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When to Put All Your Eggs in One Basket.....: When Diversification Inceases Portfolio Risk!

1 Introduction

Professor Kevin Dowd claims that a risk measure must be sub-additive: "If our risk measure is non-sub-additive, there is a danger it might suggest that diversification is a bad thing, and that would imply the laughable conclusion that putting all your eggs in one basket might be good risk management!" (*Financial Engineering News*, November/December 2004, p. 7)

Well, the sub-additivity of a risk measure doesn't depend on our choice, but on the long memory characteristics of the rates of return. In some cases that does not guarantee sub-additivity and, ridiculous as it may sound to modern portfolio managers - who all grew up with Markowitz' mean-variance analysis and the VaR of the Basel Accords - the Fama - Samuelson Proposition of 1965-66 demonstrates that it may sometimes be good risk management to put all your eggs in one basket. The following explanation is based on Chapter 12 of Los (2003).

2 Fama-Samuelson Proposition

If two distributions are stable with the same value of α_Z, their sum also is stable with the same stability exponent α_Z. This mathematical result has portfolio applications, which are, or at least should be, rather disturbing for global portfolio managers.

Proposition 1 *If the securities in a portfolio have rates of return $x(t)$ with the same stability exponent α_Z, then the portfolio itself has a rate of return $x(t)$ that is stable, with the same value of α_Z.*

Proof. Using Zolotarev's parametrization (*Cf.* Rachev and Mittnik, 2000), the logarithm of the characteristic function of the non-standardized stable distribution of the random variable $X \sim \mathbf{S}(\alpha, \beta, \gamma, \delta; 0)$ is:

$$\ln\left[E\left\{e^{j\omega x}\right\}\right] \begin{array}{l} = \left(-\gamma^{\alpha}|\omega|^{\alpha}\left[1 + j\beta \tan\frac{\pi\alpha}{2} sign(\omega)(\gamma|\omega|^{1-\alpha} - 1)\right] + j\delta\omega\right) \text{ if } \alpha \neq 1 \\ = \left(-\gamma|\omega|\left[1 + j\beta\frac{2}{\pi} sign(\omega)(\ln|\omega| + \ln\gamma)\right] + j\delta\omega\right) \text{ if } \alpha = 1 \end{array} \tag{1}$$

with the four parameters: (1) *stability exponent* $\alpha_Z \in (0, 2]$, (2) *skewness parameter* $\beta \in [-1, 1]$, (3) *scale parameter* $\gamma > 0$, and (4) *location parameter* $\delta \in \mathbb{R}$. For simplicity, we'll discuss the case of symmetric distributions when $\beta = 0$, so that

$$\ln\left[E\left\{e^{jx\omega}\right\}\right] = j\delta\omega - \gamma^{\alpha_Z}|\omega|^{\alpha_Z} \tag{2}$$

For the stable distributions of two rates of return $x_i(t), i = 1, 2$, the distribution of the weighted portfolio sum $x_p(t) = w_1 x_1(t) + w_2 x_2(t)$, with $w_1 + w_2 = 1$, has the characteristic function:

$$
\begin{aligned}
\ln E\left\{e^{jx_p\omega}\right\} &= \ln E\left\{e^{j(w_1x_1+w_2x_2)\omega}\right\} \\
&= \ln\left[E\left\{e^{jw_1x_1\omega}\right\}E\left\{e^{jw_2x_2\omega}\right\}\right], \ \textit{i.e.}, \text{ stable distributions} \\
&= \ln E\left\{e^{jx_1w_1\omega}\right\} + \ln E\left\{e^{jx_2w_2\omega}\right\} \\
&= [j\delta_1 w_1\omega \ - \gamma_1^{\alpha_Z}\left|w_1\omega\right|^{\alpha_Z}] + [j\delta_2 w_2\omega \ - \gamma_2^{\alpha_Z}\left|w_2\omega\right|^{\alpha_Z}] \\
&= j(w_1\delta_1 + w_2\delta_2)\omega \ - [w_1^{\alpha_Z}\gamma_1^{\alpha_Z} + w_2^{\alpha_Z}\gamma_2^{\alpha_Z}]\left|\omega\right|^{\alpha_Z} \\
&= j\delta_p\omega - \gamma_p^{\alpha_Z}\left|\omega\right|^{\alpha_Z}
\end{aligned} \tag{3}
$$

so that the location parameter, or mean, of the stable portfolio distribution

$$\delta_p = w_1\delta_1 + w_2\delta_2 \tag{4}$$

and its scale parameter

$$\gamma_p^{\alpha_Z} = w_1^{\alpha_Z}\gamma_1^{\alpha_Z} + w_2^{\alpha_Z}\gamma_2^{\alpha_Z} \tag{5}$$

It is easy to see that this bivariate return result generalizes, so that for stable distributions with the same stability parameter in general, for a portfolio with $i = 1, 2, ..., n$ assets, the portfolio location parameter or mean

$$\delta_p = \sum_{i=1}^{n} w_i\delta_i, \text{ where } \sum_{i=1}^{n} w_i = 1 \tag{6}$$

and the portfolio scale parameter

$$\gamma_p^{\alpha_Z} = \sum_{i=1}^{n} w_i^{\alpha_Z}\gamma_i^{\alpha_Z} \tag{7}$$

or

$$\gamma_p = \left(\sum_{i=1}^{n} w_i^{\alpha_Z}\gamma_i^{\alpha_Z}\right)^{\frac{1}{\alpha_Z}} \tag{8}$$

■

Fama (1965) and Samuelson (1967) used this proposition to adapt the portfolio theory of Markowitz (1952) for infinite or undefined variance distributions of rates of return on investments.

Their Proposition implies that the distribution of the portfolio returns is *self – affine* and scales with stability exponent α_Z as scaling exponent. In other words, the shape of the stable distribution of portfolio returns is the same as that of the underlying asset returns, no matter what the scale of portfolio variance. Only the value of the location parameter changes.

It is a peculiar fact of history that this Proposition of Fama and Samuelson has disappeared from the standard textbooks on investments and from portfolio analysis and management, although it has considerable empirical value! S&P500 stock index is often used as the market index in the Capital Asset Pricing Model (CAPM). But, as we now know, the S&P500 stock index has no finite limiting variance, since its $\alpha_Z = 1.67$, and this fact alone undermines most if not all of the stock and bond pricing results from the CAPM and from S&P500 option pricing.

Remark 2 *For Gaussian distributions, when $\alpha_Z = 2$, we have the familiar portfolio variance relationship from classical Markowitz mean - variance analysis, except that Markowitz' important diversifying correlation term is missing:*

$$\gamma_p^2 = w_1^2\gamma_1^2 + w_2^2\gamma_2^2 \tag{9}$$

For Gaussian distributions, the variance $\sigma_i^2 = 2\gamma_i^2$ (i.e., $\gamma_i^2 = \sigma_i^2/2$), so that for stable distributions with the same α_Z we have also:

$$\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 \tag{10}$$

How does the existence of stable non - Gaussian rates of return distributions affect portfolio diversification? For example, when we use uniform weights $w_i = \frac{1}{n}$,

$$\gamma_p^{\alpha_Z} = \left(\frac{1}{n}\right)^{\alpha_Z} \sum_{i=1}^{n} \gamma_i^{\alpha_Z} \tag{11}$$

we can discern three important cases:

(1) When $1 < \alpha_Z \leq 2$, the portfolio risk, as measured by the scaling parameter

$$\gamma_p = \frac{1}{n}\left(\sum_{i=1}^{n} \gamma_i^{\alpha_Z}\right)^{\frac{1}{\alpha_Z}} \tag{12}$$

decreases, as the number of assets in the portfolio, n, increases. In other words, there is a diversification effect: including more assets in the portfolio reduces the portfolio risk, despite the empirically established fact that there exists no finite limiting variance.

Remark 3 *Since most (but not all!) empirical stocks appear to have a stability exponent close to that of the S&P500's* $\alpha_Z \approx 1.67$*, diversification does reduce the non - market risk of an empirical stock investment portfolio. But this risk reduction through diversification has nothing to do with the covariances, as in Markowitz' (1952) original theory.*

(2) When $\alpha_Z = 1$

$$\gamma_p = \frac{1}{n} \sum_{i=1}^{n} \gamma_i \tag{13}$$

there is no diversification effect: adding more assets to the portfolio does not reduce the portfolio risk.

(3) When $0 < \alpha_Z < 1$, increasing the number of assets in the portfolio may actually increase the portfolio risk.[1] In this case, neither the means nor the variances of the rates of return of the assets in the portfolio exist. Neither their means nor their variances converge. In other words, when asset return rates behave like black noise, increasing the portfolio size only increases the portfolio risk!

Thus MPT-diversification to reduce non-market risk is still useful when the asset returns are non - Gaussian and they have stable distributions with the same stability $1 < \alpha_Z \leq 2$, despite the fact that these stable distributions have undefined variances. However, when $\alpha_Z = 1$, there is no diversification and when $0 < \alpha_Z < 1$, the portfolio risk can actually increase when more assets are included in the portfolio! It is very important for portfolio managers to compute the homogeneous Zolotarev alpha $\alpha_Z = \frac{1}{\alpha_L}$, to determine the degree of achievable diversification. Also, portfolio risk managers should compute the multifractal spectrum of heterogeneous of stock return stability

[1] This range of $\alpha_Z = \frac{1}{\alpha_L}$ cannot be measured by the Hurst exponent H, but can be measured by the Lipschitz α_L.

exponents $\alpha_{Zi} = \frac{1}{\alpha_{Li}}$, which lie outside the range of the usual measurement of the monofractal Hurst exponent H.

This does not necessarily mean that there does not exist a liquidity preference theorem. We can still reduce the risk in a portfolio by including more risk - free cash, even when the distributions are nonstationary but stable. In other words, it is dynamic *liquidity management* that ultimately determines the investment portfolio risk exposure of a fund manager (Bawa, Elton and Gruber, 1979). That dynamic liquidity management is very similar to the dynamic risk management of the extreme values of high risk dams!

3 Skewed - Stable Investment Opportunity Sets

The Fama - Samuelson Proposition is an example of Mandelbrot's invariance of scaling under weighted mixture. It shows why it is important to determine the stability parameters of the rates of return $x(t)$ for the assets in a portfolio and to see if they are the same. However, if the stability parameters are heterogenous, α_{Zi}, this simple generalization of Markowitz mean - variance analysis, or Modern Portfolio Theory (MPT) and its derivatives, does also no longer hold true. Or, as Peters (1994, p. 208) states:

> "....different stocks can have different Hurst exponents and different values of α_Z. Currently, there is no theory on combining distributions with different alphas. The EMH, assuming normality for all distributions, assumed $\alpha_Z = 2.0$ for all stocks, which we know [now] to be incorrect."

Huston McCulloch of Ohio State University has done some empirical work on what happens when the stability parameters α_{Z_i} for the rates of return of the assets in a portfolio are heterogeneous, *i.e.,* they are different from each other. In particular, he has produced interesting $3-$dimensional visualizations of the resulting Markowitz efficiency frontiers, which are no longer $2-$dimensional (McCulloch, 1986, 1996). In accordance with his findings, McCulloch also developed an alternative to the Black - Scholes option pricing formula, using stable distributions.

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