Risk, Uncertainty, and Option Exercise*

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Abstract

Many economic decisions can be described as an option exercise or optimal stopping problem under uncertainty. Motivated by experimental evidence such as the Ellsberg Paradox, we follow Knight (1921) and distinguish risk from uncertainty. To afford this distinction, we adopt the multiple-priors utility model. We show that the impact of ambiguity on the option exercise decision depends on the relative degrees of ambiguity about continuation payoffs and termination payoffs. Consequently, ambiguity may accelerate or delay option exercise. We apply our results to firm investment and exit problems, and show that the myopic NPV rule can be optimal for an agent having an extremely high degree of ambiguity aversion.

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1 Introduction

Many economic decisions can be described as binary choices. Examples are abundant. An investor may decide whether and when to invest in a project. A firm may decide whether and when to enter or exit an industry. It may also decide whether and when to default on debt. A worker may decide whether and when to accept a job offer. All these decisions share three characteristics. First, the decision is irreversible to some extent. Second, there is uncertainty about future rewards. Third, agents have some flexibility in choosing the timing of decisions. These three characteristics imply that waiting has positive value. Importantly, all the preceding problems can be viewed as a problem where agents decide when to exercise an "option" analogous to a financial call option – it has the right but not the obligation to buy an asset at some future time of its choosing. This real-options approach has been widely applied in economics and finance (see Dixit and Pindyck (1994)). The aim of the present paper is to analyze an option exercise problem where there is a distinction between risk and uncertainty in the sense attributed to Knight (1921), and where agents' attitudes toward uncertainty play a nontrivial role.

As argued by Dixit (1992), the standard real-options approach to investment under uncertainty can be summarized as "a theory of optimal inertia". "It says that firms that refuse to invest even when the currently available rates of return are far in excess of the cost of capital may be optimally waiting to be surer that this state of affairs is not transitory. Likewise, farmers who carry large losses may be rationally keeping their operation alive on the chance that the future may be brighter" (Dixit (1992, p.109)).

However, the standard real-options approach rules out the situation where agents are unsure about the likelihoods of states of the world. It typically adopts strong assumptions about agents' beliefs. For example, according to the rational expectations hypothesis, agents know the objective probability law of the state process and their beliefs are identical to this probability law. Alternatively, according to the Bayesian approach, an agent's beliefs are represented by a subjective probability measure or Bayesian prior. There is no meaningful distinction between *risk*, where probabilities are available to guide choice, and *uncertainty*, where information is too imprecise to be summarized adequately by probabilities. By contrast, Knight (1921) emphasizes this distinction and argues that uncertainty is more common in decision-making. Henceforth, we refer to such uncertainty as *Knightian uncertainty* or *ambiguity*. For experimental evidence, the Ellsberg Paradox suggests that people prefer to act on known rather than unknown or ambiguous probabilities.[1] Ellsberg-type behavior contradicts the Bayesian paradigm, i.e., the existence of any prior underlying choices.

To incorporate Knightian uncertainty or ambiguity, we adopt the recursive multiple-priors utility model developed by Epstein and Wang (1994). In this model, the agent's beliefs are represented by a collection of sets of one-step-ahead conditional probabilities. These sets of one-step-ahead conditionals capture both the degree of ambiguity and ambiguity aversion.[2] The axiomatic foundation for the recursive multiple-priors utility model is laid out by Epstein and Schneider (2003). The static multiple-priors utility model is first proposed and axiomatized by Gilboa and Schmeidler (1989).

We describe an ambiguity averse agent's option exercise decision as an optimal stopping problem. We then characterize the optimal stopping rules. The standard real options approach emphasizes the importance of risk in determining option value and timing of option exercise. An important implication is that an increase in risk in the sense of mean preserving spread raises option value and delays option exercise. Recognizing the difference between risk and ambiguity, we conduct comparative statics analysis with respect to the set of one-step-ahead conditionals.

In our model, the agent is ambiguous about a state process which influences the continuation and termination payoffs. Importantly, we distinguish between two cases according to whether or not the agent is still ambiguous about the termination payoff after he exercises the option. This distinction is critical since it may generate opposite comparative statics results. We show that for both cases, ambiguity lowers the option value. Moreover, if there is no uncertainty after option exercise, a more ambiguity averse agent will exercise the option earlier. However, if he is also ambiguous about termination payoffs after option exercise, he may exercise the option later. This is because ambiguity

[1]See Ellsberg (1961). One version of the story is as follows. A decision maker is a offerred a bet on drawing a red ball from two urns. The first urn contains exactly 50 red and 50 black balls. The second urn has 100 balls, either red or black, however the exact number of red or black balls is unknown. Vast majority agents choose from the first urn rather than the second.

[2]For a formal definition of ambiguity aversion, see Epstein (1999), Epstein and Zhang (2001), and Ghirardato and Marinacci (2002).

lowers the termination payoff and this effect may dominate the decrease in the option value.

We provide two applications – real investment and firm exit – to illustrate our results. The real investment decision is an example where an agent decides whether or not to exercise an option to pursue upside potential gains. Entry and job search are similar problems. Under a specification of the set of priors, we show explicitly that if the investment project generates a stream of future uncertain profits and if the agent is ambiguous about these profits, then a more ambiguity averse agent invests relatively later. The exit problem represents an example where an agent decides whether or not to exercise an option to avoid downside potential losses. Other examples include default and liquidation decisions. We show that the exit timing depends crucially on whether the firm is ambiguous about the outside value. This ambiguity may dominate the effect of ambiguity about the profit opportunities if stay in business. Consequently, an ambiguity averse firm may be hesitant to exit, even though it has lower option value. For both investment and exit problems, we solve some examples explicitly under some specification of the set of priors. We show that the myopic net present value (NPV) rule can be optimal for an agent having an extremely high degree of ambiguity aversion.

This paper adds to the literature on applications of decision theory to macroeconomics and finance surveyed recently by Backus, Routledge and Zin (2004). The idea of ambiguity aversion and the multiple-priors utility model have been applied to asset pricing and portfolio choice problems in a number of papers.[3] A different approach based on robust control theory is proposed by Hansen and Sargent and their coauthors.[4] They emphasize "model uncertainty" which is also motivated in part by the Ellsberg Paradox. We refer readers to Epstein and Schneider (2003) for further discussion on these two approaches.

Our paper is related to Nishimura and Ozaki (2003, 2004). Nishimura and Ozaki (2004) apply the Choquet expected utility model proposed by Schmeidler (1989) to study a job search problem.[5] They show that ambiguity reduces the reservation wage and speeds

[3]See Epstein and Wang (1994, 1995), Chen and Epstein (2002), Epstein and Miao (2003), Kogan and Wang (2002), Epstein and Schneider (2004a,b), and Routledge and Zin (2003).

[4]See, for example, Anderson, Hansen and Sargent (2003) and Hansen and Sargent (2000).

[5]The Choquet expected utility model is another well known model that also addresses ambiguity. It

up job acceptance. Nishimura and Ozaki (2003) apply the continuous time multiple-priors utility model developed by Chen and Epstein (2002) to study an irreversible investment problem. They show that ambiguity delays investment. Our paper reconciles these conflicting results in a general unified framework of the optimal stopping problem. An added payoff is that our framework can address a variety of option exercise problems such as exit.

The remainder of this paper proceeds as follows. Section 2 presents the model and results. Section 3 presents applications to firm investment and exit problems. Section 4 concludes. Proofs are relegated to the appendix.

2 The Model

In this section, we first introduce the multiple-priors utility model in Section 2.1. We then provide a three-period investment example in Section 2.2 to illustrate the impact of ambiguity on the option exercise decision. Next, we present a baseline setup of the optimal stopping problem in Section 2.3. After that, we present characterization and comparative statics results in Section 2.4. We finally consider extensions in Section 2.5.

2.1 Multiple-Priors Utility

Before presenting the model, we first provide some background about multiple-priors utility. The static multiple-priors utility model of Gilboa and Schmeidler (1989) can be described informally as follows. Suppose uncertainty is represented by a measurable space $(S, \mathcal{F})$. The decision-maker ranks uncertain prospects or acts, maps from S into an outcome set $\mathcal{X}$. Then the multiple-priors utility $U(f)$ of any act f has the functional form:

$$U(f) = \min_{q \in \Delta} \int u(f)\, dq,$$

where $u : \mathcal{X} \to \mathbb{R}$ is a von Neumann-Morgenstern utility index and Δ is a subjective set of probability measures on $(S, \mathcal{F})$. Intuitively, the multiplicity of priors models ambiguity

has been applied to study wage contracts by Mukerji and Tallon (2004). Also see the references therein for other applications.

about likelihoods of events and the minimum delivers aversion to such ambiguity. The standard expected utility model is obtained when the set of priors Δ is a singleton.

The Gilboa and Schmeidler model is generalized to a dynamic setting in discrete time by Epstein and Wang (1994). Their model can be described briefly as follows. The time t conditional utility from a consumption process $c = (c_t)_{t\geq 0}$ is defined by the Bellman equation

$$V_t(c) = u(c_t) + \beta \min_{q\in\mathcal{P}_t} E_t^q \left[V_{t+1}(c)\right], \tag{1}$$

where $\beta \in (0,1)$ is the discount factor, E_t^q is the conditional expectation operator with respect to measure q, and $\mathcal{P}_t$ is a set of one-step-ahead conditional probabilities, given information available at date t. An important feature of this utility is that it satisfies dynamic consistency because it is defined recursively in (1). Recently, Epstein and Schneider (2003) provide an axiomatic foundation for this model. They also develop a reformulation of utility closer to Gilboa and Schmeidler (1989) where there is a set of priors $\mathcal{R}$ over the full state space implied by all histories of events,

$$V_t(c) = \min_{q\in\mathcal{R}} E_t^q \left[\sum_{s=t}^{\infty} \beta^{s-t} u(c_s)\right]. \tag{2}$$

The key to establishing this reformulation is to note that all sets of one-step-ahead conditionals, as one varies over times and histories, determines a unique set of priors $\mathcal{R}$ over the full state space satisfying the regularity conditions defined in Epstein and Schneider (2003). The latter also establishes formally that if one defines multiple-priors utility according to (2), an added restriction on $\mathcal{R}$ is needed to ensure dynamic consistency. To avoid this complication, we will adopt the Epstein and Wang model in (1) and specify the sets of one-step-ahead conditionals as a primitive, instead of the set of priors over the full state space.

2.2 Example

To understand our model and results, an illustrative example proves useful. Consider an investor who contemplates to invest irreversibly in a project in a three period setting. Assume that the investor is risk neutral and his subjective discount factor is $\beta = 0.2$. The payoff in period 0 is certain, while the payoffs in periods 1 and 2 are uncertain.

Investment costs $I = 145$. The initial value of the project is $x = 100$. In periods 1 and 2, the value of the project may go up or down by 50% (see Figure 1). These events are independent.

[**Insert Figure 1 Here**]

In standard models, the investor views the investment as purely risky. We may suppose that up and down have equal probability. Then according to the Marshallian net present value principle, the investor will not invest at period zero since the net present value of the investment opportunity is

$$NPV = x + \beta (1.5x + 0.5x) /2 + \beta^2 (2.25x/4 + 0.75x/2 + 0.25x/4) - I = -21 < 0.$$

However, if the investor can postpone the investment, he can observe whether the value of the project goes up or not. He can avoid the downside loss by investing when the value goes up. We use backward induction to solve the problem. It is clear that the value of the investment at period 2, F_2, is positive only when the project value is $2.25x$,

$$F_2 (2.25x) = \max\{2.25x - I, 0\} = 80.$$

Next, one can show that the value of the investment at period 1, F_1, is positive only for the project value $1.5x$,

$$F_1 (1.5x) = \max \{1.5x - I, 0.5\beta F_2 (2.25x)\} = 8 > 1.5x - I.$$

Finally, the value of the investment in period 0 is

$$F_0 (x) = \max \{x - I, 0.5\beta F_1 (1.5x)\} = 0.8 > 0 > x - I.$$

Thus, the investor will wait to invest at period 2 if and only if the value of the project goes up in both periods 1 and 2. This example illustrates the key idea of the real options approach that waiting has positive value.

Now, consider the situation under which the investor is ambiguous about the project value. Suppose he thinks that the value of the project goes up with probability 1/2 or 1/4. He is risk neutral and his preferences are represented by (1). We still use backward induction. Again, in period 2, the investment has positive value only for the project value $2.25x$,

$$V_2(2.25x) = \max\{2.25x - I, 0\} = 80.$$

In period 1, the investor compares the value of immediate investment with that if waiting until period 2. However, the investor values period 2 investment by taking the worst scenario. It can be shown that the value of the investment is positive only when the project value is $1.5x$,

$$V_1(1.5x) = \max\left\{1.5x - I, \beta\frac{1}{4}V_2(2.25x)\right\} = 5 > \beta\frac{1}{4}V_2(2.25x).$$

Finally, by a similar calculation, the period zero value of the investment is

$$V_0(x) = \max\left\{x - I, \beta\frac{1}{4}V_1(1.5x)\right\} = 0.25 > x - I.$$

The above two inequalities state that an uncertainty averse investor chooses optimally to invest in period 1, when the project value goes up. Moreover, he has a lower option value than the investor with expected utility since $V_0(x) < F_0(x)$.

In summary, this example shows that waiting has positive value for both uncertainty averse agent and the expected utility maximizer. However, the option value of the investment opportunity is lower under Knightian uncertainty than in the standard model. Therefore, an uncertainty averse agent invests earlier. We next turn to the formal model.

2.3 A Baseline Setup

Consider an infinite horizon discrete time optimal stopping problem. As explained in Dixit and Pindyck (1994), the optimal stopping problem can be applied to study an agent's option exercise decision. The agent's choice is binary. In each period, he decides on whether stopping a process and taking a termination payoff, or continuing for one more period, and making the same decision in the future.

Formally, uncertainty is generated by a Markov state process $(x_t)_{t\geq 0}$ taking values in $X = [a, b] \subset \mathbb{R}$. The probability kernel of $(x_t)_{t\geq 0}$ is given by $P : X \to \mathcal{M}(X)$, where $\mathcal{M}(X)$ is the space of probability measures on X endowed with the weak convergence topology. Continuation at date t generates a payoff $\pi(x_t)$, while stopping at date t

yields a payoff $\Omega(x_t)$, where π and Ω are functions that map X into $\mathbb{R}$. The decision is irreversible in that if the agent chooses to stop, he will not make further choices. Suppose the agent is risk neutral and discount future payoff flows by $\beta \in (0,1)$.

It is important to point out that, in the preceding setup, if the agent decides to stop, uncertainty is fully resolved. He has ambiguity from waiting only. In Section 2.5, we will consider a more general case where there is uncertainty about the termination payoff and the agent is ambiguous about this payoff. We will show that depending on relative degrees of ambiguity about different sources of uncertainty, ambiguity may have different impact on the agent's option exercise decision.

In standard models, the agent's preferences are represented by time-additive expected utility. As in the rational expectations paradigm, P can be interpreted as the objective probability law governing the state process $(x_t)_{t\geq 0}$, and is known to the agent. The expectation in the utility function is taken with respect to this law. Alternatively, according to the Savage utility representation theorem, P is a subjective (one-step-ahead) prior and represents the agent's beliefs. By either approach, the standard stopping problem can be described by the following Bellman equation:

$$F(x) = \max\left\{\Omega(x), \pi(x) + \beta \int F(x') P(dx'; x)\right\}, \tag{3}$$

where the value function F can be interpreted as an option value.

To fix ideas, we make the following assumptions. These assumptions are standard in dynamic programming theory (see Stokey and Lucas (1989)).

Assumption 1 $\pi : X \to \mathbb{R}$ *is bounded, continuous, and increasing.*

Assumption 2 $\Omega : X \to \mathbb{R}$ *is bounded, continuous and increasing.*

Assumption 3 P *is increasing and satisfies the Feller property. That is,* $\int f(x') P(dx'; x)$ *is increasing in* x *for any increasing function* f *and is continuous in* x *for any bounded and continuous function* f.

The following proposition describes the solution to problem (3).

Proposition 1 *Let Assumptions 1-3 hold. Then there exists a unique bounded, continuous and increasing function F solving the dynamic programming problem (3). Moreover, if there is a unique threshold value $x^* \in X$ such that*

$$\pi(x) + \beta \int F(x') P(dx'; x) \quad > \quad (<)\Omega(x), \ \textit{for } x < x^*, \ \textit{and}$$
$$\pi(x) + \beta \int F(x') P(dx'; x) \quad < \quad (>)\Omega(x), \ \textit{for } x > x^*,$$

then the agent continues (stops) when $x < x^*$ and stops (continues) when $x > x^*$. Finally, x^* is the solution to

$$\pi(x^*) + \beta \int F(x') P(dx'; x^*) = \Omega(x^*). \tag{4}$$

This proposition is illustrated in Figure 2. The threshold value x^* partitions the set X into two regions – continuation and stopping regions.[6] The top diagram of Figure 2 illustrates the situation where

$$\pi(x) + \beta \int F(x') P(dx'; x) > \Omega(x), \ \text{for } x < x^*,$$

and

$$\pi(x) + \beta \int F(x') P(dx'; x) < \Omega(x), \ \text{for } x > x^*.$$

In this case, we say that the continuation payoff curve crosses the termination payoff curve from above. Under this condition, the agent exercises the option when the process $(x_t)_{t \geq 0}$ first reaches the point x^* from below. The continuation region is given by $\{x \in X : x < x^*\}$ and the stopping region is given by $\{x \in X : x > x^*\}$. This case describes the upside of the agent's decision such as investment. The downside aspect such as disinvestment or exit is illustrated in the bottom diagram of Figure 2. The interpretation is similar.

[Insert Figure 2 Here]

In the above model, a role for Knightian uncertainty is excluded *a priori*, either because the agent has precise information about the probability law as in the rational

[6]For ease of presentation, we do not give primitive assumptions about the structure of these regions. See Dixit and Pindyck (1994, p. 129) for such an assumption. For the applications below, our assumptions can be easily verified.

expectations approach, or because the Savage axioms imply that the agent is indifferent to it. To incorporate Knightian uncertainty and uncertainty aversion, we follow the multiple-priors utility approach (Gilboa and Schmeidler (1989), Epstein and Wang (1994)) and assume that beliefs are too vague to be represented by a single probability measure and represented instead by a set of probability measures. More formally, we model beliefs by a probability kernel correspondence $\mathcal{P} : X \rightrightarrows \mathcal{M}(X)$. Given any $x \in X$, we think of $\mathcal{P}(x)$ as the set of conditional probability measures representing beliefs about next period's state. The multivalued nature of $\mathcal{P}$ reflects uncertainty aversion of preferences.

The stopping problem under Knightian uncertainty can be described by the following Bellman equation:

$$V(x) = \max\left\{\Omega(x), \pi(x) + \beta \int V(x') \mathcal{P}(dx'; x)\right\}, \tag{5}$$

where we adopt the notation throughout

$$\int f(x') \mathcal{P}(dx'; x) \equiv \min_{Q(\cdot;x)\in\mathcal{P}(x)} \int f(x') Q(dx'; x),$$

for any Borel function $f : X \to \mathbb{R}$. Note that if $\mathcal{P} = \{P\}$, then the model reduces to the standard model (3).

To analyze problem (5), the following assumption is adopted.

Assumption 4 *The probability kernel correspondence $\mathcal{P} : X \rightrightarrows \mathcal{M}(X)$ is nonempty valued, continuous, compact-valued, and convex-valued, and $P(x) \in \mathcal{P}(x)$ for any $x \in X$. Moreover, given any $Q(\cdot; x) \in \mathcal{P}(x)$, $\int f(x') Q(dx'; x)$ is increasing in x for any increasing function $f : X \to \mathbb{R}$.*

This assumption is a generalization of Assumption 3 to correspondence. It ensures that $\int f(x') \mathcal{P}(dx'; x)$ is bounded, continuous, and increasing in x for any bounded, continuous, and increasing function $f : X \to \mathbb{R}$. Notice that this assumption is quite general in the sense that it captures the fact that ambiguity may vary with the state. However, in some examples in Section 3, we consider an IID case in order to derive closed form solutions.

2.4 Erosion of Option Value

We now analyze the implications of ambiguity and ambiguity aversion on the option exercise decision for the preceding baseline model. We first characterize the solution to problem (5) in the following proposition:

Proposition 2 *Let Assumptions 1-4 hold. Then there is a unique bounded, continuous, and increasing function V solving the dynamic programming problem (5). Moreover, if there exists a unique threshold value $x^{**} \in X$ such that*

$$\begin{aligned} \pi(x) + \beta \int V(x') \mathcal{P}(dx'; x) &> (<) \Omega(x), \text{ for } x > x^{**}, \text{ and} \\ \pi(x) + \beta \int V(x') \mathcal{P}(dx'; x) &< (>) \Omega(x), \text{ for } x < x^{**}, \end{aligned}$$

then the agent stops (continues) when $x < x^{**}$ and continues (stops) when $x > x^{**}$. Finally, x^{**} is the solution to

$$\pi(x^{**}) + \beta \int V(x') \mathcal{P}(dx'; x^{**}) = \Omega(x^{**}). \tag{6}$$

This proposition implies that the agent's option exercise decision under Knightian uncertainty has similar features to that in the standard model described in Proposition 1. It is interesting to compare the option value and option exercise time in these two models.

Proposition 3 *Let assumptions in Proposition 1 and Proposition 2 hold. Then $V \leq F$. Moreover, for both V and F, if the continuation payoff curves cross the termination payoff curves from above then $x^{**} \leq x^*$. On the other hand, if the continuation payoff curves cross the termination payoff curves from below, then $x^{**} \geq x^*$.*

In the standard model, an expected utility maximizer views the world as purely risky. For the decision problems such as investment, waiting has value because the agent can avoid the downside risk, while realizing the upside potential. Similarly, for the decision problems such as exit, waiting has value because the agent hopes there is some chance that the future may be brighter. Now, if the agent has imprecise knowledge about the likelihoods of the state of the world and hence perceives the future as ambiguous, then waiting will have less value for an uncertainty averse agent because he acts on the worst scenario.

[Insert Figure 3 Here]

The threshold value under Knightian uncertainty can be either bigger or smaller than that in the standard model, depending on the shapes of the continuation and termination payoff curves (see Figure 3). More specifically, if the continuation payoff curve crosses the termination payoff curve from above, then the threshold value under Knightian uncertainty is smaller than that in the standard model. The opposite conclusion can be obtained if the continuation payoff curve crosses the termination payoff curve from below. For both cases, an uncertainty averse agent exercises the option earlier than an agent with expected utility because the former has less option value.

The following proposition concerns comparative statics.

Proposition 4 *Let the assumptions in Proposition 2 hold. Consider two probability kernel correspondences* $\mathcal{P}_1$ *and* $\mathcal{P}_2$*. Let the corresponding value functions be* $V^{\mathcal{P}_1}$ *and* $V^{\mathcal{P}_2}$ *and the corresponding threshold values be* $x^{\mathcal{P}_1}$ *and* $x^{\mathcal{P}_2}$*. If* $\mathcal{P}_1(x) \subset \mathcal{P}_2(x)$*, then* $V^{\mathcal{P}_1} \geq V^{\mathcal{P}_2}$*. Moreover, if the continuation payoff curves cross the termination payoff curves from above (below), then* $x^{\mathcal{P}_1} \geq (\leq) x^{\mathcal{P}_2}$*.*

It is intuitive that the set of priors captures ambiguity aversion and the degree of ambiguity aversion. A larger set of priors means that the agent has more imprecise knowledge and is less confident to assign probabilities to states of the world. Hence, he is more ambiguity averse. This proposition says that the option value is lower if the agent is more ambiguity averse, and hence a more ambiguity averse agent exercises the option earlier (see Figure 4). Our interpretation of this proposition is based on the definition of absolute and comparative ambiguity aversion proposed by Ghirardato and Marinacci (2002).[7] Their theory may also provide a behavioral foundation of the interpretation that the set of priors describes the degree of ambiguity. A similar interpretation is also given in Nishimura and Ozaki (2004). Henceforth, if one increases the set of priors in the

[7] See Epstein (1999) and Epstein and Zhang (2001) for a different definition.

sense of set inclusion, we may interpret that ambiguity is increased or the agent is more ambiguity averse.

[**Insert Figure 4 Here**]

2.5 Ambiguity about Termination Payoffs

So far, we have assumed that once the agent exercises the option, uncertainty is fully resolved, and that the agent bears ambiguity from waiting only. As will be illustrated in the applications in Section 3, in reality there are many instances in which there is uncertainty about termination payoffs. We now show that if the agent is ambiguous about termination payoffs, ambiguity may have different impact on the agent's option exercise decision.

To illustrate, we first consider a simple case where the termination payoff $\Omega(x_t)$ does not depend on the state x_t. However, it is a random variable with distribution Q. In addition to ambiguity about the state process $(x_t)_{t\geq 0}$, the agent is also ambiguous about the termination payoff Ω. He has a set of priors $\mathcal{Q}$ over Ω. In this case, the agent is ambiguous about two different sources of uncertainty. Section 3.3 will show that an instance of this case is the exit problem. We now formally replace Assumption 2 with

Assumption 5 *Ω is a random variable with distribution Q and $\mathcal{Q}$ is weakly compact and contains Q.*

The agent's decision problem can be described by the following Bellman equation:

$$V(x) = \max\left\{\min_{q\in\mathcal{Q}}\int \Omega dq, \pi(x) + \beta\int V(x')\mathcal{P}(dx';x)\right\}. \tag{7}$$

When $\mathcal{Q} = \{Q\}$ and $\mathcal{P} = \{P\}$, the preceding problem reduces to the standard one for an expected utility maximizer.

One can prove a characterization proposition for (7) similar to Proposition 2. In particular, there is a threshold value such that the agent exercises the option the first time the process $(x_t)_{t\geq 0}$ falls below this value. However, there is no clear-cut result about comparative statics and the comparison with the standard model as in Propositions 3-4. This is because ambiguity lowers both the continuation payoff and the termination payoff.

The overall impact on the option exercise decision depends on which effect dominates. If we fix ambiguity about the continuation payoff and consider the impact of ambiguity about the termination payoff only, we have the following clean comparative statics result analogous to Proposition 4.

Proposition 5 *Let Assumptions 1 and 3-5 hold and fix $\mathcal{P}$. Consider two sets of priors $\mathcal{Q}_1$ and $\mathcal{Q}_2$. Let the corresponding value functions be $V^{\mathcal{Q}_1}$ and $V^{\mathcal{Q}_2}$ and the corresponding threshold values be $x^{\mathcal{Q}_1}$ and $x^{\mathcal{Q}_2}$. If $\mathcal{Q}_1 \subset \mathcal{Q}_2$, then $V^{\mathcal{Q}_1} \geq V^{\mathcal{Q}_2}$ and $x^{\mathcal{Q}_1} \geq x^{\mathcal{Q}_2}$.*

This proposition shows that although ambiguity about termination payoffs lowers the option value from continuation, the agent exercises the option later if he is more ambiguous about the termination payoffs. This is because ambiguity also lowers the termination payoff and this effect dominates.

In the previous case, there is no *future* uncertainty about termination payoffs. In reality, there are many instances in which there is ongoing uncertainty about termination payoffs after the agent exercises the option. For example, an agent decides whether and when to invest in a project which can generate a stream of future uncertain profits. Then the termination payoff $\Omega(x)$ depends on the future uncertainty about the profits generated by the project. To incorporate this case, we assume that $\Omega(x)$ satisfies the following Bellman equation

$$\Omega(x) = \Phi(x) + \beta \int \Omega(x') \mathcal{P}(dx'; x), \tag{8}$$

where the period payoff $\Phi : X \to \mathbb{R}$ is an increasing and continuous function. Note that by the Blackwell Theorem, there is a unique bounded and continuous function Ω satisfying (8). In the standard model with expected utility, we have

$$\Omega(x) = \Phi(x) + \beta \int \Omega(x') P(dx'; x).$$

That is, $\Omega(x)$ is equal to the expected discounted payoffs,

$$\Omega(x) = E\left[\sum_{t=0}^{\infty} \beta^t \Phi(x_t) \middle| x_0 = x\right].$$

When $\Omega(x)$ is given by (8), the agent's decision problem is still described by the dynamic programming equation (4). Again, we can show that ambiguity lowers the option value $V(x)$. However, since ambiguity about the state process $(x_t)_{t\geq 0}$ lowers both the option value and the termination payoff, there is no general comparative statics result about the option exercise timing. We will illustrate this point in the next section.

3 Applications

This section applies our results to two classes of problems: real investment and firm exit. The real investment decision is an example where an agent decides whether or not to exercise an option to pursue upside potential. Entry and job search are similar problems. The exit problem represents an example where an agent decides whether or not to exercise an option to avoid downside potential. Other examples include default and liquidation decisions.

3.1 Investment

A classic application of the option exercise problem is the irreversible investment decision.[8] The standard real-options approach makes the analogy of investment to the exercising of an American call option on the underlying project. Formally, consider an investment opportunity which can generate stochastic values given by a Markov process $(x_t)_{t\geq 0}$. Investment costs $I > 0$. Then we can cast the investment problem into our framework by setting

$$\Omega(x) = x - I, \ \pi(x) = 0.$$

We can also write the agent's investment decision problem as follows

$$V(x) = \max\left\{x - I, \beta \int V(x') \mathcal{P}(dx'; x)\right\}. \quad (9)$$

Note that, according to this setup, once the investor makes the investment, he obtains net rewards $x - I$ and uncertainty is fully resolved. In reality, it is often the case that investment rewards come from the uncertain future. For example, after the investor invests in a project or develops a new product, the project can generate a stream of future uncertain profits. The investor may be ambiguous about profit flows. To incorporate this

[8] See Bernanke (1983), Brennan and Schwartz (1985), and McDonald and Siegel (1986) for important early contributions. See Dixit and Pindyck (1994) for a textbook treatment.

case, we presume that the period t profit is given by x_t. Then the discounted total project value at date t is given by

$$\Omega(x_t) = x_t + \beta \int \Omega(x') \mathcal{P}(dx'; x_t), \tag{10}$$

and the investor's decision problem is formulated as

$$V(x) = \max \left\{ \Omega(x) - I, \beta \int V(x') \mathcal{P}(dx'; x) \right\}. \tag{11}$$

The standard real-options model predicts that there is an option value of waiting, because investment is irreversible and flexibility in timing has value. Another main prediction of the standard real investment model is that an increase in risk in the sense of mean-preserving spread raises the option value and delays investment (see Pindyck and Dixit (1994)). This derives from the fundamental insight behind the option pricing theory, in that firms may capture the upside gains and minimizes the downside loss by waiting for the risk of project value to be partially resolved.

[Insert Figure 5 Here]

While the standard real-options model predicts a monotonic relationship between investment and risk, our model makes an important distinction between risk and uncertainty. We argue that risk (which can be described by a single probability measure) and uncertainty (multiplicity of priors) have different effects on investment timing. Specifically, our model predicts that Knightian uncertainty lowers option value of waiting (see Figure 5). Moreover, under formulation (9), Propositions 3-4 imply that an increase in ambiguity pulls the investment trigger earlier and hence speeds up investment. These propositions also imply that the more ambiguity averse the investor is, the earlier he makes the investment.

As pointed in Section 2.5, this conclusion is not generally true if there is ongoing uncertainty about the termination payoff. For the investment problem under the formulation in (10)-(11), the investor may be ambiguous about the future profit opportunities of the investment project. He may well hesitate to invest.

We now consider a concrete parametric example to illustrate the above analysis. Recall $X = [a, b]$. Following Epstein and Wang (1994), we consider an IID $\varepsilon-$contamination specification of the set of one-step-ahead priors. That is, let

$$\mathcal{P}(x) = \{(1-\varepsilon)\mu + \varepsilon m : m \in \mathcal{M}(X)\} \text{ for all } x, \tag{12}$$

where $\varepsilon \in [0,1]$ and μ is any distribution over X. The interpretation is the following: μ represents the "true" distribution of the reward. The investor does not know this distribution precisely. With probability ε, he believes that the reward may be distributed according to some other distribution. Here ε may represent the degree of ambiguity and ambiguity aversion. This can be justified by observing that, if ε is larger, the set $\mathcal{P}(x)$ is larger in the sense of set inclusion. When $\varepsilon = 0$, $\mathcal{P}(x) = \{\mu\}$ and the model reduces to the standard one with expected utility. When $\varepsilon = 1$, the investor is completely ignorant about the "true" distribution. The following proposition characterizes the optimal investment trigger.

Proposition 6 *Assume (12). (i) For problem (9), the investment threshold x^* satisfies the equation*[9]

$$x^* - I = \frac{\beta(1-\varepsilon)}{1-\beta}\int_{x^*}^{b}(x - x^*)\, d\mu. \tag{13}$$

Moreover, x^* decreases in ε. (ii) For problem (11), the investment threshold x^* satisfies the equation

$$x^* + \frac{\beta}{1-\beta}\left((1-\varepsilon) E^{\mu}[x] + \varepsilon a\right) - I = \frac{\beta(1-\varepsilon)}{1-\beta}\int_{x^*}^{b}(x - x^*)\, d\mu. \tag{14}$$

Moreover, x^* increases in ε.

The interpretation of (13) is the following. The left-hand side of (13) represents the net benefit from investment. The right-hand side represents the opportunity cost of waiting. Because waiting has positive option value, the investment threshold exceeds the investment cost I. Equation (13) states that at the investment threshold, the investor is indifferent between investing and waiting. It is also clear from (13) that because ambiguity lowers the option value, the right-hand side of (13) is less than that in the standard model with $\varepsilon = 0$. Moreover, an increase in ε lowers the investment threshold. Thus, an

[9]We have implicitly assumed that there are parameter values such that there exists an interior solution. We will not state such an assumption explicitly both in this proposition and in Proposition 7.

increase in ambiguity speeds up investment and a more ambiguity averse investor invests relatively earlier.

The interpretation of (14) is similar. The difference is that there is future uncertainty about profit opportunities of the investment project. Under the $\varepsilon-$contamination specification in (12), using (10) one can verify that the value of the investment project is given by

$$\Omega(x) = x + \frac{\beta}{1-\beta}\left((1-\varepsilon)E^{\mu}[x] + \varepsilon a\right).$$

When ε is increased, ambiguity lowers both the project value represented by the left-hand side of (14) and the option value from waiting represented by the right-hand side of (14). Proposition 6 demonstrates that the former effect dominates so that the investor delays the investment.

It is interesting to note that when ε approaches 1, the investor has no idea about the true distribution of the profit. Ambiguity erodes away completely the option value from waiting. Specifically, for problem (9) in which there is no ambiguity about termination payoff, the investment threshold becomes $x^* = I$. For problem (11) in which there is ambiguity about both termination and continuation payoffs, the investment threshold satisfies $x^* + \frac{\beta}{1-\beta}a = I$. Note that for both problems, the investor adopts the myopic NPV investment rule. Further, if the investor is also ambiguous about the future profits from the project after investment, the investor computes the NPV of the project according to the worst case scenario, in which he believes that the cash flow in each period in the future takes the minimum value a.

3.2 Exit

Firm exit is an important problem in industrial organization and macroeconomics.[10] We may describe a stylized exit model as follows. Consider a firm in an industry. The process $(x_t)_{t\geq 0}$ could be interpreted as a demand shock or a productivity shock. Stay in business at date t generates profits $\Pi(x_t)$ and incurs a fixed cost $c_f > 0$. The firm may also exit and seek outside opportunities. Let the outside opportunity value be a constant $\gamma \geq 0$.

[10]See Hopenhayn (1992) for an industry equilibrium model of entry and exit.

Then the problem fits into our framework by setting

$$\Omega(x) = \gamma,\ \pi(x) = \Pi(x) - c_f,$$

where we assume $\Pi(\cdot)$ is increasing and continuous.

According to the standard real options approach, the exit trigger is lower than that predicted by the textbook Marshallian net present value principle. This implies that firms stay in business for a long period of time while absorbing operating losses. Only when the upside potential gain is bad enough, will the firm not absorb losses and abandon operation. The standard real options approach also predicts that an increase in risk in the sense of mean preserving spread raises the option value, and hence lowers the exit trigger. This implies that firms should stay in business longer in riskier situations, even though they suffer substantial losses. However, this prediction seems to be inconsistent with the large amount of quick exit in IT industry in recent years.

The Knightian uncertainty theory may shed light on this issue. In recent years, due to economic recessions, firms are more ambiguous about the industry demand and their productivity. They are less sure about the likelihoods of when the economy will recover. Intuitively, the set of probability measures that firms may conceive is larger in recessions. Thus, by Proposition 4, the option value of the firm is lower and the exit trigger is higher. This induces firms to exit earlier (see Figure 6).

[Insert Figure 6 Here]

The previous argument relies crucially on the fact that the outside value γ is a constant. In reality, there may be uncertainty about the outside value. For example, the outside value could represent the scrapping value of the firm and the firm is uncertain about its market value. The outside value could also represent the profit opportunity of a new business and the firm is uncertain about this opportunity. Proposition 5 shows that, *ceteris paribus*, if the firm is more ambiguous about the outside value, it will be more hesitant to exit. Thus, the overall effect of ambiguity on the exit timing depends on the relative degrees of ambiguity about different sources.

To illustrate, we consider a parametric example. We simply take $\Pi(x) = x \in X = [a, b]$. We still adopt the IID $\varepsilon-$contamination specification (12) for the process $(x_t)_{t \geq 0}$.

When the outside value γ is a constant, the firm's decision problem is described the following Bellman equation

$$V(x) = \max\left\{\gamma,\, x - c_f + \beta \int_a^b V(x')\, \mathcal{P}(dx'; x)\right\}. \tag{15}$$

When the firm is also ambiguous about the outside value, we adopt the following $\eta-$ contamination specification for the outside value $\gamma \in [\underline{\gamma}, \overline{\gamma}]$,

$$\mathcal{Q} = \left\{(1-\eta)\,\nu + \eta m :\ m \in \mathcal{M}\left([\underline{\gamma}, \overline{\gamma}]\right)\right\}, \eta \in [0,1], \tag{16}$$

where ν is a distribution over $[\underline{\gamma}, \overline{\gamma}]$ and may represent the "true" distribution of γ. The interpretation is that the firm is not sure about the true distribution of the outside value and believes other distributions are possible with probability η. Note that η can be interpreted as a parameter measuring the degree of ambiguity and ambiguity aversion about the outside value. In this case, the firm's decision problem is described by the following Bellman equation:

$$V(x) = \max\left\{\min_{q \in \mathcal{Q}} \int_{\underline{\gamma}}^{\overline{\gamma}} \gamma dq,\, x - c_f + \beta \int_a^b V(x')\, \mathcal{P}(dx'; x)\right\}. \tag{17}$$

The following proposition characterizes the solutions to problems (15) and (17).

Proposition 7 *Assume (12) and (16). (i) For problem (15), the exit threshold x^* satisfies the equation*

$$(1-\beta)\,\gamma = x^* - c_f + \beta\,(1-\varepsilon) \int_{x^*}^b (x - x^*)\, d\mu. \tag{18}$$

Moreover, x^* increases in ε. (ii) For problem (17), the exit threshold x^* satisfies the equation

$$(1-\beta)\left((1-\eta)\, E^{\nu}[\gamma] + \eta \underline{\gamma}\right) = x^* - c_f + \beta\,(1-\varepsilon) \int_{x^*}^b (x - x^*)\, d\mu. \tag{19}$$

Moreover, x^* increases in ε and decreases in η.

The interpretation of (18) is the following. The left-hand side of (18) represents the per period outside value if the firm chooses to exit. The right-hand side represents the payoff if the firm chooses to stay. In particular, the first term represents the immediate profits and the second term represents the option value of waiting. At the exit threshold value, the firm is indifferent between exit and stay. From (18), the impact of ambiguity is transparent. An increase in ε lowers the option value of stay in business by raising the exit threshold. Hence, the firm exits earlier.

The interpretation of (19) is similar. Note that, as one fixes ε and increases η, the outside value and the exit trigger are reduced. Thus, *ceteris paribus,* ambiguity about the outside value delays exit. If the firm is more ambiguous about the outside value, it exits later. If one increases both ε and η, either effect may dominate and the overall effect on the exit timing depends on the relative degrees of ambiguity about these two sources of uncertainty.

Note that as in the investment problem, when ε is equal to 1, the option value of waiting to exit is equal to zero. Thus, the firm just follows the simple myopic NPV rule, by using the worst-scenario belief.

4 Conclusion

There are many economic decisions that can be described as an option exercise problem. In these decision making settings, uncertainty plays an important role. In standard expected utility models, there is no meaningful distinction between risk and uncertainty in the sense attributed to Knight (1921). To afford this distinction, we apply the multiple-priors utility model to analyze an option exercise problem. In particular, we formulate it as a general optimal stopping problem. While the standard analysis shows that risk increases option value, we show that ambiguity lowers the option value. Moreover, the impact of ambiguity on the option exercise timing depends crucially on whether the agent has ambiguity about termination payoffs after option exercise. If uncertainty is fully resolved after option exercise, then an increase in ambiguity speeds up option exercise and a more ambiguity averse agent exercises the option earlier. However, if the agent is ambiguous about the termination payoff, then the agent may delay option exercise if this ambiguity dominates ambiguity about continuation.

We apply our general model to firm investment and exit problems. For the investment problem, we show that if the project value is modeled as a stock value and uncertainty over this value is fully resolved after investment, then ambiguity accelerates investment. However, if the project value is modeled as a discounted sum of future uncertain profit flows and the agent is ambiguous about these profits, then ambiguity may delay investment. For the exit problem, we presume that there are two sources of uncertainty – outside value and profit opportunities if stay in business. The firm may be ambiguous about both sources. We show that ambiguity may delay or accelerate exit depending on the effect of ambiguity about which source dominates. We also show that for both problems, the myopic NPV rule often recommended by the business textbooks and investment advisors may be optimal for an agent having an extremely high degree of ambiguity aversion.

Appendix

A Proofs

Proof of Proposition 1: The proof is similar to that of Proposition 2. So we omit it. Q.E.D.

Proof of Proposition 2: Let $C(X)$ denote the space of all bounded and continuous functions endowed with the sup norm. $C(X)$ is a Banach space. Define an operator T as follows:

$$Tv(x) = \max\left\{\Omega(x), \pi(x) + \beta \int v(x') \mathcal{P}(dx'; x)\right\}, \ v \in C(X).$$

Then it can be verified that T maps $C(X)$ into itself. Moreover, T satisfies the Blackwell sufficient condition and hence is a contraction mapping. By the Contraction Mapping Theorem, T has a unique fixed point $V \in C(X)$ which solves the problem (5) (see Theorems 3.1 and 3.2 in Stokey and Lucas (1989)).

Next, let $C'(X) \subset C(X)$ be the set of bounded continuous and increasing functions. One can show that T maps any increasing function $C'(X)$ into an increasing function in $C'(X)$. Since $C'(X)$ is a closed subset of $C(X)$, by Corollary 1 in Stokey and Lucas (1989, p.52), the fixed point of T, V, is also increasing. The remaining part of the proposition is trivial and follows from similar intuition illustrated in Figure 2. Q.E.D.

Remark: The Contraction Mapping Theorem also implies that $\lim_{n\to\infty} T^n v = V$ for any function $v \in C(X)$.

Proof of Proposition 3: Let $v \in C(X)$ satisfy $v \leq F$. Since $P(x) \in \mathcal{P}(x)$,

$$\int v(x') \mathcal{P}(dx'; x) = \min_{Q(\cdot;x)\in\mathcal{P}(x)} \int v(x') Q(dx'; x) \leq \int v(x') P(dx'; x) \leq \int F(x') P(dx'; x).$$

Thus,

$$\begin{aligned} Tv(x) &= \max\left\{\Omega(x), \pi(x) + \beta \int v(x') \mathcal{P}(dx'; x)\right\} \\ &\leq \max\left\{\Omega(x), \pi(x) + \beta \int F(x') P(dx'; x)\right\} \\ &= F(x). \end{aligned}$$

It follows from induction that the fixed point of T, V, must also satisfy $V \leq F$. The remaining part of the proposition follows from this fact and Figure 3. Q.E.D.

Proof of Proposition 4: Define the operator $T^{\mathcal{P}_1} : C(X) \to C(X)$ by

$$T^{\mathcal{P}_1} v(x) = \max\left\{\Omega(x), \pi(x) + \beta \int v(x') \mathcal{P}_1(dx'; x)\right\}, \ v \in C(X).$$

Similarly, define an operator $T^{\mathcal{P}_2} : C(X) \to C(X)$ corresponding to $\mathcal{P}_2$. Take any functions $v_1, v_2 \in C(X)$ such that $v_1 \geq v_2$, it can be shown that $T^{\mathcal{P}_1} v_1(x) \geq T^{\mathcal{P}_2} v_2(x)$. By induction, the fixed points $V^{\mathcal{P}_1}$ and $V^{\mathcal{P}_2}$ must satisfy $V^{\mathcal{P}_1} \geq V^{\mathcal{P}_2}$. The remaining part of the proposition follows from Figure 4. Q.E.D.

Proof of Proposition 5: First, one can use the standard dynamic programming technique similar to that used in Propositions 2-4 to show that the value functions $V^{\mathcal{Q}_1}$ and $V^{\mathcal{Q}_2}$ are increasing and $V^{\mathcal{Q}_1} \geq V^{\mathcal{Q}_2}$. To show $x^{\mathcal{Q}_1} \geq x^{\mathcal{Q}_2}$, let $G^i(x) = V(x) - \min_{q \in \mathcal{Q}_i} \int \Omega dq$ for $i = 1, 2$. Then from (7), we can derive that

$$G^i(x) = \max\left\{0, \pi(x) - (1-\beta) \min_{q \in \mathcal{Q}_i} \int \Omega dq + \beta \int G^i(x') \mathcal{P}(dx'; x)\right\}.$$

Again, by the standard dynamic programming technique, we can show that G^i is increasing and $G^2(x) \geq G^1(x)$. The threshold values $x^{\mathcal{Q}_1}$ are determined by the equation

$$0 = \pi(x) - (1-\beta) \min_{q \in \mathcal{Q}_i} \int \Omega dq + \beta \int G^i(x') \mathcal{P}(dx'; x).$$

Since $\min_{q \in \mathcal{Q}_1} \int \Omega dq \geq \min_{q \in \mathcal{Q}_2} \int \Omega dq$ and $G^2(x) \geq G^1(x)$, we have $x^{\mathcal{Q}_1} \geq x^{\mathcal{Q}_2}$. Q.E.D.

Proof of Proposition 6: Because $V(x)$ is an increasing and continuous function, the optimal investment rule is described as a trigger policy whereby the investor invests the first time the process $(x_t)_{t \geq 0}$ hits a threshold value x^*. We now determine x^* and focus on problem (9). Problem (11) can be analyzed similarly. Now, for problem (9), $V(x)$ satisfies

$$V(x) = \begin{cases} x - I & \text{if } x \geq x^*, \\ \beta \int_a^b V(x') \mathcal{P}(dx'; x) & \text{if } x < x^*. \end{cases} \tag{A.1}$$

At the threshold value x^*, we have

$$x^* - I = \beta \int_a^b V(x') \mathcal{P}(dx'; x). \tag{A.2}$$

According to the IID $\varepsilon-$contamination specification (12), we have

$$x^* - I = \beta(1-\varepsilon) \int_a^b V(x)\, d\mu + \beta\varepsilon \min_{m \in \mathcal{M}([a,b])} \int_a^b V(x)\, dm.$$

Since the minimum of $V(x)$ is $\beta \int_a^b V(x') \mathcal{P}(dx'; x)$, which is equal to $x^* - I$ by (A.2), we can rewrite the preceding equation as

$$\begin{aligned}
x^* - I &= \beta(1-\varepsilon) \int_a^b V(x)\, d\mu + \beta\varepsilon (x^* - I) \\
&= \beta(1-\varepsilon) \left[\int_a^{x^*} V(x)\, d\mu + \int_{x^*}^b V(x)\, d\mu \right] + \beta\varepsilon (x^* - I) \\
&= \beta(1-\varepsilon) \left[\int_a^{x^*} (x^* - I)\, d\mu + \int_{x^*}^b (x - I)\, d\mu \right] + \beta\varepsilon (x^* - I).
\end{aligned}$$

Note that in the last equality, we have used (A.1) and (A.2). Rearranging yields the desired result (13).

The comparative statics result follows from the implicit function theorem applied to (13). Q.E.D.

Proof of Proposition 7: The proof is similar to that of Proposition 6. We consider part (i) first. The value function $V(x)$ satisfies

$$V(x) = \begin{cases} x - c_f + \beta \int_a^b V(x') \mathcal{P}(dx'; x) & \text{if } x \geq x^*, \\ \gamma & \text{if } x < x^*. \end{cases} \tag{A.3}$$

At the threshold value x^*, we have

$$\gamma = x^* - c_f + \beta \int_a^b V(x') \mathcal{P}(dx'; x). \tag{A.4}$$

Given the IID $\varepsilon-$contamination specification (12), we can derive

$$\gamma = x^* - c_f + \beta(1-\varepsilon) \int_a^b V(x)\, d\mu + \beta\varepsilon \min_{m \in \mathcal{M}([a,b])} \int_a^b V(x)\, dm.$$

Since the minimum of $V(x)$ is γ by (A.3), we can rewrite the preceding equation as

$$\begin{aligned}\gamma &= x^* - c_f + \beta(1-\varepsilon)\int_a^b V(x)\,d\mu + \beta\varepsilon\gamma \\ &= x^* - c_f + \beta(1-\varepsilon)\left[\int_a^{x^*} V(x)\,d\mu + \int_{x^*}^b V(x)\,d\mu\right] + \beta\varepsilon\gamma \\ &= x^* - c_f + \beta(1-\varepsilon)\left[\int_a^{x^*} \gamma d\mu + \int_{x^*}^b (x+\gamma-x^*)\,d\mu\right] + \beta\varepsilon\gamma.\end{aligned}$$

Here the last equality follows from (A.3) and (A.4). Simplifying yields (18). The comparative static result follows from simple algebra.

For part (ii), given the $\eta-$contamination specification in (16), we have

$$\min_{q\in\mathcal{Q}}\int_{\underline{\gamma}}^{\overline{\gamma}} \gamma dq = (1-\eta)\,E^{\nu}[\gamma] + \eta\underline{\gamma}.$$

Equation (19) follows from a similar argument for (18). The comparative statics result follows from simple algebra. Q.E.D.

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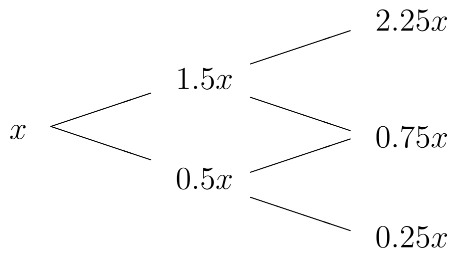

Figure 1: Investment cash flows

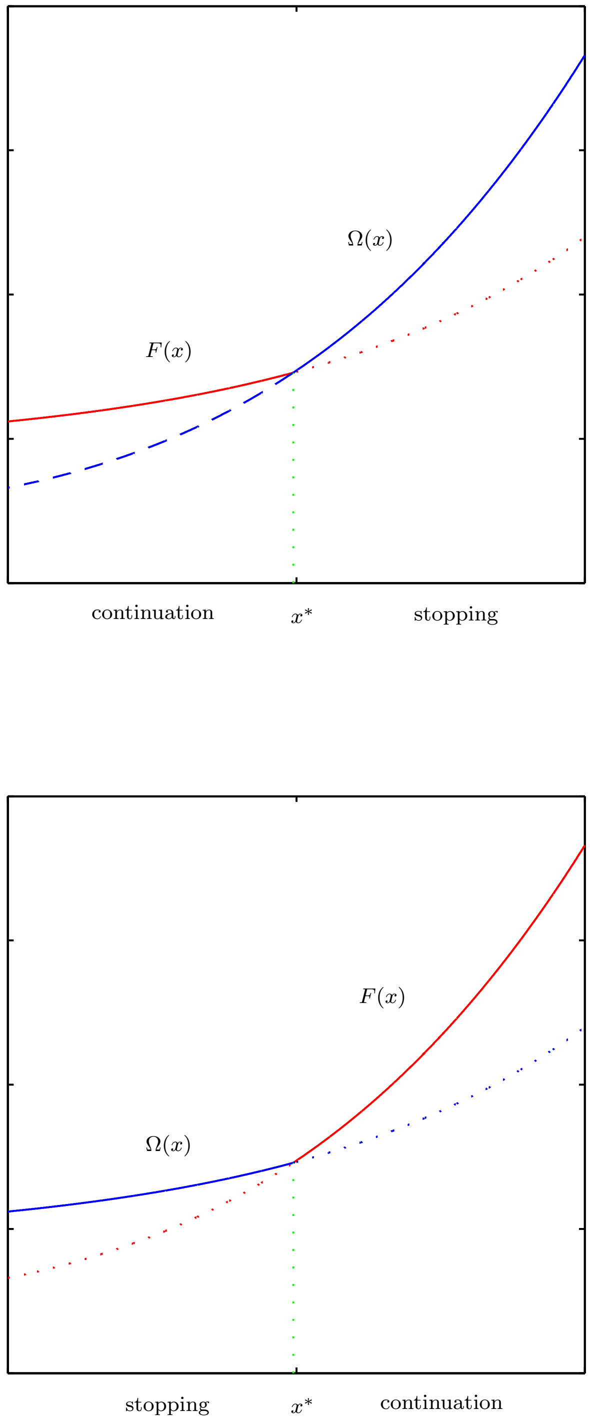

Figure 2: Value functions and exercising thresholds in the standard model. The top diagram illustrates an option exercise problem such as investment. The bottom diagram illustrates an option exercise problem such as exit.

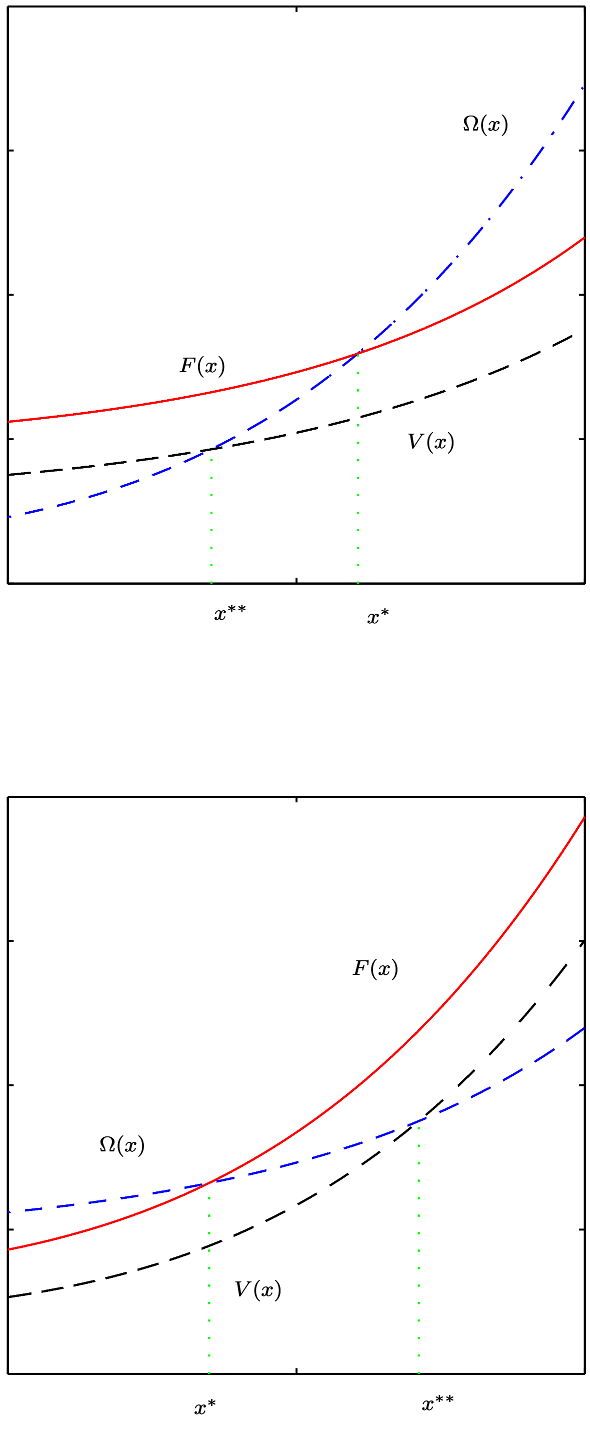

Figure 3: Comparison of the standard model and the model under Knightian uncertainty.

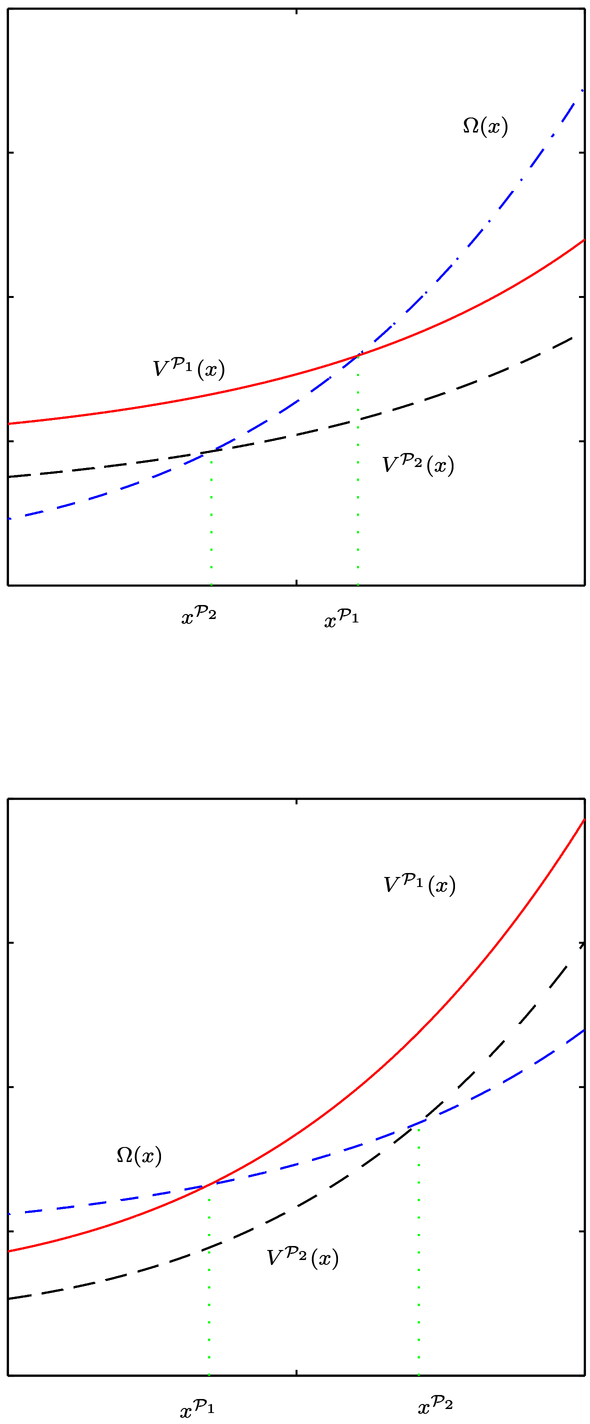

Figure 4: Option value and exercising thresholds under Knightian uncertainty for two different sets of priors $\mathcal{P}_1 \subset \mathcal{P}_2$.

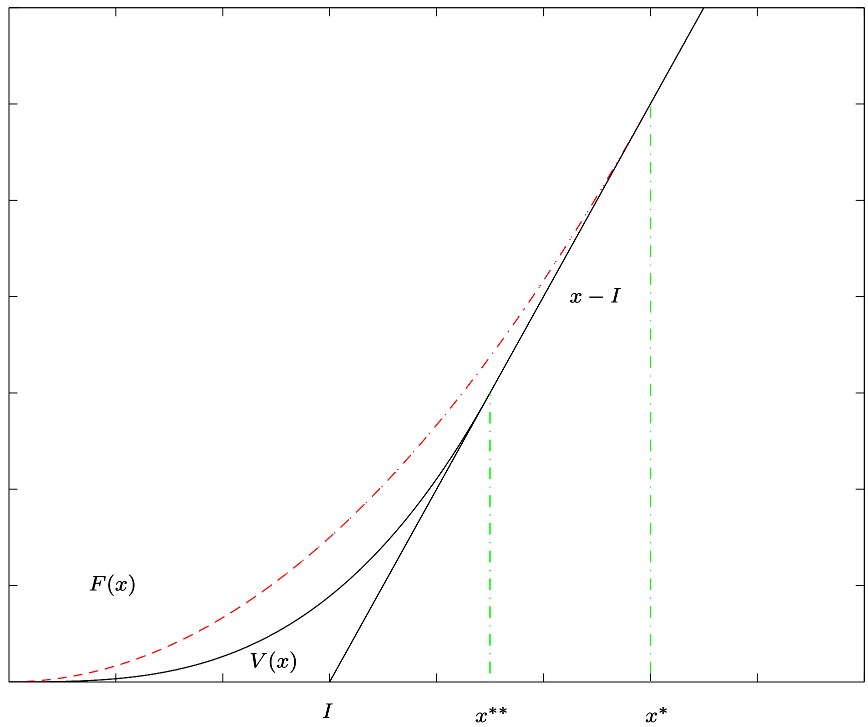

Figure 5: Investment timing under Knightian uncertainty and in the standard model. The upper (dashed) curve corresponds to the value function $F(x)$ in the standard model. The lower (solid) curve corresponds to the value function $V(x)$ under Knightian uncertainty.

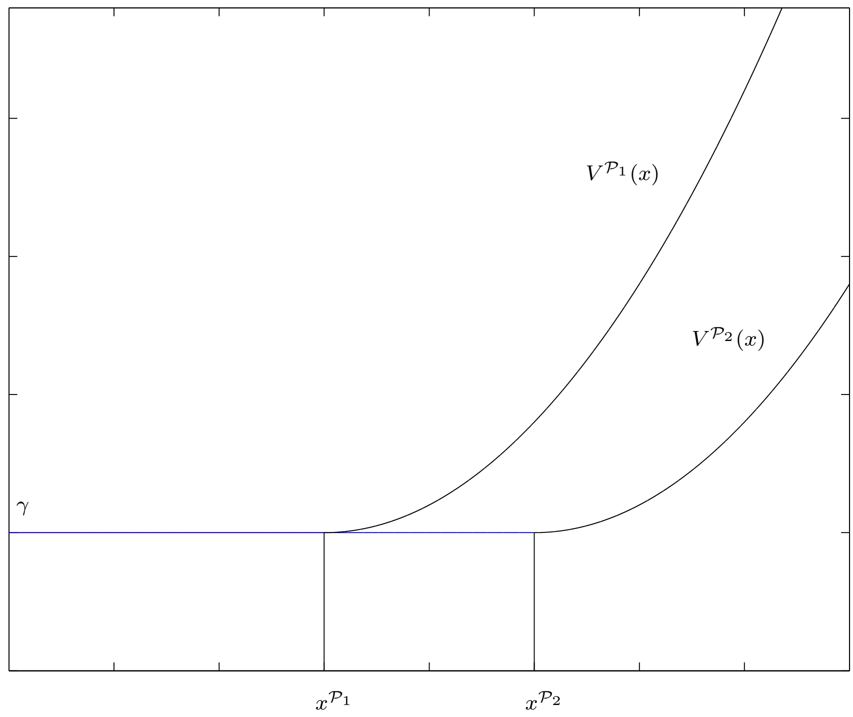

Figure 6: Firm Exit under different degrees of Knightian uncertainty. The upper (dashed) curve corresponds to the value function $V^{\mathcal{P}_1}(x)$ and the lower (solid) curve corresponds to the value function $V^{\mathcal{P}_2}(x)$ where $\mathcal{P}_1 \subset \mathcal{P}_2$.