, t, p) dt + \sum_{1 \leq j \leq d} \sigma_{ij}(s, t, p) dW_s^j(t) \right] \quad (1)$$

$$P_i(s, s, p) = p_i, i = 1, \dots, n$$

where $W_s^j(t) = W^j(t) - W^j(s)$, and,

$$dP_0(s, t, p) = P_0(s, t, p) \left[-r(s, t, p)dt - \sum_{1 \leq j \leq d} \theta_j(s, t, p) dW_s^j(t) \right] \quad (2)$$

$$P_0(s, s, p) = p_0$$

for some progressive measurable P -consistent processes with two parameters after time s_0 b_i , $\sigma_{i,j}$, r and θ_i of class $C^{0,\delta}(\mathbb{R}_+^{n+1}; \mathbb{R}^{n+1})$, for any $\epsilon > \delta > 0$. We say that b_i (for $i = 1 \dots, n$) is the *rate of return processes for the i -stock*, and that $\sigma_{i,j}$ is the *(i,j) volatility coefficient processes (for $i = 1, \dots, n, j = 1, \dots, d$)*.

Let us define $F_s(p, t) = (F_s^0(p, t), \dots, F_s^n(p, t))$ where

$$F_s^i(p, t) \equiv p_i \int_s^t b_i(s, u, p) du + p_i \int_s^t \sum_{1 \leq j \leq d} \sigma_{ij}(s, u, p) dW_s^j(u) \quad (3)$$

for $i = 1 \dots, n$, and,

$$F_s^0(p, t) \equiv -p_0 \int_s^t r(s, u, p) du - p_0 \int_s^t \sum_{1 \leq j \leq d} \theta_j(s, u, p) dW_s^j(u). \quad (4)$$

It follows that P is the unique solution of the integral equation

$$P(s, t, p) = p + \int_s^t F_s(P(s, u, p), du).$$

If it is assumed that $b_i(s, t, p) = b_i(p, t)$, $\sigma_{i,j}(s, t, p) = \sigma_{i,j}(p, t)$, $r(s, t, p) = r(p, t)$, and $\theta_i(s, t, p) = \theta_i(p, t)$ where the functions b_i , $\sigma_{i,j}$, r , and θ are deterministic functions for all i, j that are jointly continuous, and Lipschitz continuous in p , then there is a version of P that is a forward stochastic flow of homeomorphisms (Kunita [4, Theorem 4.5.1 and Theorem 4.7.1]). We also observe from Kunita [4, Lemma 4.5.6] that Kolmogorov's criterion for continuous random fields (Kunita [4, Theorem 1.4.1]) implies that if b, σ, θ , and r are of class $C^{0,1}$ (in the price variable) then P is continuous in (p, s, t) . Indeed, if b, σ, θ, r are of class $C^{k,\delta}$ for $k \geq 1$ and $\delta > 0$, then Kunita [4, Theorem 4.6.5] implies that a version of P can be chosen that is a forward stochastic flow of C^k -diffeomorphisms.

The meaning of $r(s, \cdot)$, and $\theta_i(s, \cdot)$, for $i = 1, \dots, d$, $s_0 \leq s \leq T$ is as follows; we assume that there is an imaginary or *shadow stock* whose price evolution is given by the differential equation (2); additional conditions are also imposed on σ , and r , as explained below.

We also assume that there is a two parameter *yield process* for the i -th stock, $1 \leq i \leq n$, $\{Y_i(s, t, p), p \in \mathbb{R}_+^{n+1}, s_0 \leq s \leq t \leq T\}$ that is a continuous P -consistent $C(\mathbb{R}_+^{n+1}; \mathbb{R})$ -semimartingale. We assume that the yield process satisfies the following differential equation

$$dY_i(s, t, p) = dP_i(s, t, p) + P_i(s, t, p)\delta_i(s, t, p) dt, \quad Y_i(s, s, p) = 0$$

for $p \in \mathbb{R}_+^{n+1}$, $s_0 \leq s \leq t \leq T$ where $\delta_i = \{\delta_i(s, t, p), s_0 \leq s \leq t \leq T, p \in \mathbb{R}_+^{n+1}\}$ is a progressive measurable P -consistent process with two parameters after time s_0 of class $C^{0, \epsilon'}$, for some $\epsilon' > 0$. We say that δ_i is a *dividend rate processes for the i -th stock*.

In addition, it is assumed that the random field $\theta = \{\theta(s, t, p), s_0 \leq s \leq t \leq T, p \in \mathbb{R}_+^{n+1}\}$ where $\theta'(s, t, p) = (\theta_1(s, t), \dots, \theta_d(s, t))$ for $0 \leq s \leq t \leq T$, $p \in \mathbb{R}_+^{n+1}$ is the process $\theta(s, \cdot, p) \in \ker^\perp(\sigma(s, \cdot, p))$, of class $C^{0, \delta}$ for any $\delta < \min(\epsilon, \epsilon')$ (where $\ker^\perp(\sigma(s, \cdot, p))$ denotes the orthogonal complement of the kernel of $\sigma(s, \cdot, p)$) such that

$$\begin{aligned} b(s, t, p) + \delta(s, t, p) - r(s, t, p)\mathbf{1}_n \\ - \text{proj}_{\ker(\sigma'(s, t, p))}(b(s, t, p) + \delta(s, t, p) - r(s, t, p)\mathbf{1}_n) \\ = \sigma(s, t, p)\theta(s, t, p) \end{aligned} \quad (5)$$

a.e. μ_s , for all $p \in \mathbb{R}_+^{n+1}$, and $0 \leq s \leq T$, where $\mathbf{1}'_n = (1, \dots, 1) \in \mathbb{R}^n$. Let us observe that although θ is always well defined, in the sense that there is a progressive measurable process that satisfies the above equation, it is usually not a process of class $C^{0, \delta}$ for some δ . Examples of markets where θ is of class $C^{0, \delta}$ for some δ , are those for which $\text{Im}(\sigma(s, t, p))$ is a fixed subspace. We say that the process θ is the *market price of risk*.

The P -consistent $C(\mathbb{R}_+^{n+1}; \mathbb{R}_+)$ -process $B = \{B(s, t, p)\}$ of bounded variation, whose evolution $B(s, \cdot, p)$, $p \in \mathbb{R}_+^{n+1}$, $0 \leq s \leq T$ is given by the stochastic differential equation

$$dB(s, t, p) = B(s, t, p)r(s, t, p)dt, \quad B(s, s, p) = 1, \text{ for } s_0 \leq s \leq t \leq T \quad (6)$$

will be called the *bond price process* and we say that r is the *interest rate process*.

We shall say that $\mathcal{M} = (P, b, \sigma, \delta, r, p^0)$ is a *financial market with terminal time T and initial time s_0* if $b = (b_1, \dots, b_n)$ is a vector of rate of return processes, $\sigma = (\sigma_{i,j})$ is a matrix of volatility coefficient processes, $\delta = (\delta_1, \dots, \delta_n)$ is vector of dividend rate processes, r is an interest rate process as explained above, and $p^0 \in \mathbb{R}_+^{n+1}$ is a vector of initial prices. Let us observe that if \mathcal{M} is a financial market with initial time 0 and terminal time T , then for any $0 \leq T_0 \leq T$ the restrictions (defined in the obvious way) of b , σ , δ , and r to the parameter set $T_0 \leq s \leq t \leq T$ with respect to the (two-parameter) filtration $(\mathcal{F}_{s,t})_{T_0}$, along with any $p \in \mathbb{R}_+^{n+1}$, is a financial market with initial time T_0 and terminal time T .

We define the *state price density process* to be the continuous $C(\mathbb{R}_+^{n+1}; \mathbb{R}_+)$ -semimartingale process defined by

$$H(s, t, p) = B^{-1}(s, t, p)Z(s, t, p) \quad \text{for } p \in \mathbb{R}_+^{n+1}, 0 \leq s \leq t \leq T$$

where

$$Z(s, t, p) = \exp \left\{ - \int_s^t \theta'(s, u, p) dW_s(u) - \frac{1}{2} \int_s^t \|\theta(s, u, p)\|^2 du \right\}$$

for $0 \leq s \leq t \leq T$, and $B^{-1}(s, t, p) = 1/B(s, t, p)$. From the given definitions it follows that the processes H and P_0 are related by the equations

$$P_0(s, t, p) = p_0 H(s, t, p), \quad \text{for } p \in \mathbb{R}_+^{n+1}, 0 \leq s \leq t \leq T.$$

Fix $s_0 \in [0, T]$. Assume that $\tau = \{\tau_s(x, p); s_0 \leq s \leq T, x \in \mathbb{R}, p \in \mathbb{R}_+^{n+1}\}$ is a measurable family of stopping times after time s_0 . A *wealth structure after time s_0* is a triple (X, τ, x_0) , where $x_0 \in \mathbb{R}$, and $X = \{X(s, t, x, p); x \in \mathbb{R}, p \in \mathbb{R}_+^{n+1}, s \leq t \leq \tau_s(x, p)\}$, is a family of continuous semimartingale processes with the property that $((X, P), \tau)$ is a consistent stopping structure of class $C^{0, \epsilon}(\mathbb{R} \times \mathbb{R}_+^{n+1}; \mathbb{R} \times \mathbb{R}_+^{n+1})$ for some $\epsilon = \epsilon^X > 0$ where (X, P) is the continuous process with two parameters after time s_0 , with random time τ defined as

$$(X, P) = \{(X(s, t, x, p), P(s, t, p)), x \in \mathbb{R}, p \in \mathbb{R}_+^{n+1}, s \leq t \leq \tau_s(x, p)\}.$$

We say that x_0 is the *initial value for the wealth process*, and we say that (X, τ) is a *wealth evolution structure*; we shall denote this by writing $(X, \tau) \in \mathcal{X}(\mathcal{M})$. Note that the family of processes (P_i, T) , for $i = 0, \dots, n$ and $(\{xB(s, t, p), s_0 \leq s \leq t \leq T, x \in \mathbb{R}, p \in \mathbb{R}_+^{n+1}\}, T)$ are wealth evolution structures, as any reasonable definition should account for.

Next we define a portfolio-income structure. Assume that (X, τ) is a wealth evolution structure after time s_0 . Let Γ be a continuous semimartingale process with random time τ after time s_0 of class $C^{0, \epsilon}(\mathbb{R} \times \mathbb{R}_+^{n+1}; \mathbb{R})$ (where the positive number ϵ depends on Γ) with the property that $\Gamma(s, s, x, p) = 0$ and

$$\Gamma(s, t', x, p) + \Gamma(t', t, X(s, t', x, p), P(s, t', p)) = \Gamma(s, t, x, p)$$

for all $x \in \mathbb{R}$, $p \in \mathbb{R}_+^{n+1}$, and $s_0 \leq s \leq \tau_s(x, p)$. We say that a process Γ as above is an *income evolution structure for the wealth evolution structure (X, τ)* , and we say that (X, Γ, τ) is a *wealth and income evolution structure*. If $\Gamma(s, t, x, p) \leq 0$ for all x, p , $s_0 \leq s \leq \tau_s(x, p)$ we say that Γ is a *consumption evolution structure for the wealth evolution structure (X, τ)* . Let $(\pi_0, \pi) = \{(\pi_0(s, t, x, p), \pi(s, t, x, p)); x \in \mathbb{R}, p \in \mathbb{R}_+^{n+1}, s_0 \leq s \leq t \leq \tau_s(x, p)\}$ be a (X, P) -consistent progressive measurable process of class $C^{0, \epsilon}$ for some $\epsilon > 0$ with random time τ , and $\pi_0 + \pi' \mathbf{1}_n = X$ satisfying

$$\begin{aligned} B^{-1}(s, t, p)X(s, t, x, p) &= x + \int_s^t B^{-1}(s, u, p) d\Gamma(s, u, x, p) \\ &\quad + \int_s^t B^{-1}(s, u, p) \pi'[x, p](s, u, x, p) \sigma(s, u, p) dW_s(u) \\ &\quad + \int_s^t B^{-1}(s, u, p) \pi'(s, u, x, p) (b(s, u, p) + \delta(s, u, p) - r(s, u, p) \mathbf{1}_n) du \end{aligned} \quad (7)$$

for all $x \in \mathbb{R}$, $s_0 \leq s \leq \tau_s(x, p)$, $p \in \mathbb{R}_+^{n+1}$. We say that $((\pi_0, \pi), \Gamma, \tau)$ as above is a *portfolio evolution structure with random time τ after time s_0 , financed by the income Γ* . If $x_0 \in \mathbb{R}$, we say that $((\pi_0, \pi), \Gamma, \tau, x_0)$ is a *portfolio structure for*

the random time τ after time s_0 , financed by the income Γ with initial wealth x_0 . We say that a wealth evolution structure $(X, \tau) \in \mathcal{X}(\mathcal{M})$ after time s_0 is financed by the income structure Γ , if there exists a portfolio evolution structure $((\pi_0, \pi), \Gamma, \tau)$ with random time τ after time s_0 with $\pi_0 + \pi' \mathbf{1}_n = X$. In this case we say that (X, Γ, τ) is a *hedgeable wealth-income structure after time s_0* . Whenever $s_0 = 0$ in any of the structures above, we omit explicit reference to the initial time. In case (X, Γ, τ) is a hedgeable wealth-income structure after time s_0 and $\Gamma \equiv 0$, we say that the wealth-income structure (X, τ) is a *self-financed wealth evolution structure*.

Example 1. It is possible to construct structures with the above characteristics. For instance, the simplest example is when the “relevant” parameters of the model are deterministic functions. Assume $b(p, t)$, $\sigma(p, t)$, $\delta(p, t)$, $r(p, t)$, and $\theta(p, t)$ are continuous functions in $(p, t) \in \mathbb{R}_+^{n+1} \times [0, T]$, that are locally Lipschitz continuous in p with values in \mathbb{R}^n , $L(\mathbb{R}^d: \mathbb{R}^n)$, \mathbb{R}^n , \mathbb{R} , and \mathbb{R}^d , respectively, with

$$\begin{aligned} b(p, t) + \delta(p, t) - r(p, t) \mathbf{1}_n \\ - \text{proj}_{\ker(\sigma'(p, t))}(b(p, t) + \delta(p, t) - r(p, t) \mathbf{1}_n) \\ = \sigma(p, t) \theta(p, t) \end{aligned}$$

for all p, t , where $\theta(p, \cdot)$ belongs to the orthogonal complement of the kernel of $\sigma(p, \cdot)$. It follows that the above functions define a financial market with terminal time T . Assume that Γ is the continuous $C(\mathbb{R} \times \mathbb{R}_+^{n+1}: \mathbb{R})$ -semimartingale

$$\Gamma(x, p, s, t) = \int_s^t b_\Gamma(x, p, u) du + \int_s^t \sigma_\Gamma(x, p, t) dW_s(u)$$

where b_Γ and σ_Γ are continuous functions in $(x, p, t) \in \mathbb{R} \times \mathbb{R}^{n+1} \times [0, T]$, which are locally Lipschitz continuous in (x, p) . Moreover, assume a \mathbb{R}^n -valued function $\pi(x, p, t)$ that is continuous in (x, p, t) and locally Lipschitz continuous in (x, p) . It follows that the solution of

$$\begin{aligned} X(s, t, x, p) &= x + \int_s^t X(s, t, x, p) r(p, u) du \\ &\quad + \int_s^t [b_\Gamma(x, p, u) du + \sigma_\Gamma(x, p, u) dW_s(u)] \\ &\quad + \int_s^t \pi'(x, p, u) [\sigma(p, u) dW_s(u) + (b(p, u) + \delta(p, u) - r(p, u) \mathbf{1}_n) du] \end{aligned}$$

defines a unique solution process X on any random time interval $[s, \tau']$ before explosion, for each $s \geq s_0$.

More precisely, let $F_s(x, p, t) \equiv (F_s^0(x, p, t), \dots, F_s^{n+1}(x, p, t))$ where

$$\begin{aligned} F_s^{n+1}(x, p, t) &= \int_s^t (x r(p, u) + b_\Gamma(x, p, u) + b(p, u) + \delta(p, u) - r(p, u) \mathbf{1}_n) du \\ &\quad + \int_s^t [\sigma_\Gamma(x, p, u) + \pi'(x, p, u) \sigma(p, u)] dW_s(u) \end{aligned}$$

and the processes F_s^0, \dots, F_s^n are defined by equations (3) and (4), where $b_i(s, t, p) = b_i(t, p)$, $\sigma_{i,j}(s, t, p) = \sigma_{i,j}(t, p)$, $r(s, t, p) = r(t, p)$, and $\theta_j(s, t, p) = \theta_j(t, p)$ for all i, j, s , and t . Let $((X, P), \tau)$ be the consistent stopping structure defined as the local solution to the equation

$$\tilde{X}(s, t, p) = p + \int_s^t F_s(\tilde{X}(s, u, p), du)$$

where $\tilde{X} = (X, P)$, and τ is the explosion time. Let (τ_n) be a sequence of \tilde{X} -consistent families of stopping times increasing to τ with the property that $\tau_n(s, x, p) < \tau(s, x, p)$ and $\tau_n(s, x, p) \uparrow \tau(s, x, p)$ holds a.s.. It follows that (X, Γ, τ') is a wealth-income structure for any stopping time for which there exists n , with $s_0 \leq \tau' \leq \tau_n$. The meaning is natural. Assume that a market is given and that an investor has an income after time $s \geq s_0$, that depends on his current wealth x and the state of the economy (reflected by the value of the stocks $p_{1, \dots, n} \equiv (p_1, \dots, p_n)$ and shadow stock value p_0). Assume also that the increase of the wealth is about

$$d\Gamma(x, p, s, s + \delta) \approx b_\Gamma(x, p, s) d\delta + \sigma_\Gamma(x, p, u) dW_s(s + \delta)$$

at any time s after time s_0 for small δ , where $p = (p_0, \dots, p_n)$. Moreover assume that the investor has a strategy to invest in stocks that consists in holding at time s , $\pi(x, p, s)$ worth of stocks (when he has x dollars of wealth, the vector of prices of stocks is (p_1, \dots, p_n) , and the price of the shadow stock is p_0). Then the previous example says that this completely characterizes the total wealth of the investor at any given time, as long as the initial wealth is given.

A definition of state European and state American contingent claims is given below. It is known that state completeness is equivalent to the maximality of the range of the matrix $\sigma(s_0, t, p^0)$ a.e. with respect to the Lebesgue measure for all $s_0 \leq t \leq T$. See Londoño [5].

4 State tameness and state arbitrage. Characterization

Definition 1. A $((\pi_0, \pi), \Gamma, \tau)$ portfolio evolution structure with random time τ after time s_0 financed by the income Γ is said to be state-tame, if the process $H(s, \cdot, x, p)G(s, \cdot, x, p)$ is uniformly bounded below for all $x \in \mathbb{R}$, $p \in \mathbb{R}_+^{n+1}$, (the bound could depend on x, p and s), where the process G is defined by

$$\begin{aligned} G(s, t, x, p) &= B(s, t, p) \int_s^t B^{-1}(s, u, p) \pi'(s, u, x, p) \sigma(s, u, p) dW_s(u) + \\ & B(s, t, p) \int_s^t B^{-1}(s, u, p) \pi'(s, u, x, p) (b(s, u, p) + \delta(s, u, p) - r(s, u, p) \mathbf{1}_n) du \end{aligned} \tag{8}$$

Remark 1. We point out that equation (7) implies that

$$G(s, t, x, p) = X(s, t, x, p) - xB(s, t, p) - B(s, t, p) \int_s^t B^{-1}(s, u, p) d\Gamma(s, u, x, p)$$

In other words, G is the process that measures at time s the value of the total gain using the portfolio (π_0, π) after paying interest at a rate given by the interest rate process r for the initial capital x , and after discounting the amount of money that a bank account would have paid if the income stream would have been saved in an interest rate account. Therefore it is natural to call the process G the *gain in excess process*. In case $x = 0$ and $\Gamma \equiv 0$, we obtain a gain in excess process for the self-financed portfolio as described by Karatzas and Shreve [3].

Definition 2. A self-financed state-tame portfolio evolution structure $((\pi_0, \pi), \tau)$ with random time τ is said to be a state arbitrage opportunity for the initial wealth x and initial price configuration $p \in \mathbb{R}_+^{n+1}$ if

$$\mathbf{P}[H(s, t, p)G(s, t, x, p) \geq 0] = 1, \quad \text{and} \quad \mathbf{P}[H(s, t, p)G(s, t, x, p) > 0] > 0$$

where G is the gain in excess process that corresponds to $((\pi_0, \pi), \tau)$. We say that a market \mathcal{M} is state-arbitrage-free if there are not portfolio evolution structures with wealth x and price configuration p that are state arbitrage opportunities.

Theorem 1. A market \mathcal{M} is state-arbitrage-free if and only if the P -consistent family of processes θ for all $s_0 \leq s \leq T$, $p \in \mathbb{R}_+^{n+1}$ satisfies

$$b(s, t, p) + \delta(s, t, p) - r(s, t, p)\mathbf{1} = \sigma(s, t, p)\theta(s, t, p) \quad \mu_s \text{ a.s.} \quad (9)$$

Remark 2. We observe that if the family of processes θ satisfies equation (9) then for any initial capital x , initial price p , and wealth income evolution structure (X, Γ, τ) ,

$$\begin{aligned} Y(s, t, x, p) &\equiv H(s, t, p)X(s, t, x, p) - \int_s^t H(s, t, p) d\Gamma(s, u, x, p) \\ &= x + \int_s^t H(s, u, p) [\sigma'(s, u, p)\pi(s, u, x, p) - X(s, u, x, p)\theta(s, u, p)]' dW_s(u). \end{aligned} \quad (10)$$

As a straightforward consequence of Itô's calculus, a portfolio evolution structure $((\pi_0, \pi), \Gamma, \tau)$ is state tame if and only if the process defined by the expression given on the left hand side of the last equation is uniformly bounded below (where the bound might depend on x , p , and s). In case Y is uniformly bounded below, as it happens when (X, Γ, τ) is a hedgeable wealth income structure with a state tame portfolio, $Y(s, \cdot, x, p)$ is a super-martingale, for all x , p , and s . The latter follows as a result of $Y(s, \cdot, x, p)$ being a local-martingale that is uniformly bounded below. Hence under these conditions,

$$x \geq \mathbf{E}[Y(\tau)(x, p)] > -\infty.$$

Proof [Proof of Theorem 1] First, we prove necessity. For $s_0 \leq s \leq t \leq T$, let us define the P -consistent progressive measurable process with two parameters of class $C^{0, \epsilon}(\mathbb{R}_+^{n+1}; \mathbb{R}_+^n)$ (where ϵ is an appropriate positive constant),

$$\kappa(s, t, p) = b(s, t, p) + \delta(s, t, p) - r(s, t, p)\mathbf{1}_n - \sigma(s, t, p)\theta(s, t, p),$$

Define π_0 and π to be the P -consistent processes of class $C^{0,\epsilon}$

$$\begin{aligned}\pi(s, t, p) &= \kappa(s, t, p) \\ \pi_0(s, t, p) &= B(s, t, p) \int_s^t B^{-1}(s, u, p) \kappa'(s, u, p) \kappa(s, u, p) du - \kappa'(s, t, p) \mathbf{1}_n\end{aligned}$$

It follows that (π_0, π) is a self-financed portfolio evolution structure with gain in excess process given by

$$G(s, t, p) = \int_s^t B^{-1}(s, u, p) \kappa'(s, u, p) \kappa(s, u, p) du.$$

Since $H(s, t, p)G(s, t, p) \geq 0$, the non-state-arbitrage hypothesis implies the desired results. To prove sufficiency, assume that the family of processes θ satisfies equation (9) for all s, p , and let (π_0, π) be a self-financed portfolio structure with gain in excess process G . Remark 2 implies that $H[p](s, \cdot)G[x, p](s, \cdot)$ is a local-martingale for all x, p , and s . State-tameness implies that it is also bounded below. Fatou's lemma implies that $H(s, \cdot, p)G(s, \cdot, x, p)$ is a super-martingale. The result follows. \square

5 A view of state European contingent claims.

Throughout the rest of the paper we assume that equation (9) is satisfied for all s, p , μ_s -almost surely.

We propose to extend the concepts of European contingent claim, hedgeability and completeness within the framework proposed.

Definition 3. A state European contingent claim (SECC) with expiration date τ is a wealth-income evolution structure $(X, \Gamma, \tau) \in \mathcal{X}(\mathcal{M})$ where the family of processes defined by

$$Y(s, t, x, p) \equiv H(s, t, p)X(s, t, x, p) - \int_s^t H(s, t, p) d\Gamma(s, u, x, p) \quad (11)$$

are uniformly bounded from below continuous semimartingales for all x, p and s (where the bound might depend on x, p , and s). Moreover it is assumed that

$$x = \mathbf{E}[Y_s(\tau)(x, p)]. \quad (12)$$

We shall say that Y is the discounted payoff process after time s for the SECC.

Remark 3. Under the conditions of Definition 3, equation (12) is equivalent to require that the process defined by equation (11) is a martingale for each x, p , and s .

Definition 4. A state European contingent claim (X, Γ, τ) is called hedgeable if (X, Γ, τ) is a hedgeable wealth-income evolution structure (by a state-tame portfolio evolution structure). The market model \mathcal{M} is called state complete if every state European contingent claim is hedgeable. Otherwise it is said to be state incomplete.

For the following theorem we assume $\{i_1 < \dots < i_k\} \subseteq \{1, \dots, d\}$ is a set of indexes and let $\{i_{k+1} < \dots < i_d\} \subseteq \{1, \dots, d\}$ be its complement. Let $\sigma_i(s, t, p)$, $1 \leq i \leq k$, be the i^{th} column process for the matrix valued process $(\sigma_{i,j}(s, t, p))$, $s \leq t \leq T$. Namely, $\sigma_i(s, t, p)$, for $1 \leq i \leq k$, is the \mathbb{R}^n -valued progressively measurable process whose j^{th} entry, for $1 \leq j \leq d$ agrees with $\sigma_{i,j}(s, t, p)$, for $0 \leq s \leq t \leq T$. We denote by $\sigma_{i_1, \dots, i_k}(s, t, p)$, $0 \leq s \leq t \leq T$ the $n \times k$ matrix valued process whose j^{th} -column process agrees with $\sigma_{i_j}(s, t, p)$, $0 \leq s \leq t \leq T$, for $1 \leq j \leq k$. We adopt the same notation for other processes: if $i_1 < \dots < i_{k'}$ is a set of indexes, and ψ_{i_j} for $1 \leq j \leq k'$ is a process taking values in some euclidean space $\mathbb{R}^{n'}$, then we denote by $\psi_{i_1, \dots, i_{k'}}$ the $k' \times n'$ matrix valued process, where the j^{th} -column process agrees with ψ_{i_j} . We denote by $(\mathcal{F}_{s,t}^{i_1, \dots, i_k})$ the filtration with two parameters that is the \mathbf{P} augmentation by the null sets of the natural filtration $\{\sigma(W_s^{i_1}(t), \dots, W_s^{i_k}(t)), 0 \leq s \leq t \leq T\}$. We say that a wealth and income evolution structure (X, Γ, τ) is a $(\mathcal{F}_{s,t}^{i_1, \dots, i_k})$ wealth-income evolution structure if $X(s, \cdot, x, p)$, $\Gamma(s, \cdot, x, p)$ are $\mathcal{F}_{s,t}^{i_1, \dots, i_k}$ progressive measurable processes and $\tau_s(x, p)$ is a stopping time relative to the filtration $\mathcal{F}_{s,t}^{i_1, \dots, i_k}$ for all x, p , and s . In addition if (X, Γ, τ) is a SECC then we say that it is a $\mathcal{F}_{s,t}^{i_1, \dots, i_k}$ SECC. First we prove a lemma that would be needed for the proof of necessity in Theorem 2. Compare with Karatzas and Shreve [3, Lemma 1.6.9]

Lemma 1. *There exists a function $\varphi: L(\mathbb{R}^n: \mathbb{R}^k) \rightarrow \mathbb{R}^k$ of class C^∞ when $L(\mathbb{R}^n: \mathbb{R}^k)$ is identified with $\mathbb{R}^{n \times k}$ in the natural way, with the property that $\varphi(\sigma) \in \text{Im}(\sigma')$, $\varphi(\sigma) \neq \mathbf{0}$ if $\text{Im}(\sigma') \neq \{\mathbf{0}\}$.*

Proof Let $j_1 < \dots < j_r$ be a set of indexes in $\{1, \dots, k\}$. Define $D_{j_1, \dots, j_r} \subset L(\mathbb{R}^n: \mathbb{R}^k)$ to be a set of matrices σ such that $\{\sigma'_{j_1} \dots, \sigma'_{j_r}\}$ is linearly independent, $\text{Im}(\sigma'_{j_1, \dots, j_r}) = \text{Im}(\sigma')$, and there is not a set $j'_1 < \dots < j'_r$ with $j'_r = j_r$ of indexes with $\text{Im}(\sigma'_{j'_1, \dots, j'_r}) = \text{Im}(\sigma'_{j_1, \dots, j_r})$. Define the function $\varphi^{j_1, \dots, j_r}(\sigma) = \sum_{j_i} \omega_i(\sigma'_{j_1} \dots, \sigma'_{j_i}) \sigma'_{j_i}$ on D_{j_1, \dots, j_r} , where ω_i is an i -form that computes a signed volume of the simplex formed by the the convex combination of the vectors in its arguments. Define $\varphi \mid D_{j_1, \dots, j_r} = \varphi^{j_1, \dots, j_r}$, for each sequence $j_1 < \dots < j_r$ of indexes in $\{1, \dots, k\}$. It follows that φ is a C^∞ function, that satisfies the required conditions. \square

In the following we assume that $X_1(t), \dots, X_k(t)$ is a sequence of continuous $\mathcal{F}_{0,t}^{i_1, \dots, i_k}$ semimartingales with values in $\mathfrak{X}^{0, \delta}(\mathbb{R}_+^n)$, the linear topological space of $C^{0, \delta}$ vector fields for some $\delta > 0$; see Kunita [4, Sec. 4.2] for a description of semimartingales with values in $\mathfrak{X}^k(M)$ for any non-negative integer k , and a manifold M . In addition, we also assume that $X_1(t) - X_1(s), \dots, X_k(t) - X_k(s)$ are $\mathcal{F}_{s,t}^{i_1, \dots, i_k}$ semimartingales.

Theorem 2. *Assume that for each s and p , $\theta_i(s, t, p) = 0$ μ_s a.s. for $i \notin \{i_1, \dots, i_k\}$ where $\theta(s, t, p) = (\theta_1(s, t, p), \dots, \theta_d(s, t, p))'$ is the market price of risk. Assume that $\sigma_{i_1, \dots, i_k}(s, t, p)$ is a matrix valued process of two parameters*

such that

$$\begin{aligned} \text{Range}^\perp(\sigma_{i_{k+1}, \dots, i_d}(s, t, p)) &= \text{Range}(\sigma_{i_1, \dots, i_k}(s, t, p)) \\ &= \text{gen}\{X_1(t) \cdots, X_k(t)\} \end{aligned}$$

a.s. μ_s for all p, s . In addition assume that the interest rate process B is a $(\mathcal{F}_{s,t}^{i_1, \dots, i_k})$ -process with two parameters. Then, any $\mathcal{F}_{s,t}^{i_1, \dots, i_k}$ state European contingent claim is hedgeable if and only if $\text{Rank}(\sigma_{i_1, \dots, i_k}(s, t, p)) = k$ a.s. μ_s for all p, s . In particular, a financial market \mathcal{M} is state complete if and only if for all s and p , $\sigma(s, t, p)$ has maximal range a.s. μ_s .

Proof [Proof of sufficiency] Let (X, Γ, τ) be a $\mathcal{F}_{s,t}^{i_1, \dots, i_k}$ SECC. As a consequence of equation (12), the process defined by equation (11) is a martingale for all x, p , and s . By Kunita [4, Exercise 3.2.10] there exists $\epsilon > 0$, and a $(\mathcal{F}_{s,t}^{i_1, \dots, i_k})$ -process of class $C^{0,\epsilon}(\mathbb{R} \times \mathbb{R}_+^n : \mathbb{R}^d)$, $\varphi(s, t, x, p) = (\varphi_1(s, t, x, p), \dots, \varphi_d(s, t, x, p))'$, $t \in [s, \tau_s(x, p)]$, such that

$$\begin{aligned} H(s, t, p)X(s, t, x, p) - \int_s^t H(s, u, p) d\Gamma(s, u, x, p) \\ = x + \int_s^t \varphi'(s, u, x, p) dW_s(u) \end{aligned}$$

for all x, p, s , and t , where $\varphi_i(s, t, x, p) = 0$ for $i \notin \{i_1, \dots, i_k\}$. Define $\pi(s, t, x, p)$, $t \in [s, \tau_s(x, p)]$, as the unique $C^{0,\epsilon}(\mathbb{R} \times \mathbb{R}_+^n : \mathbb{R}^n)$ process with values in $\ker^\perp(\sigma'(s, t, p))$ such that

$$\sigma'(s, t, p)\pi(s, t, x, p) = H^{-1}(s, t, p)\varphi(s, t, x, p) + X(s, t, x, p)\theta(s, t, p).$$

The existence and uniqueness of such a portfolio follows from the hypotheses. (For instance, use the Gauss-Jordan algorithm to obtain a solution of the system $\sigma'_{i_1, \dots, i_k} \pi_{i_1, \dots, i_k} = H\varphi_{i_1, \dots, i_k} + X\theta_{i_1, \dots, i_k}$ and next take the projection onto $\ker^\perp(\sigma'_{i_1, \dots, i_k}) = \text{Range}(\sigma_{i_1, \dots, i_k})$.) Define $\pi_0(s, t, x, p) = X(s, t, x, p) - \pi'(s, t, x, p)\mathbf{1}_n$. It follows using Itô's formula that (π_0, π) is a portfolio process that finances the wealth X with income Γ . \square

Proof [Proof of necessity] Let $\varphi : L(\mathbb{R}^k; \mathbb{R}^n) \mapsto \mathbb{R}^k$ be a bounded C^∞ function defined by the Lemma 1. Let us define $\psi(s, t, p)$ to be the bounded, $\mathcal{F}_{s,t}^{i_1, \dots, i_k}$ -progressively measurable process $\psi_{i_1, \dots, i_k}(s, t, p) = \varphi(\sigma_{i_{k+1}, \dots, i_d}(s, t, p))$ and $\psi_j(s, t, p) = 0$ for $j \notin \{i_1, \dots, i_k\}$. We consider the $\mathcal{F}_{s,t}^{i_1, \dots, i_k}$ -progressively measurable SECC with no income, expiration date T , and whose wealth process is defined by

$$X(s, t, x, p) = x + \int_s^t \frac{1}{H(s, u, p)} \psi'(s, u, p) dW_s(u), \quad \text{for } s \leq t \leq T.$$

Let (π_0, π) be the state tame portfolio that finances the given wealth. It follows that

$$H(s, t, p)X(s, t, x, p) = x + \int_s^t \psi'(s, u, p) dW_s(u)$$

is a martingale for all x, p and s . Using equation (10), and Kunita [4, Exercise 3.2.10] we obtain

$$\begin{aligned}\psi_{i_1, \dots, i_k}(s, t, p) &= \sigma'_{i_1, \dots, i_k}(s, t, p) \pi_{i_1, \dots, i_k}(s, t, x, p) - X(s, t, x, p) \theta_{i_1, \dots, i_k}(s, t, p) \\ &\in \text{Ker}^\perp(\sigma_{i_1, \dots, i_k}(s, t, p)) \cap \text{Ker}(\sigma_{i_1, \dots, i_k}(s, t, p)) = \{\mathbf{0}\}\end{aligned}$$

a.s μ_s for all s . The result follows. \square

6 A view of state American contingent claims.

Definition 5. Let $s_0 \in [0, T]$. Assume a wealth-income evolution structure after time s_0 , $(X, \Gamma, \tau) \in \mathcal{X}(\mathcal{M})$, such that the family of processes defined by equation (11) are uniformly bounded from below continuous semimartingales for all x, p and s (where the bound might depend on x, p , and s). A State American Contingent Claim is a wealth income structure as above with the property that for all x, p and s ,

$$x = \sup_{\tau' \in \mathcal{S}_s(X, \tau)} \mathbf{E}[Y_s(\tau')(x, p)] \quad (13)$$

where $\mathcal{S}_s(X, \tau)$ denotes, for a given wealth evolution structure (X, τ) and a given s , the family of stopping times that takes values in $[s, \tau \vee s]$ that are (X, P) -consistent after time s (here $s \vee t = \max(s, t)$ and it is assumed that $\sup \emptyset = \infty$). We call the process $Y(s, \cdot, x, p)$ the discounted payoff process after time s . We denote the quantity on the right hand side of the equation (13) as $u_s(x, p)$ and we say that $u_s(x, p)$ is the value of the consistent state American contingent claim at time s (given that the price of the stock process at time s is p and the wealth process is x); in case $s = 0$ the process $Y(x, p)$, and $u_0(x, p)$ are simply called the discounted payoff process and the value of the consistent state American contingent claim, respectively.

Remark 4. Under the conditions of Definition 5, $(X, \Gamma, \tau) \in \mathcal{X}(\mathcal{M})$ satisfies equation (13) iff the family of processes $Y(s, \cdot \wedge \tau, x, p)$ are super-martingales for all x, p and s .

In addition to properties mentioned above we assume for the remainder of this section the following condition.

Condition 1. For any $s_0 \leq s \leq T$ and all stopping times $\tau' \in \mathcal{S}_s(X, \tau)$, the function

$$\varphi_{s, \tau'}(x, p) = \mathbf{E}[Y_s(\tau' \wedge \tau)(x, p)] \quad (14)$$

is a continuous function in (x, p) , and the given family of functions is a equicontinuous set of functions on compact sets (in (x, p)). Moreover assume that there exist positive constants $\gamma \geq 1$, $\alpha_1, \alpha_2, \alpha_3, \beta_0, \dots, \beta_n$, with $\alpha_1^{-1} + \alpha_2^{-1} + \alpha_3^{-1} + \sum_{i=0}^n \beta_i^{-1} < 1$ such that the random field $Y(s, t, x, p) \triangleq Y_s(\tau)(s, t, x, p)$ satisfies

$$\begin{aligned}\mathbf{E}[|Y(x, p, s, t) - Y(x', p', s', t')|^\gamma] &\leq \\ C \left(|s - s'|^{\alpha_1} + |t - t'|^{\alpha_2} + |x - x'|^{\alpha_3} + \sum_{i=0}^n |p_i - p'_i|^{\beta_i} \right). &\quad (15)\end{aligned}$$

The previous condition is usually satisfied when X is a process that solves a stochastic differential equation. For instance, see Example 1, and Kunita [4, Lemma 4.5.6]. Equation (15) above is needed in order to obtain a continuous modification of the random field and its conditional expectation. See Kolmogorov's continuity criterion of random fields (Kunita [4, Theorem 1.4.1 and Exercise 1.4.12]). In the following, conditional expectations of stochastic processes are the continuous modifications of the given stochastic processes.

In order to state and prove a theorem for valuation of State American contingent claims is needed to define when a wealth income evolution structure outperforms other.

Definition 6. *We say that a wealth and income evolution structure (X', Γ', τ') dominates a wealth income structure (X, Γ, τ) if $\tau'_s(x, p) \geq \tau_s(x, p)$ for all s, x and p ,*

$$X'(s, t, x, p) \geq X(s, t, x, p) \quad \text{and} \quad \Gamma'(s, t, x, p) \leq \Gamma(s, t, x, p)$$

for all $s \leq t \leq \tau_s(x, p)$.

Theorem 3 below provides conditions under which every state American contingent claim is dominated by a hedgeable state American contingent claim; compare with Theorem 5.1 Londoño [5].

Theorem 3. *Assume that the hypotheses of Theorem 2 and Condition 1 hold. Then, any $\mathcal{F}_{s,t}^{i_1, \dots, i_k}$ -progressively measurable state American contingent claim (X, Γ, τ) is dominated by a hedgeable $\mathcal{F}_{s,t}^{i_1, \dots, i_k}$ -progressively measurable state American contingent claim if and only if for each s and p , $\text{Rank}(\sigma_{i_1, \dots, i_k}(s, t, p)) = k$ a.s. μ_s . In particular, a financial market \mathcal{M} is American state complete if and only if $\sigma(s, t, p)$ has maximal range a.s. μ_s for all s .*

The sufficiency is a consequence of Remark 4, and Theorem 2; the proof of necessity is given below.

First we point out an elementary fact, needed for the proof. If $\sigma \in \mathcal{S}_t(X, \tau)$ for $t \geq s$, then σ can be seen as an element of $\mathcal{S}_s(X, \tau)$ using the natural identification, namely σ is identified with the stopping structure $\sigma^s \in \mathcal{S}_s(X, \tau)$ defined by $\sigma_{t'}^s(x, p) = \sigma_{t'}(x, p)$ for $t' \geq t$, and $\sigma_{t'}(x, p) = \sigma_t(X(t', t, x, p), P(t', t, x, p))$ otherwise. We denote both elements in the same way, hoping that the meaning will be clear from context.

Lemma 2. *Assume Condition 1 and let (X, Γ, τ) be a state American contingent claim and let $s_0 \leq s \leq T$. Assume $\tau_1, \tau_2 \in \mathcal{S}_s(X, \tau)$. Then, there exists $\tau' \in \mathcal{S}_s(X, \tau)$ with the property that for any $s \leq s' \leq T$,*

$$u_{s'}(x, p) \geq \mathbf{E}[Y_{s'}(\tau')(x, p)] \geq \max\{\mathbf{E}[Y_{s'}(\tau_1)(x, p)], \mathbf{E}[Y_{s'}(\tau_2)(x, p)]\}$$

and

$$\mathbf{E}[Y_{s'}(\tau')(x, p) \mid \mathcal{F}_{s',t}] \geq \max\{\mathbf{E}[Y_{s'}(\tau_1)(x, p) \mid \mathcal{F}_{s',t}], \mathbf{E}[Y_{s'}(\tau_2)(x, p) \mid \mathcal{F}_{s',t}]\}.$$

a.s. $\mu_{s'}$.

Proof Define

$$\begin{aligned} \tau_{s'}(x, p) &= (\tau_1 \wedge \tau_2)_{s'}(x, p) \mathbf{1}_{\mathbf{E}[Y_{s'}(\tau_1 \vee \tau_2)(x, p) | \mathcal{F}_{s', t}](\tau_1 \wedge \tau_2)_{s'}(x, p) < Y_{s'}(\tau_1 \wedge \tau_2)(x, p)} \\ &\quad + (\tau_1 \vee \tau_2)_{s'}(x, p) \mathbf{1}_{\mathbf{E}[Y_{s'}(\tau_1 \vee \tau_2)(x, p) | \mathcal{F}_{s', t}](\tau_1 \wedge \tau_2)_{s'}(x, p) \geq Y_{s'}(\tau_1 \wedge \tau_2)(x, p)} \end{aligned}$$

for all $s' \geq s$ where $t \vee t' = \max(t, t')$, and $t \wedge t' = \min(t, t')$. Then τ has the required properties. \square

Proof [Proof of necessity.] Let Y be the discounted payoff structure of a state American contingent claim (X, Γ, τ) . For each x, p and s there exists a sequence of families of consistent stopping times $\sigma_n^{x, p} \in \mathcal{S}_s(X, P)$ along sets $N_s \in \mathcal{P}_s$ of zero μ_s measure such that $\mathbf{E}[Y_s(\sigma_n^{x, p})(x, p)] \uparrow x = u_s(x, p)$, with

$$\mathbf{E}[Y(\sigma_{n+1}^{x, p})(x, p) | \mathcal{F}_{s, t}] \geq \mathbf{E}[Y(\sigma_n^{x, p})(x, p) | \mathcal{F}_{s, t}] \quad \text{for } (t, \omega) \notin N_s.$$

The latter follows by Lemma 2. Without loss of generality the version chosen for the given random fields are continuous (see Kunita [4, Exercise 1.4.12]). Using a diagonalizing process and Lemma 2 again, it is possible to prove that there exists a sequence of families of consistent stopping times $\sigma_n \in \mathcal{S}_{s_0}(X, P)$ and a set with the property that for any triple of points with rational coordinates $x \in \mathbb{Q}$, $p \in \mathbb{R}_+^n \cap \mathbb{Q}^n$, and $s \in [s_0, T] \cap \mathbb{Q}$ there exists M (depending on x, p , and s) large enough with

$$\mathbf{E}[Y_s(\sigma_{n+1})(x, p) | \mathcal{F}_{s, t}] \geq \mathbf{E}[Y_s(\sigma_n)(x, p) | \mathcal{F}_{s, t}], \quad \text{for } (t, \omega) \notin N_s$$

where $\mu_s(N_s) = 0$, and

$$\mathbf{E}[Y_s(\sigma_n)(x, p)] \uparrow x = u_s(x, p).$$

Using Condition 1, and the Ascoli-Arzelà Theorem, there exist a subsequence of stopping times (that we also denote as σ_n) with $\sigma_{\tau_n} \rightarrow u$ uniformly on compact sets, where $u(s, x, p) \equiv u_s(x, p)$ is the value of the SACC, and $\varphi_{\sigma_n}(s, x, p) \equiv \varphi_{s, \sigma_n}(x, p)$. Doob's inequality shows that the stochastic process $(\mathbf{E}[Y(\sigma_n)(x, p) | \mathcal{F}_{s, t}])_{s \leq t \leq T}$ is a Cauchy sequence in the sense of uniform convergence in probability uniformly on compact sets. By completeness of the space of local-martingales, for each s there exists a local-martingale $\bar{Y}(s, t, x, p)$, $t \in [s, T]$, such that $\mathbf{E}[Y(\sigma_n)(x, p) | \mathcal{F}_{s, t}] \rightarrow \bar{Y}(s, t, x, p)$, $t \in [s, T]$, uniformly in probability, (and the convergence is uniform in compact sets). It follows by continuity that $\bar{Y}(s, t, x, p) \geq Y(s, t, x, p)$ for all s, x, p , a.s. μ_s , and clearly $\bar{Y}(x, p)(s, s) = u_s(x, p)$. Define τ_n to be the first hitting time of $\bar{Y}(s, t, x, p)$, $t \in [s, \tau]$, to the set $[-n, n]^c$. Using Kunita [4, Exercise 3.2.10] it follows that there exists a \mathbb{R}^d -valued process of class $C^{0, \varepsilon}$ for some $\varepsilon > 0$, $\varphi(s, t, x, p) = (\varphi_1(s, t, x, p), \dots, \varphi_d(s, t, x, p))'$, $t \in [s, \tau_n]$, such that

$$\bar{Y}(s, t, x, p) = x + \int_s^t \varphi'(s, t, x, p) dW_s(u)$$

where $\varphi_i(s, t, x, p) = 0$ for $i \notin \{i_1, \dots, i_k\}$. Define $\bar{X}(s, t, x, p)$ by

$$H(s, t, p) \bar{X}(s, t, x, p) - \int_s^t H(s, u, p) d\Gamma(s, u, x, p) = \bar{Y}(s, t, x, p)$$

and $\pi(s, t, x, p)$, $s \leq t \leq \tau$, as the unique \mathbb{R}^n -continuous process such that

$$\sigma'(s, t, p)\pi(s, t, x, p) = H^{-1}(s, t, p)\varphi(s, t, x, p) + \bar{X}(s, t, x, p)\theta(s, t, p).$$

The existence and uniqueness of such a portfolio follows from the hypotheses (see Lemma 1.4.7 in Karatzas and Shreve [3]). Define $\pi_0(s, t, x, p) = \bar{X}(s, t, x, p) - \pi'(s, t, x, p)\mathbf{1}_n$. Using Itô's formula it follows that (\bar{X}, Γ, τ) is a wealth income process with the desired characteristics. \square

Remark 5. We point out that the definition of consistent state American contingent claim and state American contingent claim are not equivalent. In fact the supremum that defines the value of state American contingent claim is greater or equal than the value of the consistent American contingent claim defined by (13), and *a priori* could be strictly greater. The latter follows because the class of stopping times contains the class of (X, Γ, τ) -consistent families of stopping times.

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